An Overview of the State-of-the-Art in Computational Modeling of Ice Sheets

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Motivation

- Global mean sea-level is rising at the rate of $3.2 \text{ mm/year}$ and the rate is increasing!
- Latest studies suggest possible increase in sea-level of $0.3-2.5\text{m}$ by 2100.
- Greenland and Antarctic ice sheets store most of the fresh water on Earth.

Map of North America showing 6 m sea-level rise (NASA)

Total mass loss of ice sheets between 1992-2011 (Sheperd et al. 2012)

Modeling of ice sheet (Greenland and Antarctica) dynamics is essential for providing estimates of sea-level rise, towards understanding the global and local effects of climate change.
Ice Behavior/Properties

- Ice behaves like a very viscous shear-thinning non-Newtonian fluid (similar to lava flow)
  - Source: snow packing/water freezing.
  - Sink: ice melting/calving in ocean.

- Ice sheet flows are assumed to be quasi-static relative to temperature evolution
  - Neglect inertial terms \( \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) \) in the incompressible Navier-Stokes equations.
  - Steady momentum balance equations are coupled to time-dependent equations for temperature (energy) and ice thickness (mass).
  - Highly nonlinear rheology.
## A Hierarchy of Ice Sheet Models (ISMs)

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<th>Comments</th>
<th>Validity</th>
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<td>continental scale</td>
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- Stokes flow model is “**gold standard**” but expensive.
- **Simplified models** are derived from full Stokes model and take advantage of the fact that ice sheets are thin: \(\delta \ll 1\).

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A Hierarchy of Ice Sheet Models (ISMs)

- There are also **hybrid models**: couple different models in different regions (statically or dynamically) to expand validity
  
  - SIA + SSA (HySSA): Kirchner *et al.*, 2011; Pollard *et al.* 2012; ...
  
  - Stokes + SSA (van Dongen *et al.*, 2017), ISCAL\(^1\) = Stokes + SIA (Ahlkrona *et al.* 2016)

\(^1\)Ice Sheet Coupled Approximation Levels: can be up to 9× faster than Stokes model with < 5% relative error.
Ice Sheet Dynamical Cores

- **MPAS-Albany Land Ice (MALI):** SNL
  - FO Stokes Model + Finite Elements

- **Berkeley Ice Sheet Initiative for Climate at Extreme Scales (BISICLES):** LBL
  - L1L2 Model + Finite Volume + AMR (Chombo)

- **Community Ice Sheet Model (CISM):** NCAR
  - FO Stokes, L1L2, SSA + Finite Differences or Finite Elements

- **Ice Sheet System Model (ISSM):** JPL
  - Stokes, FO Stokes, SSA + Finite Elements

- **Parallel Ice Sheet Model (PISM):** U Alaska + PIK
  - FO Stokes, hybrid SIA+SSA, SIA, SSA + Finite Differences or Finite Elements

- **Elmer/ICE:** CSC-IT Center for Science Ltd. (Finland) *et al.*
  - Stokes, ISCAL, SIA, SSA + Finite Elements

...and others!
An ESM has **six modular components**:  
1. Atmosphere model  
2. Ocean model  
3. Sea ice model  
4. Land ice model  
5. Land model  
6. Flux coupler  

**Goal of ESM**: to provide actionable scientific predictions of 21st century sea-level change due to global climate change.
ISM Computation Cost in ESM

High-res climate model processor layout

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<th>component</th>
<th>horizontal</th>
<th>vertical</th>
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<tr>
<td>25km</td>
<td>ATM/LND</td>
<td>0.8M</td>
<td>72</td>
</tr>
<tr>
<td>18-6km</td>
<td>OCN/ICE</td>
<td>3.7M</td>
<td>80</td>
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<tr>
<td>2-20km</td>
<td>AIS ISM</td>
<td>1.6M</td>
<td>10</td>
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➢ **ISM throughput:** 1 SYPD (simulated year per wallclock day)

➢ **ISM cost:** 4M core-hours per simulated year
Discretizations & Meshes

• “Legacy” ISMs employed finite difference methods on uniform grids
  ➢ Led to overkill resolution and robustness/performance issues (e.g. CISM)!
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• Most modern ISMs are based on **finite element** or **finite volume methods** with **variable resolution/adaptively refined** meshes.
  ➢ AIS 1.5km-40km variable resolution mesh can have $O(1M)$-$O(10M)$ elements (3D).

