Peridynamic model for fatigue cracks

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Outline

• Peridynamics background
• Cyclic bond strain
• Fatigue model
  • Nucleation phase
  • Growth phase
  • Heterogeneity

Related work:
• Oterkus, Guven, & Madenci, IEEE (2010)
Purpose of peridynamics*

- To unify the mechanics of continuous and discontinuous media within a single, consistent set of equations.

- Why do this?
  - Avoid coupling dissimilar mathematical systems (A to C).
  - Model complex fracture patterns.
  - Communicate across length scales.

* Peri (near) + dyn (force)
Peridynamics basics: Horizon and family

- Any point $x$ interacts directly with other points within a distance $\delta$ called the “horizon.”
- The material within a distance $\delta$ of $x$ is called the “family” of $x$, $\mathcal{H}_x$.

$\delta$ = horizon
$\mathcal{H}_x$ = family of $x$

General references
- Madenci & Oterkus, Peridynamic Theory & Its Applications (2014)
Peridynamic nonlocality: Strain energy at a point

Key assumption: the strain energy density at $W(x)$ is determined by the deformation of its family.
Potential energy minimization yields the peridynamic equilibrium equation

- Potential energy:
  \[ \Phi = \int_{\mathcal{B}} (W - b \cdot y) \, dV_x \]
  where \( W \) is the strain energy density, \( y \) is the deformation map, \( b \) is the applied external force density, and \( \mathcal{B} \) is the body.

- Euler-Lagrange equation is the equilibrium equation:
  \[ \int_{\mathcal{H}_x} f(q, x) \, dV_q + b(x) = 0 \]
  for all \( x \). \( f \) is the pairwise bond force density.
Material model determines bond forces

- Bond forces depend on the deformation of the families of both $\mathbf{x}$ and $\mathbf{q}$, together with the material models at these points.

\[ f(\mathbf{x}, \mathbf{q}, t) = T[\mathbf{x}](\mathbf{q} - \mathbf{x}) - T[\mathbf{q}](\mathbf{x} - \mathbf{q}) \]

In state notation: $f(\mathbf{q}, \mathbf{x})$ from mat model at $\mathbf{x}$ and $f(\mathbf{x}, \mathbf{q}, t)$ from mat model at $\mathbf{q}$.
The nature of internal forces

**Standard theory**
Stress tensor field
(assumes continuity of forces)

\[\sigma_{11}, \sigma_{22}, \sigma_{12}\]

\[\rho \ddot{u}(x, t) = \nabla \cdot \sigma(x, t) + b(x, t)\]

Differentiation of surface forces

**Peridynamics**
Bond forces between neighboring points
(allowing discontinuity)

\[f(q, x)\]

\[\rho \ddot{u}(x, t) = \int_{H_x} f(q, x) dV_q + b(x, t)\]

Summation over bond forces

Stress tensor maps surface normal vectors onto surface forces

Force state maps bonds onto bond forces
Peridynamic vs. local equations

- The structures of the theories are similar, but peridynamics uses nonlocal operators.

