A Discontinuous Enrichment Method (DEM) for Advection-Dominated Fluid Problems

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Albuquerque, NM
Aerosciences Department Seminar
Monday, August 23, 2010

* Joint work with Prof. Charbel Farhat and Dr. Radek Tezaur (Department of Aeronautics & Astronautics, Stanford University).
Motivation

Advection-Diffusion Equation

Discontinuous Enrichment Method (DEM)

Advection-Diffusion DEM

Numerical Experiments

Summary

Background

Education:

- Ph.D. Candidate, Institute for Computational & Mathematical Engineering, Farhat Research Group, Stanford University, Stanford, CA (expected 2011).

Current Research Interests:

- Numerical solution to PDEs
- Mixed/hybrid FEMs
- Reduced Order Modeling, CFD

Recent Relevant Work Experience:

- Graduate Technical Intern, Aerosciences Department (Org. 1515), Sandia National Laboratories, Albuquerque, NM (June 2007 – present).
  - Reduced Order Modeling of Fluid/Structure Interaction (Mentor: Matthew F. Barone, Org. 6333)
  - Modeling of Transitional and Fully Turbulent Pressure Fluctuation Loading (Mentor: Lawrence J. DeChant, Org. 1515)
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1 Motivation

2 Advection-Diffusion Equation

3 Discontinuous Enrichment Method (DEM)

4 Advection-Diffusion DEM
   - Enrichment Bases
   - Lagrange Multiplier Approximations
   - Element Design
   - Variable-Coefficient Problems
   - 3D Advection-Diffusion

5 Numerical Experiments
   - Homogeneous Boundary Layer Problem
   - Double Ramp Problem on an \( L \)-Shaped Domain
   - Inhomogeneous Rotating Field Problem on an \( L \)-shaped Domain

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Galerkin **Finite Element Method** (FEM) has a number of attractions in fluid mechanics:

- Flexibility in handling complex geometries.
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FEM is quasi-optimal optimal for elliptic (diffusion-dominated) PDEs.
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**EXPENSIVE!**

- **Goal:** build an efficient method that can accurately capture boundary layers
- **Approach:** start with simple canonical problem; then generalize.
2D Scalar Advection-Diffusion Equation

\[ \mathcal{L} u = -\kappa \Delta u + \mathbf{a} \cdot \nabla u = f \]

diffusion  advection

- Advection velocity:
  \[ \mathbf{a} = (a_1, a_2)^T = |\mathbf{a}|(\cos \phi, \sin \phi)^T. \]

- \( \phi \) = advection direction.

- \( \kappa \) = diffusivity.

- Describes many transport phenomena in fluid mechanics:
  - Heat transfer.
  - Semiconductor device modeling.
  - Usual scalar model for the more challenging Navier-Stokes equations.
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  - Usual scalar model for the more challenging Navier-Stokes equations.

- Global Péclet number (\( L = \) length scale associated with \( \Omega \)):

\[
P_{e} = \frac{\text{rate of advection}}{\text{rate of diffusion}} = \frac{L|\mathbf{a}|}{\kappa} = Re \cdot \left\{ \begin{array}{ll}
Pr & \text{(thermal diffusion)} \\
Sc & \text{(mass diffusion)}
\end{array} \right.
\]
Advection-Dominated Regime

- Typical applications: flow is advection dominated.

Figure 1: Galerkin $Q_1$ solution (color) vs. exact solution (black) as $Pe \uparrow (Pe = 10 \rightarrow 150)$

- Some classical remedies:
  - **Stabilized FEMs** (SUPG, GLS, USFEM): add weighted residual (numerical diffusion) to variational equation.
  - **RFB, VMS, PUM**: construct conforming spaces that incorporate knowledge of local behavior of solution.
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Advection-Dominated (High $Pe$) Regime

- Sharp gradients in exact solution
- Galerkin FEM inadequate: spurious oscillations (Fig. 1)

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The Discontinuous Enrichment Method (DEM)

First developed by Farhat et. al. in 2000 for the Helmholtz equation.

