Variable length scale in a peridynamic body

Stewart Silling
Sandia National Laboratories

Pablo Seleson
University of Texas, Austin

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Outline

• Peridynamics background
  • States, horizon
• Rescaling a material model (at a point)
• Variable length scale (over a region)
• Partial stress
• Local-nonlocal coupling examples
Peridynamics basics: The nature of internal forces

**Standard theory**
Stress tensor field
(assumes contact forces and smooth deformation)

\[ \rho \ddot{u}(x, t) = \nabla \cdot \sigma(x, t) + b(x, t) \]
Differentiation of contact forces

**Peridynamics**
Bond forces within small neighborhoods
(allow discontinuity)

\[ \rho \ddot{u}(x, t) = \int_{H_x} f(q, x) dV_q + b(x, t) \]
Summation over bond forces

\[ H_x = \text{Family of } x \]

Horizon \( \delta \)
Peridynamics basics: Deformation state and force state

- The **deformation state** maps each bond to its deformed image.
  \[ Y[x](q - x) = y(q) - y(x) \]
- The **force state** maps bonds to bond forces according to the constitutive model.
  \[ f(q, x) = T[x](q - x) - T[q](x - q) \]
- The **constitutive model** maps deformation states to force states.
  \[ T[x] = \hat{T}(Y[x]) \]  \[ T[q] = \hat{T}(Y[q]) \]
Scaling of a material model at a point

- Let $\epsilon$ and $\delta$ be two horizons. Denote by $\xi_\epsilon$ and $\xi_\delta$ bonds within each family.
- Suppose we have a material model with horizon $\epsilon$. Find a rescaled model with $\delta$.
- Map the bonds (undeformed and deformed):

  \[
  \frac{\xi_\epsilon}{\epsilon} = \frac{\xi_\delta}{\delta}, \quad \frac{Y_\epsilon \langle \xi_\epsilon \rangle}{\epsilon} = \frac{Y_\delta \langle \xi_\delta \rangle}{\delta}
  \]

- Require

  \[
  W_\epsilon (Y_\epsilon) = W_\delta (Y_\delta)
  \]

- It follows from definition of Frechet derivative that the force state scales according to

  \[
  \epsilon^{d+1} T_\epsilon (Y_\epsilon) \langle \xi_\epsilon \rangle = \delta^{d+1} T_\delta (Y_\delta) \langle \xi_\delta \rangle
  \]
Rescaling works fine if the horizon is independent of position

- Example: uniform strain in a 1D homogeneous bar \( (d = 1, \ F = \text{constant}) \):
  \[
  y = Fx
  \]
- If we scale the material model as derived above:
  \[
  \epsilon^2 T_\epsilon (F) \langle \xi_\epsilon \rangle = \delta^2 T_\delta (F) \langle \xi_\delta \rangle
  \]
  we are assured that the strain energy density and Young’s modulus are independent of horizon.
- Also the peridynamic equilibrium equation is satisfied.
Variable horizon: the problem

- Same example: uniform strain in a 1D homogeneous bar
  \[ y = Fx \]
- Set \( \epsilon = 1 \), define \( Z(F) = T_1(F) \).
- Let the horizon be given by \( \delta(x) \). The scaled force state is
  \[ T[x](\xi) = \delta^{-2}(x)Z\left(\frac{\xi}{\delta(x)}\right) \]
- From the previous discussion, we know \( W \) is independent of \( x \).
- There’s just one problem: this deformation isn’t a minimizer of energy.
  - That is, the uniform strain deformation is not in equilibrium.
Origin of artifacts

- The peridynamic force density operator $L(x)$ involves the force state not only at $x$ but also the force states at all points within the horizon.

$$0 = L(x) + b,$$

$$L(x) = \int_{-\infty}^{\infty} \{ T_\delta(x)[x](q - x) - T_\delta(q)[q](x - q) \} dq$$

so simply scaling the material model at $x$ is not sufficient.
“Patch test” requirement for a coupling method

- In a deformation of the form
  \[ y(x) = a + Fx \]
  where \( H \) is a constant and the material model is of the form
  \[ T[x](\xi) = \delta^{-2}(x)Z(\xi/\delta(x)) \]
  where \( \delta(x) \) is a prescribed function and \( Z \) is a state that depends only on \( F \), we require
  \[ L(x) = 0 \quad \text{for all } x. \]
Peridynamic stress tensor

- Define the 1D peridynamic stress tensor field* by

\[ v(x) = \int_{0}^{\infty} \int_{0}^{\infty} \{T[x - y](y + w) - T[x + y](-y - w)\} \, dy \, dz \]

- Identity:

\[ \frac{dv}{dx} = \int_{-\infty}^{\infty} \{T[x](q - x) - T[q](x - q)\} \, dq \]

- \( v(x) \) is the force per unit area carried by all the bonds that cross \( x \).

Partial stress field

- Under our assumption that
  \[ T[x]\langle \xi \rangle = \delta^{-2}(x)Z\langle \xi / \delta(x) \rangle \]
  one computes directly that
  \[ \nu_0(x) := \int_{-\infty}^{\infty} \xi T[x]\langle \xi \rangle \, d\xi = \int_{-\infty}^{\infty} \xi Z\langle \xi \rangle \, d\xi \]
  which is independent of \( x \), so \( d\nu_0/dx = 0 \).

