A Stable Galerkin Reduced Order Model (ROM) for Compressible Flow

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Outline

1. Motivation

2. Overview of the POD/Galerkin Method for Model Reduction
   - Step 1: Constructing the POD Modes
   - Step 2: Galerkin Projection

3. A Stable Galerkin ROM for Compressible Flow
   - Stability Definitions
   - Equations for Compressible Flow
   - Stability-Preserving “Symmetry” Inner Product for Compressible Flow

4. Numerical Examples
   - Numerical Implementation
   - Test Case 1: Purely Random Basis
   - Test Case 2: 1D Acoustic Pressure Pulse
   - Test Case 3: 2D Pressure Pulse

5. Summary & Further Work
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5 Summary & Further Work
Why Develop a Fluid Reduced Order Model (ROM)?

CFD modeling of unsteady 3D flows is expensive!
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A Reduced Order Model (ROM) is a surrogate numerical model that aims to capture the essential dynamics of a full numerical model but with far fewer dofs.
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A **Reduced Order Model (ROM)** is a surrogate numerical model that aims to capture the essential dynamics of a full numerical model but with far fewer dofs.

**Applications in Fluid Dynamics:**
- Predictive modeling across a parameter space (e.g., aeroelastic flutter analysis).
- System modeling for active flow control.
- Long-time unsteady flow analysis, e.g., fatigue of a wind turbine blade under variable wind conditions.
Use of ROMs in predictive applications raises questions about their stability & convergence.

- Projection ROM approach is an alternative discretization of the governing PDEs.
- Desired numerical properties of a ROM discretization:
  - **Consistency** (with continuous PDEs):
  - **Stability**:
  - **Convergence**: 

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Motivation for Numerical Analysis of ROMs

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  - **Stability:**
  - **Convergence:**
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  - **Stability**: numerical stability is **NOT** in general guaranteed **a priori** for a ROM!
  - **Convergence**: 
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  - **Convergence**: requires consistency and stability.

This talk focuses on how to construct a Galerkin ROM that is stable \textit{a priori}.
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5 Summary & Further Work
Step 1: Constructing the POD Modes

High-Fidelity CFD Simulations:
- Snapshot 1
- Snapshot 2
- ...  
- Snapshot N

Fluid Modal Decomposition (POD):
\[
\mathbf{u} = \sum_{k} a_k(t) \phi_k(\mathbf{x})
\]

Step 1: Constructing the POD Modes

Galerkin Projection of Fluid PDEs:
\[
(\phi_j, \frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot \mathbf{F}(\mathbf{u})) = 0
\]

Step 2: Galerkin Projection

“Small” ROM ODE System:
\[
\dot{a}_k = f(a_1, ..., a_M)
\]
Proper Orthogonal Decomposition (POD), a.k.a. “Method of Snapshots”

- **Step 1.1:** Take $N$ snapshots from full simulation: $\{u^k(x)\}_{k=1}^N$
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- **Step 1.1:** Take $N$ snapshots from full simulation: $\{u^k(x)\}_{k=1}^N$
- **Step 1.2:** Compute a reduced POD basis $\{\phi_i\}_{i=1}^M$ with $M << N$ s.t. the energy in the projection of snapshots onto $\text{span}\{\phi_i\}$ is maximized:

$$
\max_{\phi \in H(\Omega)} \frac{\langle (u, \phi)^2 \rangle}{\|\phi\|^2}
$$

(1)

where

$$(\cdot, \cdot) \equiv \text{inner product}$$

$$\langle \cdot \rangle \equiv \text{time or ensemble averaging operator}$$
Proper Orthogonal Decomposition (POD), a.k.a. “Method of Snapshots”

