Energy-Stable Galerkin Reduced Order Models for Prediction and Control of Fluid Systems

I. Kalashnikova\textsuperscript{1}, S. Arunajatesan\textsuperscript{2}, B. van Bloemen Waanders\textsuperscript{1}

\textsuperscript{1} Numerical Analysis & Applications Department, Sandia National Laboratories*, Albuquerque, NM, U.S.A.
\textsuperscript{2} Aerosciences Department, Sandia National Laboratories*, Albuquerque, NM, U.S.A.

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Target Cavity Flow Control Problem

Configuration/Plant: compressible non-linear fluid flow over open cavity containing components.
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- **Physical Control Problem**: using upstream actuation, control oscillations within cavity caused by pressure fluctuations propagating between downstream wall and shear layer.

\[ u_{\text{opt}} = \begin{pmatrix} 0 \\ f(t) \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad y_{\text{rms}} = \sqrt{\frac{1}{K} \sum_{i=1}^{K} (p(t_k) - \bar{p})^2} \]
Target Cavity Flow Control Problem

**Configuration/Plant:** compressible non-linear fluid flow over open cavity containing components.

**Physical Control Problem:** using upstream actuation, control oscillations within cavity caused by pressure fluctuations propagating between downstream wall and shear layer.

**Mathematical Control Problem:** compute optimal body-force actuation input $u_{opt}$ to minimize the RMS pressure halfway up the downstream wall.

\[
\text{input } u : \quad q^T = \begin{pmatrix} 0, f(t), 0, 0, 0 \end{pmatrix}^T \\
\text{output } y : \quad p_{rms} = \sqrt{\frac{1}{K} \sum_{i=1}^{K} (p(t_k) - \bar{p})^2}
\]
Collect snapshots from non-linear high-fidelity CFD cavity simulation

\[
\dot{x} = f(x, u_i), \quad y_i = h(x, u_i)
\]

for some set of inputs \(\{u_i(t)\}\), and construct empirical basis (POD, BPOD) from this snapshot set.

Build a ROM for the fluid system, or compute optimal controller \(u_{\text{opt}}(t)\) using ROM.

Apply ROM-based controller to non-linear cavity problem.
1. Collect snapshots from non-linear high-fidelity CFD cavity simulation

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3. Compute optimal controller \( u_{opt}(t) \) using ROM.

\[ \begin{align*}
\text{Plant (Cavity)} & \quad \text{Non-linear CFD} \\
\{u_i(t)\} & \quad \rightarrow \\
y_i(t) & \\
\text{Estimator} & \quad \text{Linear ROM} \\
\text{Controller} & \quad \text{Linear ROM}
\end{align*} \]
1. Collect snapshots from non-linear high-fidelity CFD cavity simulation

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for some set of inputs \( \{u_i(t)\} \), and construct empirical basis (POD, BPOD) from this snapshot set.

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3. Compute optimal controller \( u_{opt}(t) \) using ROM.

4. Apply ROM-based controller to non-linear cavity problem.
ROM-Based Cavity Flow Control

Road Map

1. Collect snapshots from non-linear high-fidelity CFD cavity simulation

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2. Build a ROM for the fluid system, or approximation of fluid system.

3. Compute optimal controller \(u_{opt}(t)\) using ROM.

4. Apply ROM-based controller to non-linear cavity problem.
3D Full (Non-Linear) Compressible Navier-Stokes Equations

- 3D compressible Navier-Stokes equations:
  \[
  \begin{align*}
  \rho \frac{Du_1}{dt} &= -\frac{\partial p}{\partial x_1} + \sum_{j=1}^{3} \frac{\partial}{\partial x_j} \left\{ \mu \left( \frac{\partial u_1}{\partial x_j} + \frac{\partial u_j}{\partial x_1} \right) + \lambda \delta_{1j} \nabla \cdot u \right\}, \\
  \rho \frac{Du_2}{dt} &= -\frac{\partial p}{\partial x_2} + \sum_{j=1}^{3} \frac{\partial}{\partial x_j} \left\{ \mu \left( \frac{\partial u_2}{\partial x_j} + \frac{\partial u_j}{\partial x_2} \right) + \lambda \delta_{2j} \nabla \cdot u \right\}, \\
  \rho \frac{Du_3}{dt} &= -\frac{\partial p}{\partial x_3} + \sum_{j=1}^{3} \frac{\partial}{\partial x_j} \left\{ \mu \left( \frac{\partial u_3}{\partial x_j} + \frac{\partial u_j}{\partial x_3} \right) + \lambda \delta_{3j} \nabla \cdot u \right\}, \\
  \rho C_v \frac{DT}{dt} &= -p \nabla \cdot u + \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \left( \kappa \frac{\partial T}{\partial x_i} \right), \\
  \frac{D\rho}{Dt} &= -\rho \nabla \cdot u.
  \end{align*}
  \]

- ROM approach is based on local linearization of full non-linear equations (1):
  - Full non-linear equations (1) are solved to generate snapshots in high-fidelity code.
  - Linearized approximation of (1) is projected onto reduced basis modes in building the ROM.
3D Linearized Compressible Navier-Stokes Equations

- Appropriate when a compressible fluid system can be described by viscous, small-amplitude perturbations about a steady-state mean (or base) flow.

- Linearization of full compressible Navier-Stokes equations:

\[
\begin{align*}
q^T(x,t) & \equiv (u_1, u_2, u_3, T, \rho) \\
& \equiv q^T(x) + q'^T(x,t) \in \mathbb{R}^5
\end{align*}
\]

- **Simplest linearization**: neglect \( \nabla q \) terms (uniform base flow)

\[
q',_t + A_i(q)q',_i - [K_{ij}(q)q',_j],_i = F
\]

- **More accurate linearization**: retain \( \nabla q \) terms

\[
q',_t + [A_i(q) - K_{vw}^{vw}(\nabla q)]q',_i - [K_{ij}(q)q',_j],_i + C(\nabla q)q' = F
\]

\[
\begin{align*}
A_i(q) & : \text{convective flux matrices} \\
K_{ij}(q) & : \text{diffusive flux matrices} \\
K_{vw}^{vw}(q) & : \text{viscous work matrices}
\end{align*}
\]
This talk focuses on how to construct a Galerkin ROM that is **stable a priori**

1. Stability Definitions
2. POD/Galerkin Approach to Model Reduction
3. Energy-Stable ROMs for Linearized Compressible Flow
   - Stability via Continuous Projection
   - Stability via Discrete Projection
4. Numerical Experiments
   - Implementation
   - Driven Pulse in Uniform Base Flow
   - Laminar Viscous Driven Cavity
5. Summary & Future Work
6. References
7. Appendix
Energy-Stability

- **Practical Definition:** Numerical solution does not “blow up” in finite time.
Energy-Stability

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- **More Precise Definition:** Numerical discretization does not introduce any spurious instabilities inconsistent with natural instability modes supported by the governing continuous PDEs.
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**More Precise Definition:** Numerical discretization does not introduce any spurious instabilities inconsistent with natural instability modes supported by the governing continuous PDEs.

Numerical solutions **must** maintain a proper energy balance.
**Energy-Stability**

- **Practical Definition:** Numerical solution does not “blow up” in finite time.

- **More Precise Definition:** Numerical discretization does not introduce any spurious instabilities inconsistent with natural instability modes supported by the governing continuous PDEs.

Numerical solutions **must** maintain a proper energy balance

\[
\frac{dE}{dt} \leq 0
\]

Linearized Compressible Navier-Stokes Equations:  
Non-increasing energy [5]

Compressible Navier-Stokes Equations:  
\[
\frac{d}{dt} \int_{\Omega} \rho \eta d\Omega \geq 0
\]

Clausius-Duhem Inequality  
Non-decreasing entropy [4]
Energy-Stability

- **Practical Definition**: Numerical solution does not “blow up” in finite time.

- **More Precise Definition**: Numerical discretization does not introduce any spurious instabilities inconsistent with natural instability modes supported by the governing continuous PDEs.

**Numerical solutions must maintain a proper energy balance**

- **Linearized Compressible Navier-Stokes Equations**: $\frac{dE}{dt} \leq 0$
  Non-increasing energy [5]

- **Compressible Navier-Stokes Equations**: $\frac{d}{dt} \int_\Omega \rho \eta d\Omega \geq 0$
  Clausius-Duhem Inequality
  Non-decreasing entropy [4]

- Analyzed using the **Energy Method**: Uses an equation for the evolution of numerical solution “energy” (or “entropy”) to determine stability.
Connection to Lyapunov Stability

\[ \dot{x}_N = f_N(x_N), \quad x_N \in \mathbb{R}^N \]

**Lyapunov Stability:** If there exists a **Lyapunov function** \( V \) such that

- \( V > 0 \) (positive-definite), and
- \( \frac{dV}{dt} = \frac{dV}{dx} f(x) \leq 0 \) (negative semi-definite along system trajectories)

in \( B_r(x_s) \), then \( x_s \) is **locally stable in the sense of Lyapunov** [8].

**Energy Stability:** Let

\[ E_N \equiv \frac{1}{2} ||x_N||^2 \]

denote the system energy. If

\[ \frac{dE_N}{dt} \leq 0 \]

the system is **energy-stable**.

*Manuscript in preparation.*
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denote the system energy. If

\[ \frac{dE_N}{dt} \leq 0 \]

the system is *energy-stable*.

**Remark:** System energy \( E_N \) satisfies the definition of a Lyapunov function!

*Manuscript in preparation.*
Model Reduction Approach

High-Fidelity CFD Simulations:
- Snapshot 1
- Snapshot 2
- ... Snapshot $K$

Fluid Modal Decomposition (POD):
\[ \mathbf{u}_M = \sum_{k=1}^{M} a_k(t) \phi_k(x) \]

Galerkin Projection of Fluid PDEs:
\[ (\phi_j, \dot{\mathbf{u}}_M + \nabla \cdot \mathbf{F}(\mathbf{u}_M)) = 0 \]

“Small” ROM ODE System:
\[ \dot{a}_k = f(a_1, \ldots, a_M) \]
Step 1: Constructing the Modes

High-Fidelity CFD Simulations:

- Snapshot 1
- Snapshot 2
- \ldots
- Snapshot \( K \)

Fluid Modal Decomposition (POD):

\[
\mathbf{u}_M = \sum_{k=1}^{M} a_k(t) \phi_k(\mathbf{x})
\]

Galerkin Projection of Fluid PDEs:

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(\phi_j, \dot{\mathbf{u}}_M + \nabla \cdot \mathbf{F}(\mathbf{u}_M)) = 0
\]

**POD basis** \( \{\phi_i\}_{i=1}^{M} \) with \( M \ll K \) maximizes the energy in the projection of snapshots onto \( \text{span}\{\phi_i\} \).

