The Discontinuous Enrichment Method (DEM) for Multi-Scale Transport Problems

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Outline

1. Motivation
2. Advection-Diffusion Equation
3. Discontinuous Enrichment Method (DEM)
4. DEM for Constant-Coefficient Advection-Diffusion
   - Enrichment Bases
   - Lagrange Multiplier Approximations
   - Element Design
   - Numerical Experiments
5. DEM for Variable-Coefficient Advection-Diffusion
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   - Lagrange Multiplier Approximations
   - Element Design
   - Numerical Experiments
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The Finite Element Method (FEM) in Fluid Mechanics

- Galerkin **Finite Element Method** (FEM) has a number of attractions in fluid mechanics:
  - Flexibility in handling complex geometries.
  - Ability to handle different forms of boundary conditions.

- FEM is quasi-optimal for elliptic (\textit{diffusion}-dominated) PDEs: assures good performance of the computation at any mesh resolution.

However:

- coarse mesh accuracy is not guaranteed when the flow is \textit{advection}-dominated!
- Significant mesh refinement typically needed to capture boundary layer region EXPENSIVE!

Goal:

- build an efficient method that can accurately capture boundary layers.

Approach:

- start with simple canonical equation; then generalize.
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### 2D Scalar Advection-Diffusion Equation

\[
\mathcal{L}c = -\kappa \Delta c + a \cdot \nabla c = f
\]

- **Advection velocity:**
  \[ a = (a_1, a_2)^T = |a|(\cos \phi, \sin \phi)^T. \]
- **φ** = advection direction.
- \( \kappa = \) diffusivity.

- **Describes many transport phenomena in fluid mechanics:**
  - Heat transfer.
  - Semi-conductor device modeling.
  - Usual scalar model for the more challenging Navier-Stokes equations.
2D Scalar Advection-Diffusion Equation

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- **Global Péclet number** \((L = \text{length scale associated with } \Omega)\):
  \[ Pe = \frac{\text{rate of advection}}{\text{rate of diffusion}} = \frac{L|\mathbf{a}|}{\kappa} = Re \cdot \begin{cases} Pr \quad (\text{thermal diffusion}) \\ Sc \quad (\text{mass diffusion}) \end{cases} \]
Advection-Dominated Regime

• Typical applications: flow is advection-dominated.

![Figure 1: Galerkin $Q_1$ solution (color) vs. exact solution (black) for $Pe = 150$](image)

Advection-Dominated (High $Pe$) Regime
↓
Sharp gradients in exact solution
↓
Galerkin FEM inadequate: spurious oscillations (Fig. 1)

• Some classical remedies:
  - Stabilized FEMs (SUPG, GLS, USFEM): add weighted residual (numerical diffusion) to variational equation.
  - RFB, VMS, PUM: construct conforming spaces that incorporate knowledge of local behavior of solution.
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The Discontinuous Enrichment Method (DEM)

Idea of DEM:

“Enrich” the usual Galerkin polynomial field $\mathcal{V}^P$ by the free-space solutions to the governing homogeneous PDE $\mathcal{L}c = 0$.

$$c^h = c^P + c^E \in \mathcal{V}^P \oplus (\mathcal{V}^E \setminus \mathcal{V}^P)$$

where

$$\mathcal{V}^E = \text{span}\{c : \mathcal{L}c = 0\}$$

- **Simple 1D Example:**

$$\begin{cases} u_x - u_{xx} = 1 + x, & x \in (0, 1) \\ u(0) = 0, u(1) = 1 \end{cases}$$

- **Enrichments:** $u_x^E - u_{xx}^E = 0 \Rightarrow u^E = C_1 + C_2 e^x \Rightarrow \mathcal{V}^E = \text{span}\{1, e^x\}$.

- **Galerkin FEM polynomials:** $\mathcal{V}^P_{\Omega^e=(x_j,x_{j+1})} = \text{span} \left\{ \frac{x_{j+1} - x}{h}, \frac{x - x_j}{h} \right\}$. 
History of DEM’s Success

- **Acoustic scattering problems** (Helmholtz equation) [4,5].
  - First developed by Farhat *et. al* in 2000 for the Helmholtz equation.
  - A family of 3D hexahedral DEM elements for medium frequency problems achieved the same solution accuracy as Galerkin elements of comparable convergence order using 4–8 times fewer dofs, and *up to 60 times less CPU time* [4].
  - Numerically scalable domain decomposition-based iterative solver for 2D and 3D acoustic scattering problems in medium- and high-frequency regimes has been developed [5].

- **Wave propagation in elastic media** (Navier’s equation) [6].

- **Fluid-structure interaction problems** (Navier’s equation and the Helmholtz equation) [7, 8].
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Excellent performance motivates development of DEM for other applications
→ **Fluid Mechanics**
## Two Variants of DEM

- **Two variants of DEM:** “pure DGM” vs. “true DEM”

<table>
<thead>
<tr>
<th></th>
<th>DGM</th>
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<tr>
<td>$V^h$</td>
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**Enrichment-Only “Pure DGM”:**
Contribution of the standard polynomial field is dropped entirely from the approximation.

**True or “Full” DEM:**
Splitting of the approximation into coarse (polynomial) and fine (enrichment) scales.

- Unlike PUM, VMS & RFB: enrichment field in DEM is not required to vanish at element boundaries
Two Variants of DEM

Two variants of DEM: “pure DGM” vs. “true DEM”

DGM

\[ \nabla^h \quad \nabla^E \quad c^h \quad c^E \]

DEM

\[ \nabla^P \oplus (\nabla^E \setminus \nabla^P) \quad c^P + c^E \]

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Contribution of the standard polynomial field is dropped entirely from the approximation.

True or “Full” DEM:
Splitting of the approximation into coarse (polynomial) and fine (enrichment) scales.

Unlike PUM, VMS & RFB: enrichment field in DEM is not required to vanish at element boundaries \( \Rightarrow \) DEM is discontinuous by construction!

DEM = DGM with Lagrange Multipliers
What about Inter-Element Continuity?