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**Below:** AMR grid from BISICLES FV dycore showing refinement around grounding line

**Right:** zoom-in of Greenland mesh from MALI refined based on gradient of surface velocity
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  - AIS 1.5km-40km variable resolution mesh can have $O(1M)-O(10M)$ elements (3D).
- Some recent work on **mesh-free methods** (e.g. RBF, RBF-PUM) for land-ice modeling
  - **Upshots:** does not require remeshing of entire domain, avoids repeated matrix assembly, high convergence rate (Ahlkrona *et al.* 2017; Cheng & Shcherbakov, 2018).

Below: AMR grid from BISICLES FV dycore showing refinement around grounding line

Right: zoom-in of Greenland mesh from MALI refined based on gradient of surface velocity
Discretizations & Meshes

- 3D models (Stokes, FO Stokes) typically employ meshes which are **structured (layered)** in the vertical dimension.

MALI uses dual of hexagonal mesh extruded to tetrahedra.

Variable resolution triangular mesh extruded to a (thin) tetrahedral mesh.
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- Finite element discretizations of the full Stokes equations require **inf-sup stable** finite elements or stabilization\(^1,2\).

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\(^1\) Gunzburger *et al.* 2012.  \(^2\) Ahlkrona *et al.* 2018.
Nonlinear Solvers

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Nonlinear Solvers

- **Picard iterations** have been method of choice in ice sheet modeling
- Recently, various models have switched to *Newton method*.
  - With **automatic differentiation (AD)** libraries, deriving/hand-coding Jacobian is not necessary.
  - Special start-up is required to deal with “singular” viscosity (e.g., Picard followed by Newton, **homotopy continuation**\(^1\) w.r.t. regularization parameter).

*Robustness of MALI FO Stokes nonlinear solver is improved via homotopy continuation.*

\[\gamma = 10^{-1.0}, 10^{-2.5}, 10^{-6.0}, 10^{-10}\]

Glen’s Law Viscosity:

\[
\mu = \frac{1}{2} A(T)^{-\frac{1}{3}} \left( \frac{1}{2} \sum_{ij} \dot{\varepsilon}_{ij}^2 + \gamma \right)^{(-\frac{2}{3})}
\]

\[
\gamma = \text{regularization parameter}
\]

\[
\dot{\varepsilon}_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)
\]

\(^1\)Tezaur et al. 2015.
Linear Solvers

• Krylov iterative linear solvers are employed – CG or GMRES.
  ➢ FO Stokes equations are symmetric.
Linear Solvers

- **Krylov iterative linear solvers** are employed – CG or GMRES.
  - FO Stokes equations are **symmetric**.

- **Bad aspect ratios, floating ice, and island/ice hinges** can **wreak havoc** on linear solver!
  - Specialized **algebraic multi-grid (AMG)**\(^1\) and **geometric multi-grid (GMG)**\(^2,3\) solvers have been developed to deal with these issues and are available in Trilinos/PETSc.
  - **Graph-based algorithms** for removing islands/ice hinges are being developed\(^1\).

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- Specialized **block solvers** required for full Stokes equations and/or coupled momentum-energy formulations.

Ice sheet models contain **unknown parameters**, e.g. basal friction $\beta$ and ice thickness $H$ at equilibrium.
Parameter Estimation/Initialization

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- **Idea:** estimate unknown parameters using **available data/measurements**, e.g.
  - Ice extent and surface topography
  - Surface velocity
  - Surface mass balance (accumulation – runoff)
  - Sparse measurements of thickness

**Sources of data:** satellite infrarommetry, radar, altimetry, etc.
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- **Traditional approach:** “spin-up” of ice-flow dynamics in a way that matches present-day observations – initialize model with (imperfect/unknown) present state and integrate forward until states consistent with observations are reached.
  - Can require a lot of CPU time (“spin-up time”): long timescale adjustments to past BC forcing requires a model “spin-up” of order $10^4$-$10^5$ years\(^1\).
  - “Spun-up” initial conditions can result in “shocks”, which initiate large transients that can derail dynamic ice simulations\(^1\).

\(^1\)Perego et al. 2014.
Parameter Estimation/Initialization

- **Modern approaches:** formulate/solve PDE-constrained optimization problem for unknown parameters/ice initial state, e.g.

```plaintext
minimize_{\text{ice params}} F(u, H) \\
\text{s.t. ice PDEs}
```

**Ice params:** basal friction, ice thickness, thermal diffusivities, ice viscosity, ...

\[
F(u, h) = \int_{\Gamma} \frac{1}{\sigma_u^2} |u - u^{obs}|^2 ds + \int_{\Gamma} \frac{1}{\sigma_H^2} |H - H^{obs}|^2 ds
\]

- **Lots of recent work** in this area for various ISMs: Michel *et al.* 2014\(^1\); Goldberg and Heimbach, 2013\(^1\); Pralong and Gudmundsson, 2011; Morlinghem *et al.* 2010, 2013; Gillet-Chaulet *et al.* 2012; Brinherhoff and Johnson, 2013; Perego *et al.* 2014; Petra *et al.* 2012; S. Marchenko *et al.* 2019; etc...