**Idea of DEM:**

“Enrich” the usual Galerkin polynomial field $\mathcal{V}^P$ by the free-space solutions to the governing homogeneous PDE $\mathcal{L}u = 0$.

$$u^h = u^P + u^E \in \mathcal{V}^P \oplus (\mathcal{V}^E \setminus \mathcal{V}^P)$$

where

$$\mathcal{V}^E = \text{span}\{u : \mathcal{L}u = 0\}$$

**Simple 1D Example:**

\[
\begin{align*}
\begin{cases}
  u_x - u_{xx} &= 1 + x, & x \in (0, 1) \\
  u(0) &= 0, u(1) = 1 
\end{cases}
\end{align*}
\]

- **Enrichments:** $u^E_x - u^E_{xx} = 0 \Rightarrow u^E = C_1 + C_2 e^x \Rightarrow \mathcal{V}^E = \text{span}\{1, e^x\}$

- **Galerkin FEM polynomials:** $\mathcal{V}_{\Omega^e}^{P \Omega^e} = (x_j, x_{j+1}) = \text{span}\left\{\frac{x_{j+1} - x}{h}, \frac{x - x_j}{h}\right\}$
Two Variants of DEM

- Two variants of DEM: “pure DGM” vs. “true DEM”

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<tr>
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Enrichment-Only “Pure DGM”:
Contribution of the standard polynomial field is dropped entirely from the approximation.

Genuine or “Full” DEM:
Splitting of the approximation into coarse (polynomial) and fine (enrichment) scales.

- Unlike PUM, VMS & RFB: enrichment field in DEM is not required to vanish at element boundaries
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- Genuine or “Full” DEM: Splitting of the approximation into coarse (polynomial) and fine (enrichment) scales.

- Unlike PUM, VMS & RFB: enrichment field in DEM is not required to vanish at element boundaries $\Rightarrow$ DEM is discontinuous by construction!

DEM = DGM with Lagrange Multipliers
What about Inter-Element Continuity?

- Continuity across element boundaries is enforced weakly using Lagrange multipliers $\lambda^h \in \mathcal{W}^h$:

$$
\lambda^h \approx \nabla u^E \cdot \mathbf{n}^e = -\nabla u^{E'} \cdot \mathbf{n}^{e'} \quad \text{on } \Gamma^{e,e'}
$$

- Discrete Babuška-Brezzi *inf-sup* condition:\

\[
\left\{ \begin{array}{c}
\text{# Lagrange multiplier constraint equations} \\
\text{\# enrichment equations}
\end{array} \right\} 
\]

Rule of thumb to satisfy the Babuška-Brezzi *inf-sup* condition is to limit:

\[
\begin{align*}
n^\lambda &= \left\lfloor \frac{n^E}{4} \right\rfloor \equiv \max \left\{ n \in \mathbb{Z} : n \leq \frac{n^E}{4} \right\} \\
n^\lambda &= \text{# Lagrange multipliers per edge} \\
n^E &= \text{# enrichment functions}
\end{align*}
\]

\[1\text{ Necessary condition for generating a non-singular global discrete problem.}\]
Hybrid Variational Formulation of DEM

- **Strong form:**

\[
(S) : \begin{cases} 
\text{Find } u \in H^1(\Omega) \text{ such that } \\
-\kappa \Delta u + \mathbf{a} \cdot \nabla u = f, \quad \text{in } \Omega \\
u = g, \quad \text{on } \Gamma = \partial \Omega \\
u_e - u_{e'} = 0 \quad \text{on } \Gamma^{\text{int}}
\end{cases}
\]

- **Weak hybrid variational form:**

\[
(W) : \begin{cases} 
\text{Find } (u, \lambda) \in V \times W \text{ such that: } \\
a(v, u) + b(\lambda, v) = r(v) \\
b(\mu, u) = -r_d(\mu)
\end{cases}
\]

holds \( \forall v \in V, \forall \mu \in W \).

where

\[
a(v, u) = (\kappa \nabla v + v \mathbf{a}, \nabla u)_{\tilde{\Omega}}
\]

\[
b(\lambda, v) = \sum_{e} \sum_{e' < e} \int_{\Gamma^{e,e'}} \lambda(v_{e'} - v_e) d\Gamma + \int_\Gamma \lambda v d\Gamma
\]

**Notation:**

\[
\tilde{\Omega} = \bigcup_{e=1}^{n_{el}} \Omega^e \setminus \bigcup_{e=1}^{n_{el}} \Gamma^e \setminus \bigcup_{e' < e \in \Omega^e} \Gamma^{e,e'} \\
\Gamma^e = \bigcup_{e=1}^{n_{el}} \Gamma^e \setminus \bigcup_{e' < e \in \Omega^e} \Gamma^{e,e'}
\]
Element matrix problem (uncondensed):

\[
\begin{pmatrix}
k^{PP} & k^{PE} & k^{PC} \\
k^{EP} & k^{EE} & k^{EC} \\
k^{CP} & k^{CE} & 0
\end{pmatrix}
\begin{pmatrix}
u^P \\
u^E \\
\lambda^h
\end{pmatrix}
=
\begin{pmatrix}
r^P \\
r^E \\
r^C
\end{pmatrix}
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Due to the discontinuous nature of $V^E$, $u^E$ can be eliminated at the element level by a static condensation.
Discretization & Implementation

