Finding a Covering Triangulation Whose Maximum Angle is Provably Small

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ABSTRACT

We consider the following problem: given a planar straight-line graph, find a *covering* triangulation whose maximum angle is as small as possible. A covering triangulation is a triangulation whose vertex set contains the input vertex set and whose edge set contains the input edge set. The covering triangulation problem differs from the usual Steiner triangulation problem in that we may not add a vertex on any input edge. Covering triangulations provide a convenient method for triangulating multiple regions sharing a common boundary, as each region can be triangulated independently.

We give an explicit lower bound γ_{opt} on the maximum angle in any covering triangulation of a particular input graph in terms of its local geometry. Our algorithm produces a covering triangulation whose maximum angle γ is provably close to γ_{opt}. Bounding γ by a constant times γ_{opt} is trivial: We prove something significantly stronger. Specifically, we show that

$$\pi - \gamma \geq \min(\frac{\pi - \gamma_{\text{opt}}}{2}, \frac{\pi}{6}),$$

i.e., our γ is not much closer to π than is γ_{opt}. To our knowledge, this result represents the first nontrivial bound on a covering triangulation's maximum angle. Our algorithm adds $O(n)$ Steiner points and runs in time $O(n \log^2 n)$, where n is the number of vertices of the input. We have implemented an $O(n^2)$ time version of our algorithm.

1 INTRODUCTION

In this paper, we consider a class of triangulations called *covering* triangulations. A covering triangulation is a triangulation whose vertex set contains the input vertex set and whose edge set contains the input edge set. For example, if the input is a polygon, then a covering triangulation may have additional vertices in the polygon's interior, but not on its boundary. Covering triangulations with bounded triangle shape were first considered in Mitchell [1993].

Traditionally, most triangulation algorithms generate either a *constrained* triangulation or a *Steiner* triangulation. A constrained triangulation has a vertex set that is exactly the vertex set of the input, and an edge set that contains the edge set of the input. A Steiner triangulation has a vertex set that contains the vertex set of the input, and every edge of the input is the union of some edges of the triangulation. This is also known as a conforming triangulation.

For all three of these classes we assume the triangulations are *conformal.* That is, if we consider a triangulation as a lattice of faces, then being conformal means that two faces intersect in a face of the lattice or not at all. For example, there may be no vertices in the middle (relative interior) of an edge or triangle.

1.1 PREVIOUS RESULTS

There are several algorithms for generating constrained triangulations that exactly optimize some measure. Edelsbrunner, Tan, and Waupotitsch [1990] introduce the edge insertion paradigm, which is a global generalization of local edge flip. Edge insertion can be used to find a constrained triangulation that minimizes the maximum angle. Bern, Edelsbrunner, Eppstein, Mitchell, and Tan [1992] show that edge insertion may also be used to find optimal constrained triangulations for any measure for which every triangle has at least one *anchor vertex.* This property states that for any triple of vertices of the input forming a triangle, no constrained triangulation can have measure better than that of the triangle, unless there is an edge from the anchor vertex crossing the opposite edge. Like constrained triangulations, covering triangulations have prescribed edges that cannot be crossed or subdivided. Hence, this property also has application to covering triangulations, as we show in Section 2. Measures that have the anchor property include minmax angle, maxmin height, and minmax slope for points embedded in $\Re^3$.

The most famous constrained triangulation is the (constrained) Delaunay triangulation, or CDT. It optimizes several measures, including maximizing the minimum angle (Lawson [1977]). Many algorithms exist for the CDT, following paradigms such as plane sweep, edge flip, and incremental insertion. These are summarized in Fortune [1992]. Chew[1989] presents an algorithm for finding a CDT in optimal $O(n \log n)$ time.

Triangles in a CDT that contain edges of the input play a special role for covering triangulations. In particular, they determine the minimum maximum angle possible for any covering triangulation, see Section 2 and Section 5. Also, their circumcircles help guide our triangulation in Section 3.

Steiner triangulations that exactly optimize some

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criteria may have a non-polynomial number of vertices. As such, most algorithms seek to come close to a provable measure bound, while simultaneously achieving a reasonable cardinality. Bern, Dobkin and Eppstein [1991] present algorithms that approximately maximizes the minimum height and minimize the maximum angle. They are able to bound the largest angle away from π by a constant. The cardinality of these triangulations for no large angles is a small polynomial in the number of vertices of the input, depending on whether the input is a polygon or polygon with holes, and what constant bound is desired for the largest angles.

Bern, Mitchell and Ruppert [1993] has recently given an algorithm for constructing a Steiner triangulation of a polygon with holes with all non-obtuse angles. The algorithm adds only $O(n)$ Steiner points (matching the worst case lower bounds) and runs in time $O(n \log^2 n)$. The algorithm builds upon some key ideas presented in this paper, namely how circles can be triangulated independently with a reasonable angle bound. The tight angle bound of $\pi/2$ is based on some beautiful mathematical properties of tangent circles.

Mitchell [1993b] gives an algorithm for triangulating a PSLG so that angles are at most $7\pi/8$, adding $O(n^2 \log n)$ Steiner points. The algorithm is based on refining an arbitrary triangulation.

There are many quadtree based algorithms that yield a no large angle Steiner triangulation. Bern, Eppstein and Gilbert [1990] provides an algorithm for point sets, as well as other algorithms for no small angles. Eppstein [1991] combines no large angles, and also separately no small angles, with finding an approximately minimum weight triangulation. No large and no small angles are achieved simultaneously in Baker, Gross and Rafferty [1988] and in Melissaratos and Souvaine [1992]. However, the cardinality of a triangulation with no small angles is doomed to be dependent on the input geometry. Bern and Eppstein [1992] summarizes much of the Steiner triangulation literature.

Mitchell[1993] presents the only other known algorithm for a covering triangulation with a provable bound on triangle shape. Given a PSLG, the algorithm generates a triangulation whose minimum angle is at least a constant factor times an explicit upper bound provided by the input geometry. This bound, and generating a triangulation with minimum angle close to this bound, is significantly different from their analog for maximum angle.