- **Step 1.1:** Take $N$ snapshots from full simulation: $\{u^k(x)\}_{k=1}^{N}$

- **Step 1.2:** Compute a reduced POD basis $\{\phi_i\}_{i=1}^{M}$ with $M \ll N$ s.t. the energy in the projection of snapshots onto span{\phi_i} is maximized:

$$\max_{\phi \in H(\Omega)} \frac{\langle (u, \phi)^2 \rangle}{\|\phi\|^2} \iff R\phi = \lambda \phi$$

where

- $(\cdot, \cdot) \equiv$ inner product
- $\langle \cdot \rangle \equiv$ time or ensemble averaging operator

\[ R\phi \equiv \langle u^k(u^k, \phi) \rangle \]
Proper Orthogonal Decomposition (POD), a.k.a. “Method of Snapshots”

- **Step 1.1**: Take $N$ snapshots from full simulation: $\{u^k(x)\}_{k=1}^N$

- **Step 1.2**: Compute a reduced POD basis $\{\phi_i\}_{i=1}^M$ with $M << N$ s.t. the energy in the projection of snapshots onto span$\{\phi_i\}$ is maximized:

\[
\max_{\phi \in H(\Omega)} \frac{\langle (u, \phi)^2 \rangle}{\|\phi\|^2} \quad \leftrightarrow \quad R\phi = \lambda\phi
\]

Solution to (1) is the set of $M$ eigenfunctions $\{\phi_i\}_{i=1}^M$ corresponding to the $M$ largest eigenvalues $\lambda_1 \geq \cdots \geq \lambda_M$ of $R$
Properties of the POD Basis

- POD basis $\{\phi_i\}_{i=1}^{M}$ is **orthonormal**: $(\phi_i, \phi_j) = \delta_{ij}$.

- Average **energy** of projection of the snapshot ensemble onto the $i^{th}$ mode is given by:

  $$\langle (u^k, \phi_i)^2 \rangle = \lambda_i$$

  $$\Rightarrow \text{energy of set } \{\phi_i\}_{i=1}^{M} = \sum_{j=1}^{M} \lambda_j$$

- Truncated POD basis $\{\phi_i\}_{i=1}^{M}$ describes more energy (on average) of the ensemble than any other linear basis of the same dimension.

- Given $M \ll N$ modes, ROM solution can be represented as a linear combination of these modes:

  $$u_M(x, t) = \sum_{i=1}^{M} a_i(t) \phi_i(x)$$  \hspace{1cm} (2)
Properties of the POD Basis

- **POD basis** \( \{ \phi_i \}_{i=1}^{M} \) is **orthonormal**: \( (\phi_i, \phi_j) = \delta_{ij} \).

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- Given \( M \ll N \) modes, ROM solution can be represented as a linear combination of these modes:
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  u_M(x, t) = \sum_{i=1}^{M} a_i(t) \phi_i(x)
  \] (2)
**Step 2: Galerkin Projection**

**High-Fidelity CFD Simulations:**
- Snapshot 1
- Snapshot 2
- ...
- Snapshot N

**Fluid Modal Decomposition (POD):**

\[ u = \sum_{k} a_k(t) \phi_k(x) \]

**Galerkin Projection of Fluid PDEs:**

\[
\left( \phi_j, \frac{\partial u}{\partial t} + \nabla \cdot F(u) \right) = 0
\]

**“Small” ROM ODE System:**

\[ \dot{a}_k = f(a_1, \ldots, a_M) \]
Galerkin Projection of *(Continuous!)* Equations

Governing System of PDEs:

\[
\frac{\partial u}{\partial t} = \mathcal{L} u + \mathcal{N}_2(u, u) + \mathcal{N}_3(u, u, u)
\]  

(3)
Galerkin Projection of \((Continuous!)\) Equations

**Step 2.1:** Project (3) onto the modes \(\phi_j\) in inner product \((\cdot, \cdot)\)

\[
\left( \phi_j, \frac{\partial u}{\partial t} \right) = (\phi_j, \mathcal{L}u) + (\phi_j, \mathcal{N}_2(u, u)) + (\phi_j, \mathcal{N}_3(u, u, u))
\]
Galerkin Projection of \((\text{Continuous!})\) Equations