- Element matrix problem (uncondensed):

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Due to the discontinuous nature of $\nabla E$, $u^E$ can be eliminated at the element level by a static condensation

- Statically-condensed True DEM Element:

\[
\begin{pmatrix}
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  \tilde{k}_{CP} & \tilde{k}_{CC}
\end{pmatrix}
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  \lambda^h
\end{pmatrix} =
\begin{pmatrix}
  \tilde{r}^P \\
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\]

- Statically-condensed Pure DGM Element:

\[-k_{CE}(k_{EE})^{-1}k_{EC}\lambda^h = r^C - k_{CE}(k_{EE})^{-1}r^E,\]
Discretization & Implementation

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- Due to the discontinuous nature of \( V^E \), \( u^E \) can be eliminated at the element level by a static condensation.

  Computational complexity depends on \( \dim W^h \) not on \( \dim V^E \).

- Statically-condensed True DEM Element:

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A DEM for Advection-Diffusion
Angle-Parametrized Enrichment Functions for 2D Advection-Diffusion

- Derived by solving \( \mathcal{L}u^E = \mathbf{a} \cdot \nabla u^E - \kappa \Delta u^E = 0 \) analytically (e.g., separation of variables).

\[
\begin{align*}
 u^E(x; \theta_i) &= e\left(\frac{a_1 + |\mathbf{a}| \cos \theta_i}{2\kappa}\right)(x - x_{r,i}) e\left(\frac{a_2 + |\mathbf{a}| \sin \theta_i}{2\kappa}\right)(y - y_{r,i}) \\
 \Theta^u &\equiv \{\theta_i\}_{i=1}^{n^E} \in [0, 2\pi) = \text{set of angles specifying } V^E \\
 (x_{r,i}, y_{r,i}) &\equiv \text{reference point for } u^E_i \\
 \mathbf{a}^T &\equiv (a_1, a_2) = \text{advection velocity vector}
\end{align*}
\]

The parametrization with respect to \( \theta_i \) in (1) is non-trivial!

- Enrichment functions are now specified by a set of “flow directions”.
- Without this parametrization, systematic element design would not be possible!
Plots of Enrichment Functions for Some Angles

\[ \theta_i \in [0, 2\pi) \]

\[ \phi = 0, \theta_i = 0 \]  

\[ \phi = 0, \theta_i = \frac{\pi}{2} \]  

\[ \phi = \frac{\pi}{2}, \theta_i = \frac{3\pi}{4} \]  

\[ \phi = 0, \theta_i = \pi \]  

\[ \phi = \frac{3\pi}{2}, \theta_i = \frac{5\pi}{4} \]  

\[ \phi = 0, \theta_i = \frac{3\pi}{2} \]  

Figure 2: Plots of enrichment function \( u^E(x; \theta_i) \) for several values of \( \theta_i \) \((P_e = 20)\)
What about the Lagrange Multiplier Approximations?

Figure 3: Straight edge $\Gamma^{e,e'}$ oriented at angle $\alpha^{e,e'} \in [0, 2\pi)$

- Trivial to compute given exponential enrichments:

\[
\lambda^h(s)|_{\Gamma^{e,e'}} \approx \nabla u^E \cdot n|_{\Gamma^{e,e'}} \\
= C \cdot e\left\{ \frac{|a|}{2\kappa} \left[ \cos(\phi - \alpha^{e,e'}) + \cos(\theta_k - \alpha^{e,e'}) \right] (s - s_r^{e,e'}) \right\}
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$$

Non-trivial to satisfy inf-sup condition: the set $\Theta^u$ that defines $V^E$ typically leads to too many Lagrange multiplier dofs!
Lagrange Multiplier Selection

Define:

\[ \Lambda_i^{e,e'} \equiv \frac{|a|}{2\kappa} \left[ \cos(\phi - \alpha^{e,e'}) + \cos(\theta_k - \alpha^{e,e'}) \right] \]

\[ \downarrow \]

\[ \lambda^h_{|_{\Gamma e,e'}} = \text{span} \left\{ e^{\Lambda_i^{e,e'}(s-s_{r,i})} : 0 \leq s \leq h \right\} \]

Determine # Lagrange multipliers allowed: card\{\Lambda_i^{e,e'}\} = \left\lfloor \frac{n E}{4} \right\rfloor.