<table>
<thead>
<tr>
<th>Relation</th>
<th>Peridynamic theory</th>
<th>Standard theory</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kinematics</td>
<td>$\mathbf{Y}(q-x) = y(q) - y(x)$</td>
<td>$\mathbf{F}(x) = \frac{\partial y}{\partial x}(x)$</td>
</tr>
<tr>
<td>Linear momentum balance</td>
<td>$\rho \ddot{y}(x) = \int_{\mathcal{H}} \left( t(q, x) - t(x, q) \right) dV_q + b(x)$</td>
<td>$\rho \ddot{y}(x) = \nabla \cdot \mathbf{\sigma}(x) + b(x)$</td>
</tr>
<tr>
<td>Constitutive model</td>
<td>$t(q, x) = \mathbf{T}(q-x), \quad \mathbf{T} = \hat{\mathbf{T}}(\mathbf{Y})$</td>
<td>$\mathbf{\sigma} = \hat{\mathbf{\sigma}}(\mathbf{F})$</td>
</tr>
<tr>
<td>Angular momentum balance</td>
<td>$\int_{\mathcal{H}} \mathbf{Y}(q-x) \times \mathbf{T}(q-x) dV_q = 0$</td>
<td>$\mathbf{\sigma} = \mathbf{\sigma}^T$</td>
</tr>
<tr>
<td>Elasticity</td>
<td>$\mathbf{T} = W_{\mathbf{Y}}$ (Fréchet derivative)</td>
<td>$\mathbf{\sigma} = W_{\mathbf{F}}$ (tensor gradient)</td>
</tr>
<tr>
<td>First law</td>
<td>$\dot{\mathbf{\varepsilon}} = \mathbf{T} \cdot \dot{\mathbf{Y}} + q + r$</td>
<td>$\dot{\mathbf{\varepsilon}} = \mathbf{\sigma} \cdot \dot{\mathbf{F}} + q + r$</td>
</tr>
</tbody>
</table>

$\mathbf{T} \cdot \dot{\mathbf{Y}} := \int_{\mathcal{H}} \mathbf{T}(\xi) \cdot \dot{\mathbf{Y}}(\xi) dV_\xi$
EMU numerical method

- Integral is replaced by a finite sum: resulting method is meshless and Lagrangian.

\[ \rho \ddot{y}(x, t) = \int_{\mathcal{H}} f(q, x, t) \, dV_q + b(x, t) \]

\[ \rho \ddot{y}_i^n = \sum_{k \in \mathcal{H}} f(x_k, x_i, t) \, \Delta V_k + b_i^n \]

- Looks a lot like MD.

Discretized model in the reference configuration
Bond based material models

- If each bond response is independent of the others, the resulting material model is called bond-based.
- The material model is then simply a graph of bond force density vs. bond strain.
- Damage can be modeled through bond breakage.
- Bond response is calibrated to:
  - Bulk elastic properties.
  - Critical energy release rate.
Damage due to bond breakage

- Recall: each bond carries a force.
- Damage is implemented at the bond level.
  - Bonds break irreversibly according to some criterion.
  - Broken bonds carry no force.
- Examples of criteria:
  - Critical bond strain (brittle).
  - Hashin failure criterion (composites).
  - Gurson (ductile metals).
Autonomous crack growth

- When a bond breaks, its load is shifted to its neighbors, leading to progressive failure.
Critical bond strain: Relation to critical energy release rate

If the work required to break the bond $\xi$ is $w_0(\xi)$, then the energy release rate is found by summing this work per unit crack area (J. Foster):

$$G = \int_0^\delta \int_{R_+} w_0(\xi) \, dV_\xi \, ds$$

- Can then get the critical strain for bond breakage $s^*$ in terms of $G$.
- Could also use the peridynamic J-integral as a bond breakage criterion.
Constant bond failure strain reproduces the Griffith crack growth criterion

- This confirms that the energy consumed per unit crack growth area equals the expected value from bond breakage properties.

From bond properties, energy release rate should be:

\[ G = 0.013 \]

Slope = 0.013

Total work – total strain energy

Crack tip position

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Treating discontinuities on an equal basis allows some subtleties in fracture to appear.

- Wavy crack path in a sheet (VIDEO)
- Complex crack paths in reinforced concrete
- Peeling and tearing of an adhesive membrane
Cyclic strain in a bond

- For a given bond \( \xi \), the *bond elongation* is the change in bond length:
  \[
e = |Y(\xi)| - |\xi| = |y(x + \xi) - y(x)|.
  \]

- The *bond strain* is the change in length over initial length:
  \[
s = \frac{e}{|\xi|}.
  \]

- Let \( s^+ \) and \( s^- \) be the two extremes under cyclic loading of \( \xi \).