**POD SVD problem:**

\[
\mathbf{X} = \mathbf{U} \Sigma \mathbf{V}^T \\
(\phi_1, \ldots, \phi_M) = \mathbf{U}(:, 1 : M)
\]

“Small” ROM ODE System:

\[
\dot{a}_k = f(a_1, \ldots, a_M)
\]
Step 2: Galerkin Projection

**High-Fidelity CFD Simulations:**
- Snapshot 1
- Snapshot 2
- ... Snapshot $K$

**Fluid Modal Decomposition (POD):**

\[ u_M = \sum_{k=1}^{M} a_k(t) \phi_k(x) \]

**Galerkin Projection of Fluid PDEs:**

\[ (\phi_j, \dot{u}_M + \nabla \cdot F(u_M)) = 0 \]

**“Small” ROM ODE System:**

\[ \dot{a}_k = f(a_1, \ldots, a_M) \]
Discrete vs. Continuous Projection

DISCRETE APPROACH

Governing Equations
\[ \dot{u}_t = Lu \]
\[ \dot{\mathbf{u}}_N = A_N \mathbf{u}_N \]

CFD Model

Discrete Modal Basis \( \Phi \)

Projection of CFD Model (Matrix Operation)

\[ \dot{\mathbf{a}} = \Phi^T A_N \Phi \mathbf{a} \]

CONTINUOUS APPROACH

Governing Equations
\[ \dot{u}_t = Lu \]
\[ \dot{\mathbf{u}}_N = A_N \mathbf{u}_N \]

CFD Model

Continuous Modal Basis* \( \phi_j(x) \)

Projection of Governing Equations (Numerical Integration)

\[ \dot{a}_j = (\phi_j, L \phi_k) a_k \]

* Continuous functions space is defined using finite elements.
Energy stability of the Galerkin ROM can be proven [1] following “symmetrization” the linearized compressible Navier-Stokes equations.

- Linearized compressible Navier-Stokes system is “symmetrizable” [5].
- Pre-multiply equations by symmetric positive definite matrix:

\[
H \equiv \begin{pmatrix}
\bar{\rho} & 0 & 0 & 0 & 0 \\
0 & \bar{\rho} & 0 & 0 & 0 \\
0 & 0 & \rho & 0 & 0 \\
0 & 0 & 0 & \frac{\bar{\rho} R}{T(\gamma - 1)} & 0 \\
0 & 0 & 0 & 0 & \frac{RT}{\bar{\rho}} \\
\end{pmatrix}
\Rightarrow Hq'_{t} + HA_{i} q'_{i} - H[K_{ij} q'_{i},j + \cdots = F
\]

- \(H\) is called the “symmetrizer” of the system:
  - The convective flux matrices \(HA_{i}\) are all symmetric.
  - The following augmented viscosity matrix

\[
K^{S} \equiv \begin{pmatrix}
HK_{11} & HK_{12} & HK_{13} \\
HK_{21} & HK_{22} & HK_{23} \\
HK_{31} & HK_{32} & HK_{33} \\
\end{pmatrix}
\]

is symmetric positive semi-definite.
Define the “symmetry” inner product and “symmetry” norm:

\[ (q'(1), q'(2))_{(H,\Omega)} \equiv \int_{\Omega} [q'(1)]^T H q'(2) \, d\Omega, \quad \|q'\|_{(H,\Omega)} \equiv (q', q')_{(H,\Omega)} \]  

Stability analysis reveals that the symmetry inner product (and not the \(L^2\) inner product!) is the energy inner product for this equation set.

Uniform base flow case:
non-increasing energy in Galerkin approximation

\[ q'_{M} = \sum_{i=1}^{M} a_k(t) \phi_k(x) \, dE \ \equiv \ \frac{1}{2} \frac{d}{dt} \|q'_{M}(x,t)\|_{(H,\Omega)} \leq 0 \]

General case:
Galerkin approximation satisfies same energy expression as solutions to the continuous PDEs

\[ \|q'_{M}(x,t)\|_{(H,\Omega)} \leq e^{\beta t} \|q'_{M}(x,0)\|_{(H,\Omega)} \]

Practical Implication:
Symmetry inner product ensures Galerkin projection step of the ROM is stable (provided system is in stable state) for any basis!
Define the “symmetry” inner product and “symmetry” norm:

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(q'(1), q'(2))_{(H, \Omega)} \equiv \int_{\Omega} [q'(1)]^T H q'(2) \, d\Omega, \quad ||q'||_{(H, \Omega)} \equiv (q', q')_{(H, \Omega)}
\] (2)

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Symmetry Inner Product & A Stable Galerkin ROM

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- **Uniform base flow case:** non-increasing energy in Galerkin approximation
  \[
  q'_M = \sum_{i=1}^{M} a_k(t) \phi_k(x)
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  \[
  \frac{dE_M}{dt} \equiv \frac{1}{2} \frac{d}{dt} ||q'_M(x, t)||_{(H,\Omega)} \leq 0
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Symmetry Inner Product & A Stable Galerkin ROM

- Define the “symmetry” inner product and “symmetry” norm:

\[
(q'^{(1)}, q'^{(2)})_{(H,\Omega)} \equiv \int_{\Omega} [q'^{(1)}]^T H q'^{(2)} d\Omega,
\]

\[
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**Practical Implication:**
Symmetry inner product ensures Galerkin projection step of the ROM is stable (provided system is in stable state) for any basis!
Consider linear discrete (i.e., discretized in space) stable full order system
\[ \dot{x} = Ax \]  
(3)

Lyapunov function for (3): \( V(x) = x^T P x \) where \( P \) is the solution of the Ricatti equation:
\[ A^T P + PA = -Q \]  
(4)

- S.p.d. solution to (4) exists if \( Q \) is s.p.d. and \( A \) is stable [8].
- Solution to (4) can be obtained using MATLAB control toolbox:
\[ P = \text{lyap}(A', Q, \emptyset \text{ speye}(n, n)); \]

Discrete analog of symmetry inner-product: Lyapunov inner product
\[ (x_1, x_2)_P \equiv x_1^T P x_2 \]
Consider linear discrete (i.e., discretized in space) stable full order system
\[ \dot{x} = Ax \quad (3) \]

Lyapunov function for (3): \[ V(x) = x^TPx \] where \( P \) is the solution of the Ricatti equation:
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- Solution to (4) can be obtained using MATLAB control toolbox:
  \[ P = \text{lyap}(A', Q, []) \text{ speye}(n, n); \]

Discrete analog of symmetry inner-product: **Lyapunov inner product**

\[ (x_1, x_2)_P \equiv x_1^TPx_2 \]

Can show: if ROM for (3) is constructed in Lyapunov inner product,
\[ \frac{dE_M}{dt} \equiv \frac{1}{2} \frac{d}{dt} ||x_M||_2^2 \leq 0 \]
Symmetry Inner Product
(Continuous)

\[(q^{(1)}, q^{(2)})_{\mathbf{H},\Omega} \equiv \int_{\Omega} [q^{(1)}]^T \mathbf{H} q^{(2)} d\Omega\]

Lyapunov Inner Product
(Discrete)

\[(x_1, x_2)_P \equiv x_1^T \mathbf{P} x_2\]
Symmetry Inner Product (Continuous)

\[(q'^{(1)}, q'^{(2)})_{(H,\Omega)} \equiv \int_{\Omega} [q'^{(1)}]^T H q'^{(2)} \, d\Omega\]

For linear system:

\[q'_{,t} + A_i q'_{,i} - [K_{i,j} q'_{,i}]_{,j} + \cdots = F\]

Lyapunov Inner Product (Discrete)

\[(x_1, x_2)_P \equiv x_1^T P x_2\]

For linear system:

\[x_{,t} = A x\]
Energy-Stable ROM via Discrete Projection: vs. Continuous Projection

Symmetry Inner Product (Continuous)

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(q'(1), q'(2))_{(H,\Omega)} \equiv \int_{\Omega} [q'(1)]^T H q'(2) d\Omega
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- For linear system:
  \[ q',t + A_i q'i - [K_{ij} q'_i]_j + \cdots = F \]
- Defined for unstable systems, but stability of ROM not guaranteed.

Lyapunov Inner Product (Discrete)

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(x_1, x_2)_P \equiv x_1^T P x_2
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- For linear system:
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Energy-Stable ROM via Discrete Projection: vs. Continuous Projection

Symmetry Inner Product (Continuous)

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  q_t + A_i q_i - [K_{ij} q_i]_j + \cdots = F
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- Defined for unstable systems, but stability of ROM not guaranteed.
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Lyapunov Inner Product (Discrete)

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Energy-Stable ROM via Discrete Projection: vs. Continuous Projection

**Symmetry Inner Product**

(Continuous)

\[
(q'(1), q'(2))_{(H,\Omega)} \equiv \int_{\Omega} [q'(1)]^T H q'(2) d\Omega
\]

- For linear system:
  \[q', t + A_i q'_i - [K_{ij} q'_j, j + \cdots = F\]
  Defined for unstable systems, but stability of ROM not guaranteed.
  Induced by Lyapunov function for system.
  Equation-specific (⇒ embedded algorithm).
  Known analytically in closed form.

**Lyapunov Inner Product**

(Discrete)

\[(x_1, x_2)_P \equiv x_1^T P x_2\]

- For linear system:
  \[x, t = Ax\]
  Undefined for unstable systems.
  Induced by Lyapunov function for system.
  Black-box.
  Computed numerically by solving Ricatti equation (\(O(N^3)\) ops).
Implementation

- **Stability-Preserving Discrete Implementation of ROM:**
  - ROM is implemented in a C++ code that uses distributed vector and matrix data structures and parallel eigensolvers from the Trilinos project [7].
  - POD modes defined using piecewise smooth finite elements.
  - Gauss quadrature rules of sufficient accuracy are used to compute exactly inner products with the help of libmesh library.

ROM code is potentially compatible with any CFD code that can output a mesh and snapshot data stored at the nodes of this mesh.

- **High-fidelity CFD Code: SIGMA CFD**
  - Sandia in-house finite volume flow solver derived from LESLIE3D [6], a LES flow solver originally developed in the Computational Combustion Laboratory at Georgia Tech.
  - Solves the turbulent compressible flow equations using an explicit 2-4 MacCormack scheme.
  - A hybrid scheme coupling the MacCormack scheme to flux difference splitting schemes is employed to capture shocks.
Driven Pulse in a Uniform Base Flow

Uniform base flow in \( \Omega = (-1, 1)^2 \):

\[
\bar{p} = 10.1325 \text{ Pa} \\
\bar{T} = 300 \text{ K} \\
\bar{\rho} = \frac{\bar{p}}{\bar{R} \bar{T}} = 1.17 \times 10^{-4} \text{ kg/m}^3 \\
\bar{u}_1 = \bar{u}_2 = \bar{u}_3 = 0.0 \text{ m/s} \\
\bar{c} = 347.9693 \text{ m/s}.
\]

Slip wall boundary conditions applied on all boundaries of \( \Omega \).
Driven Pulse in a Uniform Base Flow

- Uniform base flow in $\Omega = (-1, 1)^2$:
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  \overline{u}_1 = \overline{u}_2 = \overline{u}_3 = 0.0 \text{ m/s} \\
  \overline{c} = 347.9693 \text{ m/s}.
  \]

- Slip wall boundary conditions applied on all boundaries of $\Omega$.