Sample \Lambda_i^{e,e'} uniformly in the interval [\Lambda_{\text{min}}^{e,e'}, \Lambda_{\text{max}}^{e,e'}] to span space of all exponentials of the form \{e^{\Lambda_i^{e,e'}s} : \Lambda_{\text{min}}^{e,e'} \leq \Lambda_i^{e,e'} \leq \Lambda_{\text{max}}^{e,e'}\}. 
Algorithm 1. “Build your own DEM element”

Fix \( n^E \in \mathbb{N} \) (the desired number of angles defining \( \mathcal{V}^E \)).

Select a set of \( n^E \) distinct angles \( \{\theta_k\}_{k=1}^{n^E} \) between [0, 2\( \pi \)].

Let \( \Theta^u = \phi + \{\theta_i\}_{i=1}^{n^E} \).

Define the enrichment functions by:

\[
u^E(x; \Theta^u) = e^{\left(\frac{a_1 + |a| \cos \Theta^u}{2\kappa}\right)(x-x_{r,i})} e^{\left(\frac{a_2 + |a| \sin \Theta^u}{2\kappa}\right)(y-y_{r,i})}\]

Determine \( n^\lambda = \left\lfloor \frac{n^E}{4} \right\rfloor \).

**for** each edge \( \Gamma^{e,e'} \in \Gamma^{\text{int}} \)

Compute max and min of \( \frac{|a|}{2\kappa} \left[ \cos(\phi - \alpha^{e,e'}) + \cos(\theta_k - \alpha^{e,e'}) \right] \), call them \( \Lambda_{\text{min}}^{e,e'}, \Lambda_{\text{max}}^{e,e'} \).

Sample \( \{\Lambda_i^{e,e'} : i = 1, \ldots, n^\lambda\} \) uniformly in the interval \([\Lambda_{\text{min}}^{e,e'}, \Lambda_{\text{max}}^{e,e'}]\).

Define the Lagrange multipliers approximations on \( \Gamma^{e,e'} \) by:

\[
\lambda^h|_{\Gamma^{e,e'}} = \text{span} \left\{ e^{\Lambda_i^{e,e'}(s-s_{r,i}^{e,e'})} : 0 \leq s \leq h \right\}
\]

**end for**
Some DGM/DEM Elements

Notation

DGM Element: $Q - n^E - n^\lambda$
DEM Element: $Q - n^E - n^\lambda+ \equiv [Q - n^E - n^\lambda] \cup [Q_1]$

‘$Q$’: Quadrilateral
$n^E$: Number of Enrichment Functions
$n^\lambda$: Number of Lagrange Multipliers per Edge
$Q_1$: Galerkin Bilinear Quadrilateral Element

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<td>$Q - 4 - 1$</td>
<td>4</td>
<td>$\phi + \left{ \frac{m\pi}{2} : m = 0, \ldots, 3 \right}$</td>
<td>1</td>
</tr>
<tr>
<td>$Q - 8 - 2$</td>
<td>8</td>
<td>$\phi + \left{ \frac{m\pi}{4} : m = 0, \ldots, 7 \right}$</td>
<td>2</td>
</tr>
<tr>
<td>$Q - 12 - 3$</td>
<td>12</td>
<td>$\phi + \left{ \frac{m\pi}{6} : m = 0, \ldots, 11 \right}$</td>
<td>3</td>
</tr>
<tr>
<td>$Q - 16 - 4$</td>
<td>16</td>
<td>$\phi + \left{ \frac{m\pi}{8} : m = 0, \ldots, 15 \right}$</td>
<td>4</td>
</tr>
<tr>
<td>DEM elements</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$Q - 5 - 1^+$</td>
<td>5</td>
<td>$\phi + \left{ \frac{2m\pi}{5} : m = 0, \ldots, 4 \right}$</td>
<td>1</td>
</tr>
<tr>
<td>$Q - 9 - 2^+$</td>
<td>9</td>
<td>$\phi + \left{ \frac{2m\pi}{9} : m = 0, \ldots, 8 \right}$</td>
<td>2</td>
</tr>
<tr>
<td>$Q - 13 - 3^+$</td>
<td>13</td>
<td>$\phi + \left{ \frac{2m\pi}{13} : m = 0, \ldots, 12 \right}$</td>
<td>3</td>
</tr>
<tr>
<td>$Q - 17 - 4^+$</td>
<td>17</td>
<td>$\phi + \left{ \frac{2m\pi}{17} : m = 0, \ldots, 16 \right}$</td>
<td>4</td>
</tr>
</tbody>
</table>
Illustration of the Sets $\Theta^u$ for the True DEM Elements

\[ Q - 5 - 1^+ \]
\[ Q - 9 - 2^+ \]
\[ Q - 13 - 3^+ \]
\[ Q - 17 - 4^+ \]
### Computational Properties