1.2 APPLICATION MOTIVATION

Rendering of computer graphics, functional interpolation and finite element methods all require triangulations in which the largest angle is bounded away from π. See Babuška and Aziz [1976] and Barnhill[1983]. Our algorithm can be used to triangulate two intersecting regions independently, where Steiner triangulations might fail to produce a conformal triangulation. That is, most Steiner triangulation algorithms add many Steiner points on the boundary of a region. Suppose a region is triangulated, then the triangulation of an intersecting region may introduce nonconformal Steiner points on the edges of the first region. Such intersecting regions naturally occur in models of objects composed of two different materials, such as semiconductors. These regions also occur when generating a triangulation of the surface of a three dimensional polytope, such as cubes of an octree.

We give a characterization of the smallest angle possible without adding Steiner points on the boundary of a region. Thus it may be possible to preprocess the boundary between two regions by adding Steiner points, such that our bound on the smallest possible angle in a covering triangulation of either region is a constant. In general, this is a difficult problem (see Mitchell[1993b]), but for certain geometries this task is easy. If this task can be performed, then our algorithm can be used to triangulate each region independently, generating a triangulation of the entire region with largest angle no more than a constant.

1.3 ALGORITHM OVERVIEW

Our algorithm consists of two main steps. First, for each input edge E we find its *almond* A, the largest circular arc with chord E that contains no visible input vertex. See Figure 6 and Figure 8. No covering triangulation can have largest angle smaller than that subtended by E at a vertex on the almond A. Given arbitrary vertices on A, we show how to triangulate the almond so that we come close to this bound, see Figure 3 and Figure 4.

Second, after all almonds are triangulated, we have a polygonal region R inside P that is untriangulated, see Figure 7. We triangulate R using any Steiner triangulation algorithm for polygons with holes that achieves no angle larger than $5\pi/6$, see Figure 9. This introduces (arbitrary) Steiner points on some edges of triangulated almonds. We fix such nonconformal vertices by adding the edge between it and the opposite triangle vertex (inside the almond). If none of the edges of R are too large (compared to the corresponding almond), this does not affect our angle bound.

As noted, a key subroutine to our algorithm is a solution to the following problem: Given a polygon with holes, find a Steiner triangulation with a constant bound on the largest angle. As discussed in the introduction, we may chose among several known algorithms for this subroutine. Let $S(n)$ be the number of Steiner points added by the subroutine, and $T(n)$ its running time. Then our covering triangulation algorithm adds $O(n + S(n))$ Steiner points and runs in time $O(n \log n + T(n))$. Using the current best solution, that of Bern, Mitchell and Ruppert [1993], we have $S(n) = O(n)$ and $T(n) = O(n \log^2 n)$. Hence our covering triangulation adds $O(n)$ Steiner points and runs in time $O(n \log^2 n)$.

Like the CDT and unlike quadtree and dicing based algorithms, our algorithm (including the subroutine Bern, Mitchell and Ruppert [1993]) has no preferred directions and is independent of the orientation of the input.

2 THE SMALLEST ANGLE POSSIBLE

A key question is what geometric features of the input determine the smallest angle possible in any covering triangulation.

In Mitchell[1993], the largest angle possible in a covering triangulation was determined by the local geometry around a vertex of the input, that is by nearby points on a face disjoint from the vertex. We show below that our angle bound depends on local features around an input edge, that is by nearby vertices disjoint from the edge.

What the smallest angle possible locally in a constrained triangulation has been considered in Edelsbrunner, Tan and Waupotitsch [1990]. Let $\mu(T)$ de-

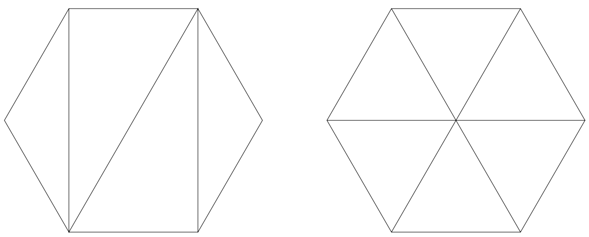

Figure 1: Any constrained triangulation of a regular m-gon has an angle of $\pi - 2\pi/m$, while the lower bound from lemma 1 is less than $\pi/3$. A covering triangulation of an m-gon with one Steiner point may have no obtuse angles.

note the maximum angle of a triangulation T, then they show the following:

Lemma 1 *Given a vertex set S, in any constrained triangulation T containing edge $\overline{UV}$, we have $\mu(T) \geq \max_{W \in S} \angle UWV$.*

This result was written in the context of constrained triangulations. For covering triangulations certain edges $\overline{VU}$ are prescribed by the input. The ability to add non-boundary Steiner points will only increase the input vertex set S. Hence the lemma gives us a lower bound on the largest angle in any covering triangulation of a planar straight line graph (PSLG). If the input P is a polygon, and we wish to triangulate only the interior of P, it is necessary to consider only vertices W such that $\triangle UWV$ is completely inside P. This may yield a smaller lower bound, that is, a smaller largest angle may be possible.

This lower bound is not always achievable for a constrained triangulation because of the global geometry, for example see Figure 1. Our results are that we can always construct a covering triangulation that comes close to achieving this lower bound.

2.1 WHAT MEASURE OF CLOSENESS TO OPTIMALITY TO USE

What do we mean by coming close to this bound? Constrained triangulations exactly achieve some optimal value. Steiner triangulations achieve no angles larger than a constant, which we have shown is impossible for covering triangulations (Figure 1).

No small angle Steiner triangulations achieve minimum angle at least a constant factor times some upper bound determined from the local input geometry. The analog of this for no large angles, that of achieving maximum angle no larger than a constant times our lower bound, is trivial because π is no more than a constant times the trivial lower bound of $\pi/3$! Hence we have decided to consider the difference between the largest angle we produce and π.

