Sampling Conditions for Clipping-free Voronoi Meshing by the VoroCrust Algorithm

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Abstract

Decomposing the volume bounded by a closed surface using simple cells is a key step in many applications. Although tetrahedral meshing is well-established with many successful implementations, many research questions remain open for polyhedral meshing, which promises significant advantages in some contexts. Unfortunately, current approaches to polyhedral meshing often resort to clipping cells near the boundary, which results in certain defects. In this talk, we present an analysis of the VoroCrust algorithm, which leverages ideas from α-shapes and the power crust algorithm to produce conforming Voronoi cells on both sides of the surface. We derive sufficient conditions for a weighted sampling to produce a topologically-correct and geometrically-accurate reconstruction, using the ϵ-sampling paradigm with a standard sparsity condition. The resulting surface reconstruction consists of weighted Delaunay triangles, except inside tetrahedra with a negative weighted circumradius where a Steiner vertex is generated close to the surface.

Generating quality meshes is an important problem in computer graphics and geometric modeling. Polyhedral meshing offers higher degrees of freedom per element than tetrahedral or hex-dominant meshing, and is more efficient in filling a space, because it produces fewer elements for the same number of nodes. Within the class of polyhedral mesh elements, Voronoi cells are particularly useful in numerical simulations for their geometric properties, e.g., planar facets and positive Jacobians.

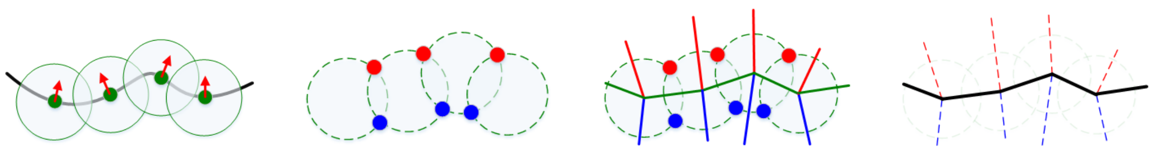
A conforming mesh exhibits two desirable properties *simultaneously*: 1) a decomposition of the enclosed volume, and 2) a reconstruction of the bounding surface. A common technique for producing boundary-conforming decomposition from Voronoi cells relies on *clipping*, i.e., intersecting and truncating, each cell by the bounding surface [3]. An alternative to clipping is to locally mirror the Voronoi generators on either side of the surface [2].

VoroCrust can be viewed as a principled mirroring technique. Similar to the power crust [1], the reconstruction is composed of the facets shared by cells on the inside and outside of the manifold. However, VoroCrust uses pairs of unweighted generators tightly hugging the surface, which allows further decomposition of the interior without disrupting the surface reconstruction. VoroCrust can also be viewed as a special case of the weighted α-shape [4]. A description of the abstract VoroCrust algorithm we analyze is provided next. Figure 1 illustrates the basic concepts in 2D.

The Abstract VoroCrust Algorithm

1. Take as input a weighted point sampling $\mathcal{P} = \{(p_i, w_i)\}$ of a closed 2-manifold $\mathcal{M}$.
2. Use weights to define a ball B_i of radius $r_i = \sqrt{w_i}$ centered at each sample p_i.
3. Find intersecting triplet of balls to obtain one corner point on either side of $\mathcal{M}$.
4. Collect the Voronoi generators $\mathcal{G}$ as the set of corner points outside all sample balls.
5. Optionally, include in $\mathcal{G}$ more generators far-inside $\mathcal{M}$ to further decompose its interior.
6. Output the Voronoi diagram of $\mathcal{G}$ as the desired decomposition, and the facets separating the inside and outside generators as the reconstructed surface $\hat{\mathcal{M}}$.

Problem Statement: We seek to characterize the locations and weights of the input samples in Step (1) to guarantee a topologically-correct and geometrically-accurate reconstruction.



(a) Weighted samples. (b) Intersection pairs. (c) Voronoi cells. (d) Reconstruction facets.

Figure 1: VoroCrust reconstruction, demonstrated on a planar curve. The weight of a point defines the radius of a ball around it. The reconstruction is the Voronoi facets separating the uncovered intersection pairs on opposite sides of the manifold.

Summary of Results: An ϵ-sampling $\mathcal{S}$ is δ-*weighted* if each sample is associated with a ball of radius $r_i = \delta\,\text{lfs}(p_i)$, with $\delta \geq \epsilon$. In addition, $\mathcal{S}$ is *conflict-free* if $\forall j \neq i, \|p_i - p_j\| \geq \epsilon \cdot \min(\text{lfs}(p_i), \text{lfs}(p_j))$. For some constants ϵ, δ, c, we show that the reconstruction $\hat{\mathcal{M}}$ is geometrically-close and topologically-correct. Moreover, $\hat{\mathcal{M}} \to \mathcal{M}$ quadratically as $\epsilon \to 0$. Specifically, for every $p \in \mathcal{M}$ with closest point $q \in \hat{\mathcal{M}}$, and for every $q \in \hat{\mathcal{M}}$ with closest point $p \in \mathcal{M}$, we have $\|pq\| < c \cdot \delta^2 \text{lfs}(p)$. The reconstruction $\hat{\mathcal{M}}$ contains every input sample as a vertex. Rather than filtering, the algorithm as outlined above naturally resolves slivers by introducing Steiner vertices.

References

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