- Force for $y$–momentum equation drives the flow:
  \[
  F_v(x, t) = (1 \times 10^{-4}) \cos(2000\pi t), \quad x \in (-0.1, 0)^2
  \]
Driven Pulse in a Uniform Base Flow

- Uniform base flow in $\Omega = (-1, 1)^2$:

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  \bar{T} = 300 \text{ K} \\
  \bar{\rho} = \frac{\bar{p}}{RT} = 1.17 \times 10^{-4} \text{ kg/m}^3 \\
  \bar{u}_1 = \bar{u}_2 = \bar{u}_3 = 0.0 \text{ m/s} \\
  \bar{c} = 347.9693 \text{ m/s}.
  \]

- Slip wall boundary conditions applied on all boundaries of $\Omega$.

- Force for $y$–momentum equation drives the flow:

  \[
  F_v(x, t) = \left(1 \times 10^{-4}\right) \cos(2000\pi t), \quad x \in (-0.1, 0)^2
  \]

- High-fidelity CFD simulation run on 3362 node mesh until time $T = 0.5$ seconds.

- 2500 snapshots (saved every $2 \times 10^{-5}$ seconds), used to construct a 20 mode POD basis.
Figure below shows:

- $t$ vs. $a_i(t)$ (ROM coefficients).
- $t$ vs. $(q^i_{CFD}(x,t), \phi_i(x))$ (projection of snapshots onto modes).

Movie on right shows $u$-velocity snapshot (top) vs. 20 mode symmetry ROM solution for $u$ (bottom).
Uncontrolled Symmetry ROM Results

- Figure below shows:
  - ○: $t$ vs. $a_i(t)$ (ROM coefficients).
  - −: $t$ vs. $(q_{CFD}^i(x, t), \phi_i(x))$ (projection of snapshots onto modes).

- Movie on right shows $u$-velocity snapshot (top) vs. 20 mode symmetry ROM solution for $u$ (bottom).
Figure below shows:

- $t$ vs. $a_i(t)$ (ROM coefficients).
- $t$ vs. $(q'_{CFD}(x,t), \phi_i(x))$ (projection of snapshots onto modes).

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- $t$ vs. $a_i(t)$ (ROM coefficients).
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Uncontrolled Symmetry ROM Results

Figure below shows:

- ○: \( t \) vs. \( a_i(t) \) (ROM coefficients).
- ⋅: \( t \) vs. \( (q'_{CFD}(x,t), \phi_i(x)) \) (projection of snapshots onto modes).

Movie on right shows \( u \)-velocity snapshot (top) vs. 20 mode symmetry ROM solution for \( u \) (bottom).
Uncontrolled Symmetry ROM Results

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- $t$ vs. $a_i(t)$ (ROM coefficients).
- $t$ vs. $(q_i^{CFD}(x,t), \phi_i(x))$ (projection of snapshots onto modes).

Movie on right shows $u$-velocity snapshot (top) vs. 20 mode symmetry ROM solution for $u$ (bottom).
Uncontrolled Symmetry ROM Results

Figure below shows:

- ▶ ○: $t$ vs. $a_i(t)$ (ROM coefficients).
- ▶ −: $t$ vs. $(q_{CFD}(x,t), \phi_i(x))$ (projection of snapshots onto modes).

Movie on right shows $u$-velocity snapshot (top) vs. 20 mode symmetry ROM solution for $u$ (bottom).
Figure below shows:

- $t$ vs. $a_i(t)$ (ROM coefficients).
- $t$ vs. $(q_{CFD}(x,t), \phi_i(x))$ (projection of snapshots onto modes).

Movie on right shows $u$-velocity snapshot (top) vs. 20 mode symmetry ROM solution for $u$ (bottom).
Uncontrolled Symmetry ROM Results

Figure below shows:

- $t$ vs. $a_i(t)$ (ROM coefficients).
- $t$ vs. $(q_{CFD}^{'}(x,t), \phi_i(x))$ (projection of snapshots onto modes).

Movie on right shows $u$-velocity snapshot (top) vs. 20 mode symmetry ROM solution for $u$ (bottom).
Figure below shows:

- $t$ vs. $a_i(t)$ (ROM coefficients).
- $t$ vs. $(q^J_{CFD}(x,t), \phi_i(x))$ (projection of snapshots onto modes).

Movie on right shows $u$-velocity snapshot (top) vs. 20 mode symmetry ROM solution for $u$ (bottom).
Figure below shows:

- ◆: $t$ vs. $a_i(t)$ (ROM coefficients).
- -: $t$ vs. $(q_{CFD}(x,t), \phi_i(x))$ (projection of snapshots onto modes).

Movie on right shows $u$-velocity snapshot (top) vs. 20 mode symmetry ROM solution for $u$ (bottom).
Uncontrolled Symmetry ROM Results

Figure below shows:

- ○: $t$ vs. $a_i(t)$ (ROM coefficients).
- -: $t$ vs. $(q'_{CFD}(x,t), \phi_i(x))$
  (projection of snapshots onto modes).

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Uncontrolled Symmetry ROM Results

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- $t$ vs. $(q_{CFD}^i(x, t), \phi_i(x))$ (projection of snapshots onto modes).

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Uncontrolled Symmetry ROM Results

- Figure below shows:
  - $t$ vs. $a_i(t)$ (ROM coefficients).
  - $t$ vs. $(q'_{CFD}(x,t), \phi_i(x))$
    (projection of snapshots onto modes).

- Movie on right shows $u$-velocity snapshot (top) vs. 20 mode symmetry ROM solution for $u$ (bottom).
Uncontrolled Symmetry ROM Results

Figure below shows:

- $t$ vs. $a_i(t)$ (ROM coefficients).
- $t$ vs. $(q_{CFD}^{F}(x, t), \phi_i(x))$ (projection of snapshots onto modes).

Movie on right shows $u$-velocity snapshot (top) vs. 20 mode symmetry ROM solution for $u$ (bottom).
Uncontrolled Symmetry ROM Results

- Figure below shows:
  - ○: $t$ vs. $a_i(t)$ (ROM coefficients).
  - -: $t$ vs. $(q_{CFD}(x,t), \phi_i(x))$ (projection of snapshots onto modes).

- Movie on right shows $u$-velocity snapshot (top) vs. 20 mode symmetry ROM solution for $u$ (bottom).
Uncontrolled Symmetry ROM Results

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Uncontrolled Symmetry ROM Results

- Figure below shows:
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  - −: $t \text{ vs. } (q_{CFD}(x,t), \phi_i(x))$ (projection of snapshots onto modes).

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Movie on right shows $u$-velocity snapshot (top) vs. 20 mode symmetry ROM solution for $u$ (bottom).
Uncontrolled Symmetry ROM Results

- Figure below shows:
  - $t$ vs. $a_i(t)$ (ROM coefficients).
  - $-t$ vs. $(q_{CFD}(x,t), \phi_i(x))$ (projection of snapshots onto modes).

- Movie on right shows $u$-velocity snapshot (top) vs. 20 mode symmetry ROM solution for $u$ (bottom).
Uncontrolled Symmetry ROM Results

Figure below shows:

- \( a_i(t) \) vs. \( t \) (ROM coefficients).
- \( t \) vs. \( \mathbf{q}_{CFD}(x, t), \phi_i(x) \) (projection of snapshots onto modes).

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- $t$ vs. $(q^{CFD}_{x,t}, \phi_i(x))$ (projection of snapshots onto modes).

Movie on right shows $u$-velocity snapshot (top) vs. 20 mode symmetry ROM solution for $u$ (bottom).
Figure below shows:

- $t$ vs. $a_i(t)$ (ROM coefficients).
- $t$ vs. $(q_{CDF}(x,t), \phi_i(x))$ (projection of snapshots onto modes).

Movie on right shows $u$-velocity snapshot (top) vs. 20 mode symmetry ROM solution for $u$ (bottom).
Uncontrolled Symmetry ROM Results

- Figure below shows:
  - o: $t$ vs. $a_i(t)$ (ROM coefficients).
  - −: $t$ vs. $(q_{CFD}^i(x,t), \phi_i(x))$ (projection of snapshots onto modes).
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Uncontrolled Symmetry ROM Results

Figure below shows:

- ○: \( t \) vs. \( a_i(t) \) (ROM coefficients).
- -: \( t \) vs. \( (q_{CFD}(x,t), \phi_i(x)) \) (projection of snapshots onto modes).

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  (projection of snapshots onto modes).

Movie on right shows $u$-velocity snapshot (top) vs. 20 mode symmetry ROM solution for $u$ (bottom).
Figure below shows:

- $a_i(t)$ (ROM coefficients).
- $t$ vs. $(q'_{CFD}(x,t), \phi_i(x))$ (projection of snapshots onto modes).

Movie on right shows $u$-velocity snapshot (top) vs. 20 mode symmetry ROM solution for $u$ (bottom).
Uncontrolled Symmetry ROM Results

- Figure below shows:
  - ●: $t$ vs. $a_i(t)$ (ROM coefficients).
  - ▶: $t$ vs. $(q'_{CFD}(x,t), \phi_i(x))$
    (projection of snapshots onto modes).
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Uncontrolled Symmetry ROM Results

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Uncontrolled Symmetry ROM Results

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- $t$ vs. $a_i(t)$ (ROM coefficients).
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Uncontrolled Symmetry ROM Results

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Uncontrolled Symmetry ROM Results

Figure below shows:

▶ ◦: \( t \) vs. \( a_i(t) \) (ROM coefficients).
▶ -: \( t \) vs. \( (q_{CFD}^i(x,t), \phi_i(x)) \) (projection of snapshots onto modes).

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Uncontrolled Symmetry ROM Results

- Figure below shows:
  - ◢: $t$ vs. $a_i(t)$ (ROM coefficients).
  - ◢$: t$ vs. $(q'_{CFD}(x, t), \phi_i(x))$
    (projection of snapshots onto modes).

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- −: $t$ vs. $(q_{CFD}^i(x,t), \phi_i(x))$ (projection of snapshots onto modes).

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Uncontrolled Symmetry ROM Results

Figure below shows:

- ◦ $t$ vs. $a_i(t)$ (ROM coefficients).
- − $t$ vs. $(q_{CFD}^{\prime}(x,t), \phi_i(x))$ (projection of snapshots onto modes).

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- $t$ vs. $(q_{CFD}^\prime(x, t), \phi_i(x))$ (projection of snapshots onto modes).

Movie on right shows $u$-velocity snapshot (top) vs. 20 mode symmetry ROM solution for $u$ (bottom).
Uncontrolled Symmetry ROM Results

Figure below shows:

- $t$ vs. $a_i(t)$ (ROM coefficients).
- $t$ vs. $(q^{CFD}_{i}(x,t), \phi_i(x))$ (projection of snapshots onto modes).