The upper bound B. Let B be the difference between π and the lower bound on a covering triangulation from Lemma 1. That is, denote the input by P, let v be its vertex set and e its edge set. Then $B = \pi - \max_{W \in v, E \in e} \angle WE$, where $\angle WE$ denotes the angle subtended by E at W. If P is a polygon, we further restrict ourselves to triangles $\triangle WE$ contained in P.

For $\angle UWV$, we define $\nu(UWV) = \pi - \angle UWV$. Similarly we define ν of a triangle to be the minimum of ν over the angles of the triangle, and ν of a triangulation to be the minimum of ν over its triangles.

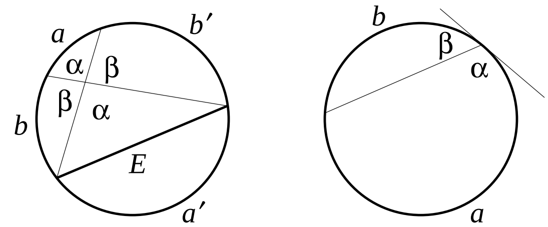

Figure 2: Left, $\alpha = (a + a')/2, \beta = (b + b')/2$, where a may be zero or negative. Right, $\alpha = a/2, \beta = b/2$.

Theorem 1 *Our triangulation T has $\nu(T) \geq min(B/2, \pi/6)$.*

Proof. The proof follows from Theorem 4 and Theorem 5 below. ∎

We also note the following lower bound on the worst case number of Steiner points needed to achieve this bound. Some input, even input consisting of just vertices and no prescribed edges, require $\Omega(n)$ Steiner points in order to achieve a given bound. Fix k, such that we require a triangulation with ν at least B/k. A regular m-gon with $m = \lceil 6k \rceil$ requires one Steiner point in its interior in order to have $\nu > B/k$, see Figure 1. Hence n/m copies of the vertices of a regular m-gon, arranged vertically and spaced far apart, is an input with n vertices requiring $\Omega(n/k)$ Steiner vertices to achieve $\nu > B/k$. Since our triangulation has $k = 2$ with $O(n)$ Steiner points, our cardinality is worst case optimal up to a constant factor.

This worst case example had a large B, but we may generalize and find a worst case construction (with prescribed edges) for any $B \leq \pi/2$. Almonds with $m = 2k + 2$ evenly spaced vertices may be used to show that for any B, achieving $\nu \geq B/k$ may require n/m Steiner points.

3 ALMONDS

We wish to find a vertex free region around each input edge, called an almond. We are able to triangulate almonds independently with reasonably small largest angle. The almonds have the property that we may triangulate outside of them arbitrarily and not affect the lower bound of Lemma 1 by much. The following theorem of Euclid is central to this notion.

Theorem 2 *Draw any circle with cord $E = \overline{VU}$. For any points X and X' on the circle and on the same side of E, we have $\angle VXU = \angle VX'U$. Any point Z inside the circle has $\angle UZV > \angle UXV$. Any point Z' outside the circle has $\angle YZ'V < \angle UXV$.*

We may restate this explicitly as follows.

Theorem 3 *The measure of the angle between two intersecting chords of a circle is equal to half the sum of arc lengths between the chords. The measure of the angle between a chord and a tangent line is equal to half the arc length between the vertices of the chord. See Figure 2.*

Almond. Consider any input edge $E = \overline{UV}$. Let W be the input vertex visible to E such that $\angle UWV$

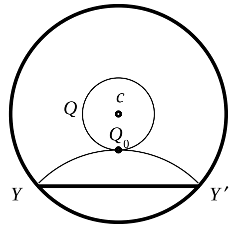

Figure 5: Rotating $\overline{YY'}$ is equivalent to rotating Q.

is maximized. If $\angle UWV \geq 2\pi/3$, we define the *almond* A for E to be the circular arc through U, V and W. Otherwise, we define the almond for E to be the circular arc in the polygon's interior such that for any point X on the arc $\angle UXV = 2\pi/3$ (this is how the constant $\pi/6$ arises in Theorem 1). We let c denote the center of the circle containing the almond arc. We define the *angle of an almond* to be the angle $\angle UXV$ for any point X on the almond, denoted $\angle opt$. We similarly define $\nu(A) = \nu(\angle opt)$. We denote the (normalized) arc length of an almond by a, and note from Theorem 3 that $a = 2\nu(A)$.

3.1 TRIANGULATING ALMONDS

We now show how to triangulate an almond with arbitrary Steiner points on it, so that no angle is much larger than the angle of the almond. The proofs are made particularly elegant by Theorem 3. To prove Theorem 4 we first need the following two technical lemmas.

The following ancient theorem can be viewed as a generalization of Theorem 2. See Figure 2.

Lemma 2 *For any point Q in a circle with center c, among all chords $\overline{YY'}$ of fixed length, $\angle YQY'$ is maximized for $\overline{cQ} \perp \overline{YY'}$.*

Proof. We view the problem as fixing the chord on the circle and rotating Q about c, see Figure 5. The trajectory of Q is a circle centered at c. Consider the maximum angle almond A' for $\overline{YY'}$ among all the vertices of the trajectory of Q. Let Q_0 be the vertex determining A'. Note Q_0 is also the point of tangency between the almond circle of A' and the trajectory of Q, and hence lies on the line between c and the almond circle center. The perpendicular bisector of $\overline{YY'}$ contains both c and the center of the almond circle, hence Q_0 lies on the perpendicular bisector of $\overline{YY'}$. ∎

Lemma 3 *Among all chords $\overline{YY'}$ through a point Q, the arc length cut by the chord is minimized for $\overline{cQ} \perp \overline{YY'}$.*

Proof. Chord length and cut arc length are monotonically decreasing functions of distance to the circle center c. The chord through Q whose distance to the circle center c is maximum has $\overline{cQ} \perp \overline{YY'}$. ∎

Theorem 4 *The convex hull of an edge $E = \overline{UV}$ and any number of vertices on its almond A may be triangulated with a covering triangulation $\mathcal{T}$ with $\nu(\mathcal{T}) \geq \nu(A)/2$. At most one Steiner point is required.*

Proof. We have two cases. In the first case there is a *long* edge of the convex hull other than E, and we may triangulate without adding a Steiner vertex. In the second case there is no long convex hull edge other than E, and we add a Steiner vertex near the "center" of the almond. We parameterize our definition of *long* by ka, and show that the optimal choice of k is $1/2$.