**Governing System of PDEs:**

\[
\frac{\partial \mathbf{u}}{\partial t} = \mathcal{L}\mathbf{u} + \mathcal{N}_2(\mathbf{u}, \mathbf{u}) + \mathcal{N}_3(\mathbf{u}, \mathbf{u}, \mathbf{u}) \tag{3}
\]

- **Step 2.1:** Project (3) onto the modes \(\phi_j\) in inner product \((\cdot, \cdot)\) is

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\]

- **Step 2.2:** Substitute the modal decomposition \(\mathbf{u}_M = \sum_{k=1}^{M} a_k(t) \phi_k(\mathbf{x})\) into (4)

\[
\dot{a}_k(t) = \sum_{l=1}^{M} a_l(\phi_k, \mathcal{L}(\phi_l)) + \sum_{l,m=1}^{M} a_l a_m (\phi_k, \mathcal{N}_2(\phi_l, \phi_m)) + \sum_{l,m,n=1}^{M} a_l a_m a_n (\phi_k, \mathcal{N}_3(\phi_l, \phi_m, \phi_n)) \tag{5}
\]
Galerkin Projection of (Continuous!) Equations

Governing System of PDEs:

\[
\frac{\partial \mathbf{u}}{\partial t} = \mathbf{L}\mathbf{u} + \mathcal{N}_2(\mathbf{u}, \mathbf{u}) + \mathcal{N}_3(\mathbf{u}, \mathbf{u}, \mathbf{u}) \quad (3)
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\]

\[
\equiv K_{kl}
\]
Continuous vs. Discrete Projection Approach

**Discrete Approach**

1. Governing Equations:
   \[ u_t = Lu \]
2. CFD Model:
   \[ \dot{u}_N = A_N u_N \]
3. Discrete Modal Basis \( \Phi \)
4. Projection of CFD Model (Matrix Operation)
   \[ \dot{a} = \Phi^T A_N \Phi a \]

**Continuous Approach**

1. Governing Equations:
   \[ u_t = Lu \]
2. CFD Model:
   \[ \dot{u}_N = A_N u_N \]
3. Continuous Modal Basis* \( \phi_j(x) \)
4. Projection of Governing Equations (Numerical Integration)
   \[ \dot{a}_j = (\phi_j, L\phi_k)a_k \]

*Continuous functions space is defined using finite elements.
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Stability Definitions

**Practical Definition:** Numerical solution does not “blow up” in finite time.
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- **More Precise Definition**: Numerical discretization does not introduce any spurious instabilities inconsistent with natural instability modes supported by the governing continuous PDEs.
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- **More Precise Definition:** Numerical discretization does not introduce any spurious instabilities inconsistent with natural instability modes supported by the governing continuous PDEs.

Analyzed with the **Energy Method:** uses an equation for the evolution of numerical solution “energy” to determine stability

\[ \| u_N(x, t) \|_E \equiv \left\{ \begin{array}{l} \text{energy of } u_N \text{ in norm } \| \cdot \|_E \\ \text{induced by inner product } (\cdot, \cdot)_E \end{array} \right\} \]
Stability Definitions

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    \text{energy of } u_N \text{ in norm } \| \cdot \|_E \\
    \text{induced by inner product } (\cdot, \cdot)_E
    \end{array} \right.
    \]

    \[
    \| u_N(x, t) \|_E \leq e^{\beta t} \| u_N(x, 0) \|_E, \quad \beta \in \mathbb{R}
    \]
3D Linearized Compressible Euler Equations

- Useful for aero-elasticity, aero-acoustics, flow instability analysis.