<table>
<thead>
<tr>
<th>Element</th>
<th>Asymptotic # of dofs</th>
<th>Stencil width for uniform $n \times n$ mesh</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q_1$</td>
<td>$n_{el}$</td>
<td>9</td>
</tr>
<tr>
<td>$Q_2$</td>
<td>$3n_{el}$</td>
<td>21</td>
</tr>
<tr>
<td>$Q_3$</td>
<td>$5n_{el}$</td>
<td>33</td>
</tr>
<tr>
<td>$Q_4$</td>
<td>$7n_{el}$</td>
<td>45</td>
</tr>
<tr>
<td>$Q - 4 - 1$</td>
<td>$2n_{el}$</td>
<td>7</td>
</tr>
<tr>
<td>$Q - 8 - 2$</td>
<td>$4n_{el}$</td>
<td>14</td>
</tr>
<tr>
<td>$Q - 12 - 3$</td>
<td>$6n_{el}$</td>
<td>21</td>
</tr>
<tr>
<td>$Q - 16 - 4$</td>
<td>$8n_{el}$</td>
<td>28</td>
</tr>
<tr>
<td>$Q - 5 - 1^+$</td>
<td>$3n_{el}$</td>
<td>21</td>
</tr>
<tr>
<td>$Q - 9 - 2^+$</td>
<td>$5n_{el}$</td>
<td>33</td>
</tr>
<tr>
<td>$Q - 13 - 3^+$</td>
<td>$7n_{el}$</td>
<td>45</td>
</tr>
<tr>
<td>$Q - 17 - 4^+$</td>
<td>$9n_{el}$</td>
<td>57</td>
</tr>
</tbody>
</table>

- Exponential enrichments ⇒ integrations computed analytically.
- $L u^E = 0$ ⇒ convert volume integrals to boundary integrals:

$$a(v^E, u^E) = \int_{\bar{\Omega}} (\kappa \nabla v^E \cdot \nabla u^E + \mathbf{a} \cdot \nabla u^E v^E) \, d\Omega = \int_{\Gamma} \nabla u^E \cdot \mathbf{n} v^E \, d\Gamma.$$
Define $V^E$ within each element as the free-space solutions to the homogeneous PDE, with locally-frozen coefficients.

- $a(x) \approx a^e = \text{constant inside each element } \Omega^e$ as $h \to 0$:

$$\left\{ a(x) \cdot \nabla u - \kappa \Delta u = f(x) \text{ in } \Omega \right\} \approx \bigcup_{\text{el}} \left\{ a^e \cdot \nabla u - \kappa \Delta u = f(x) \text{ in } \Omega^e \right\}$$

- Enrichment in each element:

$$u^E_e(x; \theta^e_i) = e^{\frac{|a^e|}{2\kappa} (\cos \phi^e + \cos \theta^e_i) (x-x^e_{r,i})} e^{\frac{|a^e|}{2\kappa} (\sin \phi^e + \sin \theta^e_i) (y-y^e_{r,i})} \in V^E_e$$
**Extension to 3D Advection-Diffusion**

- Advection direction: specified by two angles \((\theta, \phi) \in [0, 2\pi) \times [0, \pi)\)

- Advection velocity vector: \(\mathbf{a}^T = (\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi)\)

- Enrichment functions for 3D advection-diffusion:

\[
\begin{align*}
\Phi \times \Theta \equiv \{(&\theta_i, \phi_i)\}_{i=1}^{n_E} \in [0, 2\pi) \times [0, \pi) = \text{set of angles specifying } V^E
\end{align*}
\]
Outline

1 Motivation
2 Advection-Diffusion Equation
3 Discontinuous Enrichment Method (DEM)
4 Advection-Diffusion DEM
   - Enrichment Bases
   - Lagrange Multiplier Approximations
   - Element Design
   - Variable-Coefficient Problems
   - 3D Advection-Diffusion
5 Numerical Experiments
   - Homogeneous Boundary Layer Problem
   - Double Ramp Problem on an $L$–Shaped Domain
   - Inhomogeneous Rotating Field Problem on an $L$-shaped Domain
6 Summary
Homogeneous Boundary Layer Problem

- \( \Omega = (0, 1) \times (0, 1), f = 0. \)
- \( a = (\cos \phi \quad \sin \phi). \)
- Dirichlet boundary conditions are specified on \( \Gamma \) such that the exact solution to the BVP is given by

\[
\begin{align*}
    u_{ex}(x; \phi, \psi) &= e^{\frac{1}{2\kappa} \{\cos \phi + \cos \psi\}(x-1) + \{\sin \phi + \sin \psi\}(y-1)} - 1 \\
    &= e^{-\frac{1}{2\kappa} \{\cos \phi + \cos \psi + \sin \phi + \sin \psi\} - 1}
\end{align*}
\]

- \( \psi \in [0, 2\pi) : \) some flow direction (not necessarily aligned with \( \phi \)).
- Solution exhibits a sharp exponential boundary layer in the advection direction \( \phi \), whose gradient is a function of the Péclet number.