Case 1. The first case is where a long convex hull edge E' *cuts* more than k of the arc of A. That is, the arc length between the vertices of the convex hull edge E' is more than ka. Let U' be the closer vertex of E' to U, and V' the closer vertex of E' to V. Let $\mathcal{L}$ be the list of convex hull vertices between U and U', and $\mathcal{R}$ the list of those between V and V'. We triangulate by introducing an edge between U' and each vertex of $\mathcal{R}$, and an edge between V and each vertex of $\mathcal{L}$, see Figure 3.

We now show that no angle of this triangulation is large, specifically that $\nu(\mathcal{T}) \geq k\nu(A)$. Consider angle $\angle XYZ$ of any triangle. An upper bound on this angle is the larger angle between $\overline{XY}$ and a tangent line at Y (this is the limiting case as Z approaches Y). The arc length between X and Y is at most 2π minus the arc length of E', or $2\pi - ka$. Hence from Theorem 3, $\angle XYZ < \pi - ka/2$, or $\nu(\angle XYZ) > k\nu(A)$.

Case 2. The second case is where all the convex hull egdes are short. That is, the arc length between the vertices of any convex hull edge (except E) is at most ka. Let $E' = \overline{U'V'}$ be a *fictitious* chord of A, parallel to E, with arc length exactly ka, see Figure 12. We say that E' is fictitious in the sense that its vertices are not necessarily given as vertices on A, and hence are not necessarily in the construction $\mathcal{T}$. Let Q be the point where the chords $\overline{UV'}$ and $\overline{U'V}$ intersect. We triangulate by adding Steiner vertex Q and edge $\overline{XQ}$ for each of the vertices X on A, see Figure 4. We now show that no angle of this triangulation is large, specifically that $\nu(\mathcal{T}) \geq \min(k, 1-k)\nu(A)$.

Consider $\angle XYQ$ of any triangle. As in case 1, an upper bound on this angle is γ, the larger angle between $\overline{YQ}$ and the tangent at Y. Consider the chord $\overline{YY'}$ containing $\overline{YQ}$. By Theorem 3, $\gamma = \pi - b/2$, where b is the arc cut by $\overline{YY'}$. By Lemma 3, b is minimized by $\overline{YY'} \perp \overline{cQ}$. For this choice $\overline{YY'}$ is parallel to E' and closer to c, hence $b > ka$ and $\nu(\angle XYQ) > k\nu(A)$.

Consider $\angle UQV$. From Theorem 3 we have $\angle UQV = ((2\pi - a) + ka)/2 = \pi - (1-k)a/2$, or $\nu(\angle UQV) = (1-k)\nu(A)$. Consider $\angle XQY$ of any other triangle. As we increase the length of $\overline{XY}$, this angle increases monotonically. Hence an upper bound on $\angle XQY$ is achieved in the case where $\overline{XY}$ cuts arc length ka. We fix the length of $\overline{XY}$ at ka. From Lemma 2, $\angle XQY$ is maximized for $\overline{XY} \perp \overline{cQ}$. But for this choice $\overline{XY}$ is coincident with the fictitious edge E'! Thus $\angle XQY \leq \angle QE' = \angle QE$ so $\nu(\angle XQY) \geq (1-k)\nu(A)$.

Combining case 1 and 2, a lower bound on the measure of the angles in our construction $\mathcal{T}$ is $\min(k, 1-k)\nu(A)$. This is maximized at $\nu(A)/2$ for $k = 1/2$. ∎

4 TRIANGULATING P

We form a collection of polygons with holes called R, lying inside P. R is a straight line approximation to the region of P outside the visible portion of the almonds.

Lemma 4 *A point in an almond is visible to either all or none of the corresponding edge.*

Proof. By definition no input vertex visible to any point of E is interior to its almond A. Hence either

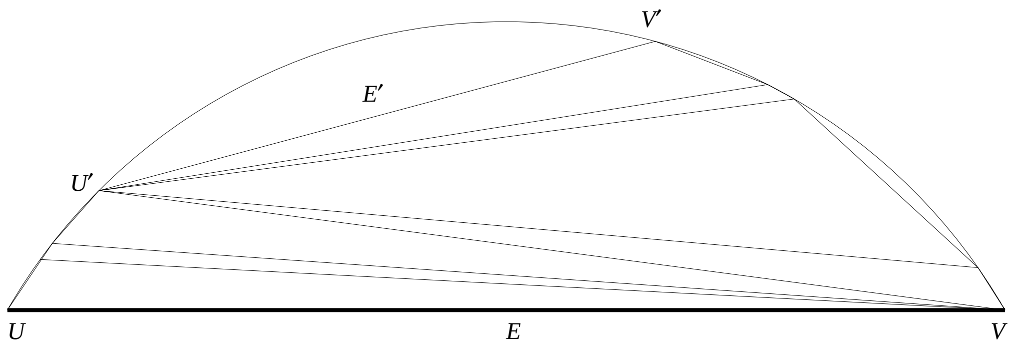

Figure 3: Triangulating an almond when a convex hull edge E' is long.