Linearization of Full Compressible Euler Equations

\[ \mathbf{q}^T(x, t) \equiv (u_1, u_2, u_3, \zeta, p) \equiv \bar{\mathbf{q}}^T(x) + \mathbf{q}'^T(x, t) \in \mathbb{R}^5 \]

\[ \Rightarrow \frac{\partial \mathbf{q}'}{\partial t} + A_i \frac{\partial \mathbf{q}'}{\partial x_i} + C \mathbf{q}' = 0 \quad (6) \]

where

\[ A_1 = \begin{pmatrix} \bar{u}_1 & 0 & 0 & 0 & \zeta \\ 0 & \bar{u}_1 & 0 & 0 & 0 \\ 0 & 0 & \bar{u}_1 & 0 & 0 \\ -\zeta & 0 & 0 & \bar{u}_1 & 0 \\ \gamma \bar{p} & 0 & 0 & 0 & \bar{u}_1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} \bar{u}_2 & 0 & 0 & 0 & 0 \\ 0 & \bar{u}_2 & 0 & 0 & \zeta \\ 0 & 0 & \bar{u}_2 & 0 & 0 \\ 0 & -\zeta & 0 & \bar{u}_2 & 0 \\ 0 & \gamma \bar{p} & 0 & 0 & \bar{u}_2 \end{pmatrix} \]

\[ A_3 = \begin{pmatrix} \bar{u}_3 & 0 & 0 & 0 & 0 \\ 0 & \bar{u}_3 & 0 & 0 & 0 \\ 0 & 0 & \bar{u}_3 & 0 & \zeta \\ 0 & 0 & -\zeta & \bar{u}_3 & 0 \\ 0 & \gamma \bar{p} & 0 & 0 & \bar{u}_3 \end{pmatrix}, \quad C = \begin{pmatrix} \frac{\partial \bar{u}_1}{\partial x_1} & \frac{\partial \bar{u}_1}{\partial x_2} & \frac{\partial \bar{u}_1}{\partial x_3} & \frac{\partial \bar{p}}{\partial x_1} & 0 \\ \frac{\partial \bar{u}_2}{\partial x_1} & \frac{\partial \bar{u}_2}{\partial x_2} & \frac{\partial \bar{u}_2}{\partial x_3} & \frac{\partial \bar{p}}{\partial x_2} & 0 \\ \frac{\partial \bar{u}_3}{\partial x_1} & \frac{\partial \bar{u}_3}{\partial x_2} & \frac{\partial \bar{u}_3}{\partial x_3} & \frac{\partial \bar{p}}{\partial x_3} & 0 \\ \frac{\partial \zeta}{\partial x_1} & \frac{\partial \zeta}{\partial x_2} & \frac{\partial \zeta}{\partial x_3} & -\nabla \cdot \bar{u} & 0 \\ \frac{\partial \bar{p}}{\partial x_1} & \frac{\partial \bar{p}}{\partial x_2} & \frac{\partial \bar{p}}{\partial x_3} & 0 & \gamma \nabla \cdot \bar{u} \end{pmatrix} \]
Symmetrized Compressible Euler Equations & Symmetry Inner Product

Energy stability can be proven following “symmetrization” of the linearized compressible Euler equations.

- Linearized hyperbolic compressible Euler system is “symmetrizable”.
- Pre-multiply equations by symmetric positive definite matrix:

\[
H = \begin{pmatrix}
\bar{\rho} & 0 & 0 & 0 & 0 \\
0 & \bar{\rho} & 0 & 0 & 0 \\
0 & 0 & \bar{\rho} & 0 & 0 \\
0 & 0 & 0 & \alpha^2 \gamma \bar{\rho}^2 \bar{p} & \bar{\rho} \alpha^2 \\
0 & 0 & 0 & \bar{\rho} \alpha^2 & \frac{1+\alpha^2}{\gamma \bar{p}}
\end{pmatrix} \Rightarrow H \frac{\partial q'}{\partial t} + \mathbf{H A}_i \frac{\partial q'}{\partial x_i} + H C q' = 0
\]

