Sampling Conditions for Clipping-free Voronoi Meshing by the VoroCrust Algorithm

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Introduction

Voronoi Meshing

Given a bounded open set $\mathcal{O}$ in Euclidean space, decompose its interior into Voronoi cells of bounded aspect-ratio. The cells should naturally \textit{conform} to the bounding surface $\mathcal{M} = \partial \mathcal{O}$.
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- The VoroCrust project developed the first solution to this problem.
- In this talk, we focus on the subproblem of surface reconstruction, assuming a set of sample points from $\mathcal{M}$ is given as input.
Surface Reconstruction for Voronoi Meshing

Given a set of sample points from a closed 2-manifold $\mathcal{M}$ in Euclidean space, decompose its interior into Voronoi cells of bounded aspect-ratio. The cells should naturally conform to a surface mesh approximating $\mathcal{M}$. Typically, cells near the boundary are clipped, introducing defects. Power crust [Amenta et al.] produces such a decomposition. However, generators (poles) are restricted to lie near the medial axis of $\mathcal{M}$. Instead of the Voronoi diagram, it is based on the power diagram. Nonetheless, rich theory with strong approximation guarantees.
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VoroCrust Intuition - A 2D Example

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(4) Keep the separating facets
Power distance and cell

For a ball $b$ centered at $c$ with radius $r$, $\pi(b, x) = \|cx\|^2 - r^2$.

$V_b = \{ x \in \mathbb{R}^d \mid \pi(b, x) \leq \pi(b', x) \forall b' \in B \}$. 

The Union of Balls and Its Dual Shape [Edelsbrunner]
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Weighted $\alpha$-complex and $\alpha$-shape

Define $\mathcal{K} = \text{Nerve}(\{V_b \cap b \mid b \in B\})$ and $S$ as the underlying space $|\mathcal{K}|$. 
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Figures from [Edelsbrunner]
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Homotopy-equivalence

The nerve theorem implies $S = |\mathcal{K}|$ has the same homotopy-type as $\bigcup B$. 

Figures from [Edelsbrunner]
Local features size (lfs)

The local feature size at a point \( x \in M \) is its distance to the medial axis of \( M \).

Medial Axis
Figure from [Wolter]
**ε-Sampling and Topological Thickening**

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**ε-sample**

A set of points \( P \) on \( M \) such that \( \forall x \in M \exists p \in P \) s.t. \( \|px\| \leq \epsilon \cdot lfs(x) \).
**ε-Sampling and Topological Thickening**

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### From balls to surfaces [Chazal, Lieutier]

Let \( P \) be an \( \varepsilon \)-sample of \( \mathcal{M} \), with \( \varepsilon < 1/160 \), and define \( b_p \) as the ball centered at \( p \in P \) with radius \( \alpha_p \cdot \text{lfs}(p) \), where \( 1/20 < \alpha_p < 1/10 \). Then, \( \mathcal{M} \) is a deformation retraction of \( \bigcup b_p \).
Beyond Homeomorphism: Isotopic Equivalence

Torus: unknot vs. knot
Figures from [Wikipedia]
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A purely topological condition [Chazal, Cohen-Steiner]

Suppose that:
- $\mathcal{M}'$ is homeomorphic to $\mathcal{M}$,
- $\mathcal{M}'$ is included in a topological thickening $\overline{\mathcal{M}}$ of $\mathcal{M}$,
- $\mathcal{M}'$ separates the sides of $\overline{\mathcal{M}}$.

Then, $\mathcal{M}'$ is isotopic to $\mathcal{M}$ in $\overline{\mathcal{M}}$. 
Start with an \( \varepsilon \)-sample \( P \subset M \) with weights \( (\delta \geq \varepsilon) \) defining the associated \( \delta \)-Ils balls \( B \).
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Collect the corner points $\mathcal{G}$ of $\bigcup B$ as a crude set of witnesses of the simplices in $\mathcal{K}$.
VoroCrust - The Abstract Algorithm

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**Sandwich theorem**

$$VC \subseteq S \subseteq \cup B.$$
VoroCrust - Ball Intersections

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Disk Caps

Each sample ball contributes exactly two caps, i.e., topological-disks, to the boundary of the union.
**VoroCrust - Sampling Conditions for Disk Caps**

**Disk Caps**
Each sample ball contributes exactly two caps, i.e., topological-disks, to the boundary of the union.

**Sampling Conditions**
For constants $\epsilon \leq \delta$, we require an $\epsilon$-sampling $P$, with associated balls of radii $r_i = \delta \cdot lfs(p_i)$ satisfying the following sparsity condition:

$$lfs(q) \geq lfs(p) \implies \|p - q\| \geq \epsilon \cdot lfs(p).$$

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\[
lfs(q) \geq lfs(p) \quad \Rightarrow \quad \|p - q\| \geq \epsilon \cdot lfs(p).
\]

Lemma
Taking \( \epsilon = 1/160 \) and \( \delta = 1/20 \), we get disk caps.
Announcements

- The VoroCrust software package is scheduled for release soon.
  - Successful implementation of more ideas than what this talk covers, e.g., sharp features, medial axis approximation, sizing estimation.
- Mohamed S. Ebeida (msebeid@sandia.gov) is the VoroCrust point-of-contact at Sandia National Labs (SNL).