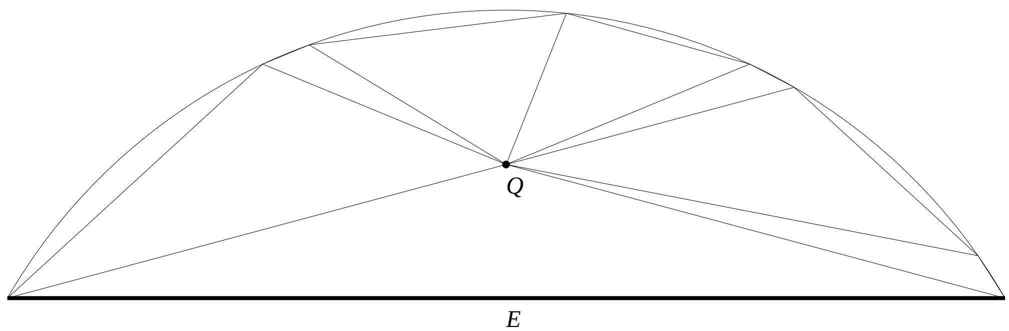

Figure 4: Triangulating an almond when all convex hull edges are short.

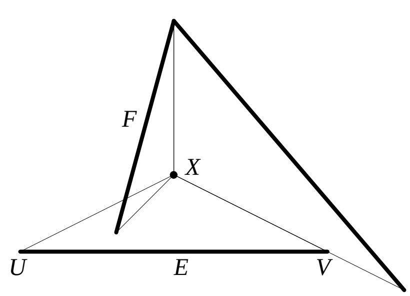

Figure 10: No point is interior and visible to three almonds. Some other edge of the input must make X not visible to any point of E.

no edge is in an almond, or there is an edge F closest to E that completely passes through A. In the latter case a point in A between E and F is visible to all of E, and a point on the far side of F is visible to none of E. ∎

Consider the collection of almonds for all edges of P. It is possible that they overlap, but not very much in the following sense.

Lemma 5 *No point is interior and visible to three almonds.*

Proof. Consider a point X strictly inside three almonds. By definition and Theorem 2, the chords (input edges) of the almonds must each subtend an angle greater than $2\pi/3$ at X. Hence for at least one of the chords $E = \overline{VU}$, there is a chord F that crosses $\overline{XV}$, see Figure 10. Hence by Lemma 4, X is not visible to any of E. ∎

The construction of R is illustrated in Figure 6 and Figure 7. Wherever two (or three) almond arcs intersect and the intersection point is also visible to each almond input edge, we introduce a Steiner vertex at that point. Lemma 5 ensures that such a Steiner point is not interior to some other almond.

If the input is a planar straight line graph, we add the convex hull edges that are not prescribed by P. We do not form almonds for these edges, but instead treat them as almond arcs: We add a Steiner vertex where such an edge intersects an almond. We ignore the almonds outside of the convex hull arising from edges of P lying on the convex hull. Vertices of P not contained in any edge of P can be handled as follows. If an isolated vertex is on the boundary of an almond, no modifications are necessary. An isolated vertex interior to R can be fattened into a small equilateral triangle and treated as part of the input P. Alternatively, the algorithm used to triangulate R can be made to handle such vertices. For example, the algorithm of Bern, Mitchell and Ruppert [1993] and also of Bern, Dobkin, and Eppstein [1991] requires only a trivial modification.

If the input is a polygon, then we ignore almonds outside of the polygon's interior. These are the only differences in our algorithm for polygons and PSLGs.

Interior edges. A given pair of almond arcs intersect at zero, one or two vertices. An example of when they intersect at one vertex is when their corresponding input edges intersect at an input vertex at an angle greater than $2\pi/3$. When a pair of almond arcs intersect at two vertices, we introduce an edge between them, called an *interior edge*. We similarly add an interior edge when an arc intersects a convex hull edge that is not prescribed by P. Interior edges lie outside of R and as such are not subdivided in the final triangulation.

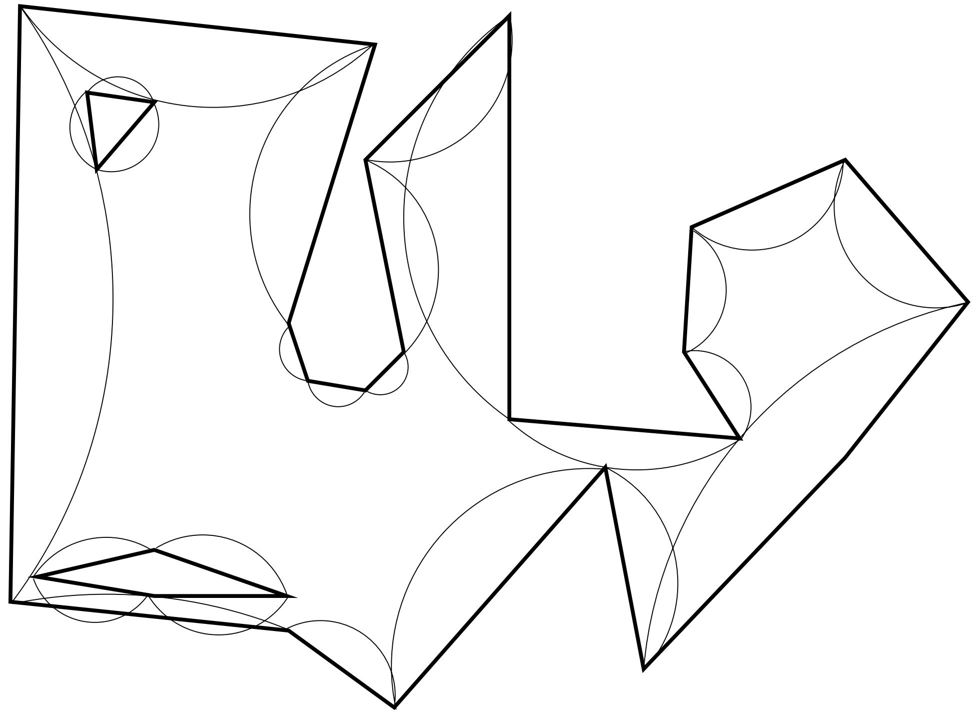

Figure 6: The almonds for a polygon.