Figure 6: \( \phi = \psi = 0 \)

Figure 7: \( \phi = \pi/7, \psi = 0 \)
Homogeneous Boundary Layer Problem

- \( \Omega = (0, 1) \times (0, 1), f = 0. \)
- \( \mathbf{a} = \begin{pmatrix} \cos \phi \\ \sin \phi \end{pmatrix}. \)
- Dirichlet boundary conditions are specified on \( \Gamma \) such that the exact solution to the BVP is given by

\[
\begin{align*}
  u_{\text{ex}}(x; \phi, \psi) &= \frac{1}{2\kappa} \left\{ [\cos \phi + \cos \psi](x-1) + [\sin \phi + \sin \psi](y-1) \right\} - 1 \\
  &\quad - \frac{1}{2\kappa} [\cos \phi + \cos \psi + \sin \phi + \sin \psi] - 1
\end{align*}
\]

- \( \psi \in [0, 2\pi) : \) some flow direction (not necessarily aligned with \( \phi \)).
- Solution exhibits a sharp exponential boundary layer in the advection direction \( \phi \), whose gradient is a function of the Péclet number.

Homogeneous problem \( \Rightarrow \) pure DGM elements sufficient
Flow Aligned with Advection Direction \((\phi = \psi)\)

- For any DGM elements, for all advection directions \(\phi\):

  \[ u_{ex} \in V^E \]

  for all DGM elements, for all advection directions \(\phi\) here.
Flow Aligned with Advection Direction \( (\phi = \psi) \)

- \( u_{ex} \in \mathcal{V}^E \) for all DGM elements, for all advection directions \( \phi \) here.
- Therefore one would expect these elements to capture the exact solution to machine precision
Flow Aligned with Advection Direction \((\phi = \psi)\)

- \(u_{\text{ex}} \in \mathcal{V}^E\) for all DGM elements, for all advection directions \(\phi\) here.

- Therefore one would expect these elements to capture the exact solution to machine precision – *but only provided* \(\nabla u_{\text{ex}} \cdot n \in \mathcal{W}^h\).

Table 1: Relative \(L^2(\Omega)\) errors, \(\approx 400\) dofs, \(P_e = 10^3\), uniform mesh: Galerkin vs. DGM elts.

<table>
<thead>
<tr>
<th>(\phi/\pi)</th>
<th>(Q_1)</th>
<th>(Q - 4 - 1)</th>
<th>(Q_2)</th>
<th>(Q - 8 - 2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(5.77 \times 10^{-1})</td>
<td>(3.43 \times 10^{-14})</td>
<td>(4.33 \times 10^{-1})</td>
<td>(2.22 \times 10^{-10})</td>
</tr>
<tr>
<td>(1/6)</td>
<td>(2.53 \times 10^{-2})</td>
<td>(1.24 \times 10^{-15})</td>
<td>(1.49 \times 10^{-2})</td>
<td>(8.38 \times 10^{-4})</td>
</tr>
<tr>
<td>(1/4)</td>
<td>(2.62 \times 10^{-2})</td>
<td>(3.19 \times 10^{-14})</td>
<td>(1.53 \times 10^{-2})</td>
<td>(5.62 \times 10^{-6})</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(\phi/\pi)</th>
<th>(Q_3)</th>
<th>(Q - 12 - 3)</th>
<th>(Q_4)</th>
<th>(Q - 16 - 4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(3.68 \times 10^{-1})</td>
<td>(5.78 \times 10^{-13})</td>
<td>(2.44 \times 10^{-1})</td>
<td>(9.75 \times 10^{-10})</td>
</tr>
<tr>
<td>(1/6)</td>
<td>(1.21 \times 10^{-2})</td>
<td>(5.50 \times 10^{-6})</td>
<td>(9.47 \times 10^{-3})</td>
<td>(3.31 \times 10^{-6})</td>
</tr>
<tr>
<td>(1/4)</td>
<td>(1.24 \times 10^{-2})</td>
<td>(4.36 \times 10^{-14})</td>
<td>(9.81 \times 10^{-3})</td>
<td>(1.27 \times 10^{-12})</td>
</tr>
</tbody>
</table>
Flow not Aligned with Advection Direction \((\phi \neq \psi)\)

- Fix \(\phi = \pi/7\), vary \(\psi\).
Flow not Aligned with Advection Direction \((\phi \neq \psi)\)

- Fix \(\phi = \pi/7\), vary \(\psi\).
- Can show that \(u_{ex} \notin \mathcal{V}^E\) for any DGM elements and advection directions tested here.