- Continuity across element boundaries is enforced weakly using Lagrange multipliers \( \lambda^h \in \mathcal{W}^h \):

\[
\lambda^h \approx \nabla c^E_e \cdot \mathbf{n}^e = -\nabla c^{E'}_{e'} \cdot \mathbf{n}^{e'} \quad \text{on } \Gamma^{e,e'}
\]

- Discrete Babuška-Brezzi \textit{inf-sup} condition\(^1\):

\[
\left\{ \begin{array}{c}
\# \text{ Lagrange multiplier constraint equations} \\
\leq \ # \text{ enrichment equations}
\end{array} \right.
\]

Rule of thumb to satisfy the Babuška-Brezzi \textit{inf-sup} condition is to limit:

\[
n^{\lambda} = \left\lfloor \frac{n^E}{4} \right\rfloor = \max \left\{ n \in \mathbb{Z} | n \leq \frac{n^E}{4} \right\}
\]

- \( n^{\lambda} \) = \# Lagrange multipliers per edge
- \( n^E \) = \# enrichment functions

\(^1\)Necessary condition for generating a non-singular global discrete problem.
Hybrid Variational Formulation of DEM

Strong form:

\[ (S) : \begin{cases} 
\text{Find } c \in H^1(\Omega) \text{ such that} \\
\quad -\kappa \Delta c + a \cdot \nabla c = f, \quad \text{in } \Omega \\
\quad c = g, \quad \text{on } \Gamma = \partial \Omega 
\end{cases} \]

Notation:

\[
\begin{align*}
\tilde{\Omega} &= \bigcup_{e=1}^{n_{\text{el}}} \Omega^e \\
\tilde{\Gamma} &= \bigcup_{e=1}^{n_{\text{el}}} \Gamma^e \\
\Gamma^{e,e'} &= \Gamma^e \cap \Gamma^{e'} \\
\Gamma^{\text{int}} &= \bigcup_{e' < e} \bigcup_{e=1}^{n_{\text{el}}} \{ \Gamma^e \cap \Gamma^{e'} \}
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\]

**Weak hybrid variational form:**

\[
(W) : \quad \begin{cases} 
\text{Find } (c, \lambda) \in V \times W \text{ such that:} \\
a(v, c) + b(\lambda, v) = r(v) \\
b(\mu, c) = -r_d(\mu) 
\end{cases}
\]

holds \( \forall c \in V, \forall \mu \in W \).

where

\[
a(v, c) = (\kappa \nabla v + \mathbf{v} \mathbf{a}, \nabla c)_{\tilde{\Omega}} \\
b(\lambda, v) = \sum_{e} \sum_{e' < e} \int_{\Gamma_{e,e'}} \lambda (v_{e'} - v_e) d\Gamma + \int_{\Gamma} \lambda v \ d\Gamma
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Discretization & Implementation

Element matrix problem (uncondensed):

\[
\begin{pmatrix}
{k_{PP}} & {k_{PE}} & {k_{PC}} \\
{k_{EP}} & {k_{EE}} & {k_{EC}} \\
{k_{CP}} & {k_{CE}} & 0
\end{pmatrix}
\begin{pmatrix}
{c^P} \\
{c^E} \\
{\lambda^h}
\end{pmatrix}
=
\begin{pmatrix}
{r^P} \\
{r^E} \\
{r^C}
\end{pmatrix}
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  Due to the discontinuous nature of $V_E$, $c^E$ can be eliminated at the element level by a static condensation.

- Statically-condensed True DEM Element:
  \[
  \begin{pmatrix}
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  \tilde{k}^{CP} & \tilde{k}^{CC}
  \end{pmatrix}
  \begin{pmatrix}
  c^P \\
  \lambda^h
  \end{pmatrix}
  =
  \begin{pmatrix}
  \tilde{r}^P \\
  \tilde{r}^C
  \end{pmatrix}
  \]

- Statically-condensed Pure DGM Element:
  \[
  -k^{CE}(k^{EE})^{-1}k^{EC}\lambda^h = r^C - k^{CE}(k^{EE})^{-1}r^E
  \]
Discretization & Implementation

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Angle-Parametrized Enrichment Functions for 2D Advection-Diffusion

- Derived by solving $\mathcal{L} c^E = \mathbf{a} \cdot \nabla c^E - \kappa \Delta c^E = 0$ analytically (e.g., separation of variables).

$$c^E(\mathbf{x}; \theta_i) = e^{\left(\frac{a_1 + |\mathbf{a}| \cos \theta_i}{2\kappa}\right)(x-x_{r,i})} e^{\left(\frac{a_2 + |\mathbf{a}| \sin \theta_i}{2\kappa}\right)(y-y_{r,i})}$$

(1)

- $\mathbf{a}^T \equiv (a_1, a_2) = \text{advection velocity vector}$
- $(x_{r,i}, y_{r,i}) = \text{reference point for } c^E_i$
- $\Theta^c \equiv \{\theta_i\}_{i=1}^{n^E} \in [0, 2\pi) = \text{set of angles specifying } V^E$

The parametrization with respect to $\theta_i$ in (1) is non-trivial!

- Enrichment functions are now specified by a set of “flow directions”.
- Parametrization enables systematic element design.
Plots of Enrichment Functions for Some Angles

$\theta_i \in [0, 2\pi)$

$\phi = 0, \theta_i = 0$

$\phi = 0, \theta_i = \frac{\pi}{2}$

$\phi = \frac{\pi}{2}, \theta_i = \frac{3\pi}{4}$

$\phi = 0, \theta_i = \pi$

$\phi = \frac{3\pi}{2}, \theta_i = \frac{5\pi}{4}$

$\phi = 0, \theta_i = \frac{3\pi}{2}$

Figure 2: Plots of enrichment function $c_E^x(\mathbf{x}; \theta_i)$ for several values of $\theta_i$ ($Pe = 20$)
What about the Lagrange Multiplier Approximations?

Figure 3: Straight edge $\Gamma_{e, e'}$ oriented at angle $\alpha_{e, e'} \in [0, 2\pi)$

- Trivial to compute given exponential enrichments:

$$\lambda^h(s)|_{\Gamma_{e, e'}} \approx \nabla c^E \cdot n|_{\Gamma_{e, e'}}$$

$$= \text{const} \cdot e^{\left\{ \frac{|a|}{2\kappa} \left[ \cos(\phi - \alpha_{e, e'}) + \cos(\theta_k - \alpha_{e, e'}) \right] (s - s_{e, e'}) \right\}}$$
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\]

Non-trivial to satisfy *inf-sup* condition: the set $\Theta^c$ that defines $V^E$ typically leads to too many Lagrange multiplier dofs!
Lagrange Multiplier Selection

- Define:
  \[ \Lambda_{i}^{e,e'} \equiv \frac{|a|}{2\kappa} \left[ \cos(\phi - \alpha^{e,e'}) + \cos(\theta_{k} - \alpha^{e,e'}) \right] \]

Illustration of Lagrange Multiplier selection for \( n^{\lambda} = 4 \)

- \( \Lambda^{e,e'}_{\min} \)
- \( \Lambda^{e,e'}_{1} \)
- \( \Lambda^{e,e'}_{2} \)
- \( \Lambda^{e,e'}_{3} \)
- \( \Lambda^{e,e'}_{\max} \)

\[ \lambda^{h}|_{\Gamma^{e,e'}} = \text{span} \left\{ e^{\Lambda_{i}^{e,e'} (s-s_{r,i}^{e,e'})}, 0 \leq s \leq h \right\} \]

- Determine # Lagrange multipliers allowed: \( \text{card}\{\Lambda_{i}^{e,e'}\} = \left\lfloor \frac{n^{E}}{4} \right\rfloor \).

- Sample \( \Lambda_{i}^{e,e'} \) uniformly in the interval \([\Lambda^{e,e'}_{\min}, \Lambda^{e,e'}_{\max}]\) to span space of all exponentials of the form \( \{e^{\Lambda_{i}^{e,e'} s} : \Lambda^{e,e'}_{\min} \leq \Lambda_{i}^{e,e'} \leq \Lambda^{e,e'}_{\max}\}\).
Mesh Independent Element Design Procedure

Algorithm 1. “Build Your Own DEM Element”

Fix \( n^E \in \mathbb{N} \) (the desired number of angles defining \( V^E \)).