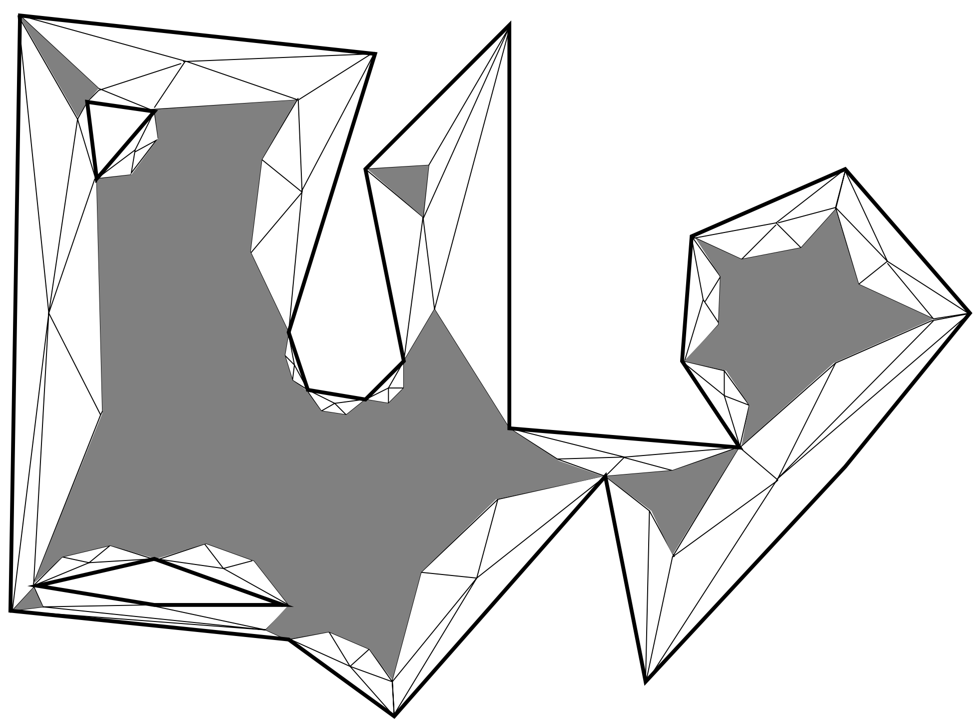

Figure 7: The triangulation of the almonds, and the region R (shaded).

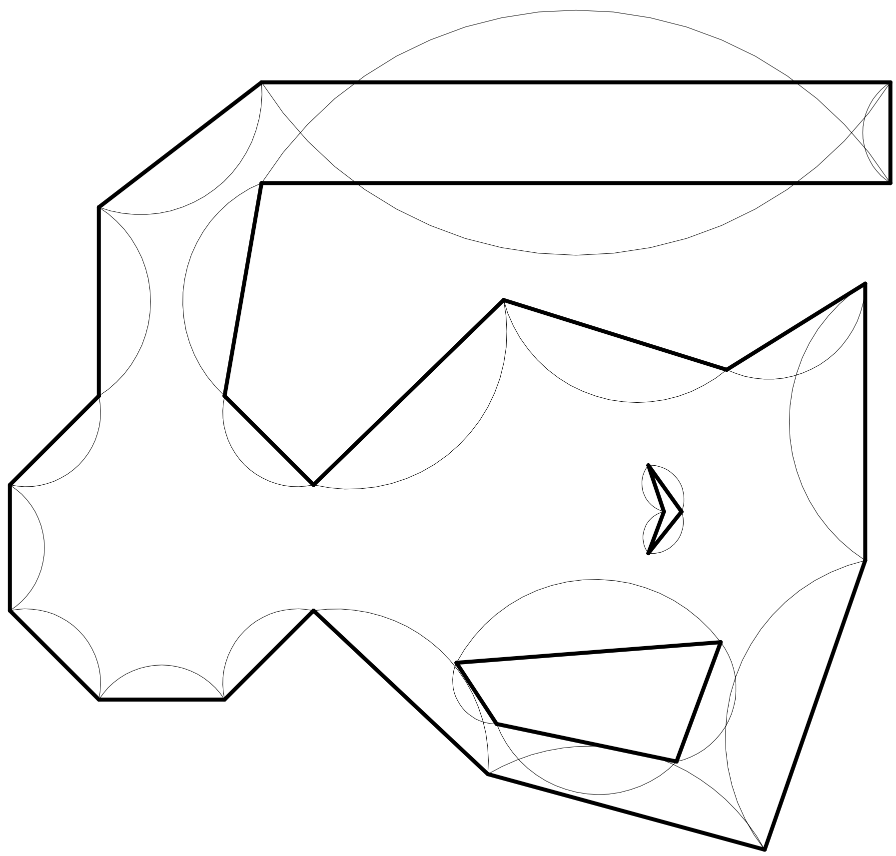

Figure 8: Almonds for an example polygon generated by our implementation.