- The *cyclic bond strain* is defined by
  \[
  \varepsilon = |s^+ - s^-|.
  \]
Structure of a crack tip field

- Let \( \varepsilon_{\text{core}}(\delta) \) be the largest cyclic strain in any bond.
- Can show by a dimensional argument \( \exists \hat{\varepsilon}_{\text{core}} \) such that

\[
\varepsilon_{\text{core}}(\delta) = \hat{\varepsilon}_{\text{core}} \frac{\Delta K}{E \sqrt{\delta}}
\]

where \( \Delta K = \) cyclic stress intensity factor and \( E = \) modulus.
Remaining life of a bond

- Each bond in the body has a remaining life $\lambda(N)$ where $N$ is the cycle number.

- The remaining life is monotonically decreasing over time.

$$\lambda(0) = 1, \quad \dot{\lambda} \leq 0.$$ 

- The bond fails at the first cycle $N$ when

$$\lambda(N) \leq 0.$$
Fatigue model

- The fatigue model specifies how the remaining life of each bond depends on the loading.
  \[
  \frac{d\lambda}{dN}(N) = -A\varepsilon^m
  \]
  where \(A\) and \(m\) are constants and \(\varepsilon\) is the cyclic bond strain.

- The constants are calibrated separately for phases I and II (nucleation and growth).
Phase I calibration from $S$-$N$ data

- Run many cyclic loading tests at different values of $\varepsilon$ (constant for each test).

- For each test, compute when damage starts:

$$\frac{d\lambda}{dN}(N) = -A\varepsilon^m \quad \implies \quad N = \frac{1}{A\varepsilon^m}.$$ 

- Compare this to data on an $\varepsilon$-$N$ plot, fit $A$ and $m$. 

![Diagram of a phase I calibration process with a log-log plot showing the relationship between strain ($\varepsilon$) and cycles to failure ($N$). The experimental data is shown as a straight line with a slope of $-1/m_1$. The diagram also illustrates the concept of phase I calibration by comparing the experimental data to the theoretical model.]
Fatigue nucleation in aluminum alloy

- Model with a fatigue limit:

\[
\frac{d\lambda}{dN}(N) = -A \left( \max(0, \varepsilon - \varepsilon_\infty) \right)^m
\]

Growth: Bonds interact with the strain field near an approaching crack.
Relate crack growth to remaining life

- Evolution of remaining life:

\[
\lambda(\delta) - \lambda(0) = \int_0^\delta \frac{d\lambda}{dz} dz = \int_0^\delta \frac{d\lambda}{dN} \frac{dN}{dz} dz.
\]

- Recall

\[
\frac{d\lambda}{dN} = -A\varepsilon^m.
\]

- Denote by \(da/dN\) the crack growth rate.

\[
1 - 0 = \frac{A}{da/dN} \int_0^\delta \varepsilon^m(z) dz
\]

- Cyclic strain ahead of a crack:

\[
\varepsilon(z) = \varepsilon_{\text{core}} f \left( \frac{z}{\delta} \right) = \frac{\Delta K}{E\sqrt{\delta}} f \left( \frac{z}{\delta} \right).
\]

- Thus, for some \(c\),

\[
\frac{da}{dN} = cA\Delta K^m
\]
Phase II calibration from Paris Law data

- Now have
  \[ \frac{da}{dN} = cA\Delta K^m \]
  where \( c \) and \( m \) are as yet unknown.

- Assume the Paris Law holds:
  \[ \frac{da}{dN} = C\Delta K^M \]
  where \( C \) and \( M \) are constants that can be found from test data.

- Conclude
  \[ m = M. \]

- Need to do one computational simulation with an assumed value \( C = 1 \) to evaluate \( A \).
Summary so far

- Each bond has a remaining life $\lambda(N)$:

  \[ \lambda(0) = 1, \quad \frac{d\lambda}{dN}(N) = -A\varepsilon^m, \quad \lambda \leq 0 \text{ means failure.} \]

- In Phase I, use $A$ and $m$ from $S-N$ data.