Select a set of \( n^E \) distinct angles \( \Theta^c = \{ \theta_k \}_{k=1}^{n^E} \) between \([0, 2\pi)\).

Define the enrichment functions by:

\[
c^E(x; \Theta^c) = e^{\left( \frac{a_1 + |a| \cos \Theta^c}{2\kappa} \right)}(x - x_r,i) e^{\left( \frac{a_2 + |a| \sin \Theta^c}{2\kappa} \right)}(y - y_r,i)
\]

Determine \( n^\lambda = \left\lfloor \frac{n^E}{4} \right\rfloor \).

for each edge \( \Gamma^{e,e'} \in \Gamma^{\text{int}} \)

Compute max and min of \( \frac{|a|}{2\kappa} \left[ \cos(\phi - \alpha^{e,e'}) + \cos(\theta_k - \alpha^{e,e'}) \right] \), call them \( \Lambda^{e,e'}_{\min}, \Lambda^{e,e'}_{\max} \).

Sample \( \{\Lambda^e_{i,e'} : i = 1, \ldots, n^\lambda \} \) uniformly in the interval \([\Lambda^e_{\min}, \Lambda^e_{\max}]\).

Define the Lagrange multipliers approximations on \( \Gamma^{e,e'} \) by:

\[
\lambda^h|_{\Gamma^{e,e'}} = \text{span} \left\{ e^{\Lambda^e_{i,e'}(s - s^e_{r,i})}, 0 \leq s \leq h \right\}
\]

end for
**Element Nomenclature**

**Notation**

DGM Element: \( Q - n^E - n^\lambda \)
DEM Element: \( Q - n^E - n^{\lambda^+} \equiv [Q - n^E - n^\lambda] \cup [Q_1] \)

’\( Q’\): Quadrilateral
\( n^E\): Number of Enrichment Functions
\( n^\lambda\): Number of Lagrange Multipliers per Edge
\( Q_1\): Galerkin Bilinear Quadrilateral Element

<table>
<thead>
<tr>
<th>Name</th>
<th>( n^E )</th>
<th>( \Theta^c )</th>
<th>( n^\lambda )</th>
</tr>
</thead>
<tbody>
<tr>
<td>DGM elements:</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( Q - 4 - 1 )</td>
<td>4</td>
<td>( \phi + { \frac{m\pi}{2} : m = 0, ..., 3 } )</td>
<td>1</td>
</tr>
<tr>
<td>( Q - 8 - 2 )</td>
<td>8</td>
<td>( \phi + { \frac{m\pi}{4} : m = 0, ..., 7 } )</td>
<td>2</td>
</tr>
<tr>
<td>( Q - 12 - 3 )</td>
<td>12</td>
<td>( \phi + { \frac{m\pi}{6} : m = 0, ..., 11 } )</td>
<td>3</td>
</tr>
<tr>
<td>( Q - 16 - 4 )</td>
<td>16</td>
<td>( \phi + { \frac{m\pi}{8} : m = 0, ..., 15 } )</td>
<td>4</td>
</tr>
<tr>
<td>DEM elements:</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( Q - 5 - 1^+ )</td>
<td>5</td>
<td>( \phi + { \frac{2m\pi}{5} : m = 0, ..., 4 } )</td>
<td>1</td>
</tr>
<tr>
<td>( Q - 9 - 2^+ )</td>
<td>9</td>
<td>( \phi + { \frac{2m\pi}{9} : m = 0, ..., 8 } )</td>
<td>2</td>
</tr>
<tr>
<td>( Q - 13 - 3^+ )</td>
<td>13</td>
<td>( \phi + { \frac{2m\pi}{13} : m = 0, ..., 12 } )</td>
<td>3</td>
</tr>
<tr>
<td>( Q - 17 - 4^+ )</td>
<td>17</td>
<td>( \phi + { \frac{2m\pi}{17} : m = 0, ..., 16 } )</td>
<td>4</td>
</tr>
</tbody>
</table>
Illustration of the Sets $\Theta^c$ for the True DEM Elements

$Q - 5 - 1^+$

$Q - 9 - 2^+$

$Q - 13 - 3^+$

$Q - 17 - 4^+$
### Computational Complexities

<table>
<thead>
<tr>
<th>Element</th>
<th>Asymptotic # of dofs</th>
<th>Stencil width for uniform $n \times n$ mesh</th>
<th>$(# \text{ dofs}) \times (\text{stencil width})$</th>
<th>$L^2$ convergence rate (a posteriori)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q_1$</td>
<td>$n_{el}$</td>
<td>9</td>
<td>$9n_{el}$</td>
<td>2</td>
</tr>
<tr>
<td>$Q_2$</td>
<td>$2n_{el}$</td>
<td>7</td>
<td>$14n_{el}$</td>
<td>2</td>
</tr>
<tr>
<td>$Q_3$</td>
<td>$3n_{el}$</td>
<td>21</td>
<td>$63n_{el}$</td>
<td>3</td>
</tr>
<tr>
<td>$Q_4$</td>
<td>$4n_{el}$</td>
<td>14</td>
<td>$56n_{el}$</td>
<td>3</td>
</tr>
<tr>
<td>$Q_5$</td>
<td>$3n_{el}$</td>
<td>21</td>
<td>$63n_{el}$</td>
<td>2 − 3</td>
</tr>
<tr>
<td>$Q_6$</td>
<td>$5n_{el}$</td>
<td>33</td>
<td>$165n_{el}$</td>
<td>4</td>
</tr>
<tr>
<td>$Q_7$</td>
<td>$6n_{el}$</td>
<td>21</td>
<td>$126n_{el}$</td>
<td>4</td>
</tr>
<tr>
<td>$Q_8$</td>
<td>$5n_{el}$</td>
<td>33</td>
<td>$165n_{el}$</td>
<td>3 − 4</td>
</tr>
<tr>
<td>$Q_9$</td>
<td>$7n_{el}$</td>
<td>45</td>
<td>$315n_{el}$</td>
<td>5</td>
</tr>
<tr>
<td>$Q_{10}$</td>
<td>$8n_{el}$</td>
<td>28</td>
<td>$224n_{el}$</td>
<td>5</td>
</tr>
<tr>
<td>$Q_{11}$</td>
<td>$7n_{el}$</td>
<td>45</td>
<td>$315n_{el}$</td>
<td>4 − 5</td>
</tr>
<tr>
<td>$Q_{12}$</td>
<td>$9n_{el}$</td>
<td>57</td>
<td>$513n_{el}$</td>
<td>4 − 5</td>
</tr>
</tbody>
</table>

**Figure 4:** $Q_1$ stencil

**Figure 5:** $Q_{-4-1}$ stencil
Summary of Computational Properties

"COMPARABLES"