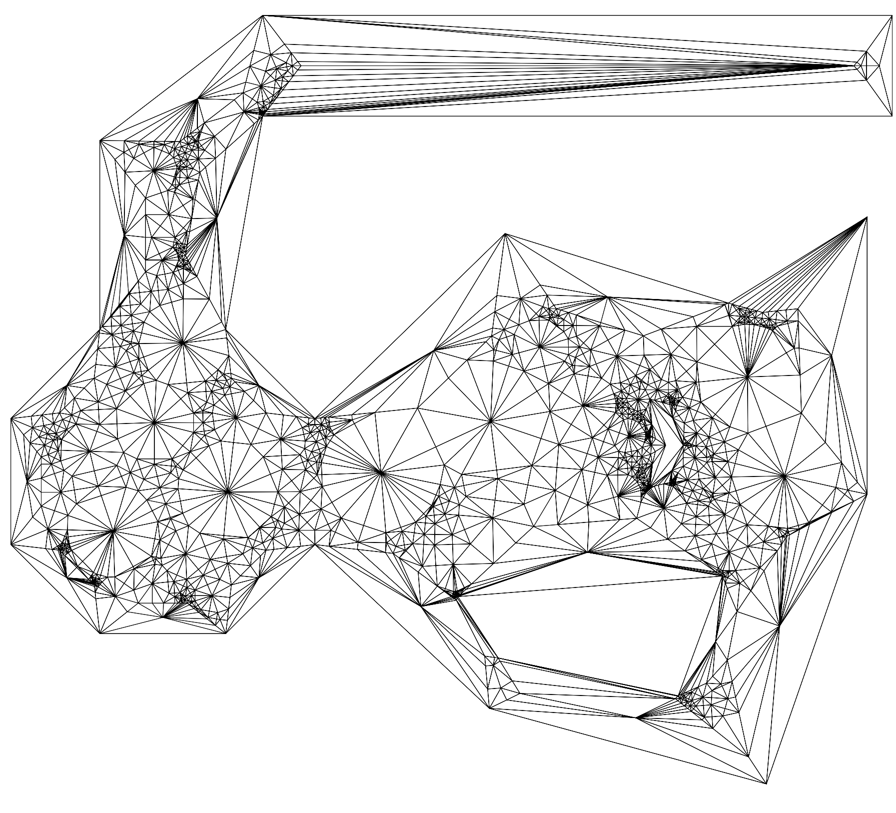

Figure 9: The final covering triangulation generated by our implementation. Maximum angle = 0.872π, minimum angle = $7.45e-05\pi$, $6.48n$ points added by our algorithm, $53.33n$ points added by Bern, Mitchell and Ruppert [1993].

Exterior edges. Consider a consecutive pair of Steiner vertices X and Y on an almond A, where X and Y arise from the intersection of *distinct* almonds with A. We wish to ensure that any pair of Steiner points are not too far apart: If the arc length of A between X and Y is more than $a/6$, we add Steiner points between X and Y to ensure that the arc length between consecutive Steiner points is at most $a/6$ (unless there is an internal edge between them). We now introduce an edge called an *exterior edge* between the (non-internal) consecutive vertices on A. The boundary of R is formed by the exterior edges, together with exterior portions of convex hull edges in the case that P is a PSLG,

We triangulate each almond together with its Steiner vertices according to Section 3.1, see Figure 7. We call such triangles *almond triangles*. We triangulate R using the algorithm of Bern, Mitchell and Ruppert [1993] for triangulating polygons with holes so that no angle is obtuse. This introduces Steiner points on exterior edges. Since exterior edges are shared with triangles interior to almonds, this makes the triangulation nonconformal. We fix this situation as follows. Let E be the almond triangle edge containing a nonconformal vertex W, and Z the vertex of the almond triangle opposite E. Then we add $\overline{WZ}$, see Figure 11 Note that any almond triangle has at most one external edge, so that none of these edges cross. If there are r vertices interior to an external edge of an almond triangle, we thus form $r+1$ triangles. The fact that we subdivided large exterior edges ensures that we do not decrease the lower bound on angle measure from Theorem 4.

Theorem 5 *Any triangle T inside an almond A has $\nu(T) \geq \nu(A)/2$.*

Proof. We have two cases, depending on whether we added a central Steiner point when we triangulated the convex hull of A.

Case 1. The first case is where we did not add a central Steiner point in the triangulation of A. Note that exterior edges all cut arcs shorter than $a/2$, so the edge E' in Figure 3 is internal and has no nonconformal Steiner points on it. Suppose $\triangle XYZ$ has a nonconformal Steiner point W on $\overline{XY}$, with $\overline{ZY}$ the longest edge, and $\overline{ZX}$ and $\overline{ZY}$ both "long" in the sense of Theorem 4. Then $\angle ZWX$ is acute. Also $\angle ZWY$ is less than the supplement of $\angle ZYX$, or $\nu(\angle ZWY) > \angle ZYX$. By Theorem 3 and Theorem 4 $\angle ZYX > a'/2 > a/4 = \nu(A)/2$, where a' is the length of the almond arc cut by E'.

Case 2. The second case is where we did add a central Steiner point Q. Let $\triangle QXY$ be an almond triangle with W on $\overline{XY}$. Angles $\angle XQW$ and $\angle WQY$ are both smaller than $\angle XQY$, an original angle already considered in Theorem 4. So it remains to bound $\angle QWY$, where without loss of generality $\angle QWY \geq \angle QWX$.

Now again $\angle QWY$ is less than the supplement of $\angle QYX$, or $\nu(\angle QWY) > \angle QYX$, see Figure 11. As we increase the distance from X to Y, $\angle QYX$ decreases, so a tight lower bound on $\angle QYX$ is achieved when $\overline{XY}$ has maximum allowable length. As stated above this length is $a/6$, but for exposition we parameterize the length as ja, then show that a choice of $j = 1/6$ is sufficient to prove the theorem.

Consider placing a fixed length $\overline{XY}$ in order to minimize $\angle QYX$. Consider chord $\overline{YY'}$ containing $\overline{YQ}$. From Theorem 3, $\angle QYX = (y - ja)/2$, where y is

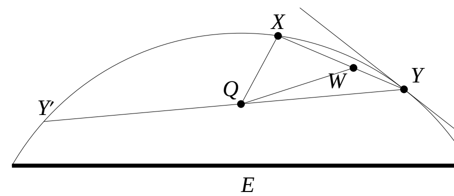

Figure 11: Fixing a nonconformal vertex W. The arc length cut by $\overline{YY'}$ minus the arc length cut by $\overline{XY}$ gives the angle at Y, which bounds the supplement of the angle at W.

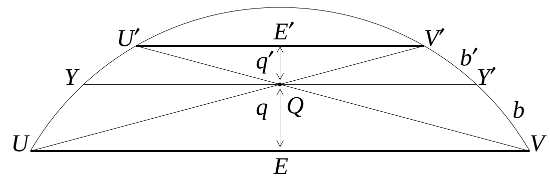

Figure 12: Bounding y, the arc length between Y and Y': $b' > a/12$.

the arc length cut by $\overline{YY'}$. Thus minimizing $\angle QYX$ is equivalent to minimizing y. From Lemma 3 the minimum y is achieved by $\overline{YY'} \perp \overline{cQ}$.