- \( \nu_0 \) is called the **partial stress** field.

- Clearly the internal force density field computed from
  \[ L_0(x) := d\nu_0/dx \]
  passes the “patch test.”

- This observation leads to the following idea...
Concept for coupling method

- Idea: within a coupling region in which $\delta$ is changing, compute the internal force density from

\[ L(x) = \frac{dv_0}{dx}(x), \quad v_0(x) = \int_{-\infty}^{\infty} \xi T[x](\xi) d\xi \]

instead of the full PD nonlocal integral.

- Here, $T[x](x)$ is determined from whatever the deformation happens to be near $x$.
  - $Z$ is no longer involved.
  - The material model has not changed from full PD, but the way of computing $L$ has.
Local-nonlocal coupling idea

- **Local region**
  \[ L(x) = \frac{dv_0}{dx} \]
  \[ v_0(x) = \sigma(F(x)) \]

- **Transition region**
  \[ L(x) = \frac{dv_0}{dx} \]
  \[ v_0(x) = \int \xi T[x] T[\xi] \, d\xi \]

- **Nonlocal region**
  \[ L(x) = \int \{T[x] T[\xi] - T[x + \xi] T[\xi] \} \, d\xi \]

**Good old-fashioned local stress**

**Partial stress (PS)**

**Full peridynamic (PD)**
Continuum patch test results

- Full PD shows artifacts, as expected.
- PS shows no artifacts, as promised.

\[ u = 0 \]  
\[ u = 0.02 \]
Continuum patch test with coupling

• No artifacts with PD-PS coupling (this was hoped for but not guaranteed).

\[ u = 0 \] \[ u = 0.02 \]
Pulse propagation test problem

- Does our coupling method work for dynamics as well as statics with variable horizon?

\[ \delta = 1 \text{ (nonlocal)} \]

\[ \delta = 0.01 \text{ (in effect local)} \]
Pulse propagation test results

- Movies of strain field evolution

Full PD everywhere

Coupled PD-PS
Pulse propagation test results

- Strain field: no artifacts appear in the coupled model the local-nonlocal transition.
Discussion

• The partial stress approach may provide a means for local-nonlocal coupling within the continuum equations.
  • Uses the underlying peridynamic material model but modifies the way internal force density is computed.
  • Expected to work in 2D & 3D, linear & nonlinear.
• PS is inconsistent from an energy minimization point of view.
  • Not suitable for a full-blown theory of mechanics and thermodynamics (as full PD is).
  • Not yet clear what implications this may have in practice.
  • We still need to use full PD for crack progression.
Extra slides
Purpose of peridynamics

- To unify the mechanics of continuous and discontinuous media within a single, consistent set of equations.

![Continuous body](image1.png)
Continuous body
![Continuous body with a defect](image2.png)
Continuous body with a defect
![Discrete particles](image3.png)
Discrete particles

- Why do this? Develop a mathematical framework that help in modeling...
  - Discrete-to-continuum coupling
  - Cracking, including complex fracture patterns
  - Communication across length scales.

![Figure 11.20](image4.png) Pull-out: (a) schematic diagram; (b) fracture surface of SiC/nm glass-ceramic reinforced with SiC fibres. (Courtesy H. S. Kim, P. S. Rogers and R. D. Rawlings.)
## Peridynamic vs. local equations

**State notation:** \( \text{State} \langle \text{bond} \rangle = \text{vector} \)

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<th>Peridynamic theory</th>
<th>Standard theory</th>
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<tr>
<td>Kinematics</td>
<td>( \underline{Y} \langle q - x \rangle = y(q) - y(x) )</td>
<td>( F(x) = \frac{\partial y}{\partial x}(x) )</td>
</tr>
<tr>
<td>Linear momentum balance</td>
<td>( \rho \ddot{y}(x) = \int_{\mathcal{H}} \left( t(q, x) - t(x, q) \right) dV_q + b(x) )</td>
<td>( \rho \ddot{y}(x) = \nabla \cdot \sigma(x) + b(x) )</td>
</tr>
<tr>
<td>Constitutive model</td>
<td>( t(q, x) = \underline{T} \langle q - x \rangle ), ( \underline{T} = \hat{T}(\underline{Y}) )</td>
<td>( \sigma = \hat{\sigma}(F) )</td>
</tr>
<tr>
<td>Angular momentum balance</td>
<td>( \int_{\mathcal{H}} \underline{Y} \langle q - x \rangle \times \underline{T} \langle q - x \rangle dV_q = 0 )</td>
<td>( \sigma = \sigma^T )</td>
</tr>
<tr>
<td>Elasticity</td>
<td>( \underline{T} = W_{\underline{Y}} ) (Fréchet derivative)</td>
<td>( \sigma = W_F ) (tensor gradient)</td>
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<tr>
<td>First law</td>
<td>( \dot{\varepsilon} = \underline{T} \cdot \dot{\underline{Y}} + q + r )</td>
<td>( \dot{\varepsilon} = \sigma \cdot \dot{F} + q + r )</td>
</tr>
</tbody>
</table>

\[ \underline{T} \cdot \dot{\underline{Y}} := \int_{\mathcal{H}} \underline{T} \langle \xi \rangle \cdot \dot{\underline{Y}} \langle \xi \rangle dV_{\xi} \]