- \( H \) is called the “symmetrizer” of the system: \( \mathbf{H A}_i \) are all symmetric.
- Define the “symmetry” inner product and “symmetry” norm:

\[
(q'^{(1)}, q'^{(2)})_{(H,\Omega)} \equiv \int_{\Omega} [q'^{(1)}]^T H q'^{(2)} \, d\Omega, \quad ||q'||_{(H,\Omega)} \equiv (q', q')_{(H,\Omega)}
\]
Stability in the Symmetry Inner Product

\[
\frac{d}{dt} \| \mathbf{q}' \|^2_{(H, \Omega)} = - \int_\Omega [\mathbf{q}']^T H \left[ A_i \frac{\partial \mathbf{q}'}{\partial x_i} + C \mathbf{q}' \right] \, d\Omega \\
= - \int_{\partial \Omega} [\mathbf{q}']^T H A_i n_i \mathbf{q}' \, dS + \int_\Omega [\mathbf{q}']^T \left( \frac{\partial}{\partial x_i} (H A_i) - H C - C^T H \right) \mathbf{q}' \, d\Omega \\
= - \int_{\partial \Omega} [\mathbf{q}']^T H A_i n_i \mathbf{q}' \, dS + \int_\Omega [\mathbf{q}']^T H^{-T/2} B H^{T/2} \mathbf{q}' \, d\Omega \\
\leq - \int_{\partial \Omega} [\mathbf{q}']^T H A_i n_i \mathbf{q}' \, dS + \beta (\mathbf{q}', \mathbf{q}')_{(H, \Omega)} \\
\leq \beta \| \mathbf{q}' \|^2_{(H, \Omega)} \quad \text{if} \quad \int_{\partial \Omega} [\mathbf{q}']^T H A_i n_i \mathbf{q}' \, dS \geq 0 \quad \text{(well-posed BCs)}
\]

where \( \beta \) is an upper bound on the eigenvalues of

\[
B \equiv H^{-T/2} \frac{\partial (H A_i)}{\partial x_i} H^{-1/2} - H^{1/2} C H^{-1/2} - (H^{1/2} C H^{-1/2})^T
\]
Stability in the Symmetry Inner Product

\[
\frac{d}{dt} \|q'(x, t)\| (H, \Omega) \leq e^{\beta t} \|q'(x, 0)\| (H, \Omega)
\]

where \(\beta\) is an upper bound on the eigenvalues of

\[
B = H^{-T/2} \frac{\partial (HA_i)}{\partial x_i} H^{-1/2} - H^{1/2} CH^{-1/2} - (H^{1/2} CH^{-1/2})^T
\]

Exact solutions to the linearized Euler equations satisfy:

\[
\frac{d}{dt} \|q'\| (H, \Omega) = - \int_\Omega [q']^T H \left[ A_i \frac{\partial q'}{\partial x_i} + C q' \right] d\Omega
\]

\[
= - \int_{\partial \Omega} [q']^T HA_i n_i q' dS + \int_\Omega [q']^T \left( \frac{\partial}{\partial x_i} (HA_i) - HC - C^T H \right) q' d\Omega
\]

\[
\leq - \int_{\partial \Omega} [q']^T HA_i n_i q' dS + \beta (q', q') (H, \Omega)
\]

\[
\leq \beta \|q'\| (H, \Omega)
\]

if \(\int_{\partial \Omega} [q']^T HA_i n_i q' dS \geq 0\) (well-posed BCs)
Stability in the Symmetry Inner Product

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- Exact solutions to the linearized Euler equations satisfy:

\[ \| \mathbf{q}'(x, t) \|_{(H, \Omega)} \leq e^{\beta t} \| \mathbf{q}'(x, 0) \|_{(H, \Omega)} \]

- It turns out that the Galerkin approximation \( \mathbf{q}'_M = \sum_{i=1}^M a_k(t) \phi_k(x) \) satisfies the same energy expression as for the continuous equations:

\[ \| \mathbf{q}'_M(x, t) \|_{(H, \Omega)} \leq e^{\beta t} \| \mathbf{q}'_M(x, 0) \|_{(H, \Omega)} \]