- In Phase II, use a different calibration from Paris law data.
Time mapping permits very large $N$

- We can avoid modeling each cycle explicitly.
- Define the *loading ratio* by
  
  $$R = \frac{s^-}{s^+} \implies \varepsilon = |s^+ - s^-| = |(1 - R)s^+|.$$  

- Map $t$ to $N$:

  $$N = e^{t/\tau}$$
  
  where $\tau$ is a constant chosen according to convenience.

- Fatigue model in terms of $t$ instead of $N$:

  $$\frac{d\lambda}{dt} = \frac{d\lambda}{dN} \frac{dN}{dt} = -|1 - R| \frac{AN}{\tau} |s^+|^m.$$
Fatigue crack growth in aluminum

Spiral crack in a rod under torsion

Front view

Rear view

Initial cavity
Mesoscale: Fatigue cracks at grain boundaries

- Recall the peridynamic fatigue model: For a given bond,
  \[ \lambda(0) = 1, \quad \frac{d\lambda}{dN} = -A\varepsilon^m \]

- Set:
  \[ A = 5 \quad \text{for bonds within a grain} \]
  \[ A = 50 \quad \text{for bonds between grains} \]

Fatigue crack growth between grains represented as Voronoi cells
Corrosion fatigue: Diffusion coupled with damage

- Let $z$ = concentration of a contaminant.
- Bond life loss rate depends on concentration:
  \[ \lambda(0) = 1, \quad \frac{d\lambda}{dN} = -A(z)\varepsilon^m \]
- Peridynamic diffusion model:
  \[ \dot{z}(x, t) = \int C(q, x)(z(q, t) - z(x, t))dV_q \]
  where $C(q, x)$ is much larger for damaged bonds than undamaged.

Example: nucleation occurs at $N = 2100$ with corrosion
$N = 12,000$ without corrosion
Corrosion fatigue:
Diffusion coupled with damage
Discussion

• Method retains the main advantages of peridynamics.
  • Autonomous crack growth
  • Includes both nucleation and growth phases
  • Permits interaction between multiple cracks
  • Arbitrary crack path in 3D.
• A simple enhancement allows a spectrum of loading frequencies
  • This is a peridynamic version of Miner’s rule.
Extra slides
Peridynamics basics: States

- A *peridynamic state* is a mapping on bonds in a family.
- We write:
  \[ u = A \langle \xi \rangle \]
  where \( \xi \) is a bond, \( A \) is a state, and \( u \) is some vector.
- States play a role in peridynamics similar to that of second order tensors in the local theory.
Peridynamics basics: Kinematics

- The *deformation state* is the function that maps each bond $\xi$ into its deformed image:

$$\overline{Y}(\xi) = y(q) - y(x)$$

where $y$ is the deformation and

$$\xi = q - x.$$
Peridynamics basics: Bonds and bond force density

- The vector from $x$ to any point $q$ in its family in the reference configuration is called a bond.

$$\xi = q - x$$

- Each bond has a pairwise force density vector that is applied at both points:

$$f(q, x, t).$$

- Equation of motion is an integro-differential equation, not a PDE:

$$\rho(x)\ddot{y}(x, t) = \int_{\mathcal{H}_x} f(q, x, t) \, dV_q + b(x, t).$$
Peridynamics basics: Elastic materials

- A peridynamic elastic material has strain energy density given by
  \[ W(Y). \]

- The force state is given by
  \[ \hat{T}(Y) = W_Y(Y) \]
  where \( W_Y \) is the Frechet derivative of the strain energy density.
Some results about peridynamics

• For any choice of horizon, we can fit material model parameters to match the bulk properties and energy release rate.
  • Using nonlocality, can obtain material model parameters from wave dispersion curves (Weckner).
• Coupled coarse scale and fine scale evolution equations can be derived for composites (Lipton and Alali).
• A set of discrete particles interacting through any multibody potential can be represented exactly as a peridynamic body.
• Well posedness has been established under certain conditions (Mangesha, Du, Gunzburger, Lehoucq).