Recent Extensions of the Discontinuous Enrichment Method (DEM) to Advection-Dominated Fluid Mechanics Problems

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Motivation

- Advection Dominated Flow Equation
  \[ \frac{\partial u}{\partial t} + \nabla \cdot (u \mathbf{u}) = f \]
  - Advection velocity: \( \mathbf{u} = (u, v, w) \)
  - \( u, v, w \): velocity components
  - \( f \): source term

- Discrete many transport phenomena in fluid mechanics.
- Usual scalar model for the more challenging Navier-Stokes equations
- Global Peclet number (L - length scale associated with 3D) is:
  \[ \frac{Pl = \frac{\text{rate of advection}}{\text{rate of diffusion}}}{\text{Pr} = \frac{\text{thermical diffusion}}{\text{mass diffusion}}} \]

- Advection-Dominated Regime
  - Sharp gradients in exact solution
  - Galerkin FEM inadequate: spurious oscillations (Fig. 1)

- Some Classical Remedies
  - Subgrid FE, MUSCL, WENO
  - Adding weighted residuals (numerical diffusion) to variational equation to damp out oscillations.

Discontinuous Enrichment Method

- First proposed and developed by Farhat et. al [1] for the solution of the Helmholtz equation.
- Enrich the usual Galerkin polynomial field \( \phi^p \) by the free-space solutions to the governing constant-coefficient homogeneous PDE.
  \[ \phi^e(x, y) = \frac{1}{2\pi} \ln \left( \frac{r^2}{2\pi} \right) \]

- Unlike MUSCL, WENO enrichment field in DEM is not required to vanish at element boundaries
- Continuity across element boundaries is enforced weakly using Lagrange multipliers \( \lambda^e \) in \( \phi^e \).

Two Variants of DEM: True DEM vs. Pure DGM

- Primal unknown \( \phi \) vs. \( \phi^e \) has one of the two forms:

  -True DEM
    - Elements enriched with the imposed solution (\( \phi^e \))
    - For enrichment-by-pair formulation, splitting of the approximation into coarse (polynomial) and fine (enrichment) scales

  -Pure DGM
    - Elements enriched with the imposed solution (\( \phi^e \))
    - For enrichment-by-pair formulation, splitting of the approximation into coarse (polynomial) and fine (enrichment) scales

Implementation

- Element matrix problem (uncondensed):
  \[ \begin{bmatrix} \mathbf{K}^p & \mathbf{M}^p & \mathbf{K}^e \end{bmatrix} \begin{bmatrix} \mathbf{u}^p \mathbf{u}^e \end{bmatrix} = \begin{bmatrix} \mathbf{f}^p \end{bmatrix} \]

- Due to the discontinuous nature of \( \phi^e \), \( \mathbf{K}^e \) can be eliminated at the element level by a static condensation.

- Statically-enriched true DEM element system:
  \[ \begin{bmatrix} \mathbf{K}^p \mathbf{M}^p \mathbf{K}^e \end{bmatrix} \begin{bmatrix} \mathbf{u}^p \mathbf{u}^e \end{bmatrix} = \begin{bmatrix} \mathbf{f}^p \end{bmatrix} \]

- Statically-enriched pure DGM element system:
  \[ \begin{bmatrix} \mathbf{K}^p \mathbf{M}^p \mathbf{K}^e \end{bmatrix} \begin{bmatrix} \mathbf{u}^p \mathbf{u}^e \end{bmatrix} = \begin{bmatrix} \mathbf{f}^p \end{bmatrix} \]

- Discrete Babuška-Brezzi inf-sup condition: a.e. in the mesh
  \[ \exists \alpha \geq 0 \quad \| \mathbf{u}^p \|_{1, \mathbf{H}^1} \geq \alpha \| \mathbf{u}^e \|_{0, \mathbf{L}^2} \]

- Boundary is necessary but is general not a sufficient condition for ensuring a non-singular global interface problem.

- Numerical tests (Fig. 4) suggest to form:
  \[ \mathbf{K}^e = \mathbf{H}^e \mathbf{H}^e \mathbf{K}^e \mathbf{H}^e \]

- The set \( \mathbf{H}^e \) that defines the enrichment field typically leads to too many Lagrange multipliers.

Conclusion & Ongoing Work

- DGM/DEM Elements outperform their Galerkin and stabilized Galerkin counterparts of comparable complexity by at least one (and sometimes many) orders of magnitude difference.
- For \( Pl = 10^5 \), to achieve a 0.1% level of relative error:
  - Q0 - 2 and Q2 - 2 elements: reduce the dof requirement of the Q2 element by a factor of \( 5 \)
  - Q1 - 2 and Q1 - 4 elements: reduce the dof requirement of the Q1 element by a factor of \( 5 \)
  - Q2 - 2 and Q2 - 4 elements: reduce the dof requirement of the Q2 element by a factor of \( 5 \)

- In high-Peclet regime, DGM and DEM solutions are almost completely oscillation-free, in contrast with the Galerkin solutions.
- Results presented herein demonstrate the potential of DEM for realistic advection-dominant transport problems in fluid mechanics.

Numerical Results

- Double Ramp Problem on an I-shaped Domain
  - Homogeneous Dirichlet boundary conditions are prescribed on all sides of the I-shaped domain.
  - Advection direction: \( \phi = 0 \)
  - Strong outflow boundary layer along the line \( y = 1 \)
  - Two crosed-flow boundary layers along \( y = 0 \) and \( y = 1 \)
  - A crossed internal layer along \( y = 0.5 \)

- Algorithm 1: DEM/DEM Element Design
  - Two Lagrange multipliers \( \Lambda(x, y) \) and \( \Lambda(x, y') \) defined on a straight edge are redundant.
  - \( \Lambda(x, y) = C \Lambda(x, y') \) for some real constant \( C \).

- Key Observations:
  - The set of angles that are clustered around \( \theta = \frac{\pi}{2} \) is the necessary angle condition for vanishing of the shear derivative of \( \Lambda \).

- Some DGM/DEM Elements
  - Enrichment Functions:
    \[ \phi^e = \phi^p + \sum_{r=0}^{R} \sum_{\alpha=0}^{N_r} a_{r,\alpha} \phi_{r,\alpha} \]
  - Lagrange multipliers:
    \[ \lambda^e = \frac{1}{2} \nabla \phi^e \cdot \mathbf{n} \]

- Conclusion

- References

References