i.e., it is stable.
Stability in the Symmetry Inner Product

\[
\frac{d}{dt} \|q'\|_{(H,\Omega)}^2 = - \int_{\Omega} [q']^T H \left[ A_i \frac{\partial q'}{\partial x_i} + C q' \right] d\Omega \\
= - \int_{\partial \Omega} [q']^T HA_i n_i q' dS + \int_{\Omega} [q']^T \left( \frac{\partial}{\partial x_i} (HA_i) - HC - CT H \right) q' d\Omega \\
= - \int_{\partial \Omega} [q']^T HA_i n_i q' dS + \int_{\Omega} [q']^T H^{-T/2} BH^{T/2} q' d\Omega \\
\leq - \int_{\partial \Omega} [q']^T HA_i n_i q' dS + \beta (q', q')_{(H,\Omega)} \\
\leq \beta \|q'\|_{(H,\Omega)}^2 \quad \text{if } \int_{\partial \Omega} [q']^T HA_i n_i q' dS \geq 0 \text{ (well-posed BCs)}
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where \(\beta\) is an upper bound on the eigenvalues of

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\]

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- For uniform base flow, the Galerkin scheme satisfies the strong stability estimate:

\[
\|q'_M(x, t)\|_{(H,\Omega)} \leq \|q'_M(x, 0)\|_{(H,\Omega)}
\]
Stability in the Symmetry Inner Product (cont’d)

- Stability analysis dictates that we use the symmetry inner product

\[
\left( q'(1), q'(2) \right)_{(H, \Omega)} \equiv \int_{\Omega} [q'(1)]^T H q'(2) d\Omega \\
= \int_{\Omega} \left[ \bar{\rho} u'(1) \cdot u'(2) + \alpha^2 \gamma \bar{\rho}^2 \zeta'(1) \zeta'(2) \\
+ \frac{1+\alpha^2}{\gamma \bar{\rho}} + \alpha^2 \bar{\rho} \left( \zeta'(2) p'(1) + \zeta'(1) p'(2) \right) \right] d\Omega
\]

to compute the POD modes and perform the Galerkin projection.

Practical Implication of Stability Analysis

Symmetry inner product ensures that any “bad” modes will not introduce spurious non-physical numerical instabilities into the Galerkin approximation.

- Galerkin projection step is stable for any basis in the symmetry inner product!
Steps to Obtain a Stable Compressible Fluid ROM

- Galerkin-project the equations in the symmetry inner product (7):

\[
\left( \phi_j, \frac{\partial q_M'}{\partial t} \right)_{(H,\Omega)} + \left( \phi_j, A_i \frac{\partial q_M'}{\partial x_i} \right)_{(H,\Omega)} + \left( \phi_j, Cq_M' \right)_{(H,\Omega)} = 0 \tag{8}
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\]

- Integrate second term in (8) by parts

\[
\left( \phi_j, \frac{\partial q'_M}{\partial t} \right)_{(H,\Omega)} = \int_{\Omega} \left[ \frac{\partial}{\partial x_i} [\phi_j^T H A_i] - \phi_j^T H C \right] q'_M d\Omega - \int_{\partial \Omega} \phi_j^T H A_n q'_M dS
\]
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- Integrate second term in (8) by parts and apply boundary conditions:

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\left( \phi_j, \frac{\partial q'_M}{\partial t} \right)_{(H,\Omega)} = \int_{\Omega} \left[ \frac{\partial}{\partial x_i} [\phi_j^T H A_i] - \phi_j^T H C \right] q'_M d\Omega - \int_{\partial \Omega} \phi_j^T H A_i n_i q'_M dS
\]

Insert boundary conditions into boundary integrals (weak implementation)

* Energy stability is maintained if the boundary conditions are such that

\[
\int_{\partial \Omega} \phi_j^T H A_i n_i q'_M dS \geq 0.
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\]

- Substitute modal decomposition \( q'_M = \sum_k a_k(t) \phi_k(x) \) to obtain an \( M \times M \) linear dynamical system of the form \( \dot{a} = Ka \)
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   ■ Equations for Compressible Flow
   ■ Stability-Preserving “Symmetry” Inner Product for Compressible Flow

4 Numerical Examples
   ■ Numerical Implementation
   ■ Test Case 1: Purely Random Basis
   ■ Test Case 2: 1D Acoustic Pressure Pulse
   ■ Test Case 3: 2D Pressure Pulse

5 Summary & Further Work
So far, all analysis is for continuous and smooth basis functions, and exact evaluation of inner product integrals.