A priori in computational cost:
- DGM with \( n \) LMs and \( Q_n \)
- DEM with \( n \) LMs and \( Q_{n+1} \)

A posteriori in convergence rate:
- DGM with \( n \) LMs and \( Q_n \)
- DEM with \( n \) LMs and \( Q_n / Q_{n+1} \)

- Exponential enrichments \( \Rightarrow \) integrations can be computed analytically.
- \( \mathcal{L}c^E = 0 \Rightarrow \) convert volume integrals to boundary integrals:

\[
\begin{align*}
    a(v^E, c^E) &= \int_{\Omega} (\kappa \nabla v^E \cdot \nabla c^E + a \cdot \nabla c^E v^E) \, d\Omega \\
    &= \int_{\Gamma} \nabla c^E \cdot n v^E \, d\Gamma
\end{align*}
\]
Homogeneous Boundary Layer Problem

\[ \Omega = (0, 1) \times (0, 1), f = 0. \]
\[ a = (\cos \phi, \sin \phi). \]

Dirichlet boundary conditions are specified on \( \Gamma \) such that the exact solution to the BVP is given by

\[
c_{\text{ex}}(x; \phi, \psi) = \frac{1}{e^{2\kappa}} \left\{ \left[ \cos \phi + \cos \psi \right](x - 1) + \left[ \sin \phi + \sin \psi \right](y - 1) \right\} - 1 \]
\[
e^{-\frac{1}{2\kappa}} \left[ \cos \phi + \cos \psi + \sin \phi + \sin \psi \right] - 1
\]

\( \psi \in [0, 2\pi) \) : some flow direction (not necessarily aligned with \( \phi \)).

Solution exhibits a sharp exponential boundary layer in the advection direction \( \phi \), whose gradient is a function of the Péclet number.

Figure 6: \( \phi = \psi = 0 \)

Figure 7: \( \phi = \pi/7, \psi = 0 \)
Homogeneous Boundary Layer Problem

- \( \Omega = (0, 1) \times (0, 1), f = 0. \)
- \( \mathbf{a} = (\cos \phi, \sin \phi). \)
- Dirichlet boundary conditions are specified on \( \Gamma \) such that the exact solution to the BVP is given by

\[
\begin{align*}
  c_{\text{ex}}(x; \phi, \psi) &= \frac{1}{e^{2\kappa}} \left\{ [\cos \phi + \cos \psi](x-1) + [\sin \phi + \sin \psi](y-1) \right\} - 1 \\
  &= e^{-\frac{1}{2\kappa} [\cos \phi + \cos \psi + \sin \phi + \sin \psi]} - 1
\end{align*}
\]

- \( \psi \in [0, 2\pi) \): some flow direction (not necessarily aligned with \( \phi \)).
- Solution exhibits a sharp exponential boundary layer in the advection direction \( \phi \), whose gradient is a function of the Péclet number.

Homogeneous problem \( \Rightarrow \) pure DGM elements sufficient
Non-trivial Test Case: Flow *not* Aligned with Advection Direction \((\phi \neq \psi)\)

- Set \(\phi = \pi/7\); vary \(\psi\).
- Can show that \(c_{ex} \notin \mathcal{V}^E\) for any DGM elements and advection directions tested here.

<table>
<thead>
<tr>
<th>(\psi/\pi)</th>
<th>(Q_1)</th>
<th>(Q - 4 - 1)</th>
<th>(Q_2)</th>
<th>(Q - 8 - 2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.45 \times 10^{-2}</td>
<td>1.65 \times 10^{-3}</td>
<td>5.92 \times 10^{-3}</td>
<td>1.79 \times 10^{-3}</td>
</tr>
<tr>
<td>1/4</td>
<td>1.52 \times 10^{-2}</td>
<td>9.38 \times 10^{-4}</td>
<td>6.06 \times 10^{-3}</td>
<td>2.54 \times 10^{-4}</td>
</tr>
<tr>
<td>1/2</td>
<td>1.51 \times 10^{-2}</td>
<td>9.23 \times 10^{-4}</td>
<td>5.97 \times 10^{-3}</td>
<td>2.12 \times 10^{-4}</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(\psi/\pi)</th>
<th>(Q_3)</th>
<th>(Q - 12 - 3)</th>
<th>(Q_4)</th>
<th>(Q - 16 - 4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>4.34 \times 10^{-3}</td>
<td>1.10 \times 10^{-4}</td>
<td>3.23 \times 10^{-3}</td>
<td>2.30 \times 10^{-5}</td>
</tr>
<tr>
<td>1/4</td>
<td>4.46 \times 10^{-3}</td>
<td>1.23 \times 10^{-5}</td>
<td>3.29 \times 10^{-3}</td>
<td>8.82 \times 10^{-7}</td>
</tr>
<tr>
<td>1/2</td>
<td>4.36 \times 10^{-3}</td>
<td>1.11 \times 10^{-5}</td>
<td>3.18 \times 10^{-3}</td>
<td>1.59 \times 10^{-6}</td>
</tr>
</tbody>
</table>
To achieve for this problem the relative error of 0.1% for $Pe = 10^{3}$:

- $Q - 4 - 1$ requires 4.4 times fewer dofs than $Q_1$.
- $Q - 8 - 2$ requires 4.5 times fewer dofs than $Q_2$.
- $Q - 12 - 3$ requires 14.7 times fewer dofs than $Q_3$.
- $Q - 16 - 4$ requires 15.1 times fewer dofs than $Q_4$.

<table>
<thead>
<tr>
<th>Element</th>
<th>Rate of convergence</th>
<th># dofs to achieve $10^{-3}$ error</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q_1$</td>
<td>1.90</td>
<td>63,266</td>
</tr>
<tr>
<td>$Q - 4 - 1$</td>
<td>1.99</td>
<td>14,320</td>
</tr>
<tr>
<td>$Q_2$</td>
<td>2.38</td>
<td>24,300</td>
</tr>
<tr>
<td>$Q - 8 - 2$</td>
<td>3.27</td>
<td>5400</td>
</tr>
<tr>
<td>$Q_3$</td>
<td>3.48</td>
<td>12,500</td>
</tr>
<tr>
<td>$Q - 12 - 3$</td>
<td>3.88</td>
<td>850</td>
</tr>
<tr>
<td>$Q_4$</td>
<td>4.41</td>
<td>8600</td>
</tr>
<tr>
<td>$Q - 16 - 4$</td>
<td>5.19</td>
<td>570</td>
</tr>
</tbody>
</table>
Solution Plots for Homogeneous BVP

Figure 8: $\phi = \psi = 0$, $Pe = 10^3$, $\approx 1600$ dofs

Figure 9: $\phi = \pi / 7$, $\psi = 0$, $Pe = 10^5$, $\approx 1600$ dofs
Double Ramp Problem on an $L$–Shaped Domain

Homogeneous Dirichlet boundary conditions are prescribed on all six sides of $L$–shaped domain $\Omega$.

Advection direction: $\phi = 0$.

Source: $f = 1$.

Strong outflow boundary layer along the line $x = 1$.