- Optimization is often performed using **adjoints**, used to efficiently compute gradients of mismatch function and sensitivities (e.g., Cheng & Lotstedt, 2019\(^1\)).

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1 In all but these references, a steady optimization (vs. transient) optimization was considered.
Uncertainty Quantification

**Goal:** uncertainty bounds for expected **sea level rise (SLR)** during 21st century.

- **Numerous sources of uncertainty** affect SLR:
  - Climate forcings (e.g., surface mass balance)
  - Basal friction
  - Bedrock topography (noisy/sparse data)
  - Geothermal heat flux
  - Modeling errors
  - Model parameters (e.g., Glen’s flow law exponent)
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- **Traditional approach:** assess impact of parametric uncertainty using *large ensemble analysis*:
  - Model is run for different values of the parameters and the uncertainty in the projections is estimated from the **spread** in the model runs (Golledge *et al.* 2015; DeConto and Pollard, 2016; Aschwanden *et al.* 2019).

*Bottom figure from Aschwanden *et al.* 2019: time series of various anomalies due to mass changes of the GIS.*
Uncertainty Quantification

• Recent UQ work has adopted **probabilistic approaches**:
  
  ➢ **Sampling methods (e.g. MCMC)**: explore input parameter space according to statistical distributions specified for each input (Heimbach and Bugnion, 2009) → very costly, as many samples are typically required!
  
  ➢ **Bayesian inference**: stochastic Newton method + log-normal assumption (Petra et al. 2013).
  
  ➢ **Stochastic sensitivity analysis**: understand which input parameter uncertainty has largest influence on SLR uncertainty (Larour et al. 2012; Schlegel et al. 2018; Bulthuis et al. 2018; Robel et al. 2019; Cheng & Shcherbakov, 2018, Cheng & Lotstedt, 2019).

There are **numerous challenges** in UQ for ice/climate: high-dimensional parameter fields, large computational cost, unknown prior uncertainties, etc.
Ice Sheet Evolution

### Momentum Balance

\[-\nabla \cdot (2\mu(u,T)\d(e(u))) = -\rho g \nabla_{x,y}(H + b), \text{ in } \Omega_H\]

### Conservation of Mass

\[\frac{\partial H}{\partial t} + \nabla \cdot (\bar{u}H) = \dot{b}\]

### Energy Balance

\[\rho c \frac{\partial T}{\partial t} + \rho c \bar{u} \cdot \nabla T = \nabla \cdot (k \nabla T) + 2\mu\varepsilon(u):\varepsilon(u)\]

- **Historical approach**: evolution of ice geometry is typically modeled explicitly (e.g., using a forward Euler scheme) with very simplistic numerical treatments of ice advance and retreat, which leads to:
  - A requirement of very small time-steps (subannual to daily depending on spatial resolution) to ensure stability.
  - **Low accuracy** with respect to the simulated position of the ice sheet margin and/or grounding line (transition from grounded to floating ice).
Ice Sheet Evolution

Ongoing R&D to mitigate these issues

- **Semi-implicit** and **fully-implicit** $H-u$ coupling:
  - **Semi-implicit coupling + adaptive time-stepping**: CG Paper I; Perego et al. 2015.
  - Formulating thickness evolution as an **obstacle problem** makes it amenable to implicit discretization: Bondzio et al. 2016.

- Improved representation of **grounding line**:
  - **AMR at grounding line**: Cornford et al. 2013; Drouet et al. 2013; Gladstone et al. 2010.
  - **Embedded boundary + cut-cell approach**: Martin et al. 2015.
  - **Subgrid parametrizations** of grounding line: Seroussi et al. 2014; Feldmann et al. 2014; CG Paper VI.
Summary of Numerical and Computational Challenges in ISM

- Domains are very large and shallow
- Saddle-point nature of full Stokes problem
- Localized areas with high velocity gradients
- Nonlinear rheology
- Movement of grounding line
- Nonlinear basal boundary conditions

- Phase changes in temperature equation
- Thickness evolution
- Ice advancement/retreat
- Ice calving
- Subglacial hydrology
- Ice sheet initialization/UQ
- Porting to new architectures
- Coupling with other climate components

Thank you! Questions?