Movie on right shows $u$-velocity snapshot (top) vs. 20 mode symmetry ROM solution for $u$ (bottom).
Uncontrolled Symmetry ROM Results

Figure below shows:

- ○: $t$ vs. $a_i(t)$ (ROM coefficients).
- -: $t$ vs. $(q'_{C_F\Delta}(x,t), \phi_i(x))$ (projection of snapshots onto modes).

Movie on right shows $u$-velocity snapshot (top) vs. 20 mode symmetry ROM solution for $u$ (bottom).
Uncontrolled Symmetry ROM Results

Figure below shows:

- $t$ vs. $a_i(t)$ (ROM coefficients).
- $t$ vs. $(q_{i, CFD}(x, t), \phi_i(x))$
  (projection of snapshots onto modes).

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- $t$ vs. $(q'_{CFD}(x,t), \phi_i(x))$ (projection of snapshots onto modes).

Movie on right shows $u$-velocity snapshot (top) vs. 20 mode symmetry ROM solution for $u$ (bottom).
Uncontrolled Symmetry ROM Results

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Figure below shows:

- $t$ vs. $a_i(t)$ (ROM coefficients).

- $t$ vs. $q_i'$ CFD $(x,t)$, $\phi_i(x)$ (projection of snapshots onto modes).

Movie on right shows $u$-velocity snapshot (top) vs. 20 mode symmetry ROM solution for $u$ (bottom).
Control problem: compute actuation that will minimize $p'$ at $(x, y) = (1, 0)$.

- Compute **LQR controller** feedback law $u_M = -Kx_M$ to minimize quadratic cost functional using ROM*: 

$$J = \frac{1}{T} \int_0^T [p'^2(1, 0; t) + \tau u^2] dt$$

* The computation of $K = R^{-1}B^TX$ requires solution of algebraic Ricatti equation 
$$A^TX +XA - \frac{1}{\tau}XBB^TX + C^TC = 0$$ [8].
LQR Control of Driven Pulse

**Control problem:** compute actuation that will minimize $p'$ at $(x, y) = (1, 0)$.

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* The computation of $K = R^{-1}B^T X$ requires solution of algebraic Ricatti equation $A^T X + XA - \frac{1}{\tau} XBB^T X + C^T C = 0$ [8].
Laminar Viscous Driven Cavity Problem

- Mach = 0.6, Re = 1898 (laminar regime).

High-fidelity CFD simulation was run on 343,408 node mesh until time $T = 0.202$ seconds. 101 snapshots were saved (every $2 \times 10^{-4}$ seconds), to construct a 20 mode POD basis.

Inherently non-linear problem! High-fidelity solution obtained by solving full non-linear Navier-Stokes equations.
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Expected ROM Performance

ROM based on Navier-Stokes equations *linearized* around snapshot mean.
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Non-linear dynamics of flow are captured in POD reduced basis modes.

\[ u \text{ mode 1 (24.9\% energy)} \]
\[ u \text{ mode 2 (23.7\% energy)} \]
\[ u \text{ mode 3 (6.93\% energy)} \]
Expected ROM Performance

ROM based on Navier-Stokes equations *linearized* around snapshot mean.

Non-linear dynamics of flow *are* captured in POD reduced basis modes. Non-linear dynamics of the flow *are not* fully captured in equations projected onto POD modes.

\[ u \text{ mode 1 (24.9\% energy)} \]
\[ u \text{ mode 2 (23.7\% energy)} \]
\[ u \text{ mode 3 (6.93\% energy)} \]
ROM based on Linearized Navier-Stokes Neglecting $\nabla \bar{q}$ Terms

$$q'_t + A_i(\bar{q})q'_i - [K_{ij}(\bar{q})q'_{,j}]{,i} = F$$

- Figure below shows:
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  - $t$ vs. $a_i(t)$ (ROM coefficients).
  - $t$ vs. $(q'_{CFD}(x,t), \phi_i(x))$ (projection of snapshots onto modes).

Movie on right shows $v$-velocity snapshot (top) vs. 20 mode symmetry ROM solution $v$ (bottom).
ROM based on Linearized Navier-Stokes Neglecting $\nabla \bar{q}$ Terms

\[ q'_t + A_i(\bar{q}) q'_i - [K_{ij}(\bar{q}) q'_j],_i = F \]

- Figure below shows:
  - $\circ$: $t$ vs. $a_i(t)$ (ROM coefficients).
  - $\times$: $t$ vs. $(q'_{CFD}(x,t), \phi_i(x))$ (projection of snapshots onto modes).

- Movie on right shows $v$-velocity snapshot (top) vs. 20 mode symmetry ROM solution $v$ (bottom).
ROM based on Linearized Navier-Stokes Neglecting $\nabla \bar{q}$ Terms

\[ q'_t + A_i(\bar{q})q'_i - [K_{ij}(\bar{q})q'_j]_i = F \]

- Figure below shows:
  - $t$ vs. $a_i(t)$ (ROM coefficients).
  - $t$ vs. $(q'_i_{CFD}(x,t), \phi_i(x))$ (projection of snapshots onto modes).
- Movie on right shows $v$-velocity snapshot (top) vs. 20 mode symmetry ROM solution $v$ (bottom).
ROM based on Linearized Navier-Stokes Neglecting $\nabla \bar{q}$ Terms

$$q'_t + A_i(\bar{q})q'_i - [K_{ij}(\bar{q})q'_{,j}],_i = F$$

Figure below shows:

- $\circ$: $t$ vs. $a_i(t)$ (ROM coefficients).
- $\rightarrow$: $t$ vs. $(q'_{CFD}(x,t), \phi_i(x))$ (projection of snapshots onto modes).

Movie on right shows $v$-velocity snapshot (top) vs. 20 mode symmetry ROM solution $v$ (bottom).
ROM based on Linearized Navier-Stokes Neglecting $\nabla \bar{q}$ Terms

\[ q'_t + A_i(\bar{q})q'_i - [K_{ij}(\bar{q})q'_j]_i = F \]

Figure below shows:

- $\circ$: $t$ vs. $a_i(t)$ (ROM coefficients).
- $\triangleright$: $t$ vs. $(q'_C F D(x, t), \phi_i(x))$ (projection of snapshots onto modes).

Movie on right shows $v$-velocity snapshot (top) vs. 20 mode symmetry ROM solution $v$ (bottom).
ROM based on Linearized Navier-Stokes Neglecting $\nabla \bar{q}$ Terms

\[ q'_t + A_i(\bar{q})q'_i - [K_{ij}(\bar{q})q'_{j}]_i = F \]

- Figure below shows:
  - $t$ vs. $a_i(t)$ (ROM coefficients).
  - $t$ vs. $(q'_{CFD}(x, t), \phi_i(x))$ (projection of snapshots onto modes).

- Movie on right shows $v$-velocity snapshot (top) vs. 20 mode symmetry ROM solution $v$ (bottom).
ROM based on Linearized Navier-Stokes Neglecting $\nabla \bar{q}$ Terms

$$q'_t + A_i(\bar{q})q'_i - [K_{ij}(\bar{q})q'_j]_i = F$$

- Figure below shows:
  - $t$ vs. $a_i(t)$ (ROM coefficients).
  - $t$ vs. $(q'_{CFD}(x,t), \phi_i(x))$ (projection of snapshots onto modes).

- Movie on right shows $v$-velocity snapshot (top) vs. 20 mode symmetry ROM solution $v$ (bottom).
ROM based on Linearized Navier-Stokes Neglecting $\nabla \bar{q}$ Terms

\[ q'_t + A_i(\bar{q})q'_i - \left[ K_{ij}(\bar{q})q'_{j, i} \right]_i = F \]

- Figure below shows:
  - ◦ $t$ vs. $a_i(t)$ (ROM coefficients).
  - − $t$ vs. $(q'_{CFD}(x, t), \phi_i(x))$ (projection of snapshots onto modes).

- Movie on right shows $v$-velocity snapshot (top) vs. 20 mode symmetry ROM solution $v$ (bottom).
ROM based on Linearized Navier-Stokes Neglecting $\nabla \bar{q}$ Terms

$q'_t + A_i(\bar{q})q'_i - [K_{ij}(\bar{q})q'_j],i =F$

Figure below shows:

- $\circ$: $t$ vs. $a_i(t)$ (ROM coefficients).
- $\triangleright$: $t$ vs. $(q'_C F D(x,t), \phi_i(x))$ (projection of snapshots onto modes).

Movie on right shows $v$-velocity snapshot (top) vs. 20 mode symmetry ROM solution $v$ (bottom).
ROM based on Linearized Navier-Stokes Neglecting $\nabla \bar{q}$ Terms

$$q'_t + A_i(\bar{q})q'_i - [K_{ij}(\bar{q})q'_{ij}],i = F$$

Figure below shows:

- $\circ$: $t$ vs. $a_i(t)$ (ROM coefficients).
- $\triangleright$: $t$ vs. $(q'_C(t),\phi_i(x))$ (projection of snapshots onto modes).

Movie on right shows $v$-velocity snapshot (top) vs. 20 mode symmetry ROM solution $v$ (bottom).
ROM based on Linearized Navier-Stokes Neglecting $\nabla \bar{q}$ Terms

$q'_{t} + A_{i}(\bar{q})q'_{i} - [K_{ij}(\bar{q})q'_{j}], i = F$

Figure below shows:
- $\circ$: $t$ vs. $a_{i}(t)$ (ROM coefficients).
- $\triangleright$: $t$ vs. $(q'_{CFD}(x, t), \phi_{i}(x))$ (projection of snapshots onto modes).

Movie on right shows $v$-velocity snapshot (top) vs. 20 mode symmetry ROM solution $v$ (bottom).
ROM based on Linearized Navier-Stokes Neglecting $\nabla \bar{q}$ Terms

$$q'_t + A_i(\bar{q})q'_i - [K_{ij}(\bar{q})q'_j]_i = F$$

Figure below shows:

- ◦: $t$ vs. $a_i(t)$ (ROM coefficients).
- -: $t$ vs. $(q'_i(\text{CFD}(x, t), \phi_i(x))$ (projection of snapshots onto modes).

Movie on right shows $v$-velocity snapshot (top) vs. 20 mode symmetry ROM solution $v$ (bottom).
ROM based on Linearized Navier-Stokes Neglecting $\nabla \bar{q}$ Terms

$$q'_t + A_i(\bar{q})q'_i - [K_{ij}(\bar{q})q'_{j}],_i = F$$

- Figure below shows:
  - $\circ$: $t$ vs. $a_i(t)$ (ROM coefficients).
  - $\triangledown$: $t$ vs. $q'_{CFD}(x, t), \phi_i(x)$ (projection of snapshots onto modes).

- Movie on right shows $v$-velocity snapshot (top) vs. 20 mode symmetry ROM solution $v$ (bottom).
ROM based on Linearized Navier-Stokes Neglecting $\nabla \bar{q}$ Terms

$$q'_t + A_i(\bar{q}) q'_i - [K_{ij}(\bar{q}) q'_j],_i = F$$

Figure below shows:

- $t$ vs. $a_i(t)$ (ROM coefficients).
- $t$ vs. $(q'_{CFD}(x, t), \phi_i(x))$ (projection of snapshots onto modes).