Two crosswind boundary layers along $y = 0$ and $y = 1$.

A crosswind internal layer along $y = 0.5$.

Figure 10: $L$-shaped domain for double ramp problem
No oscillations can be seen in the computed DGM and DEM solutions.

Would expect: DEM elements to outperform DGM elements for this *inhomogeneous* problem.

In fact: DGM elements experience some difficulty along the $y = 0.5$ line, the location of the crosswind internal layer.
Cross Sectional Solution Plots

**Figure 12:** Solution along the line $x = 0.9$ with 7600 dofs

- **Galerkin**
- **DGM**
- **DEM**

**Figure 13:** Solution along the line $y = 0.5$ with 7600 dofs

- **Galerkin**
- **DGM**
- **DEM**
Relative Errors ($Pe = 10^3$, Uniform Mesh)

<table>
<thead>
<tr>
<th># elements</th>
<th>$Q_3$</th>
<th>$Q - 12 - 3$</th>
<th>$Q - 9 - 2^+$</th>
</tr>
</thead>
<tbody>
<tr>
<td>300</td>
<td>$1.49 \times 10^{-1}$</td>
<td>$1.11 \times 10^{-1}$</td>
<td>$4.11 \times 10^{-2}$</td>
</tr>
<tr>
<td>1200</td>
<td>$6.57 \times 10^{-2}$</td>
<td>$5.00 \times 10^{-2}$</td>
<td>$8.47 \times 10^{-3}$</td>
</tr>
<tr>
<td>4800</td>
<td>$2.36 \times 10^{-2}$</td>
<td>$1.02 \times 10^{-2}$</td>
<td>$1.65 \times 10^{-3}$</td>
</tr>
<tr>
<td>10,800</td>
<td>$1.08 \times 10^{-2}$</td>
<td>$4.54 \times 10^{-3}$</td>
<td>$7.43 \times 10^{-4}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th># elements</th>
<th>$Q_4$</th>
<th>$Q - 16 - 4$</th>
<th>$Q - 13 - 3^+$</th>
</tr>
</thead>
<tbody>
<tr>
<td>300</td>
<td>$9.58 \times 10^{-2}$</td>
<td>$8.32 \times 10^{-2}$</td>
<td>$2.80 \times 10^{-2}$</td>
</tr>
<tr>
<td>1200</td>
<td>$3.78 \times 10^{-2}$</td>
<td>$1.33 \times 10^{-2}$</td>
<td>$4.71 \times 10^{-3}$</td>
</tr>
<tr>
<td>4800</td>
<td>$1.03 \times 10^{-2}$</td>
<td>$9.17 \times 10^{-3}$</td>
<td>$8.24 \times 10^{-4}$</td>
</tr>
<tr>
<td>10,800</td>
<td>$3.70 \times 10^{-3}$</td>
<td>$4.92 \times 10^{-4}$</td>
<td>$9.75 \times 10^{-5}$</td>
</tr>
</tbody>
</table>

- DEM elements outperform DGM elements.
- Both DGM and DEM elements outperform Galerkin elements.
Outline

1. Motivation
2. Advection-Diffusion Equation
3. Discontinuous Enrichment Method (DEM)
   - Enrichment Bases
   - Lagrange Multiplier Approximations
   - Element Design
   - Numerical Experiments
4. DEM for Constant-Coefficient Advection-Diffusion
   - Enrichment Bases
   - Lagrange Multiplier Approximations
   - Element Design
   - Numerical Experiments
5. DEM for Variable-Coefficient Advection-Diffusion
   - Enrichment Bases
   - Lagrange Multiplier Approximations
   - Element Design
   - Numerical Experiments
6. Extension of DEM to Unsteady, Non-Linear Problems
7. Summary
Extension to Variable-Coefficient Problems

- Define $\mathcal{V}^E$ *within each element* as the free-space solutions to the homogeneous PDE, with locally-*frozen* coefficients.

- $\mathbf{a}(\mathbf{x}) \approx \mathbf{a}^e =$ constant inside each element $\Omega^e$ as $h \to 0$:

  $\{ \mathbf{a}(\mathbf{x}) \cdot \nabla \mathbf{c} - \kappa \Delta \mathbf{c} = f(\mathbf{x}) \text{ in } \Omega \} \approx \bigcup_{e=1}^{n_e} \{ \mathbf{a}^e \cdot \nabla \mathbf{c} - \kappa \Delta \mathbf{c} = f(\mathbf{x}) \text{ in } \Omega^e \}$.

- Enrichment in each element:

  $c^E_e(\mathbf{x}; \theta^e_i) = e^{|\mathbf{a}^e|/2\kappa} (\cos \phi^e + \cos \theta^e_i)(x - x^e_{r,i}) e^{|\mathbf{a}^e|/2\kappa} (\sin \phi^e + \sin \theta^e_i)(y - y^e_{r,i}) \in \mathcal{V}^E_e$
Given \( \mathbf{a}(\mathbf{x}) \in C^1(\Omega^e) \), Taylor expand \( \mathbf{a}(\mathbf{x}) \) around an element’s midpoint \( \bar{x}^e \):

\[
\mathbf{a}(\mathbf{x}) = \mathbf{a}(\bar{x}^e) + \nabla \mathbf{a}|_{\mathbf{x}=\bar{x}^e} \cdot (\mathbf{x} - \bar{x}^e) + \mathcal{O}(\mathbf{x} - \bar{x}^e)^2 \quad \text{in} \quad \Omega^e
\]
Relation Between Local Enrichment and Governing PDE

Given $a(x) \in C^1(\Omega^e)$, Taylor expand $a(x)$ around an element's midpoint $\bar{x}^e$:

$$a(x) = a(\bar{x}^e) + \nabla a|_{x=\bar{x}^e} \cdot (x - \bar{x}^e) + O(x - \bar{x}^e)^2$$ in $\Omega^e$

Operator governing the PDE inside the element $\Omega^e$ takes the form

$$a(x) \cdot \nabla c - \kappa \Delta c = L_e c + f(c)$$ in $\Omega^e$

where

$$L_e c \equiv a(\bar{x}^e) \cdot \nabla c - \kappa \Delta c$$

$$f(c) \equiv \left[ \nabla a|_{x=\bar{x}^e} \cdot (x - \bar{x}^e) + O(x - \bar{x}^e)^2 \right] \cdot \nabla c$$
Relation Between Local Enrichment and Governing PDE

- Given \( a(\mathbf{x}) \in C^1(\Omega^e) \), Taylor expand \( a(\mathbf{x}) \) around an element’s midpoint \( \bar{x}^e \):

\[
a(\mathbf{x}) = a(\bar{x}^e) + \nabla a_{|x=\bar{x}^e} \cdot (\mathbf{x} - \bar{x}^e) + O((\mathbf{x} - \bar{x}^e)^2) \quad \text{in } \Omega^e
\]