### Table 2: Relative \(L^2(\Omega)\) errors, \(\approx 1600\) dofs, unstructured mesh, \(\phi = \pi/7\), \(Pe = 10^3\): Galerkin vs. DGM elts.

<table>
<thead>
<tr>
<th>(\psi/\pi)</th>
<th>(Q_1)</th>
<th>(Q - 4 - 1)</th>
<th>(Q_2)</th>
<th>(Q - 8 - 2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(1.45 \times 10^{-2})</td>
<td>(1.65 \times 10^{-3})</td>
<td>(5.92 \times 10^{-3})</td>
<td>(1.79 \times 10^{-3})</td>
</tr>
<tr>
<td>1/4</td>
<td>(1.52 \times 10^{-2})</td>
<td>(9.38 \times 10^{-4})</td>
<td>(6.06 \times 10^{-3})</td>
<td>(2.54 \times 10^{-4})</td>
</tr>
<tr>
<td>1/2</td>
<td>(1.51 \times 10^{-2})</td>
<td>(9.23 \times 10^{-4})</td>
<td>(5.97 \times 10^{-3})</td>
<td>(2.12 \times 10^{-4})</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(\psi/\pi)</th>
<th>(Q_3)</th>
<th>(Q - 12 - 3)</th>
<th>(Q_4)</th>
<th>(Q - 16 - 4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(4.34 \times 10^{-3})</td>
<td>(1.10 \times 10^{-4})</td>
<td>(3.23 \times 10^{-3})</td>
<td>(2.30 \times 10^{-5})</td>
</tr>
<tr>
<td>1/4</td>
<td>(4.46 \times 10^{-3})</td>
<td>(1.23 \times 10^{-5})</td>
<td>(3.29 \times 10^{-3})</td>
<td>(8.82 \times 10^{-7})</td>
</tr>
<tr>
<td>1/2</td>
<td>(4.36 \times 10^{-3})</td>
<td>(1.11 \times 10^{-5})</td>
<td>(3.18 \times 10^{-3})</td>
<td>(1.59 \times 10^{-6})</td>
</tr>
</tbody>
</table>
Solution Plots for Homogeneous BVP

Figure 8: $\phi = \psi = 0$, $Pe = 10^3$, $\approx 1600$ dofs

Figure 9: $\phi = \pi/7$, $\psi = 0$, $Pe = 10^5$, $\approx 1600$ dofs
Convergence Analysis

Figure 10: Convergence Rates ($\phi = \pi/7, \psi = 0, Pe = 10^2$, unstructured mesh)
Double Ramp Problem on an $L$–Shaped Domain

- Homogeneous Dirichlet boundary conditions are prescribed on all six sides of $L$–shaped domain $\Omega$.
- Advection direction: $\phi = 0$.
- Source: $f = 1$.
- Strong outflow boundary layer along the line $x = 1$.
- Two crosswind boundary layers along $y = 0$ and $y = 1$.
- A crosswind internal layer along $y = 0.5$.

Figure 11: $L$-shaped domain for double ramp problem.
No oscillations can be seen in the computed DGM and DEM solutions.

Would expect: DEM elements to outperform DGM elements for this *inhomogeneous* problem.

In fact: DGM elements experience some difficulty along the \( y = 0.5 \) line, the location of the crosswind internal layer.
Cross Sectional Solution Plots

Figure 13: Solution along the line $x = 0.9$ with 7600 dofs

![Graphs showing solution along the line $x = 0.9$.](image)

- Galerkin
- DGM
- DEM

Figure 14: Solution along the line $y = 0.5$ with 7600 dofs

![Graphs showing solution along the line $y = 0.5$.](image)

- Galerkin
- DGM
- DEM
Relative Errors

Table 3: $L^2(\Omega)$ errors relative to a reference solution*: uniform mesh, $Pe = 10^3$