We now find a lower bound on y for this construction, see Figure 12. Without loss of generality let Y be between U and U', where $\overline{U'V'}$ is the fictitious edge E' defining Q.

Let q be the distance from Q to E and q' the distance from Q to E'. By construction $\triangle QUV$ and $\triangle QU'V'$ are similar, so $q'/q = |E'|/|E|$, the ratio of edge lengths. Since E cuts a longer arc than E' we have that $|E'|/|E|$ is greater than the ratio of the cut arc lengths, $1/2$. Hence $q'/q > 1/2$.

Let b be the arc length between Y and V and b' the arc length between Y and V'. Since E is closer to c than is E', we also have that $b'/b > q'/q > 1/2$. By construction $b + b' = a/4$, so $b' > a/12$.

Since $y = a/2 + 2b'$, we have $\angle QYX = (a/2 + 2b' - ja)/2 > (\frac{2}{3} - j)\nu(A)$. Taking $j = 1/6$ is sufficient to complete the proof. We also note that the above inequalities are tight as $\nu(A)$ approaches zero, so that taking $j \leq 1/6$ is also necessary. ∎

There is a slight tradeoff between the maximum angle bound and the cardinality of the triangulation: From the proof of Theorem 5 bounding the maximum length of an external edge by any value in $[\frac{1}{6}, \frac{2}{3})a$ is sufficient to guarantee that the angles in our triangulation are all less than a constant factor times $\nu(A)$. In our implementation figures we have chosen $a/3$, which guarantees that $\nu(T) > \nu(A)/3$ for all triangles in almonds, and yields reasonable cardinality.

5 STEINER-VERTEX AND TIME BOUNDS

Theorem 6 *The cardinality of our triangulation is at most $17n + S(15n)$, where n is the number of vertices in the input P, and $S(n) = O(n)$ is the cardinality of the triangulation of R.*

Proof. We first consider our algorithm for polygons. Recall that no point is interior and visible to three almonds. For every pair of intersecting almonds regions, we pick a point p common to both almond regions. We may draw a line segment from p to the midpoint of each of the two corresponding input edges. This forms a graph, with the input edges corresponding to vertices of the graph, and intersecting almonds corresponding to edges of the graph. The graph is planar, and hence, by Euler's formula, the number of edges is at most $3n$. Each graph edge corresponds to an internal edge (or an isolated vertex). Hence there are at most $6n$ Steiner points on almond arcs, plus those added to subdivide large external edges. An external edge must cut more than one sixth of an almond arc in order to be considered large, so at most $5n$ Steiner points are added in this way. Hence, R has at most $11n$ vertices. Since each almond region requires at most one center Steiner point to triangulate, triangulating the almond regions adds at most n center Steiner vertices. Hence the cardinality of our triangulation of a polygon is at most $12n + S(11n)$. The algorithm of Bern, Mitchell and Ruppert [1993] triangulates R with only $S(11n) = O(n)$ vertices.

For PSLG, there are two almonds for each input edge. We may require $5n$ more Steiner points to subdivide long exterior edges, and n more center vertices to triangulate almonds, for a total of at most $17n + S(15n)$ vertices. ∎

It is interesting to compare this with the fact that a Steiner triangulation of a PSLG with a constant upper bound on the largest angle may require $\Omega(n^2)$ Steiner points. This fact follows from an example due to Paterson in Bern, Dobkin and Eppstein[1991].

Theorem 7 *The running time of the algorithm is* $O(n \log n) + T(15n)$, *where* $T(n) = O(n \log^2 n)$ *is the running time of the algorithm used to triangulate* R.

Proof. First for each edge E we must determine the vertex V that defines the almond A at E, assuming the almond has angle less than $2\pi/3$. The key observation is that $\triangle VE$ is a triangle of the constrained Delaunay triangulation of P! The almond contains no input vertex visible to V. By definition $\triangle VE$ is a triangle of the constrained Delaunay triangulation if the circle through V with chord E contains no vertex visible to both E and V. The region of the circle outside of the almond is not visible to V, and hence the characterizations coincide. Thus the almonds may be determined by computing the constrained Delaunay triangulation of the input in time $O(n \log n)$ via Chew[1989].

The next step is to compute the intersection of the n almonds. There are only a linear number of intersection points, and the almonds are circular arcs. Hence this can be done in optimal $O(n \log n)$ time using plane sweep (see Preparata and Shamos[1985]).

The order of Steiner vertices along almond arcs is determined by sorting, after which it takes only linear time to triangulate the almonds. In linear time it is easy to fix nonconformal triangles created by the triangulation of R. ∎

6 OPEN PROBLEMS

There are several other measures for which near optimal covering triangulations are desirable. Most notably, a covering triangulation that has guaranteed minimum height is needed in Mitchell and Vavasis[1992] in the triangulation of the surface of an octree box, in order to guarantee good three dimensional aspect ratio of tetrahedra. In fact, any three dimensional triangulation algorithm with bounded aspect ratio implicitly generates a two dimensional Steiner triangulation with bounded height on the surface of a small sphere centered at any input vertex.

Like minmax angle, maxmin height is also a measure which has the anchor property, so one might suppose the present work could be extended to maxmin height as well. A rectangle with semicircular ends is the locus of points determining the maximum height possible for an input edge, and is the analog of the almonds for minmax height. However, we conjecture that it is not possible to triangulate such a rectangle given arbitrary Steiner points on its boundary, and achieve triangle height within a constant factor of the height of the rectangle. Thus our results do not immediately generalize to maxmin height.

Also open is the existence of a covering triangulation algorithm that optimizes a measure that is dependent on both the largest and smallest angles. It appears impossible to generate a covering triangulation that simultaneously achieves minimum angle and maximum angle close to optimal.

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