- Operator governing the PDE inside the element \( \Omega^e \) takes the form

\[
a(\mathbf{x}) \cdot \nabla c - \kappa \Delta c = \mathcal{L}_e c + f(c) \quad \text{in } \Omega^e
\]

where

\[
\mathcal{L}_e c \equiv a(\bar{x}^e) \cdot \nabla c - \kappa \Delta c
\]

\[
f(c) \equiv \left[ \nabla a_{|x=\bar{x}^e} \cdot (\mathbf{x} - \bar{x}^e) + O((\mathbf{x} - \bar{x}^e)^2) \right] \cdot \nabla c
\]

- “Residual” advection equation acts as source-like term \( \Rightarrow \) suggests true DEM elements are in general more appropriate than pure DGM elements for variable-coefficient problems.
Relation Between Local Enrichment and Governing PDE

- Given \( a(x) \in C^1(\Omega^e) \), Taylor expand \( a(x) \) around an element’s midpoint \( \bar{x}^e \):

\[
a(x) = a(\bar{x}^e) + \nabla a|_{x=\bar{x}^e} \cdot (x - \bar{x}^e) + O(x - \bar{x}^e)^2 \quad \text{in } \Omega^e
\]

- Operator governing the PDE inside the element \( \Omega^e \) takes the form

\[
a(x) \cdot \nabla c - \kappa \Delta c = L_e c + f(c) \quad \text{in } \Omega^e
\]

where

\[
L_e c \equiv a(\bar{x}^e) \cdot \nabla c - \kappa \Delta c
\]

\[
f(c) \equiv [\nabla a|_{x=\bar{x}^e} \cdot (x - \bar{x}^e) + O(x - \bar{x}^e)^2] \cdot \nabla c
\]

- “Residual” advection equation acts as source-like term \( \Rightarrow \) suggests true DEM elements are in general more appropriate than pure DGM elements for variable-coefficient problems.

Can we build a better pure DGM element for variable-coefficient problems?
Additional “First Order” Enrichment Functions

Are we missing any free-space solutions to \( \mathbf{a}^e \cdot \nabla c^E - \kappa \Delta c^E = 0 \)?
Additional “First Order” Enrichment Functions

- Are we missing any free-space solutions to $\mathbf{a}^e \cdot \nabla c^E - \kappa \Delta c^E = 0$?

- Yes! Polynomial free-space solutions to $\mathcal{L}c^E_{e,n} = \mathbf{a}^e \cdot \nabla c^E_{e,n} - \Delta c^E_{e,n} = 0$
  (of any desired degree $n$) can be derived as well.
Are we missing any free-space solutions to $\mathbf{a}^e \cdot \nabla c^E - \kappa \Delta c^E = 0$?

Yes! Polynomial free-space solutions to $\mathcal{L}c^E_{e,n} = \mathbf{a}^e \cdot \nabla c^E_{e,n} - \Delta c^E_{e,n} = 0$ (of any desired degree $n$) can be derived as well.

$$c^E_{e,1}(\mathbf{x}) = |\mathbf{a}^e \times \mathbf{x}|$$
Additional “First Order” Enrichment Functions

Are we missing any free-space solutions to $\mathbf{a}^e \cdot \nabla c^E - \kappa \Delta c^E = 0$?

Yes! Polynomial free-space solutions to $\mathcal{L} c_{e,n}^E = \mathbf{a}^e \cdot \nabla c_{e,n}^E - \Delta c_{e,n}^E = 0$ (of any desired degree $n$) can be derived as well.

$$c_{e,1}^E(x) = |\mathbf{a}^e \times \mathbf{x}|$$

$$c_{e,2}^E(x) = |\mathbf{a}^e \times \mathbf{x}|^2 + 2(\mathbf{a}^e \cdot \mathbf{x})$$
Additional “First Order” Enrichment Functions

- Are we missing any free-space solutions to $\mathbf{a}^e \cdot \nabla c^E - \kappa \Delta c^E = 0$?

- Yes! Polynomial free-space solutions to $\mathcal{L}c^E_{e,n} = \mathbf{a}^e \cdot \nabla c^E_{e,n} - \Delta c^E_{e,n} = 0$
  (of any desired degree $n$) can be derived as well.

  $c^E_{e,1}(\mathbf{x}) = |\mathbf{a}^e \times \mathbf{x}|$

  $c^E_{e,2}(\mathbf{x}) = |\mathbf{a}^e \times \mathbf{x}|^2 + 2(\mathbf{a}^e \cdot \mathbf{x})$

  $c^E_{e,3}(\mathbf{x}) = |\mathbf{a}^e \times \mathbf{x}|^3 + 6|\mathbf{a}^e \times \mathbf{x}|(\mathbf{a}^e \cdot \mathbf{x})$

  $\vdots$
“Higher Order” Enrichment Functions

- Linearize \( a(x) \) to second order, instead of to first order:

\[
a(x) \approx a(\bar{x}^e) + \nabla a|_{x=\bar{x}^e} \cdot (x - \bar{x}^e) \quad \text{in } \Omega^e
\]

- Enrich with free-space solutions to

\[
[Ax + b] \cdot \nabla c^E - \kappa \Delta c^E = 0 \tag{2}
\]

where \( A \equiv \nabla a|_{x=\bar{x}^e}, \) \( b \equiv (a(\bar{x}^e) - \nabla a|_{x=\bar{x}^e} \bar{x}^e). \)

- Solutions to (2) are given by:

\[
c^E(x) = \int_0^{v_i \cdot x} \exp \left\{ \frac{\sigma_i w^2}{2} + (v_i \cdot b) w \right\} dw
\]

\[
\sigma_i = \text{eigenvalue of } \nabla a|_{x=\bar{x}^e} \\
v_i = \text{eigenvector of } \nabla a|_{x=\bar{x}^e}
\]
Enrichment Function Bank

Exponential Family

\[ c^E_e(x; \theta_i) = e^{\left( \frac{a^e_1 + |a^e| \cos \theta_i}{2 \kappa} \right)(x - x_{r,i})} e^{\left( \frac{a^e_2 + |a^e| \sin \theta_i}{2 \kappa} \right)(y - y_{r,i})} \]
“Enrichment Function Bank”

**Exponential Family**

\[ c^E_e(x; \theta_i) = e^{\left( \frac{a^e_1 + |a^e| \cos \theta_i}{2\kappa} \right) (x - x_{r,i})} e^{\left( \frac{a^e_2 + |a^e| \sin \theta_i}{2\kappa} \right) (y - y_{r,i})} \]

**Polynomial Family**

\[ c^E_{e,0}(x) = 1 \]
\[ c^E_{e,1}(x) = |a^e \times x| \]
\[ c^E_{e,2}(x) = |a^e \times x|^2 + 2 (a^e \cdot x) \]
\[ c^E_{e,3}(x) = |a^e \times x|^3 + 6 |a^e \times x| (a^e \cdot x) \]
\[ \vdots \]
"Enrichment Function Bank"

**Exponential Family**

\[ c^E_e(x; \theta_i) = e^{\left( \frac{a_1^e + |a^e| \cos \theta_i}{2\kappa} \right)(x-x_{ri})} e^{\left( \frac{a_2^e + |a^e| \sin \theta_i}{2\kappa} \right)(y-y_{ri})} \]