Stability-Preserving Discrete Implementation:

- Define solution snapshots and POD basis functions using a piecewise smooth finite element representation:

\[ q_e^h(x) = \sum_{i=1}^{N_n} N_i(x) q_i' \]

- Apply Gauss quadrature rules \( \left( \int_\Omega f(x) d\Omega = \sum_{j=1}^{n^{quad}} \omega_j f(x_j) \right) \) of sufficient accuracy to exactly integrate the inner products:

\[ (u, v)_{(H, \Omega^e)} = \int_{\Omega^e} [u]^T H v d\Omega^e = [u^h]^T W^e v^h \]

where \( w_{kl}^e I \) with \( w_{kl}^e = \sum_{j=1}^{n^{quad}} H_e^h N^e_k(x_j) N^e_l(x_j) \omega_j \) is the \( (k, l)^{th} \) block of \( W^e \).
Numerical Implementation of Fluid ROM (cont’d)

- AERO-F was used to generate the CFD simulations, using unstructured tetrahedral meshes.
- Piecewise-linear finite elements were used to represent snapshot data and POD modes.
- $\mathbf{H}$ was taken to be piecewise constant over each element.
- A computer code was written that reads in the snapshot data written by AERO-F, assembles the necessary finite element representation of the snapshots, computes the numerical quadrature for evaluation of inner products, and projects the equations onto the modes.
- ROMs integrated in time using RK-4 scheme with same time step that was used in the CFD computation.
Numerical Stability & Convergence Tests

To test \textit{a posteriori} the \textbf{stability} of a ROM dynamical system \( \dot{a} = Ka \), check the Lyapunov condition:

\[
\max_i \Re \{ \lambda_i(K) \} \leq 0?
\]

To test \textit{a posteriori} the \textbf{convergence} of a ROM solution \( q'_M \rightarrow q'_{CFD} \) as \( M \rightarrow \infty \), check:

- \[
(q'_M, \phi_j)(\mathbf{H},\Omega) = \left( \sum_{i=1}^{M} a_i \phi_i, \phi_j \right)_{(\mathbf{H},\Omega)} = a_j \rightarrow (q'_{CFD}, \phi_j)(\mathbf{H},\Omega)?
\]

- \[
\langle \|q'_M - q'_{exact}\|_{(\mathbf{H},\Omega)} \rangle \rightarrow \langle \|q'_{CFD} - q'_{exact}\|_{(\mathbf{H},\Omega)} \rangle?
\]
Test Case 1: Purely Random Basis

- Uniform base flow: physically stable to any linear disturbance.
- Each mode is a random disturbance field that decays to 0 at the domain boundaries.
- Model problem for modes dominated by numerical error: extreme case of “bad” modes.
Test Case 2: 1D Acoustic Pressure Pulse

- 1D acoustic pressure pulse prescribed as the initial condition in $\Omega = (0, 20) \times (-5, 5) \times (0, 1)$:

$$p'|_{t=0} = -\bar{\rho}\bar{c}e^{-(x-5)^2}, \quad u_1'|_{t=0} = u_3'|_{t=0} = 0$$

- Uniform base flow, $M_\infty \equiv \bar{u}/\bar{c} = 0.5$ in the $x-$direction (pulse propagates in $x$-direction with velocity $\bar{u} + \bar{c}$).

- Slip wall boundary conditions applied on constant $y$ and $z$ boundaries.
POD Modes for 1D Acoustic Pressure Pulse Example

- CFD simulation run until $T_{tot} = 5.25$ (non-dimensional time) using 512 time steps.
- Snapshots taken every 8 time steps ($N = 64$ snapshots).
- $M = 4$ POD modes captured 85.5% of energy; $M = 8$ POD modes captured 99.5% of total ensemble energy.
Stability for 1D Acoustic Pressure Pulse Example

Four Galerkin schemes:

1. Symmetry inner product with BCs.
2. Symmetry inner product without BCs.
3. $L^2$ inner product with BCs.
4. $L^2$ inner product without BCs.