Movie on right shows $v$-velocity snapshot (top) vs. 20 mode symmetry ROM solution $v$ (bottom).
ROM based on Linearized Navier-Stokes Neglecting $\nabla \bar{q}$ Terms

\[ q'_t + A_i(\bar{q})q'_i - [K_{ij}(\bar{q})q'_j],_i = F \]

Figure below shows:
- $t$ vs. $a_i(t)$ (ROM coefficients).
- $-t$ vs. $(q'_{CFD}(x, t), \phi_i(x))$ (projection of snapshots onto modes).

Movie on right shows $v$-velocity snapshot (top) vs. 20 mode symmetry ROM solution $v$ (bottom).
ROM based on Linearized Navier-Stokes Neglecting $\nabla \bar{q}$ Terms

\[ q'_{t} + A_{i}(\bar{q})q'_{i} - [K_{ij}(\bar{q})q'_{j}],i = F \]

Figure below shows:

- $t$ vs. $a_{i}(t)$ (ROM coefficients).
- $t$ vs. $(q'_{CFD}(x,t), \phi_{i}(x))$ (projection of snapshots onto modes).

Movie on right shows $v$-velocity snapshot (top) vs. 20 mode symmetry ROM solution $v$ (bottom).
ROM based on Linearized Navier-Stokes Neglecting $\nabla \bar{q}$ Terms

$$q'_t + A_i(\bar{q})q'_i - [K_{ij}(\bar{q})q'_{j}],_i = F$$

Figure below shows:

- ◼: $t$ vs. $a_i(t)$ (ROM coefficients).
- -: $t$ vs. $(q'_CFD(x, t), \phi_i(x))$ (projection of snapshots onto modes).

Movie on right shows $v$-velocity snapshot (top) vs. 20 mode symmetry ROM solution $v$ (bottom).
ROM based on Linearized Navier-Stokes Neglecting $\nabla \bar{q}$ Terms

$$q'_t + A_i(q)q'_i - [K_{ij}(\bar{q})q'_{j,i}], i = F$$

- Figure below shows:
  - $\circ$: $t$ vs. $a_i(t)$ (ROM coefficients).
  - $\nabla$: $t$ vs. $(q'_{CFD}(x, t), \phi_i(x))$ (projection of snapshots onto modes).

- Movie on right shows $v$-velocity snapshot (top) vs. 20 mode symmetry ROM solution $v$ (bottom).
ROM based on Linearized Navier-Stokes Neglecting $\nabla \bar{q}$ Terms

Figure below shows:
- ◦: $t$ vs. $a_i(t)$ (ROM coefficients).
- -: $t$ vs. $(q'^{CFD}(x,t), \phi_i(x))$ (projection of snapshots onto modes).

Movie on right shows $v$-velocity snapshot (top) vs. 20 mode symmetry ROM solution (bottom).
ROM based on Linearized Navier-Stokes Retaining $\nabla \bar{q}$ Terms

$$q'_t + [A_i(\bar{q}) - K_{i}^{uw}(\nabla \bar{q})]q'_i - [K_{ij}(\bar{q})q'_j],i + C(\nabla \bar{q})q' = F$$

Figure below shows:

- •: $t$ vs. $a_i(t)$ (ROM coefficients).
- -: $t$ vs. $(q'_C F D(x, t), \phi_i(x))$ (projection of snapshots onto modes).

Movie on right shows $v$-velocity snapshot (top) vs. 20 mode symmetry ROM solution $v$ (bottom).
ROM based on Linearized Navier-Stokes Retaining $\nabla \bar{q}$ Terms

\[
q'_t + \left[ A_i(\bar{q}) - K_{iw}(\nabla \bar{q}) \right] q'_i - \left[ K_{ij}(\bar{q}) q'_j \right]_i + C(\nabla \bar{q}) q' = F
\]

- Figure below shows:
  - ○ $t$ vs. $a_i(t)$ (ROM coefficients).
  - ▶ $t$ vs. $(q'_{CFD}(x, t), \phi_i(x))$ (projection of snapshots onto modes).

- Movie on right shows $v$-velocity snapshot (top) vs. 20 mode symmetry ROM solution $v$ (bottom).
ROM based on Linearized Navier-Stokes Retaining $\nabla \bar{q}$ Terms

\[
q'_t + [A_i(\bar{q}) - K_{i}^{uw}(\nabla \bar{q})]q'_i - [K_{ij}(\bar{q})q'_j],i + C(\nabla \bar{q})q' = F
\]

- Figure below shows:
  - $\circ$: $t$ vs. $a_i(t)$ (ROM coefficients).
  - $\circ$: $t$ vs. $(q'_{CFD}(x,t), \phi_i(x))$ (projection of snapshots onto modes).
- Movie on right shows $v$-velocity snapshot (top) vs. 20 mode symmetry ROM solution $v$ (bottom).
ROM based on Linearized Navier-Stokes Retaining $\nabla \bar{q}$ Terms

$$q_{t}^{'} + [A_{i}(\bar{q}) - K_{i}^{uw}(\nabla \bar{q})]q_{i}^{'} - [K_{ij}(\bar{q})q_{j}^{'}],i + C(\nabla \bar{q})q^{'} = F$$

Figure below shows:

- $\circ$: $t$ vs. $a_{i}(t)$ (ROM coefficients).
- $\bullet$: $t$ vs. $(q_{CFD}^{'}(x,t), \phi_{i}(x))$ (projection of snapshots onto modes).

Movie on right shows $v$-velocity snapshot (top) vs. 20 mode symmetry ROM solution $v$ (bottom).
ROM based on Linearized Navier-Stokes Retaining $\nabla \bar{q}$ Terms

$$q', t + [A_i(\bar{q}) - K^{uw}_i(\nabla \bar{q})]q', i - [K_{ij}(\bar{q})q', j], i + C(\nabla \bar{q})q' = F$$

Figure below shows:
- $\circ$: $t$ vs. $a_i(t)$ (ROM coefficients).
- $\triangleright$: $t$ vs. $(q'_C F D(x, t), \phi_i(x))$ (projection of snapshots onto modes).

Movie on right shows $v$-velocity snapshot (top) vs. 20 mode symmetry ROM solution $v$ (bottom).
ROM based on Linearized Navier-Stokes Retaining $\nabla \bar{q}$ Terms

\[ q'_{,t} + [A_i(\bar{q}) - K_i^{vw}(\nabla \bar{q})]q'_{,i} - [K_{ij}(\bar{q})q'_{,j},i] + C(\nabla \bar{q})q' = F \]

Figure below shows:

- ○: $t$ vs. $a_i(t)$ (ROM coefficients).
- -: $t$ vs. $(q'_{CFD}(x,t), \phi_i(x))$ (projection of snapshots onto modes).

Movie on right shows $v$-velocity snapshot (top) vs. 20 mode symmetry ROM solution $v$ (bottom).
ROM based on Linearized Navier-Stokes Retaining $\nabla \bar{q}$ Terms

\[
q'_t + [A_i(\bar{q}) - K^w_i(\nabla \bar{q})]q'_i - [K_{ij}(\bar{q})q'_{j}],_i + C(\nabla \bar{q})q' = F
\]

Figure below shows:

- o: $t$ vs. $a_i(t)$ (ROM coefficients).
- -: $t$ vs. ($q'_{CFD}(x, t), \phi_i(x)$) (projection of snapshots onto modes).

Movie on right shows $v$-velocity snapshot (top) vs. 20 mode symmetry ROM solution $v$ (bottom).
ROM based on Linearized Navier-Stokes Retaining $\nabla \bar{q}$ Terms

$$q'_t + [A_i(\bar{q}) - K_{i}^{uw}(\nabla \bar{q})]q'_i - [K_{ij}(\bar{q})q'_j],i + C(\nabla \bar{q})q' = F$$

Figure below shows:

- $\circ$: $t$ vs. $a_i(t)$ (ROM coefficients).
- $\rightarrow$: $t$ vs. $(q'_i(x), t), \phi_i(x))$ (projection of snapshots onto modes).

Movie on right shows $v$-velocity snapshot (top) vs. 20 mode symmetry ROM solution $v$ (bottom).
ROM based on Linearized Navier-Stokes Retaining $\nabla \bar{q}$ Terms

$$q'_{t} + [A_{i}(\bar{q}) - K^{vw}_{i}(\nabla \bar{q})]q'_{i} - [K_{ij}(\bar{q})q'_{j}],_{i} + C(\nabla \bar{q})q' = F$$

Figure below shows:

- ◦: $t$ vs. $a_{i}(t)$ (ROM coefficients).
- -: $t$ vs. $(q'_{CFD}(x, t), \phi_{i}(x))$ (projection of snapshots onto modes).

Movie on right shows $v$-velocity snapshot (top) vs. 20 mode symmetry ROM solution $v$ (bottom).
ROM based on Linearized Navier-Stokes Retaining $\nabla \bar{q}$ Terms

\[
q'_{t} + [A_{i}(\bar{q}) - K_{i}^{uw}(\nabla \bar{q})]q'_{i} - [K_{ij}(\bar{q})q'_{j}],i + C(\nabla \bar{q})q' = F
\]

- Figure below shows:
  - $\circ$: $t$ vs. $a_{i}(t)$ (ROM coefficients).
  - $\circ$: $t$ vs. $(q'_{CFD}(x, t), \phi_{i}(x))$ (projection of snapshots onto modes).

- Movie on right shows $v$-velocity snapshot (top) vs. 20 mode symmetry ROM solution $v$ (bottom).
ROM based on Linearized Navier-Stokes Retaining $\nabla \bar{q}$ Terms

\[ q'_t + [A_i(\bar{q}) - K^{vw}_i(\nabla \bar{q})]q'_i - [K_{ij}(\bar{q})q'_j],i + C(\nabla \bar{q})q' = F \]

- Figure below shows:
  - $\circ$: $t$ vs. $a_i(t)$ (ROM coefficients).
  - $\triangledown$: $t$ vs. $(q'_{CFD}(x, t), \phi_i(x))$ (projection of snapshots onto modes).
- Movie on right shows $v$-velocity snapshot (top) vs. 20 mode symmetry ROM solution $v$ (bottom).
ROM based on Linearized Navier-Stokes Retaining $\nabla \bar{q}$ Terms

\[ q'_t + \left[ A_i(q) - K_i^{uw}(\nabla \bar{q}) \right] q'_i - \left[ K_{ij}(\bar{q}) q'_j \right]_i + C(\nabla \bar{q}) q' = F \]

Figure below shows:

- $\circ$: $t$ vs. $a_i(t)$ (ROM coefficients).
- $\triangledown$: $t$ vs. $(q'_{CFD}(x, t), \phi_i(x))$ (projection of snapshots onto modes).