**Polynomial Family**

\[ c^E_{e,0}(x) = 1 \]
\[ c^E_{e,1}(x) = |a^e \times x| \]
\[ c^E_{e,2}(x) = |a^e \times x|^2 + 2(a^e \cdot x) \]
\[ c^E_{e,3}(x) = |a^e \times x|^3 + 6|a^e \times x|(a^e \cdot x) \]
\[ \vdots \]

**"Higher Order" Enrichment**

\[ c^E(x) = \int_0^{\nu_i \cdot x} \exp \left\{ \frac{\sigma_j w^2}{2} + (\nu_i \cdot b) w \right\} dw \]
Modification of the Lagrange Multiplier Field

Figure 14: Straight edge $\Gamma^{e,e'}$ oriented at angle $\alpha^{e,e'} \in [0, 2\pi)$

- LM approximations arising from exponential enrichments:
  \[
  \lambda^h|_{\Gamma^{e,e'}} = \text{span}\left\{ e^{\Lambda^e_{i,e'}(s-s^e_{r,e'})}, \ 0 \leq s \leq h, \ 1 \leq i \leq n^{exp} \right\}
  \]
  where $\Lambda^e_{i,e'} \equiv \frac{|a|}{2\kappa} \left[ \cos(\phi - \alpha^{e,e'}) + \cos(\theta_i - \alpha^{e,e'}) \right]$.

- LM approximations arising from polynomial enrichments:
  \[
  \lambda^h|_{\Gamma^{e,e'}} = \text{span}\left\{ s^k, \ 0 \leq s \leq h, \ 0 \leq k \leq n^{pol} - 1 \right\}
  \]
### New DGM Elements

#### Notation

New DGM Elements:  
\[
\begin{cases}
    Q - (n_{\text{pol}}, n_{\text{exp}}) - n^\lambda \\
    Q - (n_{\text{pol}}, n_{\text{exp}})^* - n^\lambda
\end{cases}
\]

- \( Q \): Quadrilateral
- \( n_{\text{pol}} \): Number of Polynomial Enrichment Functions
- \( n_{\text{exp}} \): Number of Exponential Enrichment Functions
- \( n^\lambda \): Number of Lagrange Multipliers per Edge
- \( \ast \): Element Augmented by “Higher Order” Enrichment

<table>
<thead>
<tr>
<th>DGM elements</th>
<th>Name</th>
<th>( n^E )</th>
<th>( \Theta^c )</th>
<th>( n^\lambda )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( Q - (4, 5) - 2 )</td>
<td>9</td>
<td>( \phi + { \frac{2m\pi}{5} : m = 0, \ldots, 4 } )</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>( Q - (4, 5)^* - 2 )</td>
<td>10</td>
<td>( \phi + { \frac{2m\pi}{5} : m = 0, \ldots, 4 } )</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>( Q - (4, 9) - 3 )</td>
<td>13</td>
<td>( \phi + { \frac{2m\pi}{9} : m = 0, \ldots, 8 } )</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>( Q - (4, 9)^* - 3 )</td>
<td>14</td>
<td>( \phi + { \frac{2m\pi}{9} : m = 0, \ldots, 8 } )</td>
<td>3</td>
</tr>
</tbody>
</table>

Polynomial enrichment fields of new DGM elements contain \( n_{\text{pol}} = 4 \) polynomial free-space solutions of degrees 0, 1, 2 and 3.
## Computational Complexities

<table>
<thead>
<tr>
<th>Element</th>
<th>Asymptotic # of dofs</th>
<th>Stencil width for uniform ( n \times n ) mesh</th>
<th>((# \text{ dofs}) \times (\text{stencil width}))</th>
<th>(L^2) convergence rate (a posteriori)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Q_1)</td>
<td>(n_{el})</td>
<td>9</td>
<td>(9n_{el})</td>
<td>2</td>
</tr>
<tr>
<td>(Q - 4 - 1)</td>
<td>(2n_{el})</td>
<td>7</td>
<td>(14n_{el})</td>
<td>2</td>
</tr>
<tr>
<td>(Q_2)</td>
<td>(3n_{el})</td>
<td>21</td>
<td>(63n_{el})</td>
<td>3</td>
</tr>
<tr>
<td>(Q - 4 - 8)</td>
<td>(4n_{el})</td>
<td>14</td>
<td>(56n_{el})</td>
<td>3</td>
</tr>
<tr>
<td>(Q - (4, 5) - 2)</td>
<td>(4n_{el})</td>
<td>14</td>
<td>(63n_{el})</td>
<td>3</td>
</tr>
<tr>
<td>(Q - (4, 5)^* - 2)</td>
<td>(4n_{el})</td>
<td>14</td>
<td>(63n_{el})</td>
<td>3</td>
</tr>
<tr>
<td>(Q - 5 - 1^+)</td>
<td>(3n_{el})</td>
<td>21</td>
<td>(63n_{el})</td>
<td>2 - 3</td>
</tr>
<tr>
<td>(Q_3)</td>
<td>(5n_{el})</td>
<td>33</td>
<td>(165n_{el})</td>
<td>4</td>
</tr>
<tr>
<td>(Q - 12 - 3)</td>
<td>(6n_{el})</td>
<td>21</td>
<td>(126n_{el})</td>
<td>4</td>
</tr>
<tr>
<td>(Q - (4, 9) - 3)</td>
<td>(6n_{el})</td>
<td>21</td>
<td>(126n_{el})</td>
<td>4</td>
</tr>
<tr>
<td>(Q - (4, 9)^* - 3)</td>
<td>(6n_{el})</td>
<td>21</td>
<td>(126n_{el})</td>
<td>4</td>
</tr>
<tr>
<td>(Q - 9 - 2^+)</td>
<td>(5n_{el})</td>
<td>33</td>
<td>(165n_{el})</td>
<td>3 - 4</td>
</tr>
<tr>
<td>(Q_4)</td>
<td>(7n_{el})</td>
<td>45</td>
<td>(315n_{el})</td>
<td>5</td>
</tr>
<tr>
<td>(Q - 16 - 4)</td>
<td>(8n_{el})</td>
<td>28</td>
<td>(224n_{el})</td>
<td>5</td>
</tr>
<tr>
<td>(Q - 13 - 3^+)</td>
<td>(7n_{el})</td>
<td>45</td>
<td>(315n_{el})</td>
<td>4 - 5</td>
</tr>
<tr>
<td>(Q - 17 - 4^+)</td>
<td>(9n_{el})</td>
<td>57</td>
<td>(513n_{el})</td>
<td>4 - 5</td>
</tr>
</tbody>
</table>
Inhomogeneous Rotating Advection Problem on an $L$–Shaped Domain

Homogeneous Dirichlet boundary conditions are prescribed on all six sides of $L$–shaped domain $\Omega$.

Source: $f = 1$.

$\mathbf{a}^T(\mathbf{x}) = (1 - y, x)$.