Only the symmetry inner product with BCs produces a stable ROM for all $M$

\[
\left( \max_i \mathcal{R}\{\lambda_i(K)\} < 10^{-9} \right)
\]
Convergence of the ROM for the 1D Acoustic Pressure Pulse Example

Figure shows symmetry ROM (with BCs) coefficients $a_i$ vs. $(q'_C F D, \phi_i)(H,\Omega)$ [- - 4 mode ROM; – 8 mode ROM; ○ CFD solution].

Symmetry ROM (with BCs) appears to be convergent as the number of modes increases.

Convergence check:

$$q'_M = \sum_{i=1}^{M} a_i(t) \phi_i(x)$$

$$\Pi_M q'_C F D = \sum_{i=1}^{M} (q'_C F D, \phi_i)(H,\Omega) \phi_i(x)$$
Test Case 3: 2D Pressure Pulse

- Reflection of cylindrical Gaussian pressure pulse in \( \Omega = (0, 20) \times (-5, 5) \times (0, 1) \):

\[
p'|_{t=0} = e^{-(x-10)^2-(y+1)^2}, \quad u'|_{t=0} = 0
\]

- Uniform base flow, \( M_\infty = 0.25 \) in \( x \)-direction.
- Slip wall boundary conditions applied on constant \( y \) and \( z \) boundaries.
Results for the 2D Pressure Pulse Example

- CFD simulation run until $T_{tot} = 6.4$ (non-dimensional time) using 624 time steps.
- Snapshots taken every 4 time steps starting at time $t = t_0 = 0.57$.
- 6 mode basis captures 97.4% of total ensemble energy.
- Good qualitative agreement between CFD solution and 6 mode symmetry ROM (with BCs) on large scale.
- Excellent agreement between CFD solution and 14 mode symmetry ROM (with BCs).
- Symmetry ROM (with BCs) is stable – vs. $L^2$ ROM, which experienced instability when more than 6 or 7 modes were used.

Pressure contours at $t - t_0 = 5.0$. 

![Pressure contours for CFD, 6 mode ROM, and 14 mode ROM at $t - t_0 = 5.0$.]
Convergence of the ROM for the 2D Pressure Pulse Example

\[ a_i \, \text{vs.} \, (q'_{CFD}, \phi_i)(H, \Omega) \, \text{for} \, M = 12 \]

\(-12\) mode ROM; \(\circ\) CFD solution

Tests demonstrate numerically the convergence of the symmetry ROM with BCs.

For \(M \geq 12\), ROM gives solution trajectory that is slightly closer to exact solution than the CFD solution.

Time-average error of the symmetry ROM solution as a function of \(M\), compared with the time-average error in the CFD solution.
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5 Summary & Further Work
Summary

- A Galerkin ROM in which the continuous equations are projected onto the modal basis in a continuous inner product is proposed.
- For this continuous Galerkin projection approach, the choice of inner product is crucial to stability.
- For linearized, compressible flow, Galerkin projection in the “symmetry” inner product leads to an approximation that is numerically stable for any choice of basis.
- A weak enforcement of the boundary conditions preserves stability, provided they are well-posed.
- A numerical implementation using finite elements that preserves stability is presented.
- Numerical stability of some POD/Galerkin ROMs constructed using this scheme is examined on several model problems.
Further Work

- A structure ROM governed by the non-linear plate equations was also developed (Segalman *et al.*).

- ROM convergence was examined mathematically, and *a priori* error estimates for the ROM solution error were derived (Kalashnikova & Barone 2010 *in press*).

- Extension of symmetry inner product methods to non-linear equations using an interpolation procedure to handle efficiently the non-linear terms (e.g., “best points interpolation procedure” of Peraire, Nguyen, *et al.*).
References


(can be downloaded from my website: www.stanford.edu/~irinak/pubs.html)
References


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Thank You!


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