Movie on right shows $v$-velocity snapshot (top) vs. 20 mode symmetry ROM solution $v$ (bottom).
ROM based on Linearized Navier-Stokes Retaining $\nabla \bar{q}$ Terms

$$q'_{t} + [A_{i} (\bar{q}) - K_{i}^{uw} (\nabla \bar{q})]q'_{i} - [K_{ij} (\bar{q})q'_{j}],i + C(\nabla \bar{q})q' = F$$

Figure below shows:

- $\circ$: $t$ vs. $a_{i}(t)$ (ROM coefficients).
- $\triangleright$: $t$ vs. $(q'_{CFD}(x, t), \phi_{i}(x))$ (projection of snapshots onto modes).

Movie on right shows $v$-velocity snapshot (top) vs. 20 mode symmetry ROM solution $v$ (bottom).
ROM based on Linearized Navier-Stokes Retaining $\nabla \bar{q}$ Terms

$$q'_t + [A_i(\bar{q}) - K_{i}^{vw}(\nabla \bar{q})]q'_i - [K_{ij}(\bar{q})q'_j]_i + C(\nabla \bar{q})q' = F$$

Figure below shows:

- $\circ$: $t$ vs. $a_i(t)$ (ROM coefficients).
- $\times$: $t$ vs. $(q'_{CFD}(x,t), \phi_i(x))$ (projection of snapshots onto modes).

Movie on right shows $v$-velocity snapshot (top) vs. 20 mode symmetry ROM solution $v$ (bottom).
ROM based on Linearized Navier-Stokes Retaining $\nabla \bar{q}$ Terms

$$q'_t + [A_i(\bar{q}) - K_i^{vw}(\nabla \bar{q})]q'_i - [K_{ij}(\bar{q})q'_{j},i] + C(\nabla \bar{q})q' = F$$

Figure below shows:

- $\circ$: $t$ vs. $a_i(t)$ (ROM coefficients).
- $\triangleright$: $t$ vs. $(q'_{CFD}(x,t), \phi_i(x))$ (projection of snapshots onto modes).

Movie on right shows $v$-velocity snapshot (top) vs. 20 mode symmetry ROM solution $v$ (bottom).
ROM based on Linearized Navier-Stokes Retaining $\nabla \bar{q}$ Terms

\[ q'_t + [A_i(\bar{q}) - K_{iw}^v(\nabla \bar{q})]q'_i - [K_{ij}(\bar{q})q'_j],i + C(\nabla \bar{q})q' = F \]

Figure below shows:

- $\circ$: $t$ vs. $a_i(t)$ (ROM coefficients).
- $\times$: $t$ vs. $(q'_\text{CFD}(x,t), \phi_i(x))$ (projection of snapshots onto modes).

Movie on right shows $v$-velocity snapshot (top) vs. 20 mode symmetry ROM solution $v$ (bottom).
ROM based on Linearized Navier-Stokes Retaining $\nabla \bar{q}$ Terms

\[
q_t' + \left[ A_i(\bar{q}) - K_i^{vw}(\nabla \bar{q}) \right] q_i' - \left[ K_{ij}(\bar{q})q_j' \right],i + C(\nabla \bar{q})q' = F
\]

Figure below shows:
- $\circ$: $t$ vs. $a_i(t)$ (ROM coefficients).
- $\blacksquare$: $t$ vs. $(q'_C, \phi_i(x))$ (projection of snapshots onto modes).

Movie on right shows $v$-velocity snapshot (top) vs. 20 mode symmetry ROM solution $v$ (bottom).
ROM based on Linearized Navier-Stokes Retaining $\nabla \bar{q}$ Terms

$$q'_{t} + [A_{i}(\bar{q}) - K^{vw}_{i}(\nabla \bar{q})]q'_{i} - [K_{ij}(\bar{q})q'_{j}],i + C(\nabla \bar{q})q' = F$$

- Figure below shows:
  - ◦ $t$ vs. $a_i(t)$ (ROM coefficients).
  - − $t$ vs. $(q'_C(x, t), \phi_i(x))$ (projection of snapshots onto modes).

- Movie on right shows $v$-velocity snapshot (top) vs. 20 mode symmetry ROM solution $v$ (bottom).
ROM based on Linearized Navier-Stokes Retaining $\nabla \bar{q}$ Terms

$q'_t + [A_i(\bar{q}) - K_{i}^{uw}(\nabla \bar{q})]q'_i - [K_{ij}(\bar{q})q'_j]_i + C(\nabla \bar{q})q' = F$

Figure below shows:

- $\circ$: $t$ vs. $a_i(t)$ (ROM coefficients).
- $\triangledown$: $t$ vs. $(q'_{CFD}(x, t), \phi_i(x))$ (projection of snapshots onto modes).

Movie on right shows $v$-velocity snapshot (top) vs. 20 mode symmetry ROM solution $v$ (bottom).
ROM based on Linearized Navier-Stokes Retaining $\nabla \bar{q}$ Terms

$q_{t} + [A_i(\bar{q}) - K_i^{uw}(\nabla \bar{q})]q_{i} - [K_{ij}(\bar{q})q_{j}]_i + C(\nabla \bar{q})q' = F$

Figure below shows:
- $\circ$: $t$ vs. $a_i(t)$ (ROM coefficients).
- $\triangleright$: $t$ vs. $(q'_{CFD}(x, t), \phi_i(x))$ (projection of snapshots onto modes).

Movie on right shows $v$-velocity snapshot (top) vs. 20 mode symmetry ROM solution $v$ (bottom).
ROM based on Linearized Navier-Stokes Retaining $\nabla \bar{q}$ Terms

\[ q',_t + [A_i(\bar{q}) - K_i^{vw}(\nabla \bar{q})]q',_i - [K_{ij}(\bar{q})q',_j],_i + C(\nabla \bar{q})q' = F \]

Figure below shows:

- $\circ$: $t$ vs. $a_i(t)$ (ROM coefficients).
- $\cdot$: $t$ vs. $(q'_C(x, t), \phi_i(x))$ (projection of snapshots onto modes).

Movie on right shows $v$-velocity snapshot (top) vs. 20 mode symmetry ROM solution $v$ (bottom).
ROM based on Linearized Navier-Stokes Retaining $\nabla \bar{q}$ Terms

$$q'_t + [A_i(\bar{q}) - K_i^{uv}(\nabla \bar{q})]q'_i - [K_{ij}(\bar{q})q'_j]_i + C(\nabla \bar{q})q' = F$$

- Figure below shows:
  - $\circ$: $t$ vs. $a_i(t)$ (ROM coefficients).
  - $\triangleright$: $t$ vs. $(q'_i (x, t), \phi_i (x))$ (projection of snapshots onto modes).

- Movie on right shows $v$-velocity snapshot (top) vs. 20 mode symmetry ROM solution $v$ (bottom).
ROM based on Linearized Navier-Stokes Retaining $\nabla \bar{q}$ Terms

$$q'_t + [A_i(\bar{q}) - K_{iw}^{vw}(\nabla \bar{q})]q'_i - [K_{ij}(\bar{q})q'_j],i + C(\nabla \bar{q})q' = F$$

Figure below shows:

- $\circ$: $t$ vs. $a_i(t)$ (ROM coefficients).
- $\triangledown$: $t$ vs. $(q'_C(x, t), \phi_i(x))$ (projection of snapshots onto modes).

Movie on right shows $v$-velocity snapshot (top) vs. 20 mode symmetry ROM solution $v$ (bottom).
ROM based on Linearized Navier-Stokes Retaining $\nabla \bar{q}$ Terms

\[
q'_{t, i} + [A_i(\bar{q}) - K_{iw}^{uw}(\nabla \bar{q})]q'_{i, i} - [K_{ij}(\bar{q})q'_{j, i}] + C(\nabla \bar{q})q' = F
\]

Figure below shows:
- $\circ$: $t$ vs. $a_i(t)$ (ROM coefficients).
- $\times$: $t$ vs. $(q'_{CFD}(x, t), \phi_i(x))$ (projection of snapshots onto modes).

Movie on right shows $v$-velocity snapshot (top) vs. 20 mode symmetry ROM solution $v$ (bottom).
ROM based on Linearized Navier-Stokes Retaining $\nabla \bar{q}$ Terms

$q'_{t} + [A_i(\bar{q}) - K^{vw}_i(\nabla \bar{q})]q'_i - [K_{ij}(\bar{q})q'_j],_i + C(\nabla \bar{q})q' = F$

- Figure below shows:
  - ◦: $t$ vs. $a_i(t)$ (ROM coefficients).
  - -: $t$ vs. $(q'_{CFD}(x, t), \phi_i(x))$ (projection of snapshots onto modes).

- Movie on right shows $v$-velocity snapshot (top) vs. 20 mode symmetry ROM solution $v$ (bottom).
ROM based on Linearized Navier-Stokes Retaining $\nabla \bar{q}$ Terms

$$q'_{t} + [A_i(\bar{q}) - K_i^{vw}(\nabla \bar{q})]q'_{,i} - [K_{ij}(\bar{q})q'_{,j}],_i + C(\nabla \bar{q})q' = F$$

Figure below shows:

- $\circ$: $t$ vs. $a_i(t)$ (ROM coefficients).
- $\triangleright$: $t$ vs. $(q'_{CFD}(x, t), \phi_i(x))$ (projection of snapshots onto modes).

Movie on right shows $v$-velocity snapshot (top) vs. 20 mode symmetry ROM solution $v$ (bottom).
ROM based on Linearized Navier-Stokes Retaining $\nabla \bar{q}$ Terms

$$q_{t}^{'} + \left[ A_{i}(\bar{q}) - K_{i}^{uw}(\nabla \bar{q}) \right] q_{i}^{'} - \left[ K_{ij}(\bar{q})q_{j}^{'} \right]_{,i} + C(\nabla \bar{q})q^{'} = F$$

- Figure below shows:
  - $\circ$: $t$ vs. $a_{i}(t)$ (ROM coefficients).
  - $\triangledown$: $t$ vs. $(q'^{CFD}(x, t), \phi_{i}(x))$ (projection of snapshots onto modes).

- Movie on right shows $v$-velocity snapshot (top) vs. 20 mode symmetry ROM solution $v$ (bottom).
ROM based on Linearized Navier-Stokes Retaining $\nabla \bar{q}$ Terms

$$q'_t + [A_i(\bar{q}) - K_{iw}(\nabla \bar{q})]q'_i - [K_{ij}(\bar{q})q'_j],i + C(\nabla \bar{q})q' = F$$

Figure below shows:

- $\circ$: $t$ vs. $a_i(t)$ (ROM coefficients).
- $\triangleright$: $t$ vs. $(q'_{CFD}(x,t), \phi_i(x))$ (projection of snapshots onto modes).

Movie on right shows $v$-velocity snapshot (top) vs. 20 mode symmetry ROM solution $v$ (bottom).
ROM based on Linearized Navier-Stokes Retaining $\nabla \bar{q}$ Terms

\[ q'_t + [A_i(q) - K_{iw}(\nabla \bar{q})]q'_i - [K_{ij}(q)q'_j],i + C(\nabla \bar{q})q' = F \]

- Figure below shows:
  - $t$ vs. $a_i(t)$ (ROM coefficients).
  - $t$ vs. $(q'_{CFD}(x,t), \phi_i(x))$ (projection of snapshots onto modes).