Outflow boundary layer along the line $y = 1$.

Second boundary layer that terminates in the vicinity of the re-entrant corner $(x, y) = (0.5, 0.5)$.

Figure 15: $L$-shaped domain and rotating velocity field (curved lines indicate streamlines)
Solutions Plots for $Pe = 10^3$ with $\approx 3000$ dofs

* “Stabilized $Q_1$” is upwind stabilized bilinear finite element proposed by Harari et al.
Convergence Analysis & Results

L-shaped rotating field, inhomogeneous problem, $Pe = 1000$

<table>
<thead>
<tr>
<th>Element</th>
<th>Rate of convergence</th>
<th># dofs to achieve $10^{-2}$ error</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q_2$</td>
<td>1.94</td>
<td>62,721</td>
</tr>
<tr>
<td>$Q - 5 - 1^+$</td>
<td>1.55</td>
<td>21,834</td>
</tr>
<tr>
<td>$Q_3$</td>
<td>2.67</td>
<td>33,707</td>
</tr>
<tr>
<td>$Q - 9 - 2^+$</td>
<td>2.37</td>
<td>7,568</td>
</tr>
<tr>
<td>$Q_4$</td>
<td>3.50</td>
<td>20,796</td>
</tr>
<tr>
<td>$Q - 13 - 3^+$</td>
<td>3.23</td>
<td>5,935</td>
</tr>
<tr>
<td>$Q - 17 - 4^+$</td>
<td>3.26</td>
<td>4,802</td>
</tr>
</tbody>
</table>

* “Stabilized $Q_1$” is upwind stabilized bilinear finite element proposed by Harari et al.

To achieve for this problem the relative error of 1% for $Pe = 10^3$:
- $Q - 5 - 1^+$ requires 2.9 times fewer dofs than $Q_2$.
- $Q - 9 - 2^+$ requires 4.5 times fewer dofs than $Q_3$.
- $Q - 13 - 3^+$ requires 3.5 times fewer dofs than $Q_4$. 
Lid-Driven Cavity Flow Problem

\[ \Omega = (0, 1) \times (0, 1), \quad f = 0. \]

\[ \mathbf{a}(\mathbf{x}) \] computed numerically by solving the incompressible Navier-Stokes equations for lid-driven cavity flow problem (stationary sides and bottom, tangential movement of top).

Advection field reconstructed using interpolation with bilinear shape functions \( \phi_i^e \):

\[ \mathbf{a}^e(\xi) = \sum_{i=1}^{\# \text{ nodes of } \Omega^e} a_i^e \phi_i^e(\xi) \]

\[ c(\mathbf{x}) \] represents temperature in cavity.

\[ \frac{\partial c}{\partial \mathbf{n}} = 0 \]

\[ \frac{\partial c}{\partial x} = 0 \quad \frac{\partial c}{\partial y} = 0 \]
Convergence Analysis & Results \((\kappa = 0.01, \text{Pe} \approx 260)\)

- New pure DGM elements without "higher order" enrichment outperform Galerkin comparables.

Cavity Flow DGM Relative Errors, \(\kappa = 0.01\)

\[\begin{align*}
Q_2 & \quad \cdot \cdot \cdot \quad Q-(4,5)-2 & \quad Q_3 & \quad Q-(4,9)-3
\end{align*}\]
New pure DGM elements without “higher order” enrichment outperform Galerkin comparables.

Further improvement in computation by adding “higher order” enrichment.
Outline

1. Motivation
2. Advection-Diffusion Equation
3. Discontinuous Enrichment Method (DEM)
   - Enrichment Bases
   - Lagrange Multiplier Approximations
   - Element Design
   - Numerical Experiments
4. DEM for Constant-Coefficient Advection-Diffusion
   - Enrichment Bases
   - Lagrange Multiplier Approximations
   - Element Design
   - Numerical Experiments
5. DEM for Variable-Coefficient Advection-Diffusion
   - Enrichment Bases
   - Lagrange Multiplier Approximations
   - Element Design
   - Numerical Experiments
6. Extension of DEM to Unsteady, Non-Linear Problems
7. Summary
DEM for the Viscous Burgers Equation

- Non-linear unsteady version of advection-diffusion equation = **viscous Burgers equation**:

\[
    u_t + uu_x - \kappa u_{xx} = 0
\]
DEM for the Viscous Burgers Equation

- Non-linear unsteady version of advection-diffusion equation = **viscous Burgers equation**:

\[ u_t + uu_x - \kappa u_{xx} = 0 \]

- Semi-discrete form of PDE (with semi-implicit Euler) at time \( n \):

\[
\frac{u^{n+1} - u^n}{\Delta t} + u^n u_x^{n+1} - \kappa u_{xx}^{n+1} = 0
\]
DEM for the Viscous Burgers Equation

- Non-linear unsteady version of advection-diffusion equation = \textbf{viscous Burgers equation}:

  \[ u_t + uu_x - \kappa u_{xx} = 0 \]

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  \[
  \frac{u^{n+1} - u^n}{\Delta t} + u^n u_x^{n+1} - \kappa u_{xx}^{n+1} = 0
  \]

- Enrichment functions inside each element at time step \( n \) are the free-space solutions to steady version of the equation above:

  \[
  \mathcal{V}_{e}^{E,n} = \text{span}\{ u^n(x) : u^{n-1}(\bar{x}_e)u^n_x - \kappa u^n_{xx} = 0, x \in \Omega^e \}
  \]

The enrichment field inside element \( \Omega^e \) at time step \( n \)

\[ \bar{x}_e \equiv \text{midpoint of element } \Omega^e \]
Outline

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**Summary**

**Discontinuous Enrichment Method (DEM) =**
efficient, competitive alternative to stabilized FEMs for advection-diffusion in a high Péclet regime.

- Parametrization of exponential basis enables systematic design of DEM elements of arbitrary orders.

- Augmentation of enrichment space with additional free-space solutions can improve further the approximation.

- For all test problems, enriched elements outperform their Galerkin and stabilized Galerkin counterparts of comparable computational complexity, sometimes by many orders of magnitude.

- In a high Péclet regime, DGM and DEM solutions are almost completely oscillation-free, in contrast with the Galerkin solutions.

- Advection-diffusion work generalizable to more complex equations in fluid mechanics (e.g., non-linear, unsteady, 3D).

- Future work: DEM for incompressible Navier-Stokes.


Background, Interests, Experience
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Education:
- Ph.D. Candidate, Institute for Computational & Mathematical Engineering, Farhat Research Group, Stanford University, Stanford, CA (expected June 2011).

Other Research Experience:
- Graduate Technical Intern, Aerosciences Department, Sandia National Laboratories, Albuquerque, NM (June 2007 – August 2010).
- Reduced Order Modeling of Fluid/Structure Interaction (Mentor: Matthew F. Barone).
- Modeling of Transitional and Fully Turbulent Pressure Fluctuation Loading (Mentor: Lawrence J. DeChant).

Current Research Interests:
- Finite Element Methods
- Numerical Solution to PDEs
- Reduced Order Modeling