<table>
<thead>
<tr>
<th># dofs</th>
<th>$Q_2$</th>
<th>$Q - 8 - 2$</th>
<th>$Q - 9 - 2^+$</th>
</tr>
</thead>
<tbody>
<tr>
<td>900</td>
<td>$2.72 \times 10^{-1}$</td>
<td>$1.19 \times 10^{-1}$</td>
<td>$7.22 \times 10^{-2}$</td>
</tr>
<tr>
<td>3600</td>
<td>$1.23 \times 10^{-1}$</td>
<td>$6.07 \times 10^{-2}$</td>
<td>$1.51 \times 10^{-2}$</td>
</tr>
<tr>
<td>14,400</td>
<td>$5.26 \times 10^{-2}$</td>
<td>$2.81 \times 10^{-2}$</td>
<td>$3.10 \times 10^{-3}$</td>
</tr>
<tr>
<td>32,400</td>
<td>$2.92 \times 10^{-2}$</td>
<td>$1.54 \times 10^{-2}$</td>
<td>$1.80 \times 10^{-3}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th># dofs</th>
<th>$Q_3$</th>
<th>$Q - 12 - 3$</th>
<th>$Q - 13 - 3^+$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1500</td>
<td>$1.49 \times 10^{-1}$</td>
<td>$1.11 \times 10^{-1}$</td>
<td>$5.62 \times 10^{-2}$</td>
</tr>
<tr>
<td>6000</td>
<td>$6.57 \times 10^{-2}$</td>
<td>$5.00 \times 10^{-2}$</td>
<td>$6.90 \times 10^{-3}$</td>
</tr>
<tr>
<td>24,000</td>
<td>$2.36 \times 10^{-2}$</td>
<td>$1.02 \times 10^{-2}$</td>
<td>$8.45 \times 10^{-4}$</td>
</tr>
<tr>
<td>54,000</td>
<td>$1.08 \times 10^{-2}$</td>
<td>$4.54 \times 10^{-3}$</td>
<td>$2.48 \times 10^{-4}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th># dofs</th>
<th>$Q_4$</th>
<th>$Q - 16 - 4$</th>
<th>$Q - 17 - 4^+$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2100</td>
<td>$9.58 \times 10^{-2}$</td>
<td>$8.32 \times 10^{-2}$</td>
<td>$4.66 \times 10^{-2}$</td>
</tr>
<tr>
<td>8400</td>
<td>$3.78 \times 10^{-2}$</td>
<td>$1.33 \times 10^{-2}$</td>
<td>$3.08 \times 10^{-3}$</td>
</tr>
<tr>
<td>33,600</td>
<td>$1.03 \times 10^{-2}$</td>
<td>$9.17 \times 10^{-3}$</td>
<td>$2.04 \times 10^{-4}$</td>
</tr>
<tr>
<td>75,600</td>
<td>$3.70 \times 10^{-3}$</td>
<td>$4.92 \times 10^{-4}$</td>
<td>$4.16 \times 10^{-5}$</td>
</tr>
</tbody>
</table>

* Since an analytical solution to this problem is not available, in computing the relative error, we use in place of the exact solution a reference solution, computed using a Galerkin $Q_6$ polynomial element on a $43,200 = 3 \times (120 \times 120)$ element mesh.
Inhomogeneous Rotating Advection Problem on an $L$–Shaped Domain

- Homogeneous Dirichlet boundary conditions are prescribed on all six sides of $L$–shaped domain $\Omega$
- $a^T(x) = (1 - y, x)$
- Source: $f = 1$
- Outflow boundary layer along the line $y = 1$
- Second boundary layer that terminates in the vicinity of the re-entrant corner $(x, y) = (0.5, 0.5)$.

Figure 15: $L$-shaped domain and rotating velocity field (curved lines indicate streamlines).
Solutions Plots for $Pe = 10^3$ with $\approx 3000$ dofs

* "Stabilized $Q_1$" is upwind stabilized bilinear finite element proposed by Harari * et al. *
Convergence Analysis & Results

- To achieve for this problem the relative error of 0.1%:
  - $Q - 5 - 1^+$ requires 6.4 times fewer dofs than $Q_1$.
  - $Q - 9 - 2^+$ requires 8.3 times fewer dofs than $Q_2$.
  - $Q - 13 - 3^+$ requires 5.7 times fewer dofs than $Q_3$.
  - $Q - 17 - 4^+$ requires 4.3 times fewer dofs than $Q_4$.

* “Stabilized $Q_1$” is upwind stabilized bilinear finite element proposed by Harari et al.
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6 Summary
Summary

Discontinuous Enrichment Method (DEM) = efficient, competitive alternative to stabilized FEMs for advection diffusion in a high Péclet regime.

- Parametrization makes possible systematic design of DEM elements of arbitrary orders.
- For all test problems, the enriched elements outperform their Galerkin and stabilized Galerkin counterparts of comparable computational complexity by at least one (and sometimes many) orders of magnitude difference.
- In a high Péclet regime, DGM and DEM solutions are almost completely oscillation-free, in contrast with the Galerkin solutions.
- Advection-diffusion work generalizable to more complex equations in fluid mechanics (e.g., 3D, non-linear, unsteady).
Publications

(www.stanford.edu/~irinak/pubs.html)

DEM:


ROM:

