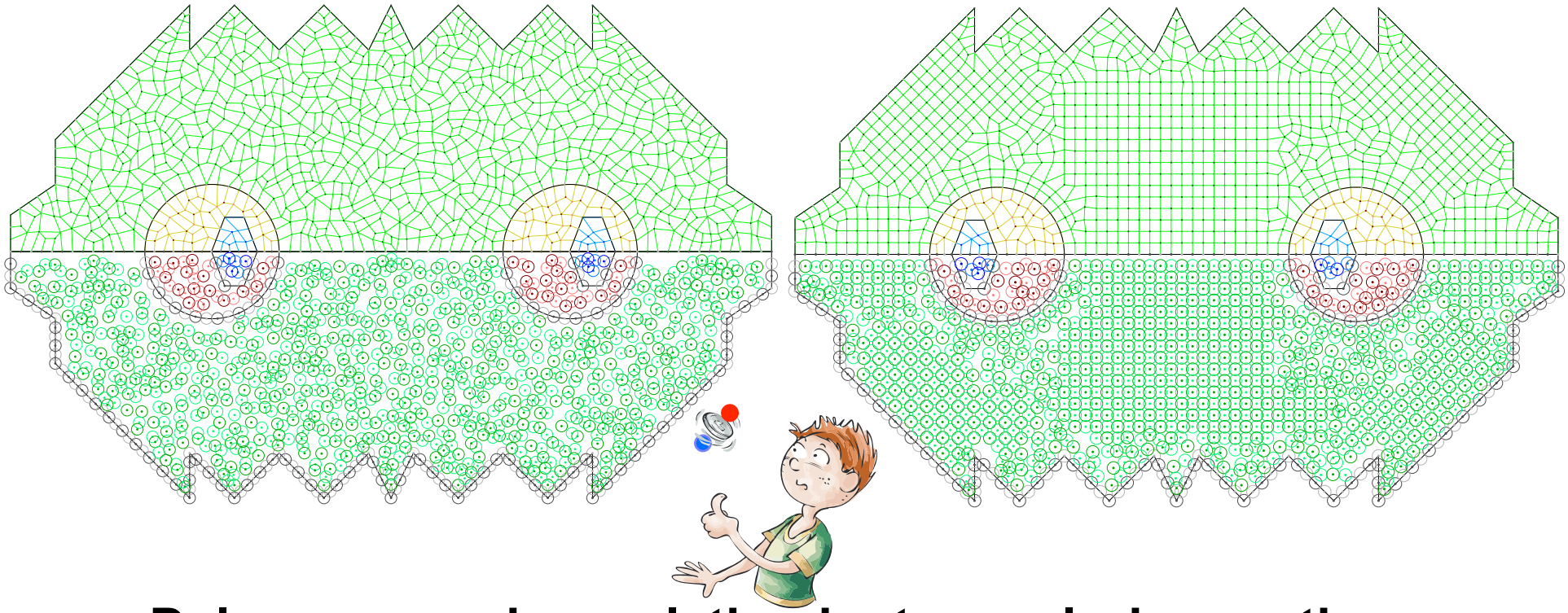


# Delaunay quadrangulation by two-coloring vertices

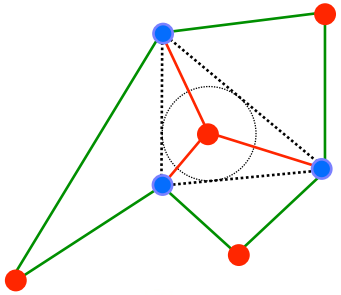
## or

## How to turn a triangle mesh into a quad one with no global problems


**Scott A. Mitchell, Mohammed Amr Mohammed, Ahmed Hassan Mahmoud, Mohamed Salah Ebeida**

**Computing Research, Sandia National Labs**

**Alexandria University**

**23rd International Meshing Roundtable (IMR23)**

**London, 15 Oct 2014**

**11:30-11:55am Wed, 20 minutes**


Sandia is a multiprogram laboratory operated by Sandia Corporation, a Lockheed Martin Company, for the United States Department of Energy's National Nuclear Security Administration under contract DE-AC04-94AL85000.

# Outline

- Algorithm on one page
- Two-coloring idea
  - Contrast to Triangle Pairing
  - Even-sided polygons
  - Templates for provable quality (appendix on website)
    - Fixing by constrained incircle refinement
      - QTran
    - Constrained median refinement
- (Random Algorithm)
  - Well-spacedness properties, achieved by
  - Delaunay refinement triangulation as input
  - MPS (sphere packing) for provable quality triangulation
    - center director asks Mohamed, “can you do this for quad meshes?” no, but two years later...
  - Generating multi-class blue noise
  - Ideal spacing
  - Heuristics for better quality
- Example meshes
  - Dare to show raw output, before cleanup
- Advancing Front Algorithm
  - Row, column, repeat. Reseed.
- Conclusions
  - Three centers: Circumcenter, Incenter, Centroid
    - orthocenter feeling left out
  - Some quad meshes are not two-colorable
  - Not for hex meshes

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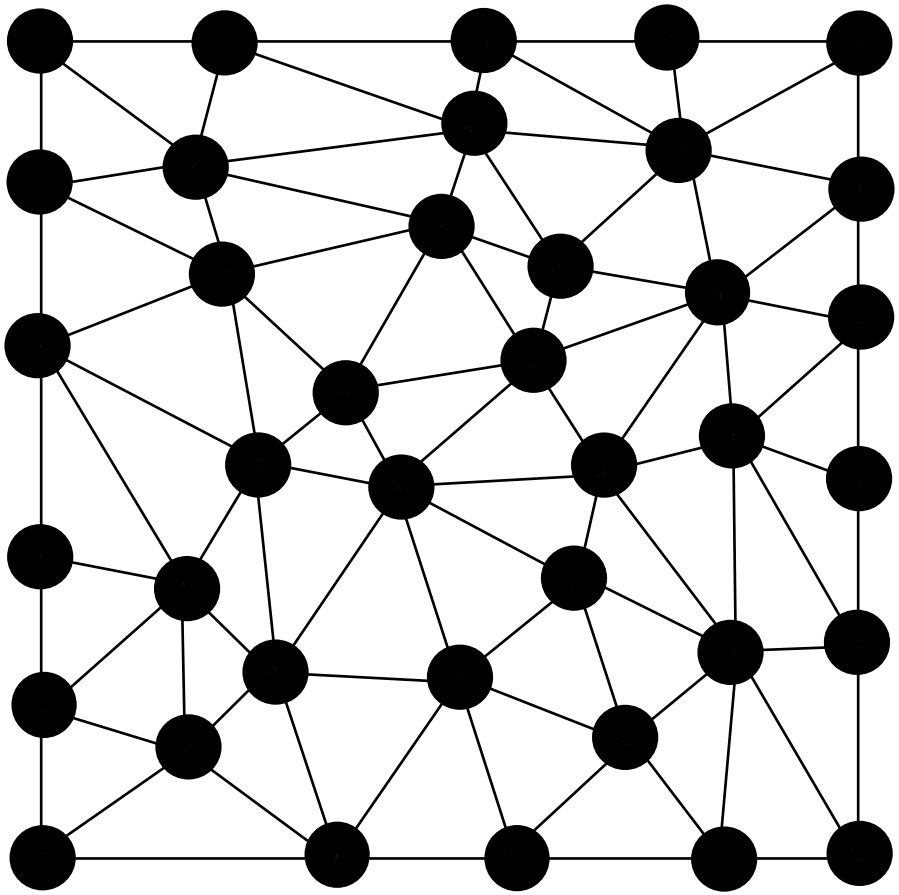
# Convert tri mesh to quads
# One-slide Algorithm

- **Generate (or given) well-spaced points**
- **Delaunay triangulate**
- Color points red or blue
  - intersperse colors
- Discard red-red and blue-blue edges
- Quads mostly
  - good quality, some large angles
- 6, 8, 10 sided polygons sometimes
  - constrained incircle refinement
  - median template for reflex quads
- All quads with provable quality

- Coloring and position heuristics improve quality in practice



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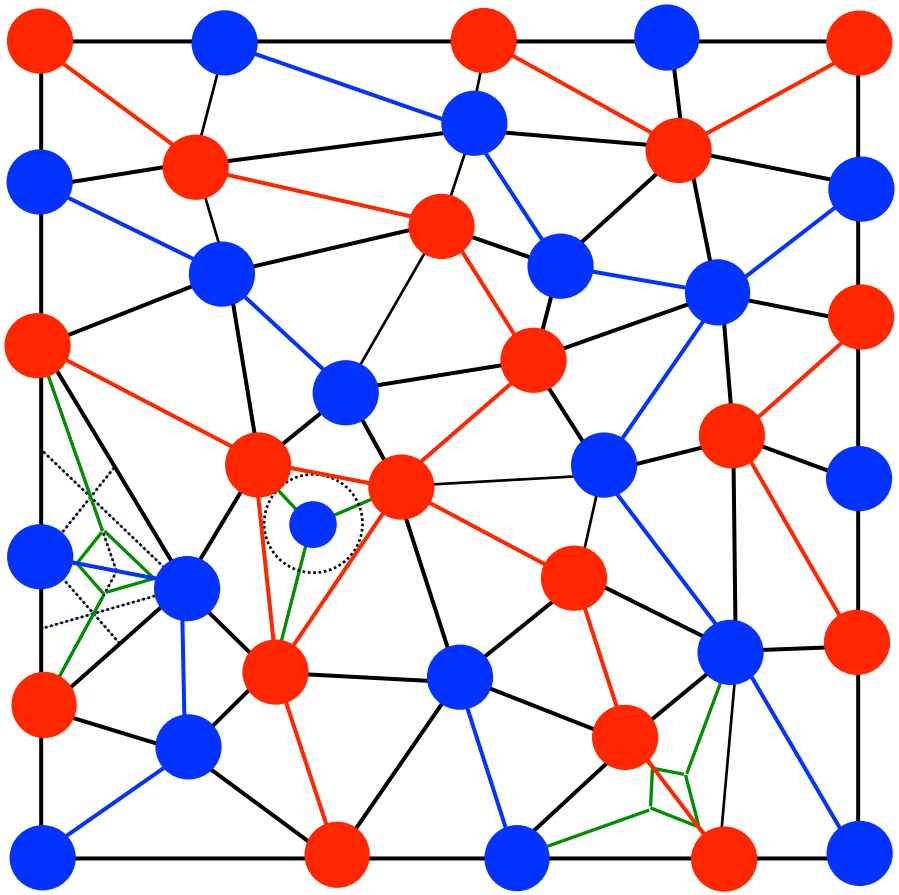

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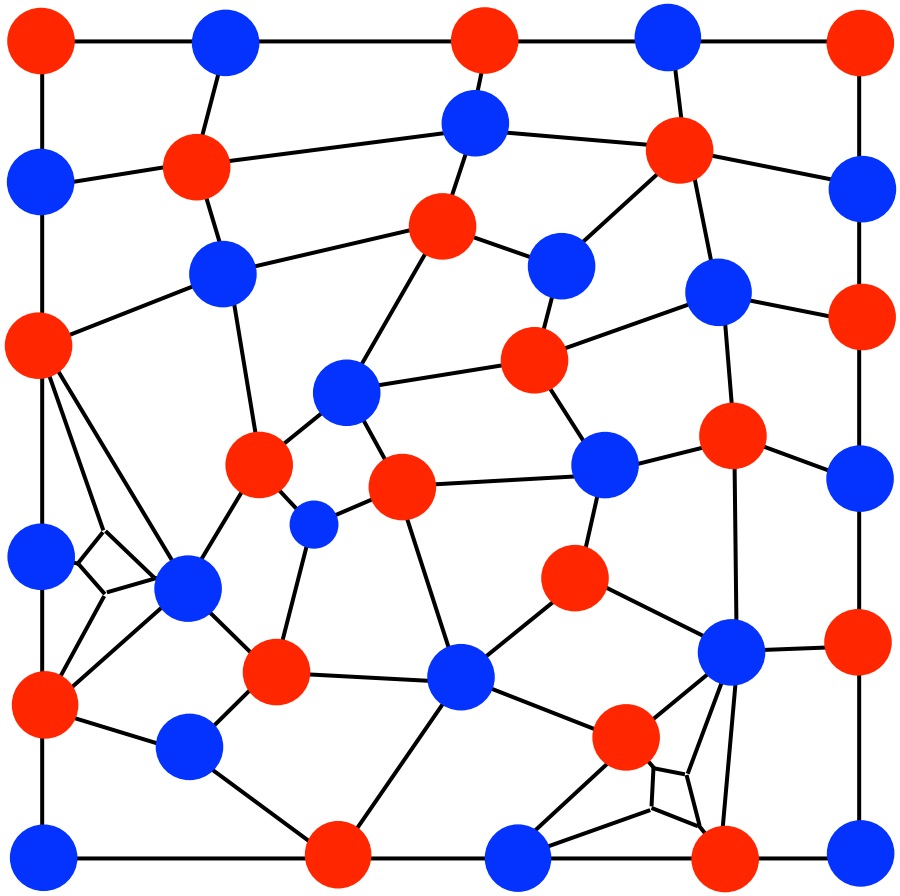

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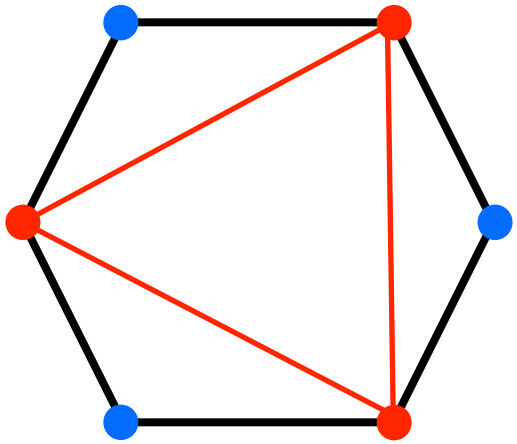
# Why does this work?

- Any planar quad mesh can be two-colored
  - converse, convert a two-colored graph into quads
- Two-coloring makes even-sided cells
  - 4, 6, 8, 10...
- Any triangulation of 6+ polygon with monochromatic edges has a monochromatic triangle
  - avoid or remove mono-triangles for all quads
- Input tri good quality
  - template quads good quality (proofs in 8 page appendix, available online)
    - http://www.cs.sandia.gov/~samitch/papers/delaunayquadproof.pdf
    - google: mitchell sandia. click on papers



This is the only hexagon triangulation using only red edges, since every blue vertex must be in an ear

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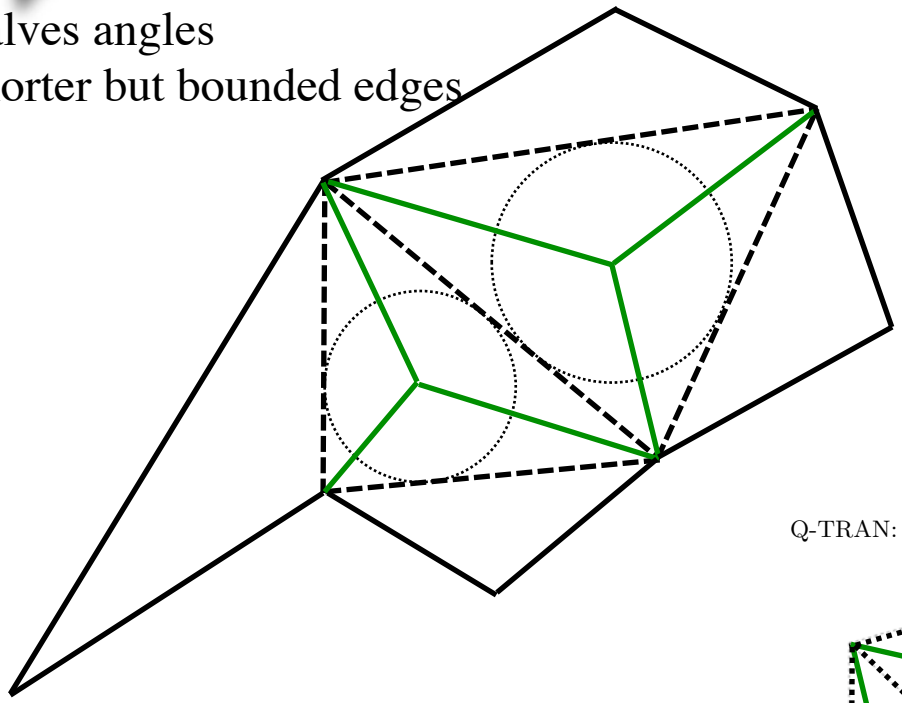
# Incircle refinement details

halves angles
shorter but bounded edges



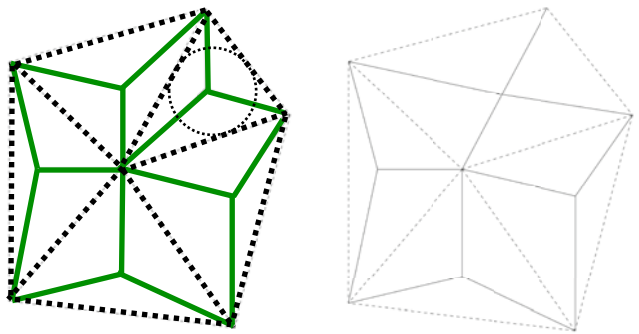
2n-gon has n-2 mono-triangles

6 -> 1
8 -> 2
10 -> 3...

adjacent mono-triangles OK
makes a red quad

Q-TRAN: Transform Triangular Meshes into Quadrilateral Meshes 27

Compare to Q-TRAN



(a) Input (b) Output

**Fig. 2.** Topology clean-up using face collapse to reduce the number of irregular vertices. A quadrilateral face is collapsed converting two irregular vertices into a regular one. The triangular tessellation is shown using dotted lines in both figures.

[5] M. S. Ebeida, K. Karamete, E. Mestreau, S. Dey, Q-TRAN: a new approach to transform triangular meshes into quadrilateral meshes locally, in: International Meshing Roundtable, volume 19, Sandia National Laboratories, 2010, pp. 23–34.

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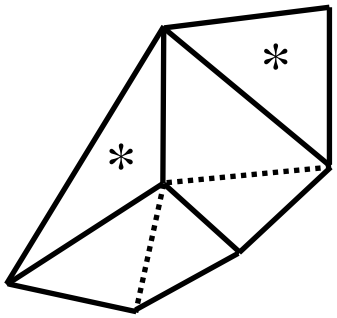
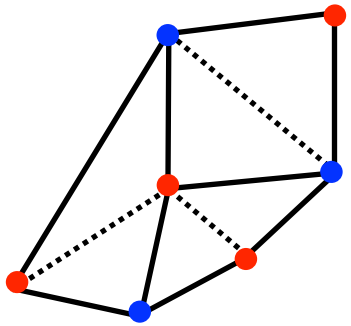
# Pairing different than two-coloring
## (unless all-quad without refinement)

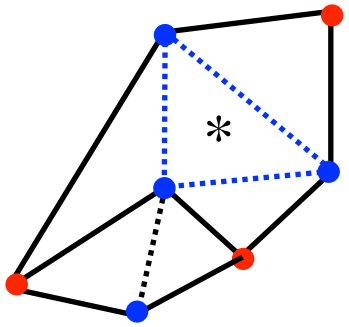
The stuck configurations are very different



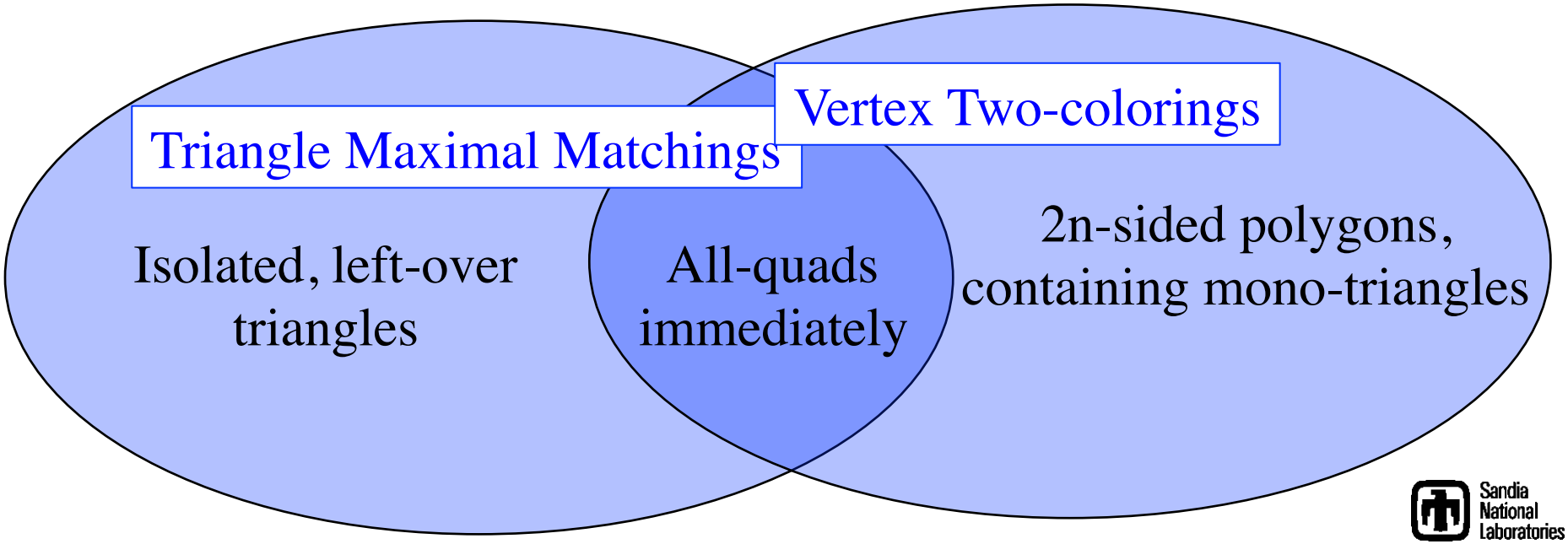

Two isolated
Unpaired triangles



All-quad
Perfect Matching <-> Two-coloring




6-sided polygon

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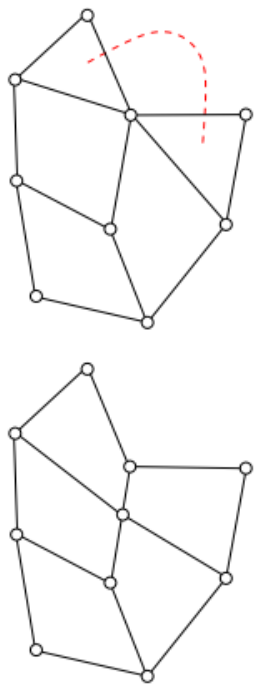
OK, it works and is different. Why would you *want* to use it?

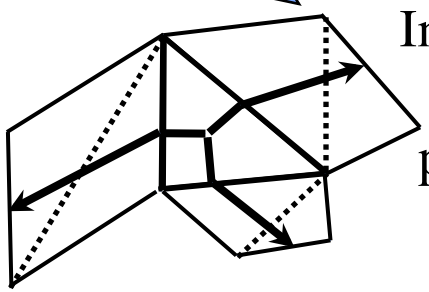
# Why two-coloring vertices?
# Why not matching triangles?

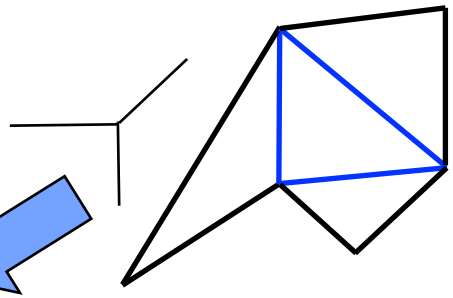
Locality!

must make a change



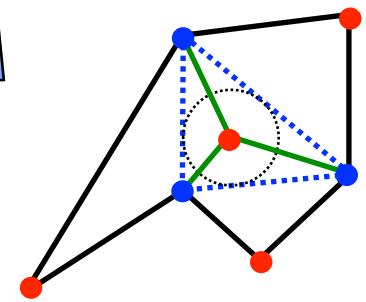
Pairing





Impossible topology
for
pairing or coloring

Coloring



local template
deterministic, provable quality

global refine chords
steering heuristics

Blossom-quad
local refinements

good (best?) triangle pairing algorithm

[21] J.-F. Remacle, J. Lambrechts, B. Seny, E. Marchandise, A. Johnen, C. Geuzainet, Blossom-quad: A non-uniform quadrilateral mesh generator using a minimum-cost perfect-matching algorithm, International Journal for Numerical Methods in Engineering 89 (2012) 1102–1119.

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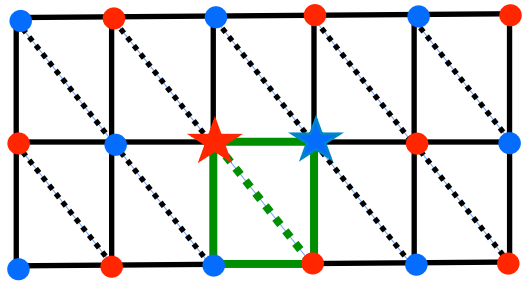
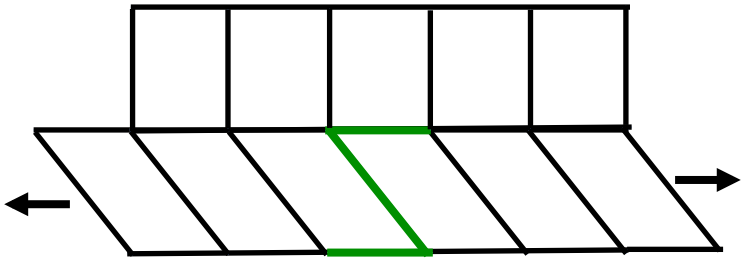
# Why two-coloring vertices?
# Why not matching triangles?

want a change



swap diagonal

Pairing

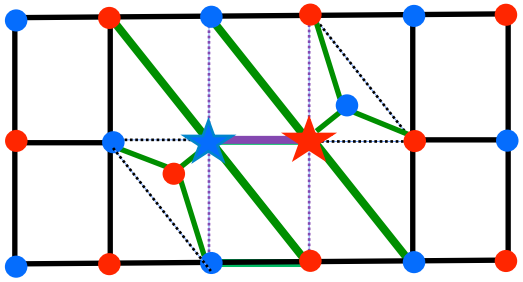


cascade of swaps

- **local changes have global scope**

alternative is local refinements

Coloring



star neighborhood of starred vertices

**+ local changes have local scope**

# Why two-coloring vertices?
# Why not matching triangles?

**Matching triangles**

- + match for quad quality
- – slow
  - - global matching alg
  - - quadratic runtime
- – rare isolated tri (unmatched)
  - tri 1->3 quad refine
    - + fixed vertices
    - - global propagation
  - + alternative local refine
    - (complicated, several rules)
- – local pair swap
  global cascade
- – global difficulties

**Two-coloring vertices**

- – colors don't measure quality
- + fast
  - + local coloring alg
  - + near linear runtime
- – rare isolated tri (monchromatic)
  - tri 1->3 tri refine
  - - adds vertices
  - + no propagation
- + local color flip
  local change
- + local difficulties

"-" means a negative feature
"+" means a positive feature

# Interval Assignment Interpretation

- Two-coloring is a local assignment
  - ensures even-number of boundary edges inherent global constraint for quad meshes
- Now let's use coloring for spatial positions

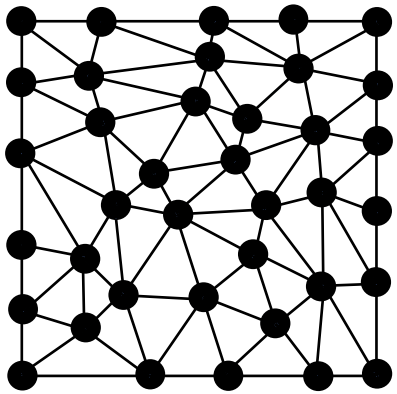
# Back to the algorithm

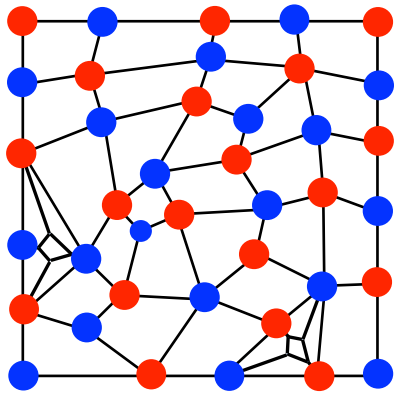
- **Generate (or given) well-spaced points**
- Delaunay triangulate
- Color points red or blue
  - intersperse colors
- Discard red-red and blue-blue edges
- Quads mostly
  - good quality, some large angles
- 6, 8, 10 sided polygons sometimes
  - constrained incircle refinement
  - median template for reflex quads
- All quads with provable quality

- **Coloring and positions improve quality in practice**

**We can take any Packing or Delaunay Refinement triangulation and two-color vertices arbitrarily.**

**We can do better!**





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# Sphere packing, better control than Delaunay Refinement

- **Delaunay Refinement**
  - **build quality, packing results**
  - **If triangle has bad quality**
    - **Then add a point**
  - **On termination, we have a sphere packing**
- **MPS**
  - **build packing, quality results**
  - **If packing is not maximal**
    - **Then add a point**
  - **On termination, the Delaunay triangulation will have good quality.**

- **Equivalent in *theory***

  **bad quality = empty sphere is large (non-maximal) compared to edge length (empty-disk)**

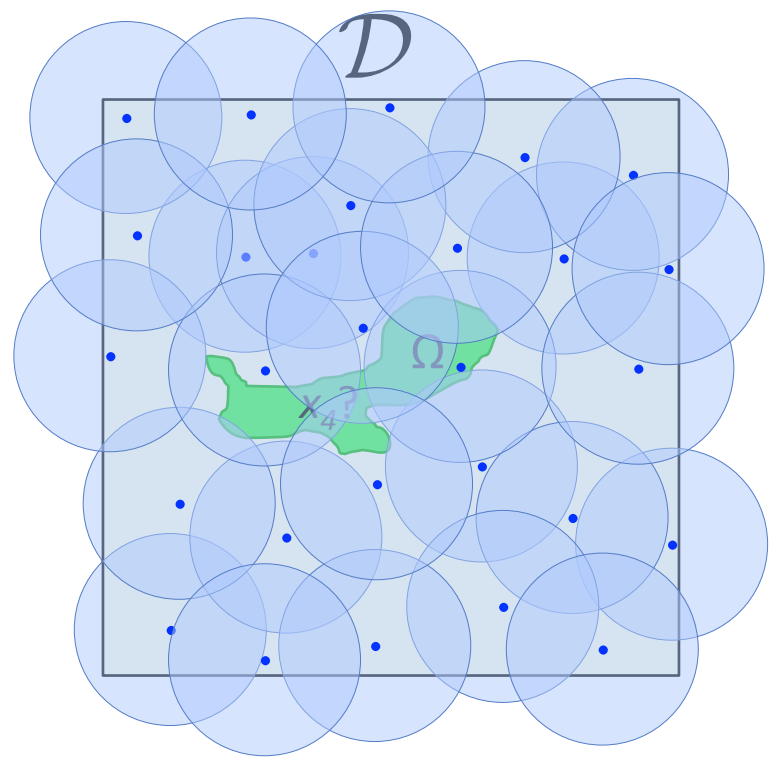
Empty disk: $\forall x_i, x_j \in X, x_i \neq x_j : ||x_i - x_j|| \geq r$

Maximal: $\forall x \in \mathcal{D}, \exists x_i \in X : ||x - x_i|| < r$

Provable angle bounds by Central Angle Theorem

- **We claim in *practice,* sphere packing has better (direct) spacing control**

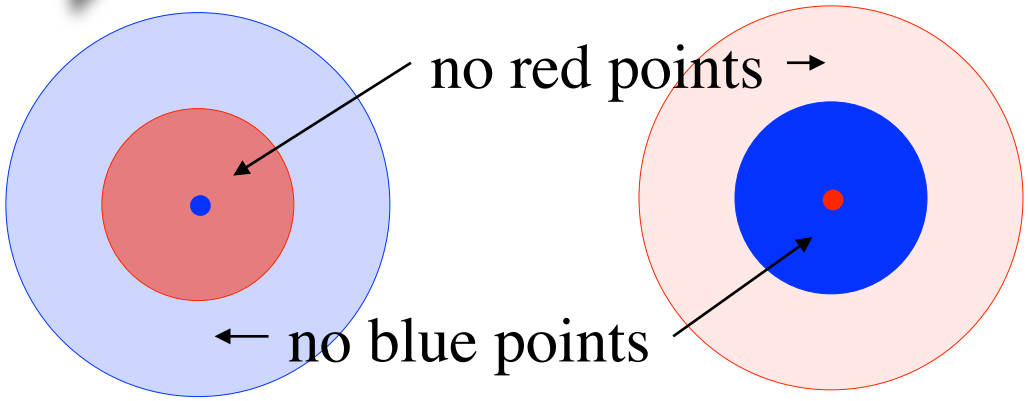
- **What is MPS? Sphere packing, output of**
  - **Insert random points**
    - **With "Poisson" process, and rejection**



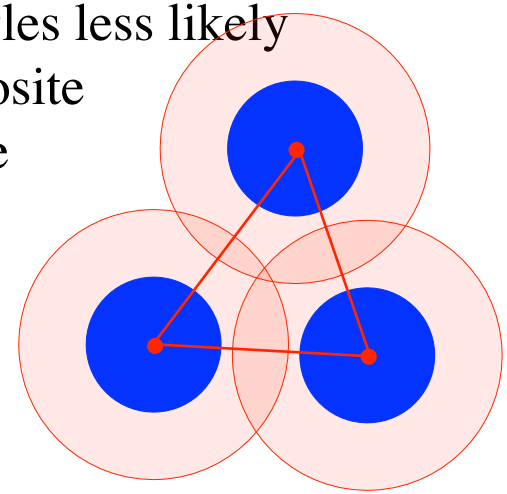

- **sphere packing algorithms are practical, our 2011-2013 work**
- **see Dafna Talmor thesis for quality theory**

# Reducing the Frequency of Mono-tris Sphere Packing with Two Colors

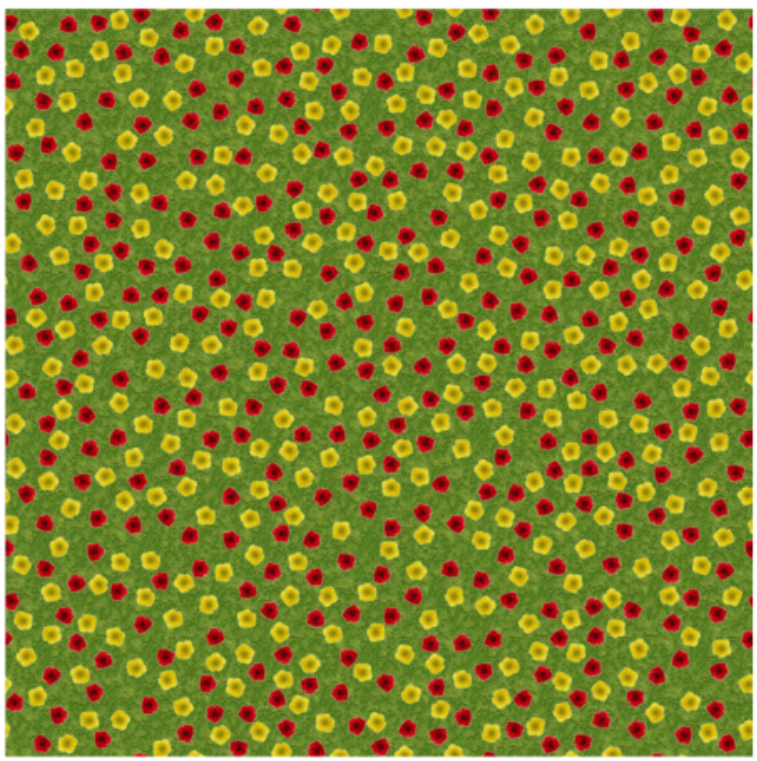
[17] L.-Y. Wei, Multi-class blue noise sampling, ACM Trans. Graph. 29 (2010) 79:1–79:8.

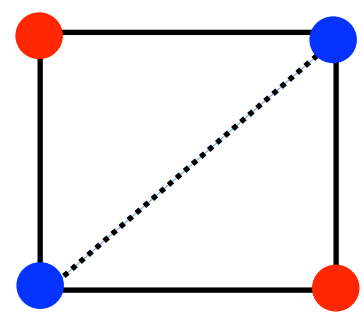



Blue-blue farther than blue-red
Mono-triangles less likely
can fit opposite
color inside



2 classes of objects





discard long diagonals
for square-like quads

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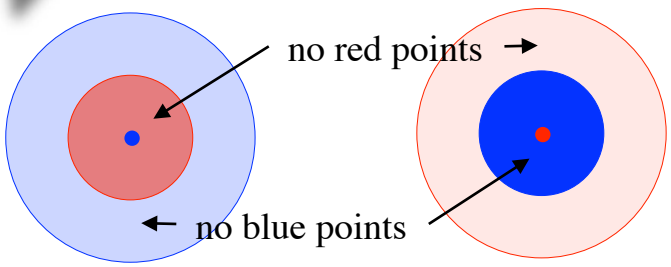
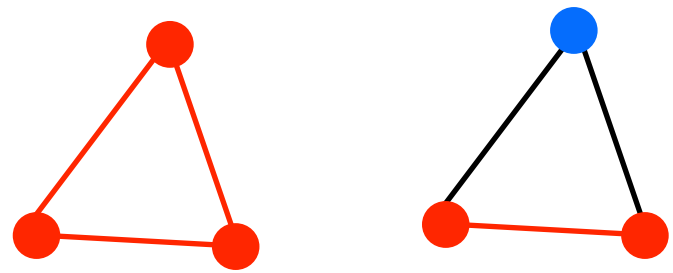
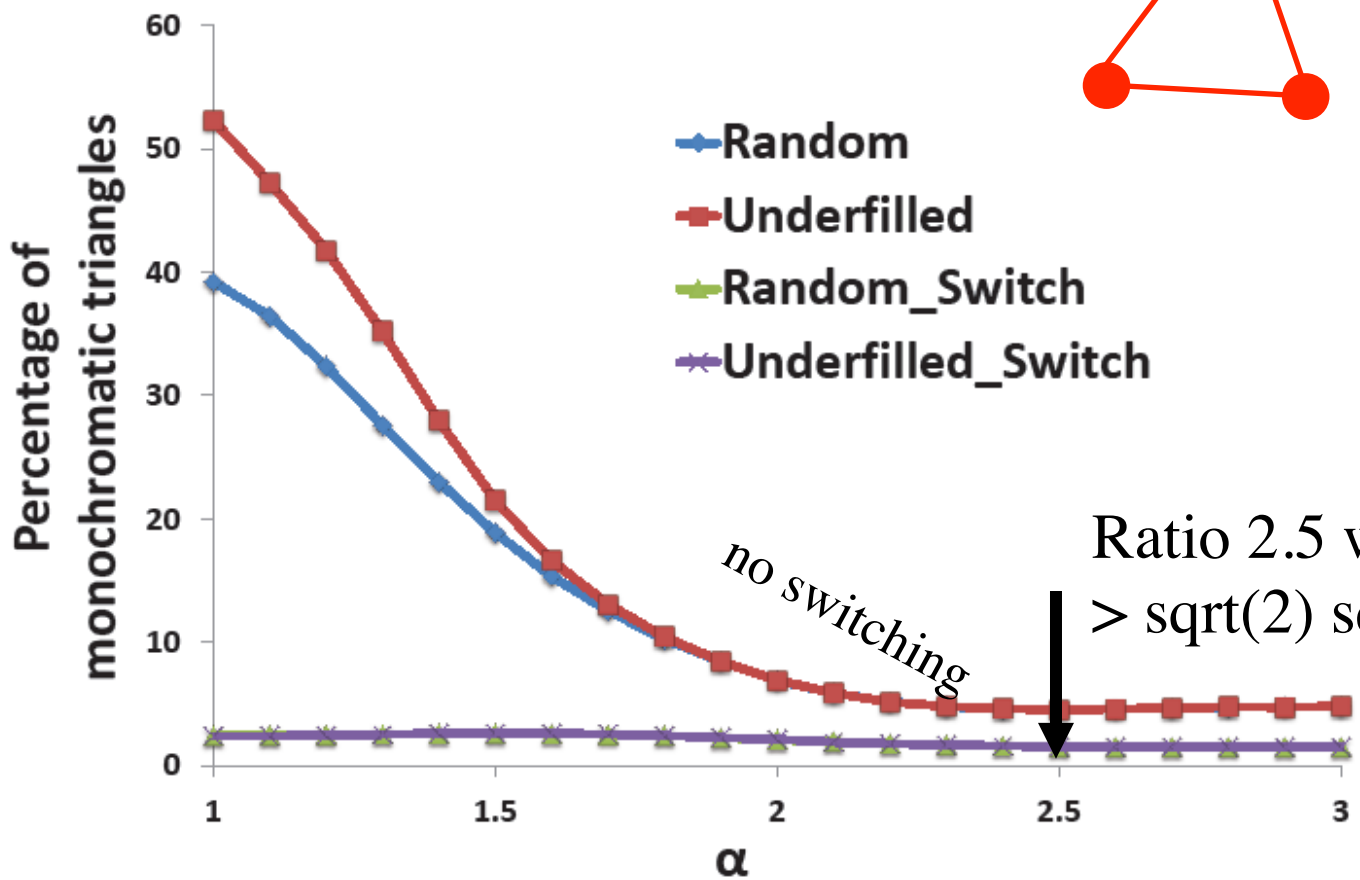
# Reducing the Frequency of Mono-tris
# Two-radii Plus Color Switching




Color switching works even better!
if mono-tri, change color of one vertex



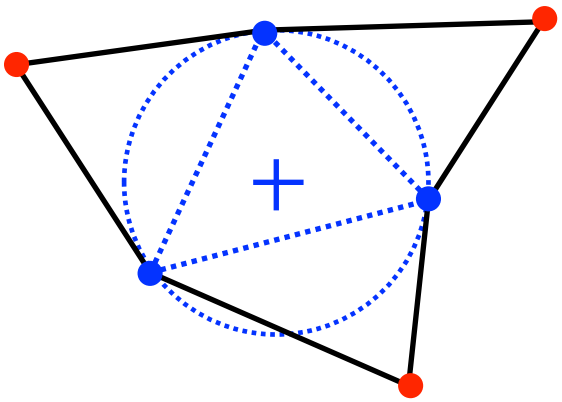


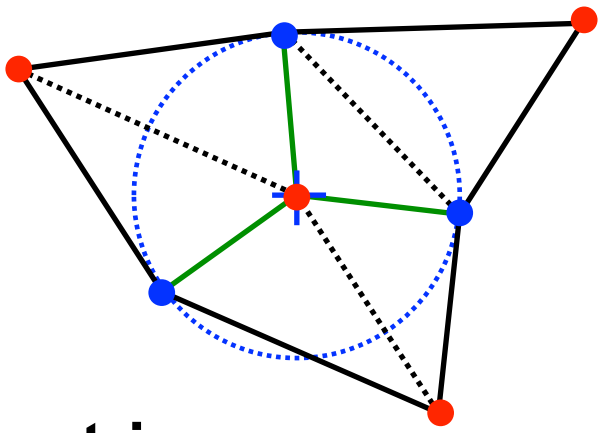

Ratio 2.5 works well,
> sqrt(2) square diagonal

switching, stubborn 2%

# Resolving Stubborn Mono-tris

## Circumcircle Delaunay Refinement



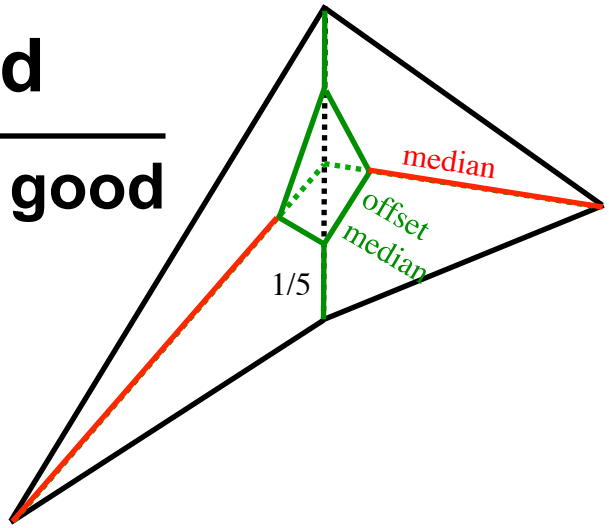


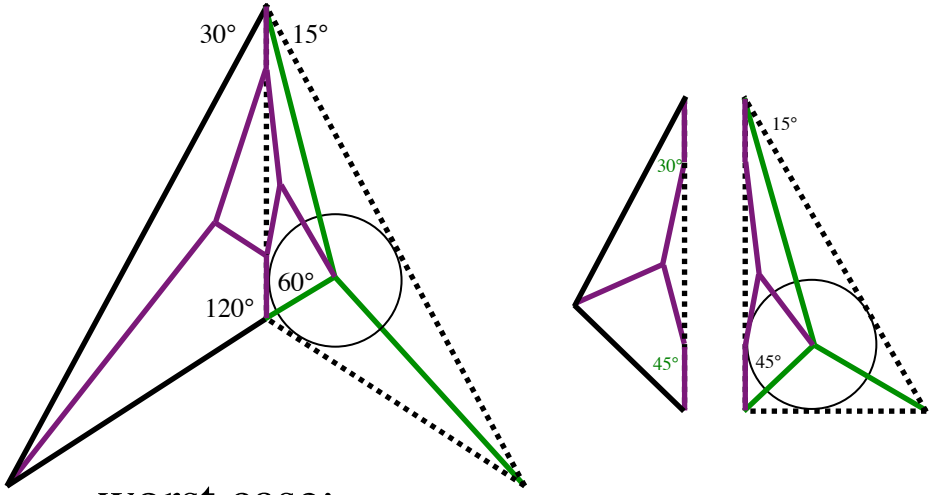
**Removes the mono-tri**

- **Rare cases produce another mono-tri, requiring more refinement...**

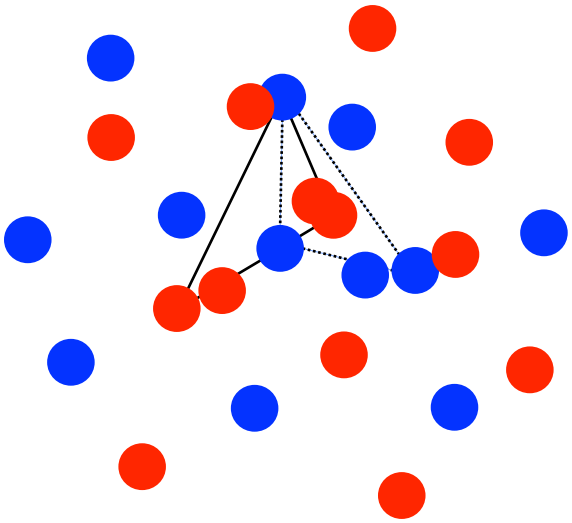
# Avoiding Reflex Quad

- **Median-refinement template is provably good**
  - **but not that good, 10-174 degrees**
- **Practical alternative**
  - **remove vertices and resample locally**
  - **works every time in practice**




worst case:
incircle followed by reflex refine



# Heuristic Summary

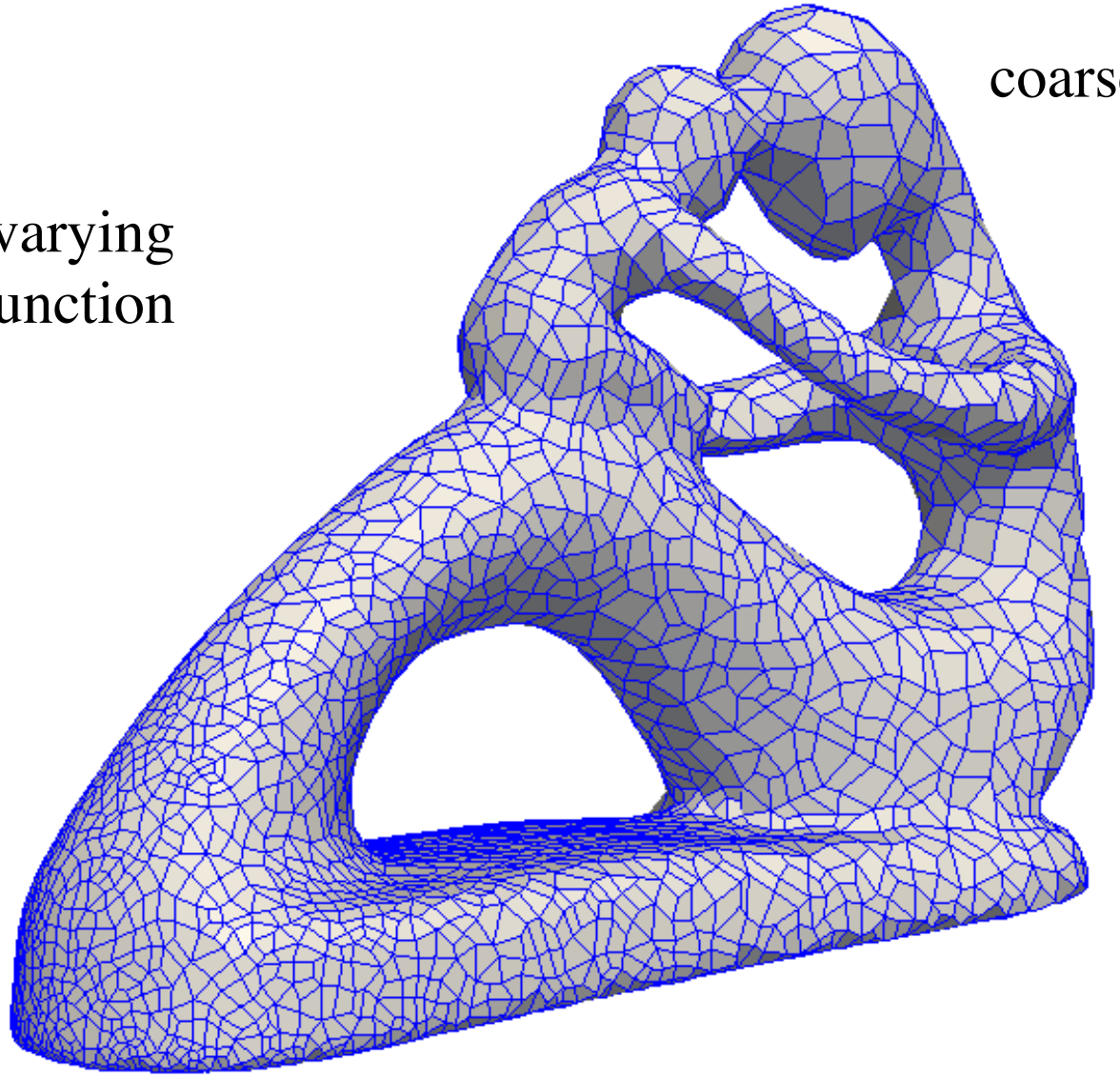
- Reduce frequency of mono-tris
  - Two-color multiclass sampling with radii ratio 2.5
  - Color switching
- Resolve mono-tris
  - Delaunay incircle refinement
- Avoid reflex quads
  - Local resampling

- Traditional cleanup may also be applied post-process
  - we provide ok quality, convex-element starting point

# Curved Surfaces
# Mesh Size may Vary

coarse

slowly varying
sizing function



fine

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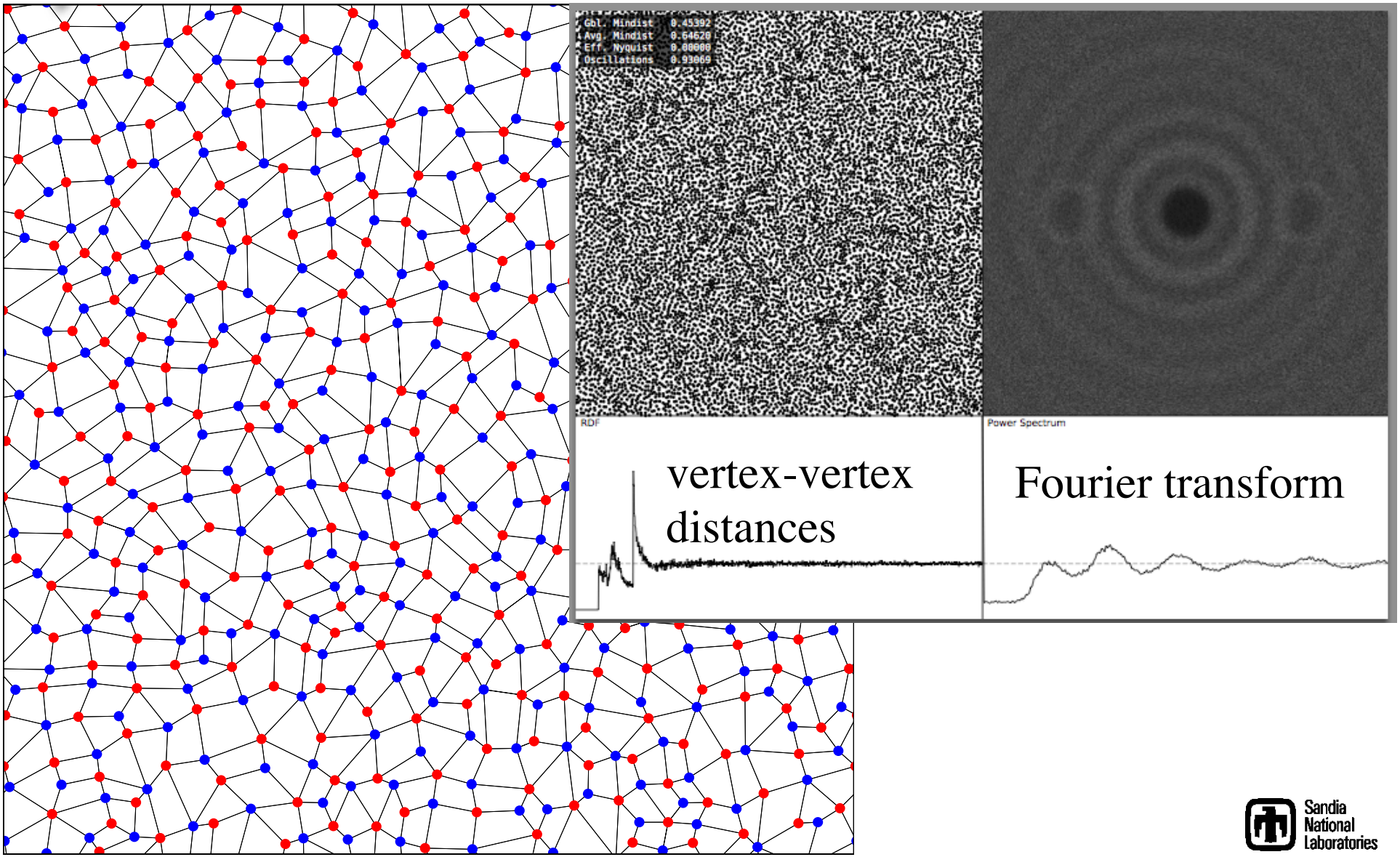
# Fourier Spectrum

Some graphics applications rely on random positions to avoid artifacts

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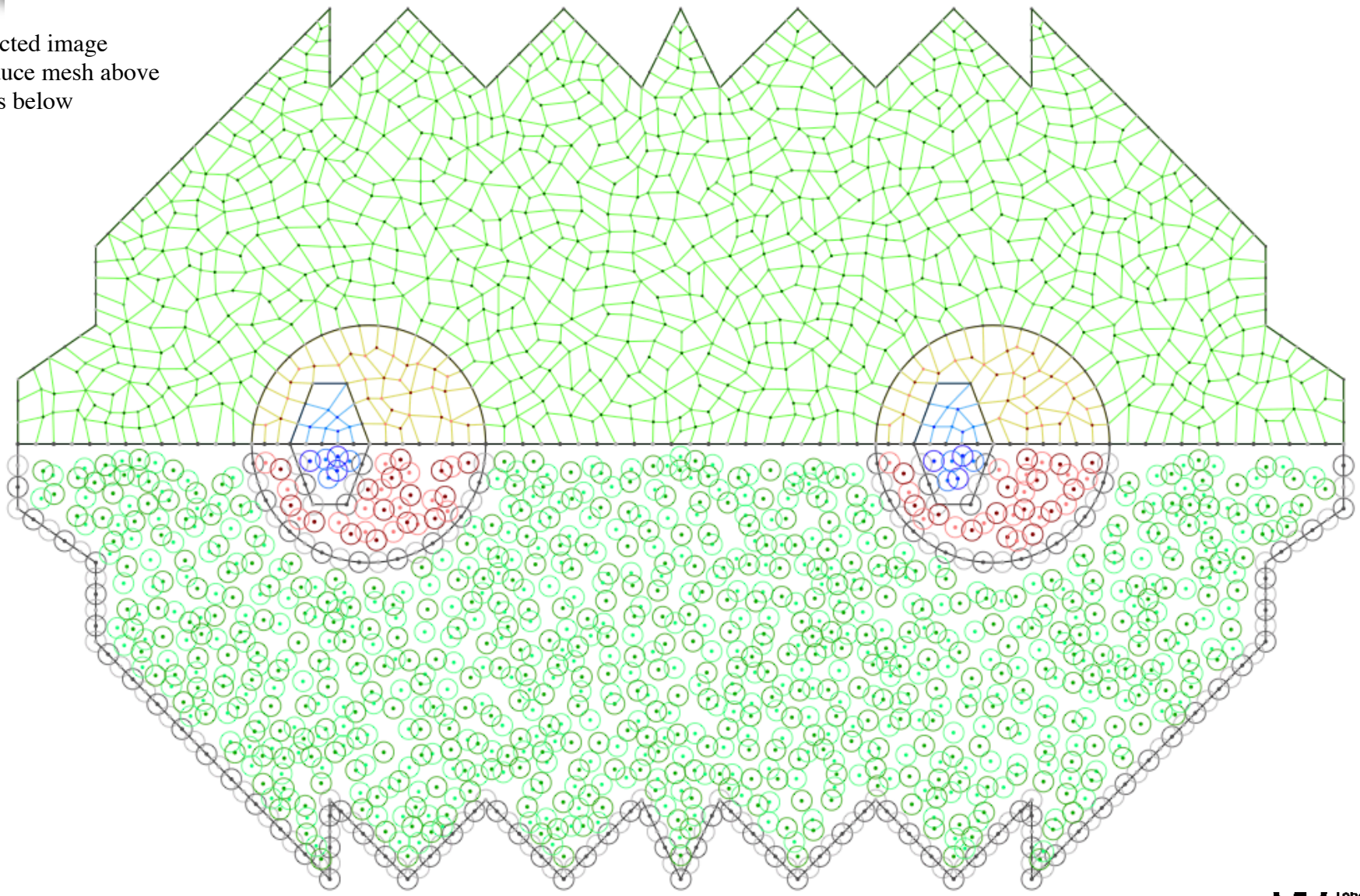
# Random

reflected image
produce mesh above
disks below



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onal
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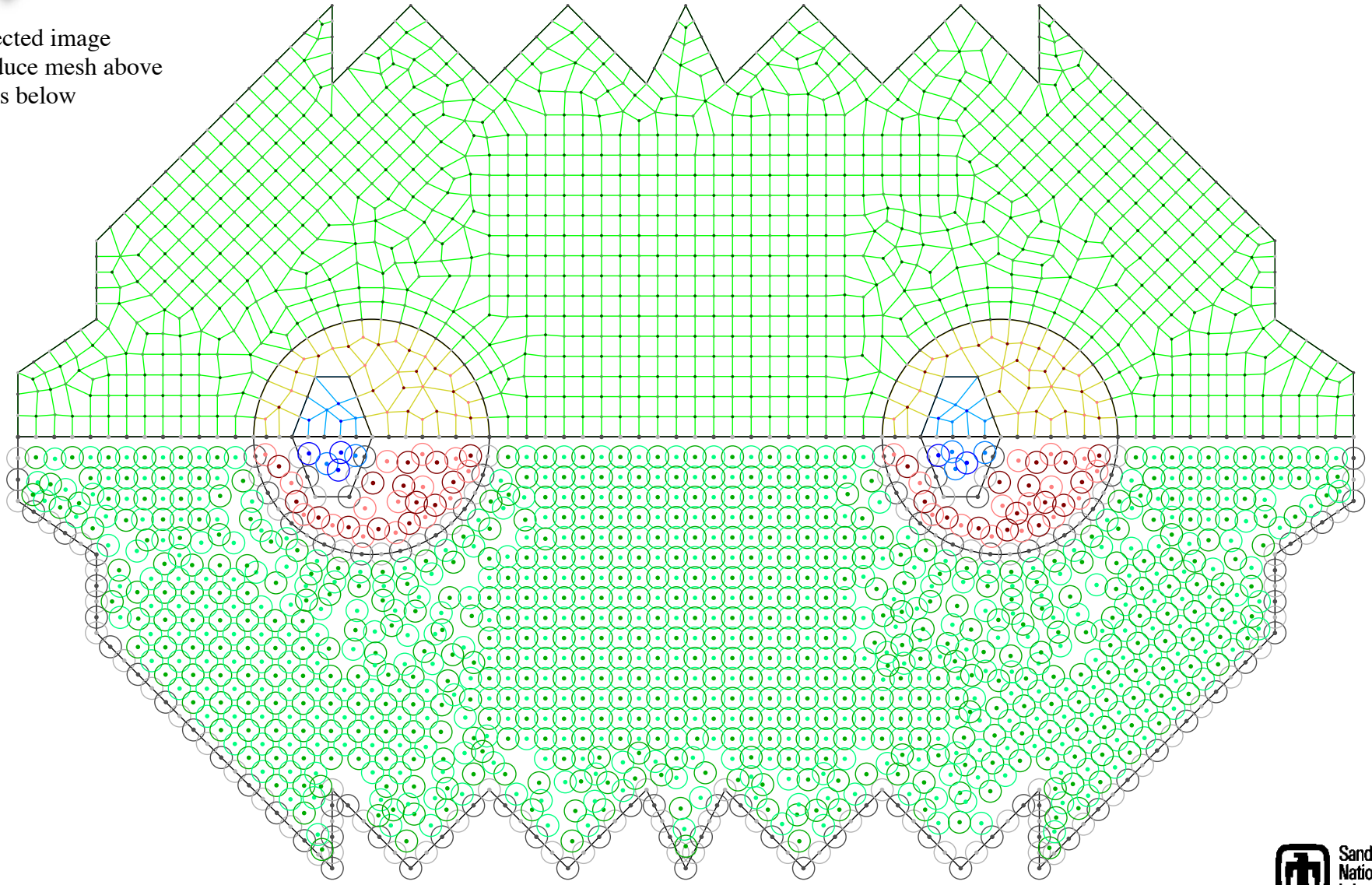
# Structured

reflected image
produce mesh above
disks below



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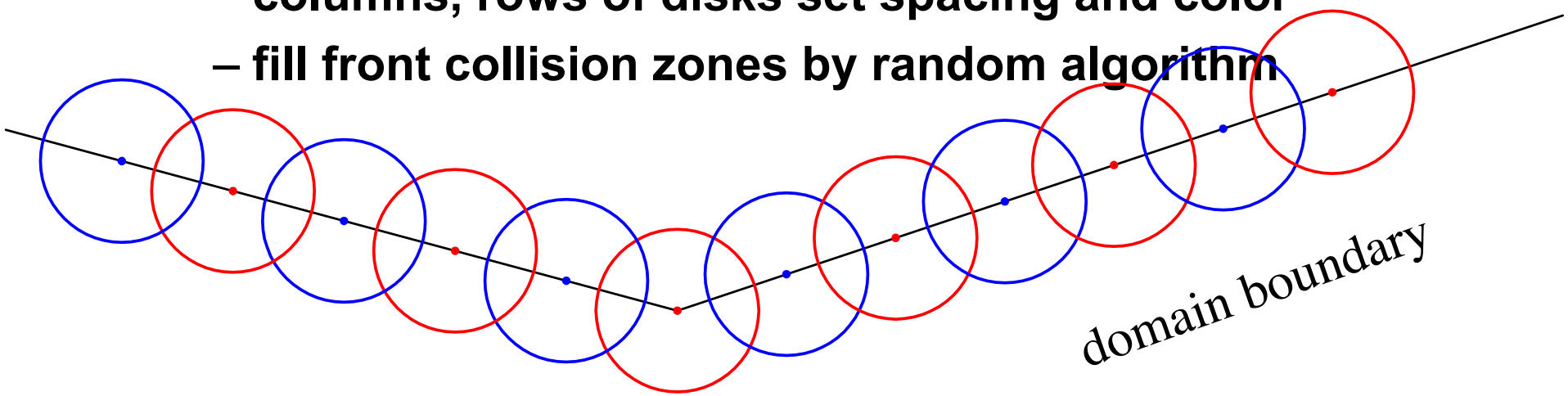
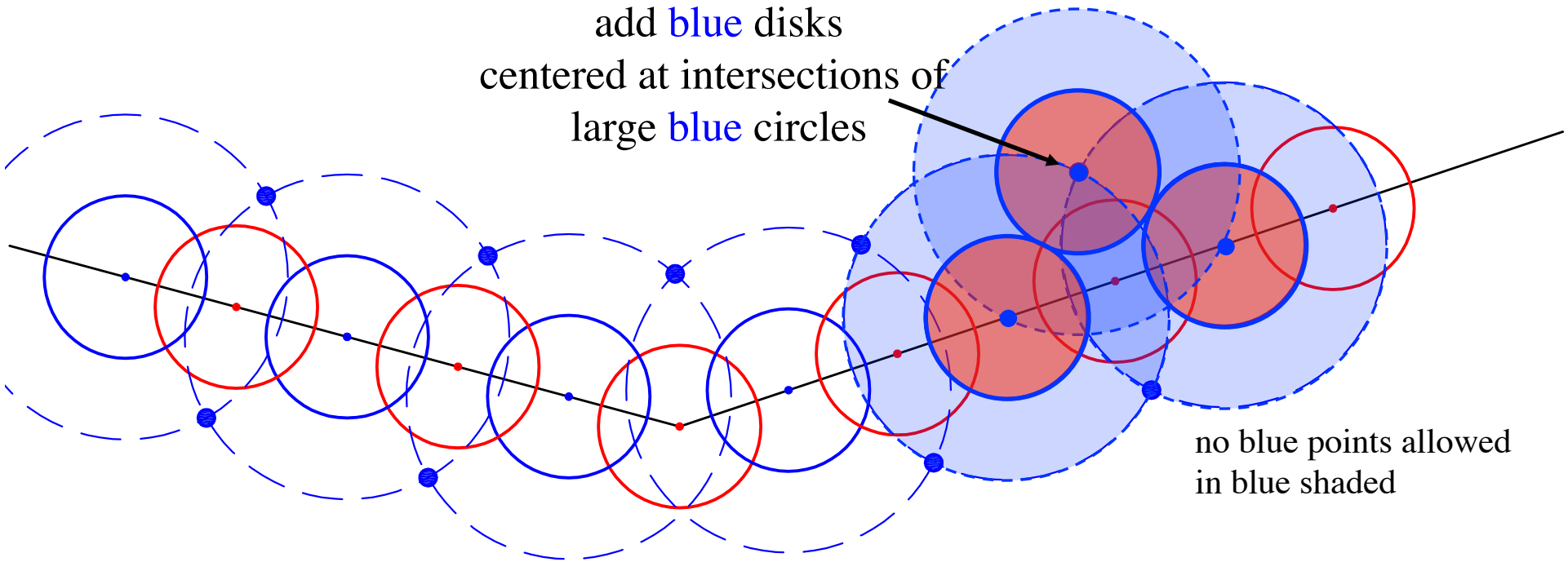
# Advancing Front for Structured

- Advance front
  - columns, rows of disks set spacing and color
  - fill front collision zones by random algorithm



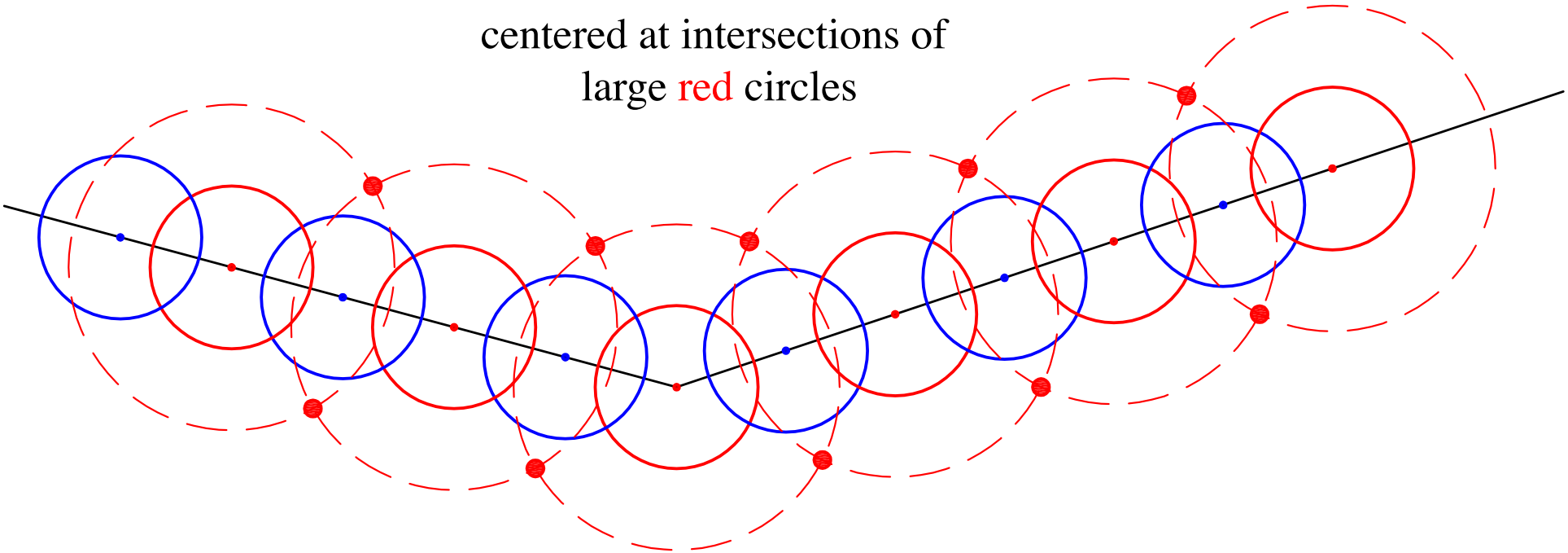
# Advancing Front for Structured




This is the closest we can place blue disks
and not violate the blue-blue separation distance

# Advancing Front for Structured

add red disks
centered at intersections of
large red circles



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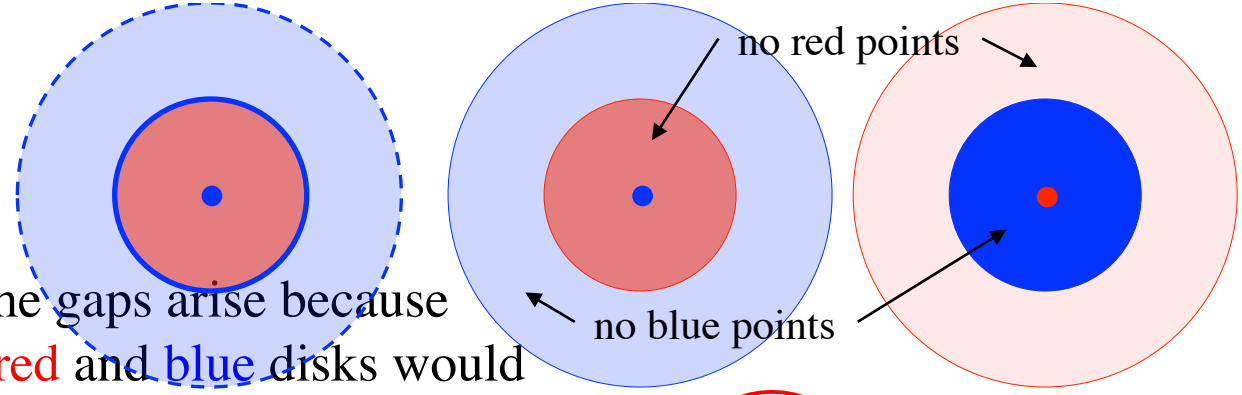
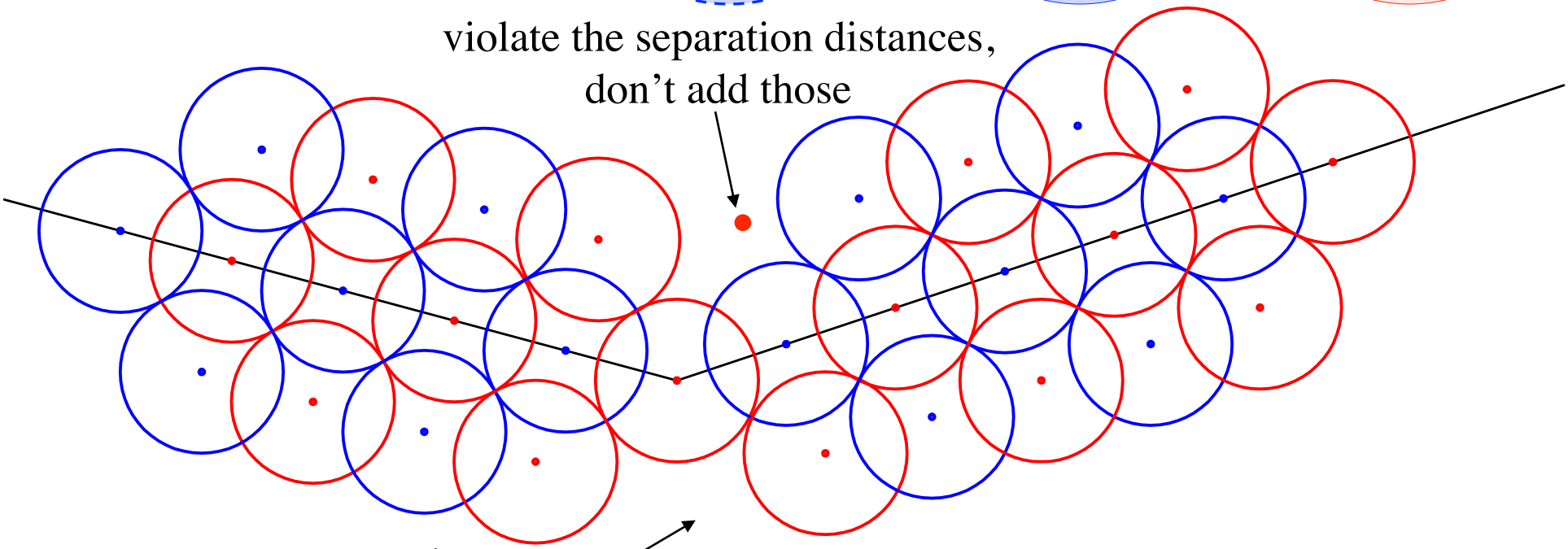
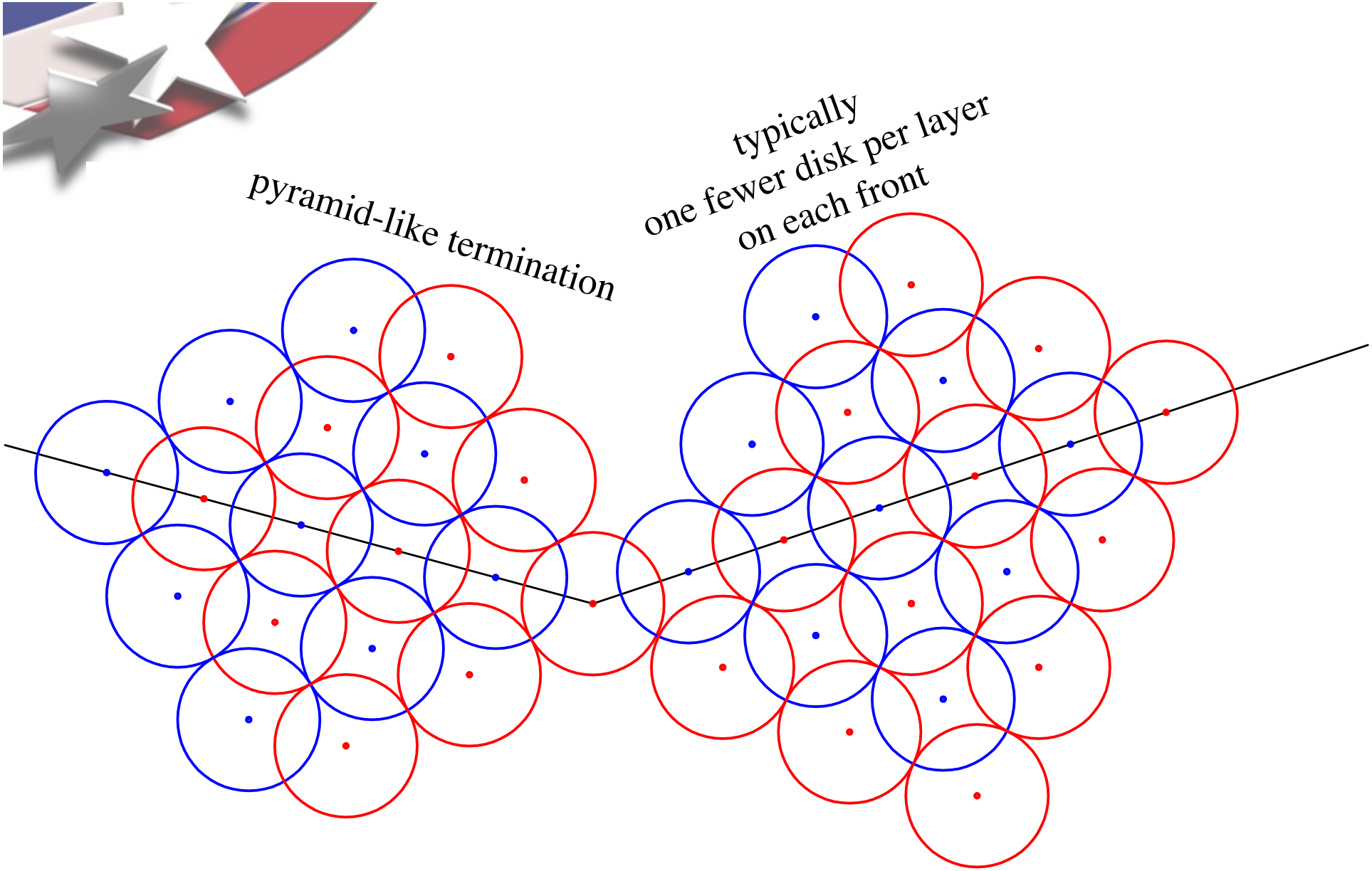



some gaps arise because
some red and blue disks would
violate the separation distances,
don't add those



some gaps arise
from diverging fronts
e.g. large blue circles didn't intersect

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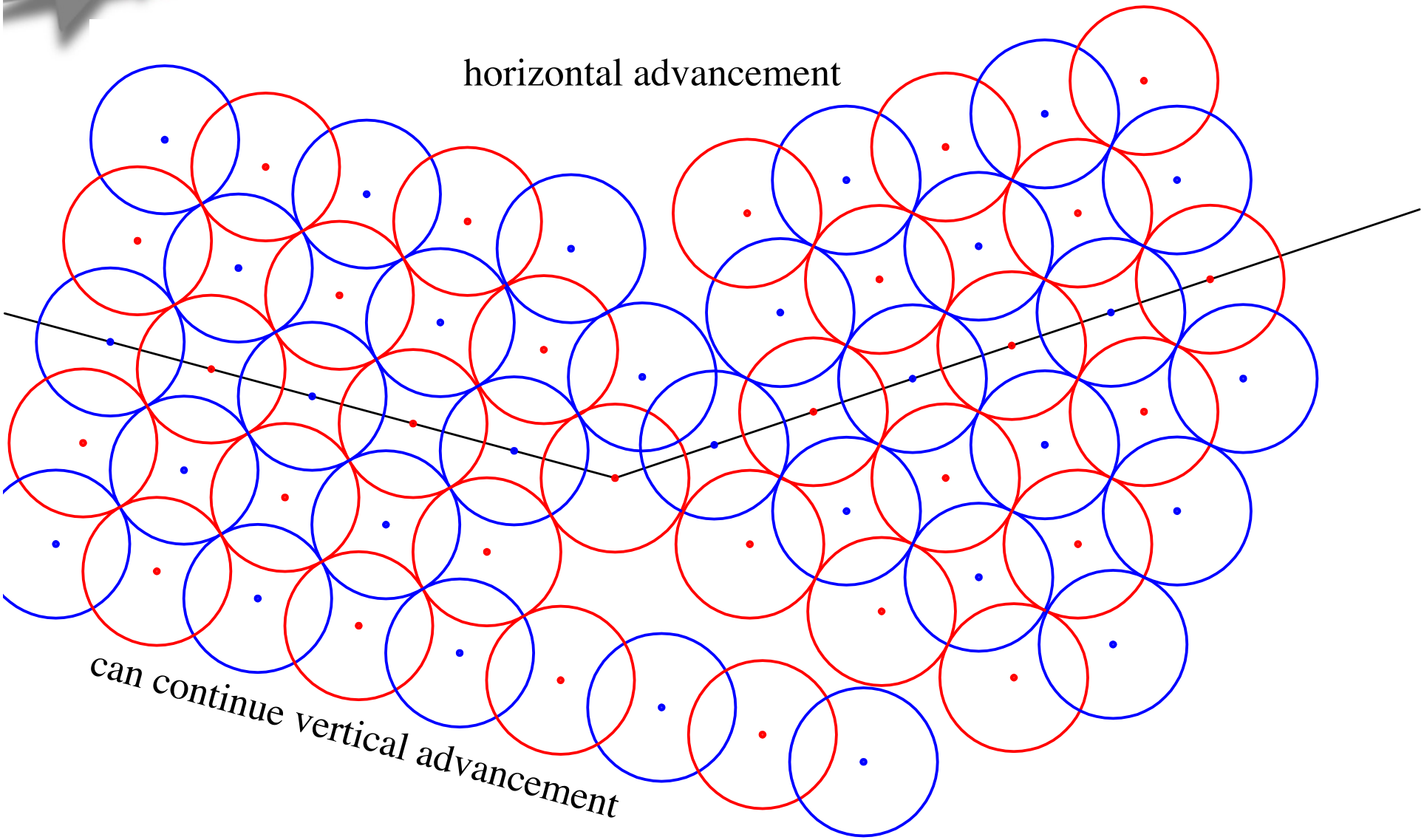


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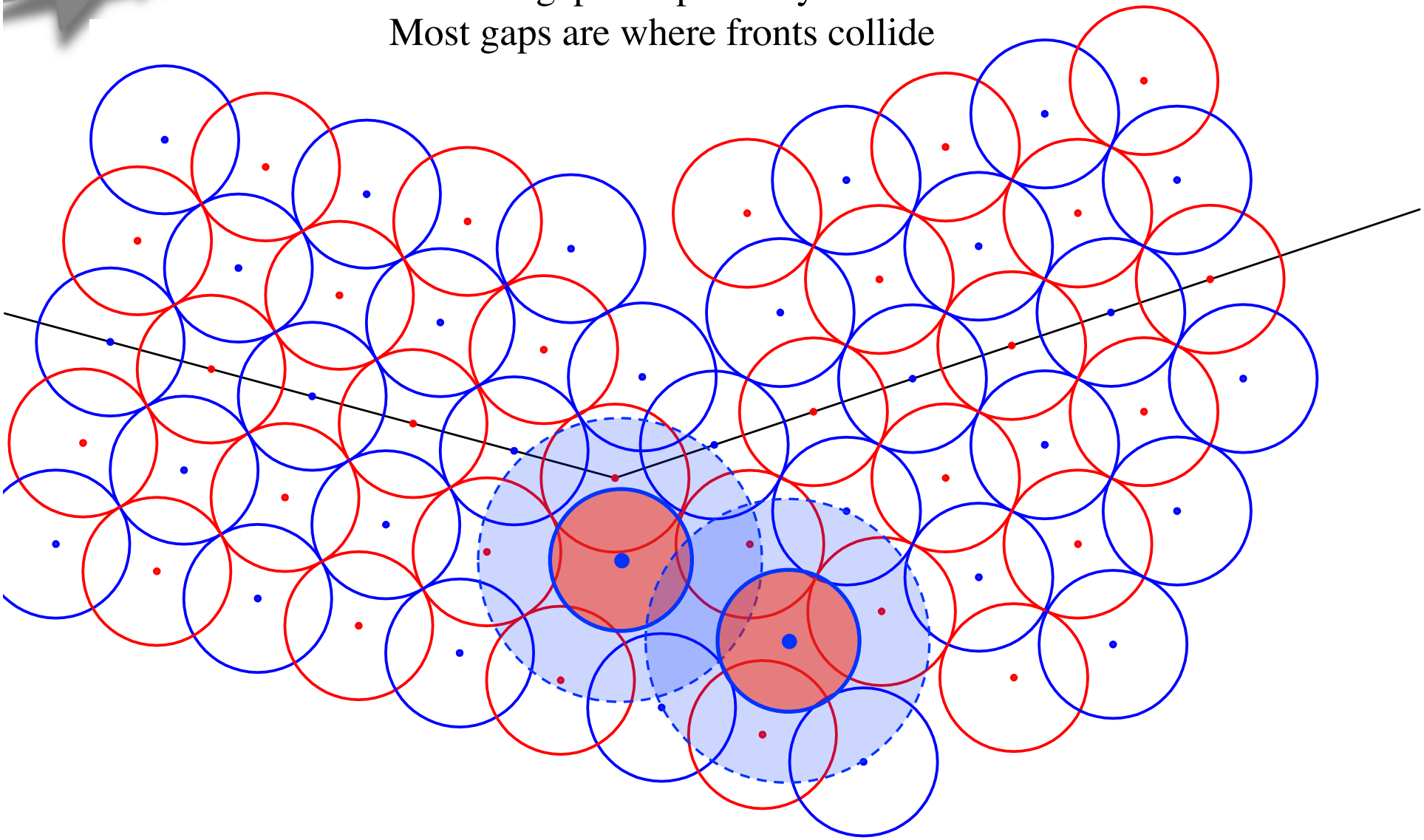
fill large gaps with random algorithm
small gaps are provably OK
Most gaps are where fronts collide



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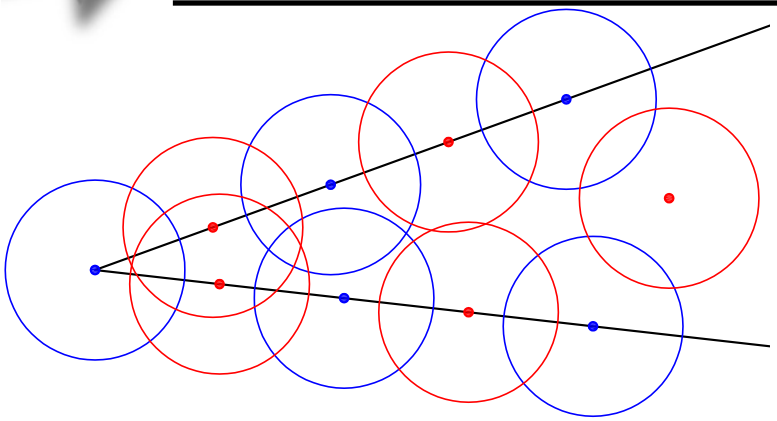
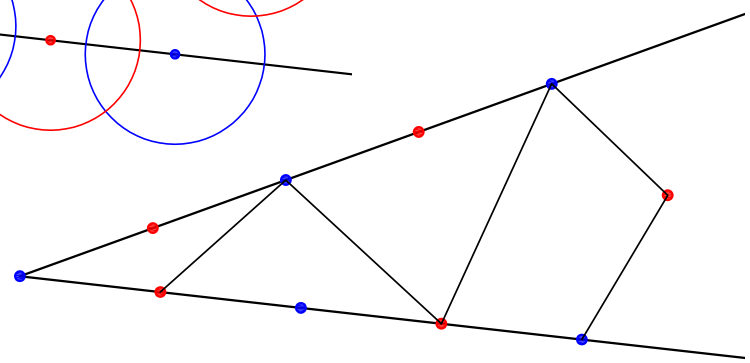
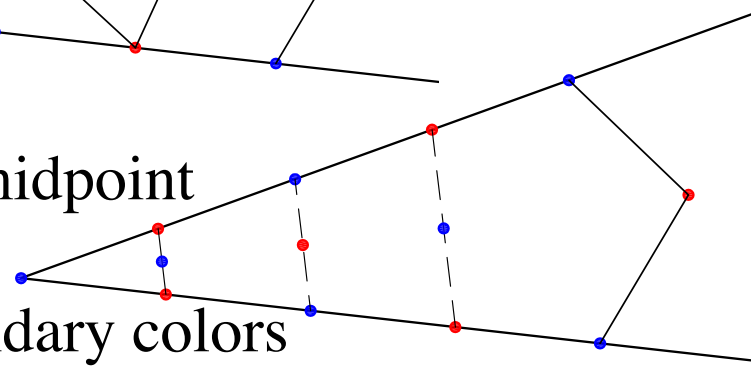
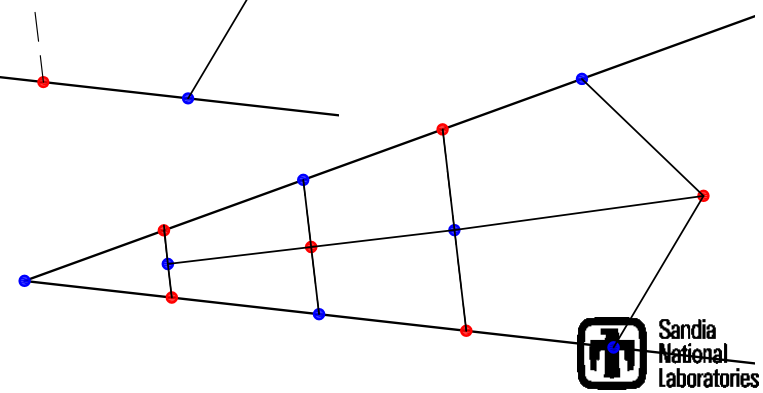
# Sharp Corners



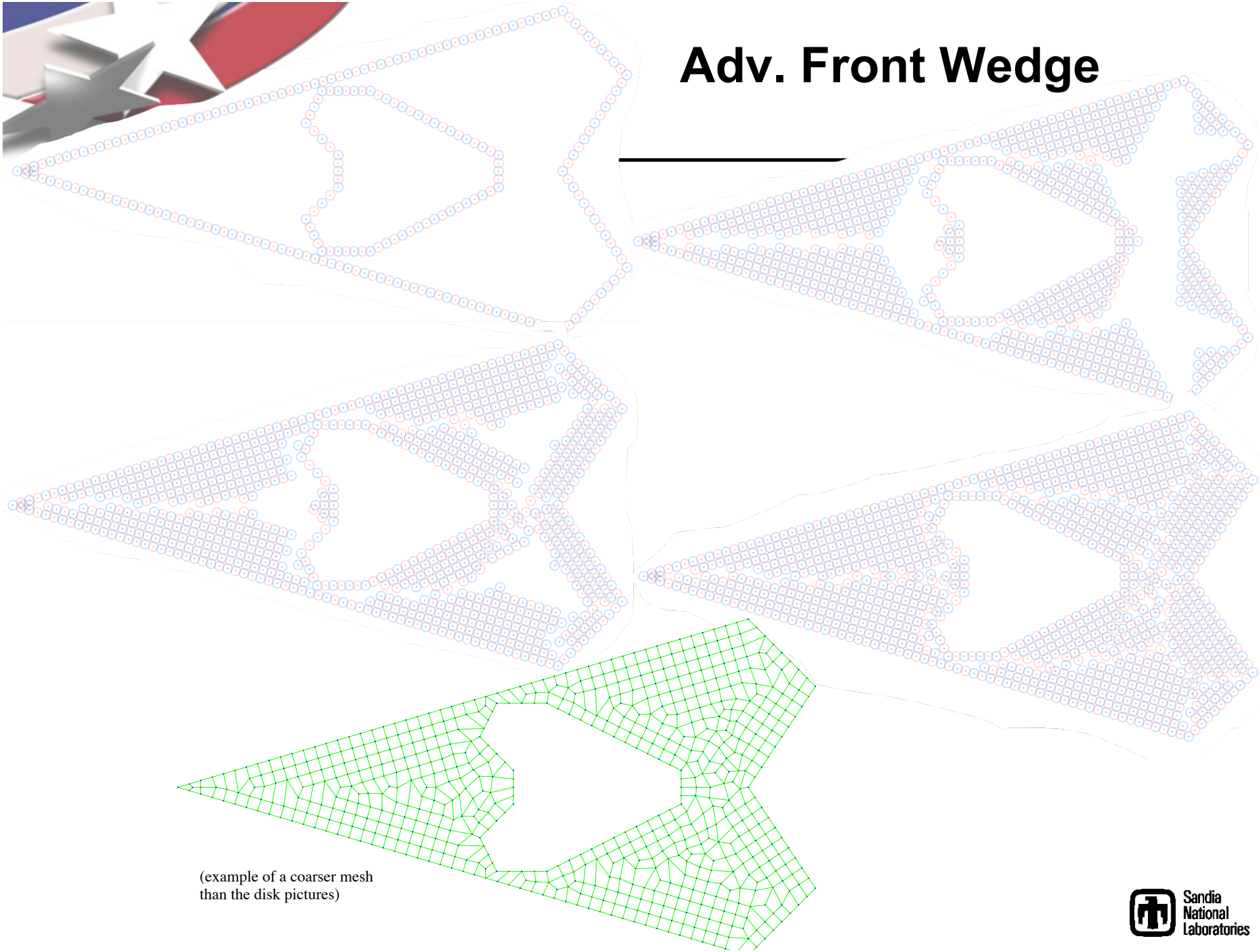




Add opposite-colored midpoint
on mono-edges
exploit alternating boundary colors



# Adv. Front Wedge



(example of a coarser mesh than the disk pictures)

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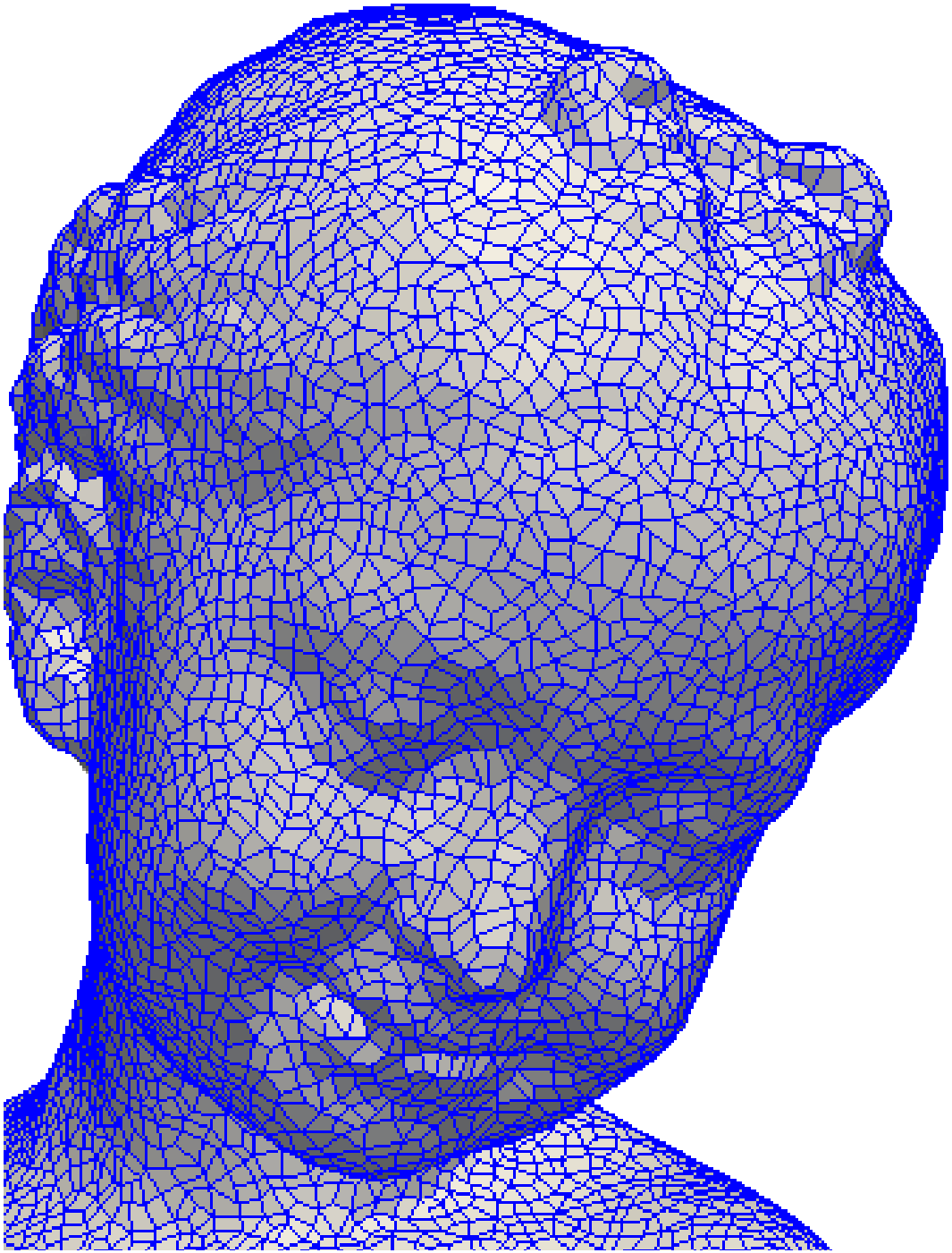
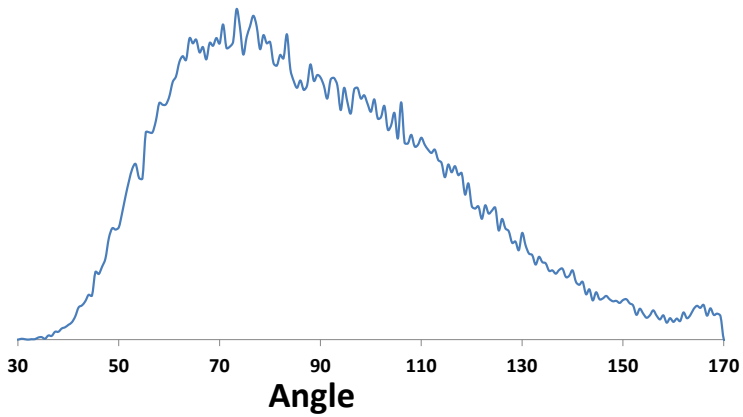
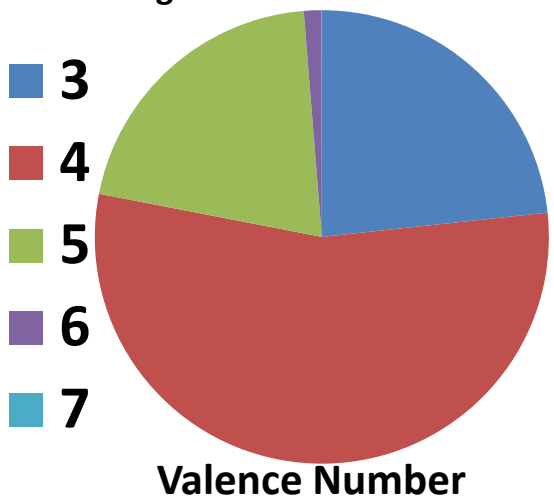
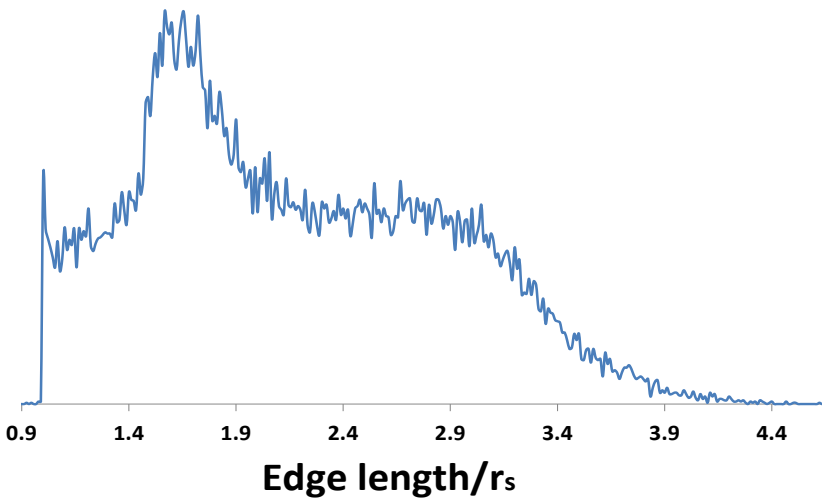
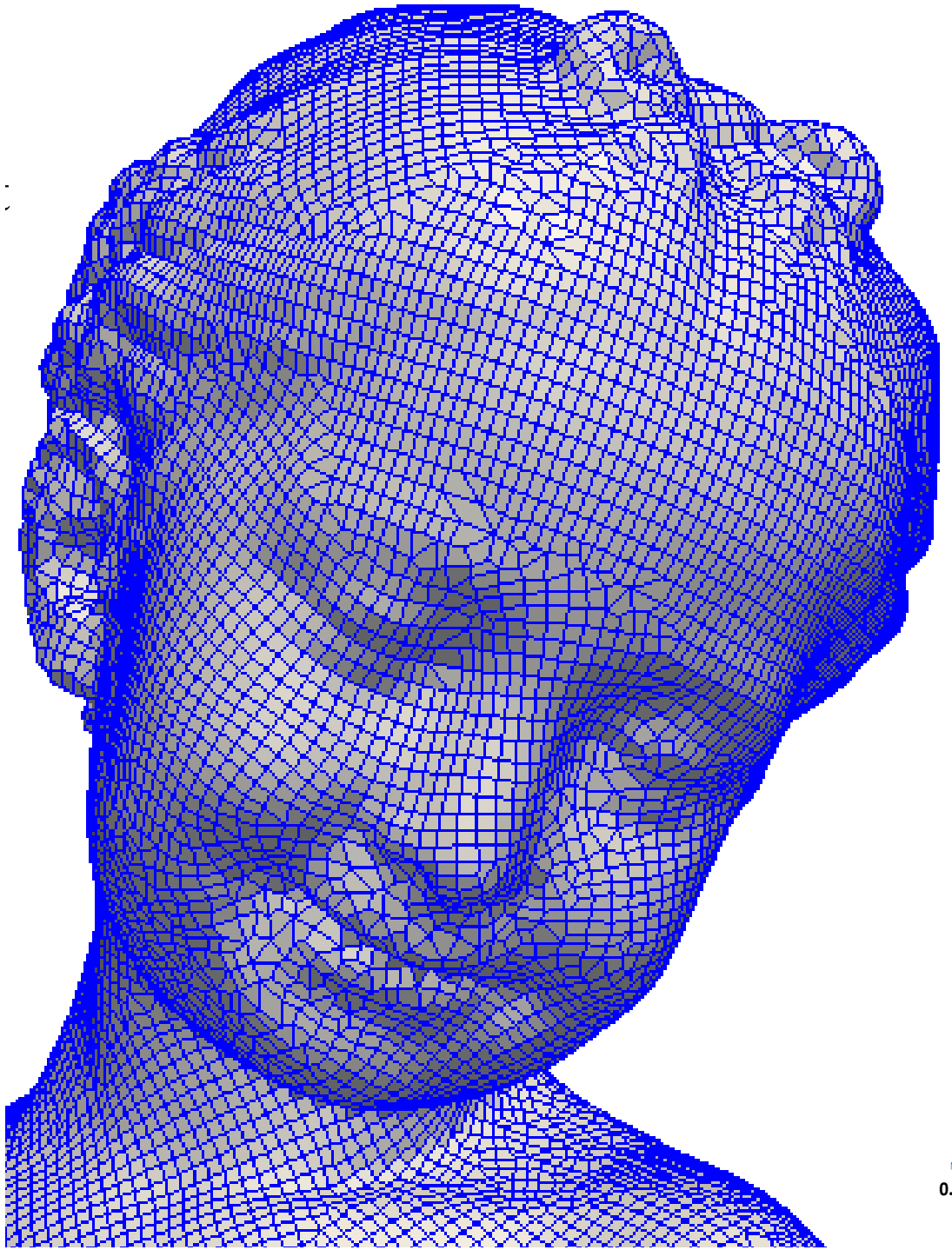
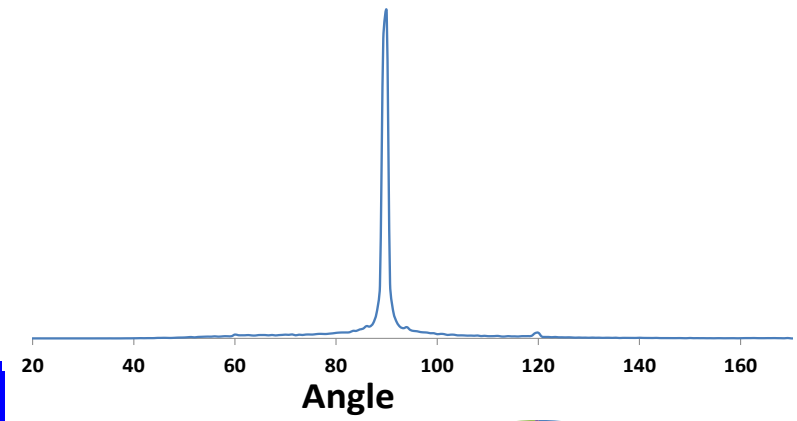


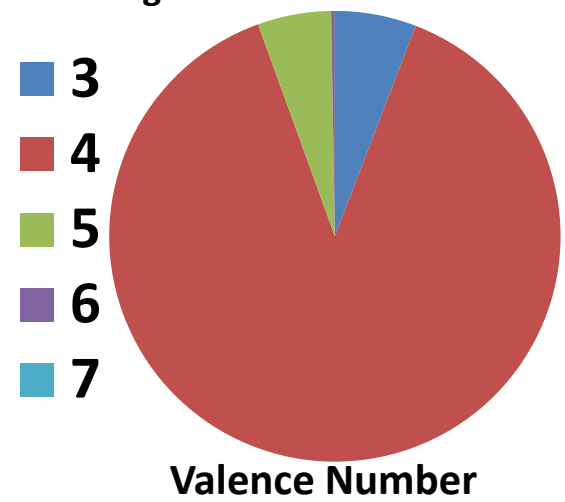

30
50
70
90
110
130
150
170
Angle

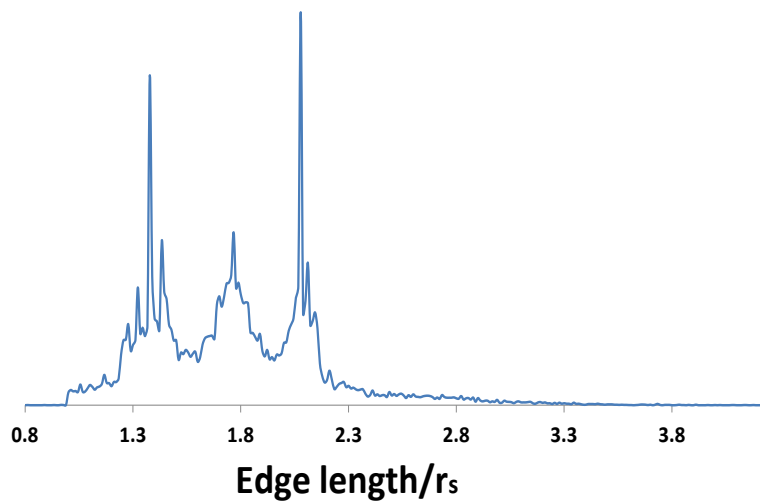

3
4
5
6
7
Valence Number


0.9
1.4
1.9
2.4
2.9
3.4
3.9
4.4
Edge length/rs

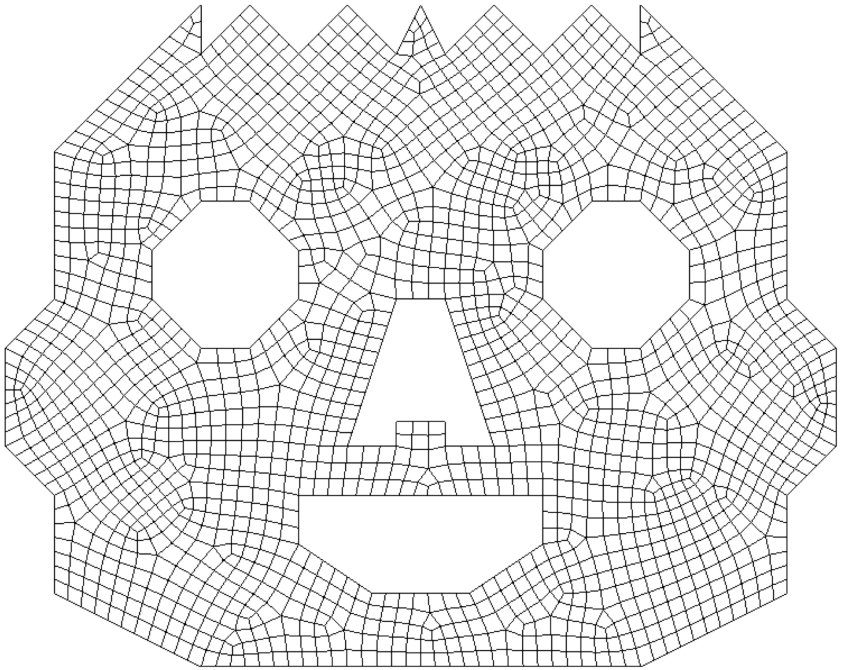



20
40
60
80
100
120
140
160
Angle

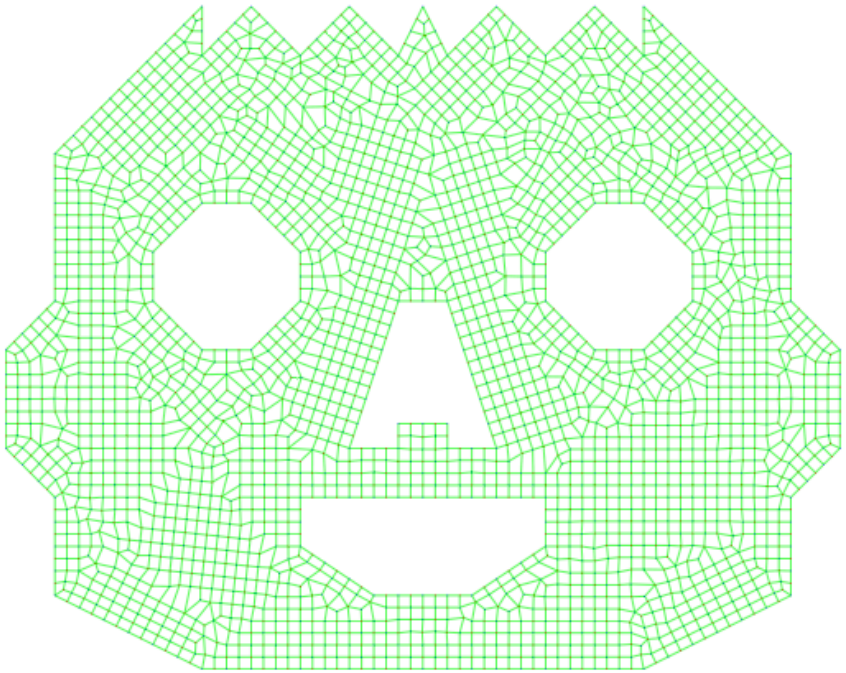

3
4
5
6
7
Valence Number


0.8
1.3
1.8
2.3
2.8
3.3
3.8
Edge length/rs

# Qualitative comparison to a production code



Cubit paver



Two-color adv. front no cleanup

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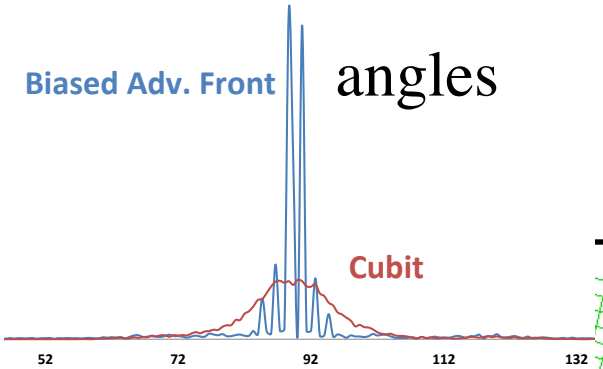
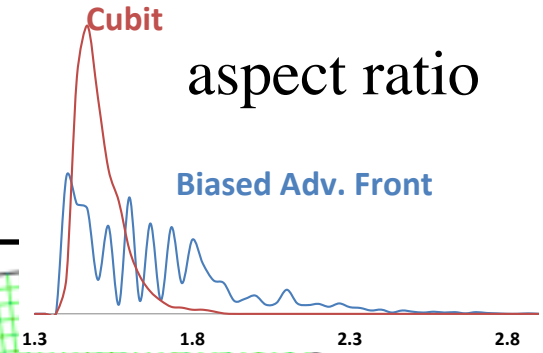
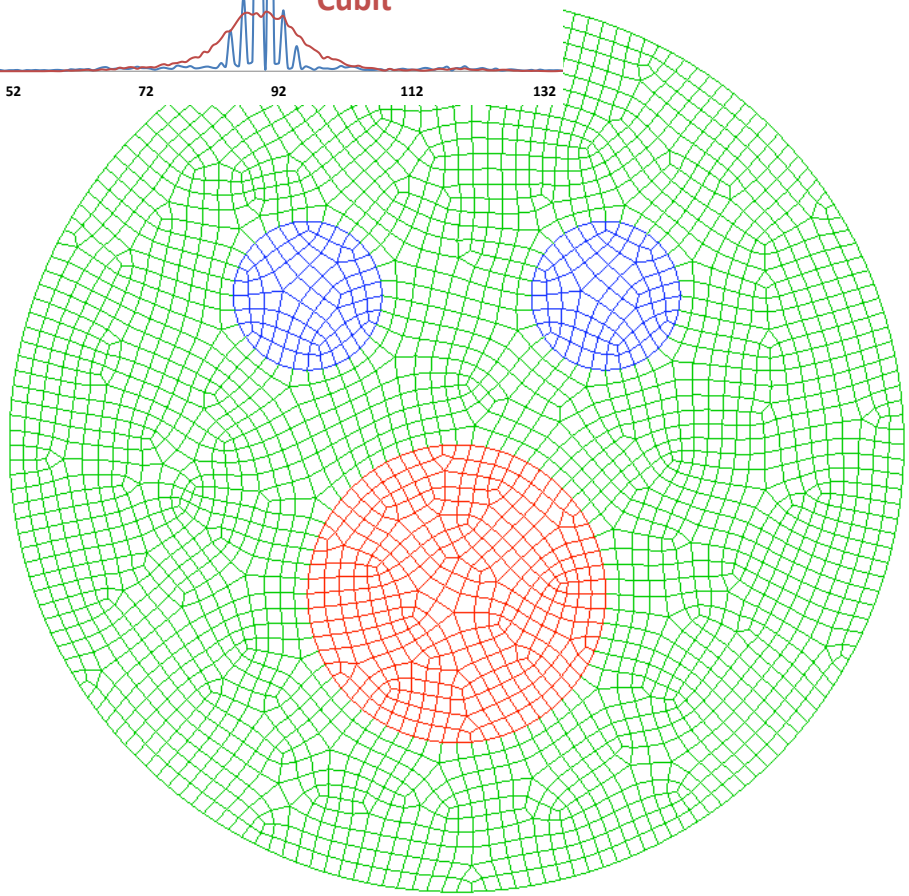
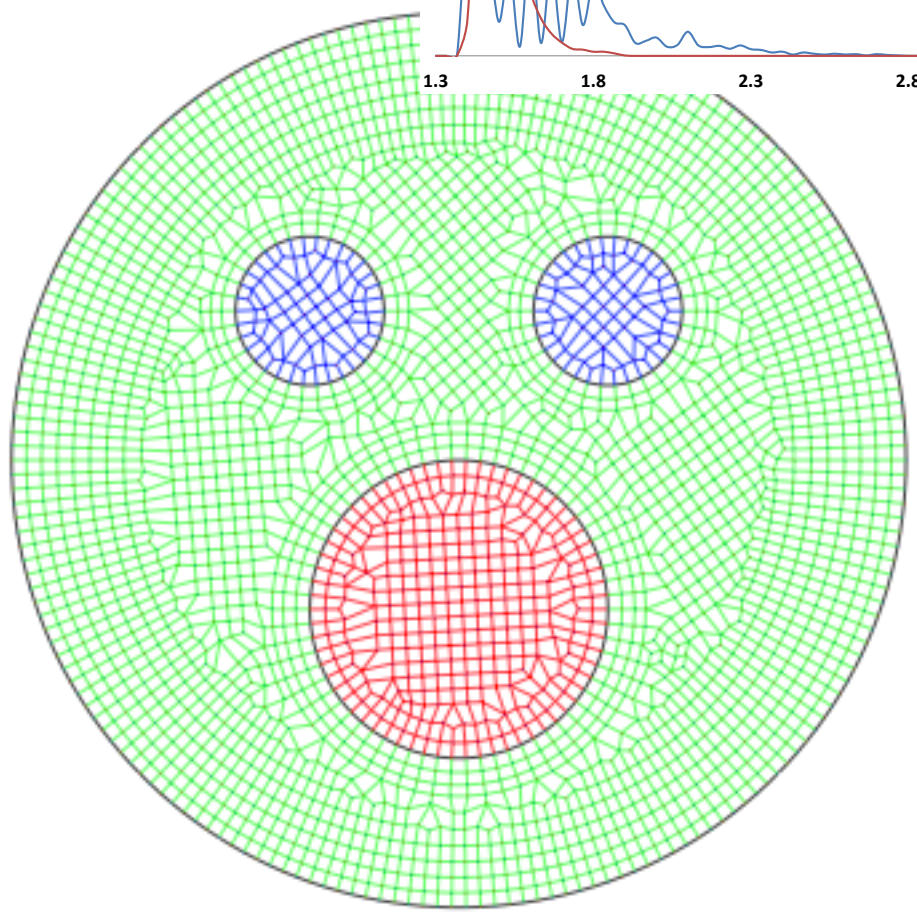
# Qualitative comparison to a production code



**Cubit paver**



**Two-color "biased" adv. front no cleanup**

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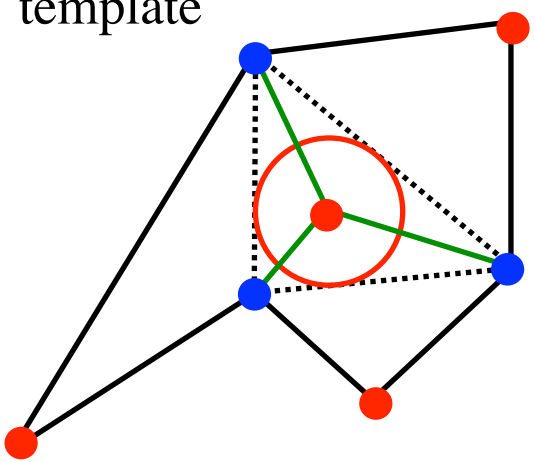
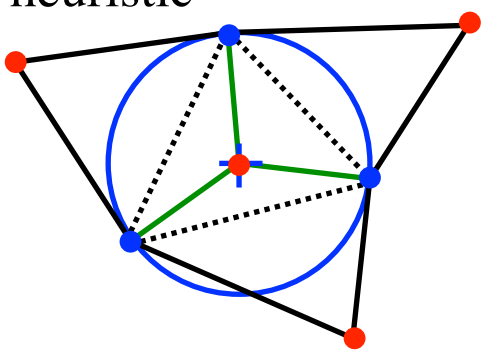
# Closing Thoughts

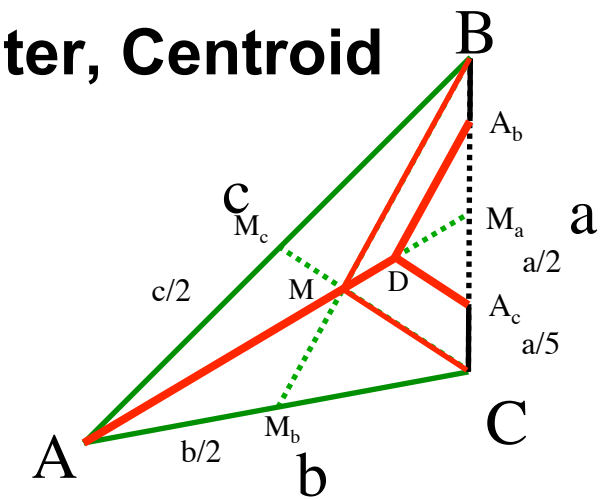
- **Three centers: Circumcenter, Incenter, Centroid**

Incenter for mono-tris template

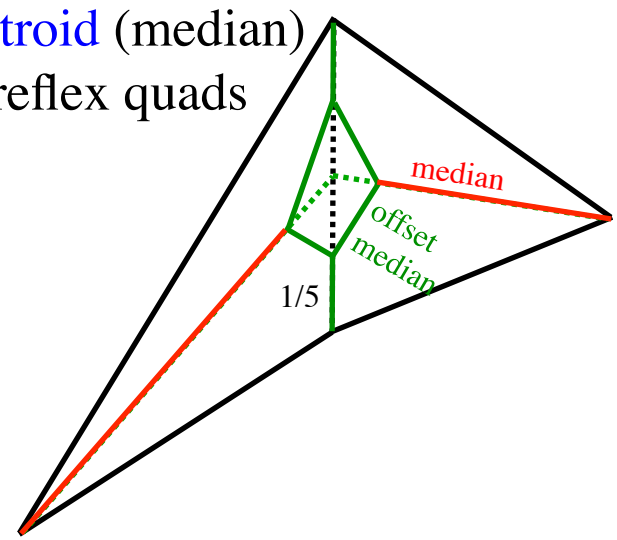


Circumcenter for mono-tris heuristic






Centroid (median) for reflex quads




Orthocenter feeling left out

deviantArt Free Ai Vector Smiley Pack
by MathieuBerenguer

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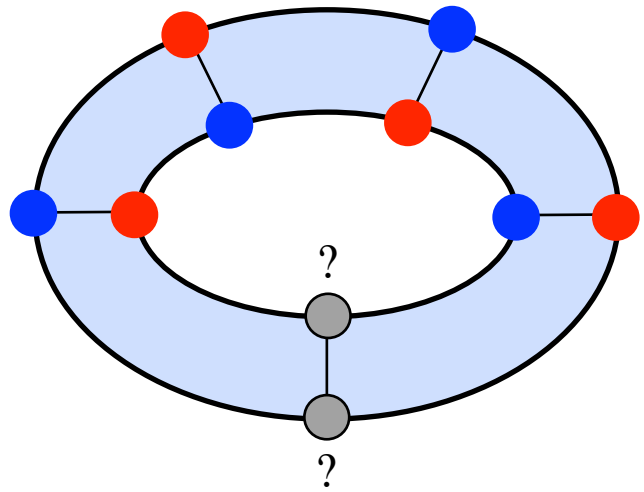
# Closing Thoughts

- **We create two-colorable quad meshes**
  - **For non-disk domain topologies, some given quad meshes are not two-colorable**

odd-cycle of edges
non-disk domain topology



# Closing thoughts

– Can we create hex meshes by coloring?

no.

2d, a chain of edges of even length -> 2n-gon

3d, a chain of edges of even length
!= edge cycles bounding hexes

Pairings: combine tets to form hexes?

Closest idea: graphics and meshing literature on 3d cross-fields and connecting hex-duals

# Reasonable Extensions

- Questions
  - Varying-size advancing front?
  - How fast can we vary the size?
    - Bounds for two-radii DT known

# Summary

- Three key ideas
  - Graph theory, two-coloring of vertices
  - Random sphere-packings (better control than Delaunay refine)
  - Advancing front

- Features
  - Robust
  - Simple
  - Local
  - Provable quality
  - Heuristics for good quality in practice
  - What more do you want?
- Quality is reasonable before cleanup
  - someone could build a production tool based on this
  - add traditional cleanup

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