- Movie on right shows $v$-velocity snapshot (top) vs. 20 mode symmetry ROM solution $v$ (bottom).
ROM based on Linearized Navier-Stokes Retaining $\nabla \bar{q}$ Terms

$q'_t + [A_i(\bar{q}) - K^{uw}_i(\nabla \bar{q})]q'_i - [K_{ij}(\bar{q})q'_j],i + C(\nabla \bar{q})q' = F$

Figure below shows:
- $\circ$: $t$ vs. $a_i(t)$ (ROM coefficients).
- $\triangledown$: $t$ vs. $(q'_C F D(x),t),\phi_i(x))$ (projection of snapshots onto modes).

Movie on right shows $v$-velocity snapshot (top) vs. 20 mode symmetry ROM solution $v$ (bottom).
ROM based on Linearized Navier-Stokes Retaining $\nabla \bar{q}$ Terms

\[ q'_{t} + [A_{i}(\bar{q}) - K_{i}^{uw}(\nabla \bar{q})]q'_{i} - [K_{ij}(\bar{q})q'_{j}]_{,i} + C(\nabla \bar{q})q' = F \]

Figure below shows:

- $\circ$: $t$ vs. $a_i(t)$ (ROM coefficients).
- $\therefore$: $t$ vs. $(q'_{CFD}(x,t), \phi_i(x))$ (projection of snapshots onto modes).

Movie on right shows $v$-velocity snapshot (top) vs. 20 mode symmetry ROM solution $v$ (bottom).
ROM based on Linearized Navier-Stokes Retaining $\nabla \bar{q}$ Terms

\[ q'_{t} + [A_i(\bar{q}) - K_i^{uw}(\nabla \bar{q})]q'_{i} - [K_{ij}(\bar{q})q'_{j}],i + C(\nabla \bar{q})q' = F \]

- Figure below shows:
  - $\circ$: $t$ vs. $a_i(t)$ (ROM coefficients).
  - $\triangledown$: $t$ vs. $(q'_{CFD}(x, t), \phi_i(x))$ (projection of snapshots onto modes).

- Movie on right shows $v$-velocity snapshot (top) vs. 20 mode symmetry ROM solution $v$ (bottom).
ROM based on Linearized Navier-Stokes Retaining $\nabla \overline{q}$ Terms

$$q_{t} + [A_{i}(\overline{q}) - K_{i}^{vw}(\nabla \overline{q})]q_{i} - [K_{ij}(\overline{q})q_{j}]_{,i} + C(\nabla \overline{q})q' = F$$

Figure below shows:

- $\circ$: $t$ vs. $a_{i}(t)$ (ROM coefficients).
- $\triangledown$: $t$ vs. $(q'_{CFD}(x, t), \phi_{i}(x))$ (projection of snapshots onto modes).

Movie on right shows $v$-velocity snapshot (top) vs. 20 mode symmetry ROM solution $v$ (bottom).
ROM based on Linearized Navier-Stokes Retaining $\nabla \bar{q}$ Terms

$$q'_t + [A_i(\bar{q}) - K_i^{uw}(\nabla \bar{q})]q'_i - [K_{ij}(\bar{q})q'_j],_i + C(\nabla \bar{q})q' = F$$

Figure below shows:

- $t$ vs. $a_i(t)$ (ROM coefficients).
- $t$ vs. $(q'_{CFD}(x, t), \phi_i(x))$ (projection of snapshots onto modes).

Movie on right shows $v$-velocity snapshot (top) vs. 20 mode symmetry ROM solution $v$ (bottom).
ROM based on Linearized Navier-Stokes Retaining $\nabla \bar{q}$ Terms

\[ q_t' + [A_i(\bar{q}) - K_{iw}^v(\nabla \bar{q})]q_i' - [K_{ij}(\bar{q})q_j',i] + C(\nabla \bar{q})q' = F \]

Figure below shows:

- $\circ$: $t$ vs. $a_i(t)$ (ROM coefficients).
- $\triangleright$: $t$ vs. $(q'_{CFD}(x, t), \phi_i(x))$ (projection of snapshots onto modes).

Movie on right shows $v$-velocity snapshot (top) vs. 20 mode symmetry ROM solution $v$ (bottom).
ROM based on Linearized Navier-Stokes Retaining $\nabla \bar{q}$ Terms

$$q'_t + [A_i(\bar{q}) - K^w_i(\nabla \bar{q})]q'_{,i} - [K_{ij}(\bar{q})q'_{,j},i] + C(\nabla \bar{q})q' = F$$

Figure below shows:

- $\circ$: $t$ vs. $a_i(t)$ (ROM coefficients).
- $\triangleright$: $t$ vs. $(q'_{CFD}(x,t), \phi_i(x))$ (projection of snapshots onto modes).

Movie on right shows $v$-velocity snapshot (top) vs. 20 mode symmetry ROM solution $v$ (bottom).
ROM based on Linearized Navier-Stokes Retaining $\nabla \bar{q}$ Terms

$$q'_t + \left[ A_i(\bar{q}) - K_{iw}(\nabla \bar{q}) \right] q'_{,i} - \left[ K_{ij}(\bar{q})q'_{,j} \right]_{,i} + C(\nabla \bar{q})q' = F$$

- Figure below shows:
  - ◦: $t$ vs. $a_{i}(t)$ (ROM coefficients).
  - −: $t$ vs. $(q'_{CFD}(x, t), \phi_{i}(x))$ (projection of snapshots onto modes).

- Movie on right shows $v$-velocity snapshot (top) vs. 20 mode symmetry ROM solution $v$ (bottom).
ROM based on Linearized Navier-Stokes Retaining $\nabla \vec{q}$ Terms

$$q'_t + [A_i(\bar{q}) - K_{i}^{vw}(\nabla \bar{q})]q'_i - [K_{ij}(\bar{q})q'_{j,i}] + C(\nabla \bar{q})q' = F$$

Figure below shows:

- $\circ$: $t$ vs. $a_i(t)$ (ROM coefficients).
- $\bullet$: $t$ vs. $(q'_{CFD}(x,t), \phi_i(x))$ (projection of snapshots onto modes).

Movie on right shows $v$-velocity snapshot (top) vs. 20 mode symmetry ROM solution $v$ (bottom).
ROM based on Linearized Navier-Stokes Retaining $\nabla \tilde{q}$ Terms

$$q'_{,t} + [A_i(\tilde{q}) - K_{iw}^v(\nabla \tilde{q})]q'_{,i} - [K_{ij}(\tilde{q})q'_{,j}]_{,i}$$
$$+ C(\nabla \tilde{q})q' = F$$

Figure below shows:

- $\circ$: $t$ vs. $a_i(t)$ (ROM coefficients).
- $\triangleright$: $t$ vs. $(q'_{CFD}(x, t), \phi_i(x))$ (projection of snapshots onto modes).

Movie on right shows $v$-velocity snapshot (top) vs. 20 mode symmetry ROM solution $v$ (bottom).
ROM based on Linearized Navier-Stokes Retaining $\nabla \bar{q}$ Terms

\[
q'_t + [A_i(\bar{q}) - K^{uw}_i(\nabla \bar{q})]q'_i - [K^{ij}_i(\bar{q})q'_j]_i + C(\nabla \bar{q})q' = F
\]

Figure below shows:

- $\circ$: $t$ vs. $a_i(t)$ (ROM coefficients).
- $\triangleright$: $t$ vs. $(q'^{CFD}_i(x,t), \phi_i(x))$ (projection of snapshots onto modes).

Movie on right shows $v$-velocity snapshot (top) vs. 20 mode symmetry ROM solution $v$ (bottom).
ROM based on Linearized Navier-Stokes Retaining $\nabla \bar{q}$ Terms

\[ q', t + [A_i(\bar{q}) - K_{i}^{w} (\nabla \bar{q})]q', i - [K_{ij}(\bar{q})q', j]i + C(\nabla \bar{q})q' = F \]

Figure below shows:
- ◆: \( t \) vs. \( a_i(t) \) (ROM coefficients).
- ▲: \( t \) vs. \( (q'_{CFD}(x,t), \phi_i(x)) \) (projection of snapshots onto modes).

Movie on right shows \( v \)-velocity snapshot (top) vs. 20 mode symmetry ROM solution \( v \) (bottom).
ROM based on Linearized Navier-Stokes Retaining $\nabla \bar{q}$ Terms

$$q'_t + \left[ A_i(\bar{q}) - K_{vw}^i(\nabla \bar{q})\right] q'_i - \left[ K_{ij}(\bar{q})q'_{j,i}\right],i + C(\nabla \bar{q})q' = F$$

Figure below shows:

- $\circ$: $t$ vs. $a_i(t)$ (ROM coefficients).
- $\triangle$: $t$ vs. $(q'_{CFD}(x, t),\phi_i(x))$ (projection of snapshots onto modes).

Movie on right shows $v$-velocity snapshot (top) vs. 20 mode symmetry ROM solution $v$ (bottom).
ROM based on Linearized Navier-Stokes Retaining $\nabla \bar{q}$ Terms

$$q'_t + [A_i(\bar{q}) - K_{i\nu}(\nabla \bar{q})]q'_i - [K_{ij}(\bar{q})q'_j]_i + C(\nabla \bar{q})q' = F$$

Figure below shows:
- $\circ$: $t$ vs. $a_i(t)$ (ROM coefficients).
- $\triangledown$: $t$ vs. $(q'_{CFD}(x, t), \phi_i(x))$ (projection of snapshots onto modes).

Movie on right shows $v$-velocity snapshot (top) vs. 20 mode symmetry ROM solution $v$ (bottom).
ROM based on Linearized Navier-Stokes Retaining $\nabla \bar{q}$ Terms

$$q'_{t} + [A_{i}(\bar{q}) - K^{vw}_{i}(\nabla \bar{q})]q'_{i} - [K_{ij}(\bar{q})q'_{j}],i + C(\nabla \bar{q})q' = F$$

- Figure below shows:
  - ●: $t$ vs. $a_{i}(t)$ (ROM coefficients).
  - ▶: $t$ vs. $(q'_{CFD}(x, t), \phi_{i}(x))$ (projection of snapshots onto modes).

- Movie on right shows $v$-velocity snapshot (top) vs. 20 mode symmetry ROM solution $v$ (bottom).
ROM based on Linearized Navier-Stokes Retaining $\nabla \bar{q}$ Terms

$$q_{t} + [A_{i}(ar{q}) - K_{i}^{vw} (\nabla \bar{q})]q_{i} - [K_{ij} (\bar{q})q_{j},i] + C(\nabla \bar{q})q' = F$$

Figure below shows:

- $t$ vs. $a_{i}(t)$ (ROM coefficients).
- $t$ vs. $(q'_{CFD}(x,t), \phi_{i}(x))$ (projection of snapshots onto modes).

Movie on right shows $v$-velocity snapshot (top) vs. 20 mode symmetry ROM solution $v$ (bottom).
q′,t + [A_i(\bar{q}) - K^{qw}_i(\nabla \bar{q})]q′,i - [K_{ij}(\bar{q})q′,j],i + C(\nabla \bar{q})q′ = F

Figure below shows:

- •: t vs. a_i(t) (ROM coefficients).
- •−: t vs. (q′_{CFD}(x, t), \phi_i(x)) (projection of snapshots onto modes).

Movie on right shows v-velocity snapshot (top) vs. 20 mode symmetry ROM solution v (bottom).
ROM based on Linearized Navier-Stokes Retaining $\nabla \bar{q}$ Terms

\[ q'_t + [A_i(\bar{q}) - K_i^{vw}(\nabla \bar{q})]q'_i - [K_{ij}(\bar{q})q'_{j}],i + C(\nabla \bar{q})q' = F \]

Figure below shows:

- ○: $t$ vs. $a_i(t)$ (ROM coefficients).
- -: $t$ vs. $(q'_{CFD}(x, t), \phi_i(x))$ (projection of snapshots onto modes).

Movie on right shows $v$-velocity snapshot (top) vs. 20 mode symmetry ROM solution $v$ (bottom).
ROM based on Linearized Navier-Stokes Retaining $\nabla \bar{q}$ Terms

$q''_t + [A_i(\bar{q}) - K^{uw}_i(\nabla \bar{q})]q'_i - [K_{ij}(\bar{q})q'_j],i + C(\nabla \bar{q})q' = F$

- Figure below shows:
  - ○: $t$ vs. $a_i(t)$ (ROM coefficients).
  - -: $t$ vs. $(q'_{CFD}(x, t), \phi_i(x))$ (projection of snapshots onto modes).

- Movie on right shows $v$-velocity snapshot (top) vs. 20 mode symmetry ROM solution $v$ (bottom).
ROM based on Linearized Navier-Stokes Retaining $\nabla \bar{q}$ Terms

$q'_t + [A_i(\bar{q}) - K_i^{vw}(\nabla \bar{q})]q'_i - [K_{ij}(\bar{q})q'_{,j}]_i + C(\nabla \bar{q})q' = F$

Figure below shows:

- o: $t$ vs. $a_i(t)$ (ROM coefficients).
- -: $t$ vs. $(q'_{CFD}(x, t), \phi_i(x))$ (projection of snapshots onto modes).

Movie on right shows $v$-velocity snapshot (top) vs. 20 mode symmetry ROM solution $v$ (bottom).
ROM based on Linearized Navier-Stokes Retaining $\nabla \bar{q}$ Terms

\[
q'_t + [A_i(\bar{q}) - K^w_i(\nabla \bar{q})]q'_i - [K_{ij}(\bar{q})q'_j]_i + C(\nabla \bar{q})q' = F
\]

- Figure below shows:
  - ○: $t$ vs. $a_i(t)$ (ROM coefficients).
  - -: $t$ vs. ($q'_C F D(x,t), \phi_i(x)$) (projection of snapshots onto modes).

- Movie on right shows $v$-velocity snapshot (top) vs. 20 mode symmetry ROM solution $v$ (bottom).
ROM based on Linearized Navier-Stokes Retaining $\nabla \bar{q}$ Terms
Summary & Future Work

- A Galerkin ROM in which the \textit{continuous} equations are projected onto the basis modes in a \textit{continuous} inner product is proposed.
- The choice of inner product for the Galerkin projection step is crucial to stability of the ROM.
  - For linearized compressible flow, Galerkin projection in the “symmetry” inner product leads to a ROM that is stable for any choice of basis.
  - Continuous “symmetry” inner product has discrete counterpart that can be determined in a black box fashion for \textit{any} stable linear system.
- Extensions to non-linear compressible flows based on a local linearization of the governing equations prior to projection is described.
- Performance of the proposed POD/Galerkin ROM is examined on a linear as well as a non-linear test case.
  - LQR controller design/performance demonstrated on linear test case (driven inviscid pulse).
  - Importance of retaining velocity gradient terms in ROM equations illustrated on non-linear test case (driven cavity)

\textbf{Future Work:} Controller design for non-linear cavity problems
References
(www.sandia.gov/~ikalash)


References
(www.sandia.gov/~ikalash)


Thank you! Questions?
ikalash@sandia.gov
Linearized ROM System Matrices

\[ A_1 = \begin{pmatrix} \bar{u}_1 & 0 & 0 & R & \frac{RT}{\bar{\rho}} \\ 0 & \bar{u}_1 & 0 & 0 & 0 \\ 0 & 0 & \bar{u}_1 & 0 & 0 \\ \bar{T}(\gamma - 1) & 0 & 0 & \bar{u}_1 & 0 \\ \bar{\rho} & 0 & 0 & 0 & \bar{u}_1 \end{pmatrix} , \quad A_2 = \begin{pmatrix} \bar{u}_2 & 0 & 0 & 0 & 0 \\ 0 & \bar{u}_2 & 0 & R & \frac{RT}{\bar{\rho}} \\ 0 & 0 & \bar{u}_2 & 0 & 0 \\ 0 & \bar{T}(\gamma - 1) & 0 & \bar{u}_2 & 0 \\ \bar{\rho} & 0 & 0 & 0 & \bar{u}_2 \end{pmatrix} \\
A_3 = \begin{pmatrix} \bar{u}_3 & 0 & 0 & 0 & 0 \\ 0 & \bar{u}_3 & 0 & 0 & 0 \\ 0 & 0 & \bar{u}_3 & R & \frac{RT}{\bar{\rho}} \\ 0 & 0 & \bar{T}(\gamma - 1) & \bar{u}_3 & 0 \\ \bar{\rho} & 0 & 0 & \bar{u}_3 \end{pmatrix} \]

\[ K_{11} \equiv \frac{1}{\bar{\rho}Re} \begin{pmatrix} 2\mu + \lambda & 0 & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 & 0 \\ 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & \frac{\gamma \kappa}{Pr} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} , \]

\[ K_{12} \equiv \frac{1}{\bar{\rho}Re} \begin{pmatrix} 0 & 0 & \lambda & 0 & 0 \\ \mu & 0 & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 & 0 \\ 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 \end{pmatrix} , \]

\[ K_{13} \equiv \frac{1}{\bar{\rho}Re} \begin{pmatrix} 0 & 0 & \lambda & 0 & 0 \\ \mu & 0 & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 & 0 \\ 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 \end{pmatrix} \]

\[ K_{21} \equiv \frac{1}{\bar{\rho}Re} \begin{pmatrix} 0 & \lambda & 0 & 0 & 0 \\ \mu & 0 & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 & 0 \\ 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 \end{pmatrix} , \quad K_{22} \equiv \frac{1}{\bar{\rho}Re} \begin{pmatrix} \mu & 0 & 0 & 0 & 0 \\ 0 & 2\mu + \lambda & 0 & 0 & 0 \\ 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 \end{pmatrix} \]
Linearized ROM System Matrices (continued)

\[
K_{23} \equiv \frac{1}{\bar{\rho}R_e} \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda & 0 & 0 \\
0 & \mu & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad K_{31} \equiv \frac{1}{\bar{\rho}R_e} \begin{pmatrix}
0 & 0 & \mu & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\lambda & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
K_{32} \equiv \frac{1}{\bar{\rho}R_e} \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mu & 0 & 0 \\
0 & \lambda & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \mu & 0
\end{pmatrix}, \quad K_{33} \equiv \frac{1}{\bar{\rho}R_e} \begin{pmatrix}
0 & 0 & \mu & 0 & 0 \\
0 & 0 & 0 & \mu & 0 \\
0 & 0 & 0 & 0 & \mu + \lambda \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{\gamma \kappa}{Pr}
\end{pmatrix}
\]

\[
K_{vw1} \equiv \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\frac{(\gamma - 1)}{R \bar{\rho}} \bar{\tau}_{11} & \frac{(\gamma - 1)}{R \bar{\rho}} \bar{\tau}_{12} & \frac{(\gamma - 1)}{R \bar{\rho}} \bar{\tau}_{13} & 0 & 0 \\
0 & 0 & \frac{(\gamma - 1)}{R \bar{\rho}} \bar{\tau}_{21} & \frac{(\gamma - 1)}{R \bar{\rho}} \bar{\tau}_{22} & \frac{(\gamma - 1)}{R \bar{\rho}} \bar{\tau}_{23} \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad K_{vw2} \equiv \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\frac{(\gamma - 1)}{R \bar{\rho}} \bar{\tau}_{31} & \frac{(\gamma - 1)}{R \bar{\rho}} \bar{\tau}_{32} & \frac{(\gamma - 1)}{R \bar{\rho}} \bar{\tau}_{33} & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
K_{vw3} \equiv \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\frac{(\gamma - 1)}{R \bar{\rho}} \bar{\tau}_{31} & \frac{(\gamma - 1)}{R \bar{\rho}} \bar{\tau}_{32} & \frac{(\gamma - 1)}{R \bar{\rho}} \bar{\tau}_{33} & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]
\[ C = \begin{pmatrix}
\frac{\partial \bar{u}}{\partial x} & \frac{\partial \bar{u}}{\partial y} & \frac{\partial \bar{u}}{\partial z} & \frac{R}{\rho} \frac{\partial \bar{\rho}}{\partial x} & \frac{1}{\rho} \left( \bar{u} \cdot \nabla \bar{u} + R \frac{\partial T}{\partial x} \right) \\
\frac{\partial \bar{v}}{\partial x} & \frac{\partial \bar{v}}{\partial y} & \frac{\partial \bar{v}}{\partial z} & \frac{R}{\rho} \frac{\partial \bar{\rho}}{\partial y} & \frac{1}{\rho} \left( \bar{u} \cdot \nabla \bar{v} + R \frac{\partial T}{\partial y} \right) \\
\frac{\partial \bar{w}}{\partial x} & \frac{\partial \bar{w}}{\partial y} & \frac{\partial \bar{w}}{\partial z} & \frac{R}{\rho} \frac{\partial \bar{\rho}}{\partial z} & \frac{1}{\rho} \left( \bar{u} \cdot \nabla \bar{w} + R \frac{\partial T}{\partial z} \right) \\
\frac{\partial \bar{T}}{\partial x} & \frac{\partial \bar{T}}{\partial y} & \frac{\partial \bar{T}}{\partial z} & (\gamma - 1) \nabla \cdot \bar{u} & 0 \\
\frac{\partial \bar{\rho}}{\partial x} & \frac{\partial \bar{\rho}}{\partial y} & \frac{\partial \bar{\rho}}{\partial z} & 0 & \frac{1}{\rho} \left( \bar{u} \cdot \nabla \bar{T} + (\gamma - 1) \bar{T} \nabla \cdot \bar{u} \right) \\
\frac{\partial \bar{\rho}}{\partial x} & \frac{\partial \bar{\rho}}{\partial y} & \frac{\partial \bar{\rho}}{\partial z} & 0 & \nabla \cdot \bar{\bar{u}}
\end{pmatrix} \]