

Sampling Conditions for Conforming Voronoi Meshing by the VoroCrust Algorithm

Ahmed Abdelkader

University of Maryland, College Park MD, USA
akader@cs.umd.edu

Chandrajit L. Bajaj¹

University of Texas, Austin TX, USA

Mohamed S. Ebeida

Sandia National Laboratories, Albuquerque NM, USA

Ahmed H. Mahmoud

University of California, Davis CA, USA

Scott A. Mitchell

Sandia National Laboratories, Albuquerque NM, USA

John D. Owens

University of California, Davis CA, USA

Ahmad A. Rushdi

University of California, Davis CA, USA

Abstract

We study the problem of decomposing a volume bounded by a smooth surface into a collection of Voronoi cells. Unlike the dual problem of conforming Delaunay meshing, a principled solution to this problem for generic smooth surfaces remained elusive. VoroCrust leverages ideas from α -shapes and the power crust algorithm to produce unweighted Voronoi cells conforming to the surface, yielding the first provably-correct algorithm for this problem. Given an ϵ -sample on the bounding surface, with a weak σ -sparsity condition, we work with the balls of radius δ times the local feature size centered at each sample. The corners of this union of balls are the Voronoi sites, on both sides of the surface. The facets common to cells on opposite sides reconstruct the surface. For appropriate values of ϵ , σ and δ , we prove that the surface reconstruction is isotopic to the bounding surface. With the surface protected, the enclosed volume can be further decomposed into an isotopic volume mesh of fat Voronoi cells by generating a bounded number of sites in its interior. Compared to state-of-the-art methods based on clipping, VoroCrust cells are full Voronoi cells, with convexity and fatness guarantees. Compared to the power crust algorithm, VoroCrust cells are not filtered, are unweighted, and offer greater flexibility in meshing the enclosed volume by either structured grids or random samples.

2012 ACM Subject Classification Theory of computation \rightarrow Randomness, geometry and discrete structures \rightarrow Computational geometry

Keywords and phrases sampling conditions, surface reconstruction, polyhedral meshing, Voronoi

Related Version This is an extended version of the original paper which appeared in the Symposium on Computational Geometry [1], and includes appendices with proofs that the original paper did not. See also the accompanying multimedia contribution [2].

¹ Supported in part by a contract from Sandia, #1439100, and a grant from NIH, R01-GM117594

Funding This material is based upon work supported by the U.S. Department of Energy, Office of Science, Office of Advanced Scientific Computing Research (ASCR), Applied Mathematics Program. Sandia National Laboratories is a multi-mission laboratory managed and operated by National Technology and Engineering Solutions of Sandia, LLC., a wholly owned subsidiary of Honeywell International, Inc., for the U.S. Department of Energy’s National Nuclear Security Administration under contract DE-NA0003525. The views expressed in the article do not necessarily represent the views of the U.S. Department of Energy or the United States Government.

Acknowledgements We thank Tamal Dey for helpful discussions about surface reconstruction.

1 Introduction

Mesh generation is a fundamental problem in computational geometry, geometric modeling, computer graphics, scientific computing and engineering simulations. There has been a growing interest in polyhedral meshes as an alternative to tetrahedral or hex-dominant meshes [48]. Polyhedra are less sensitive to stretching, which enables the representation of complex geometries without excessive refinement. In addition, polyhedral cells have more neighbors even at corners and boundaries, which offers better approximations of gradients and local flow distributions. Even compared to hexahedra, fewer polyhedral cells are needed to achieve a desired accuracy in certain applications. This can be very useful in several numerical methods [18], e.g., finite element [42], finite volume [39], virtual element [17] and Petrov-Galerkin [41]. In particular, the accuracy of a number of important solvers, e.g., the two-point flux approximation for conservation laws [39], greatly benefits from a conforming mesh which is *orthogonal* to its dual as naturally satisfied by Voronoi meshes. Such solvers play a crucial role in hydrology [51], computational fluid dynamics [22] and fracture modeling [20].

VoroCrust is the first provably-correct algorithm for generating a volumetric Voronoi mesh whose boundary conforms to a smooth bounding surface, and with quality guarantees. A conforming volume mesh exhibits two desirable properties *simultaneously*: (1) a decomposition of the enclosed volume, and (2) a reconstruction of the bounding surface.

Conforming Delaunay meshing is well-studied [28], but Voronoi meshing is less mature. A common practical approach to polyhedral meshing is to dualize a tetrahedral mesh and *clip*, i.e., intersect and truncate, each cell by the bounding surface [35,43,47,52,55]. Unfortunately, clipping sacrifices the important properties of convexity and connectedness of cells [35], and may require costly constructive solid geometry operations. Restricting a Voronoi mesh to the surface before *filtering* its dual Delaunay facets is another approach [7,33,56], but filtering requires extra checks complicating its implementation and analysis; see also Figure 4. An intuitive approach is to locally mirror the Voronoi sites on either side of the surface [34,57], but we are not aware of any robust algorithms with approximation guarantees in this category. In contrast to these approaches, VoroCrust is distinguished by its simplicity and robustness at producing true unweighted Voronoi cells, leveraging established libraries, e.g., Voro++ [50], without modification or special cases.

VoroCrust can be viewed as a principled mirroring technique, which shares a number of key features with the power crust algorithm [13]. The power crust literature [7,8,10,12,13] developed a rich theory for surface approximation, namely the ϵ -sampling paradigm. Recall that the power crust algorithm uses an ϵ -sample of unweighted points to place weighted sites, so-called *poles*, near the medial axis of the underlying surface. The surface reconstruction is the collection of facets separating power cells of poles on the inside and outside of the enclosed volume.

Regarding samples and poles as primal-dual constructs, power crust performs a *primal-dual-dual-primal dance*. VoroCrust makes a similar dance where weights are introduced differently; the samples are weighted to define unweighted sites tightly hugging the surface, with the reconstruction arising from their unweighted Voronoi diagram. The key advantage is the freedom to place more sites within the enclosed volume without disrupting the surface reconstruction. This added freedom is essential to the generation of graded meshes; a primary virtue of the proposed algorithm. Another virtue of the algorithm is that all samples appear as vertices in the resulting mesh. While the power crust algorithm does not guarantee that, some variations do so by means of filtering, at the price of the reconstruction no longer being the boundary of power cells [7, 11, 32].

The main construction underlying VoroCrust is a suitable union of balls centered on the bounding surface, as studied in the context of non-uniform approximations [26]. Unions of balls enjoy a wealth of results [15, 24, 37], which enable a variety of algorithms [13, 23, 30].

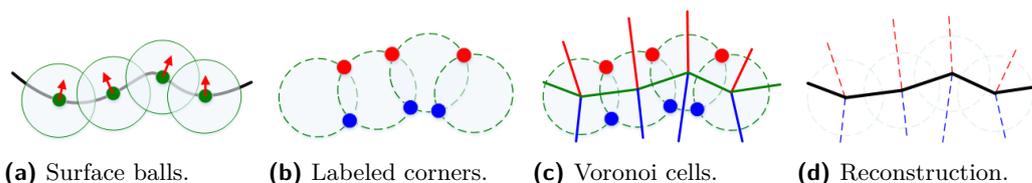
Similar constructions have been proposed for meshing problems in the applied sciences with heuristic extensions to 3D settings; see [40] and the references therein for a recent example. Aichholzer et al. [6] adopt closely related ideas to construct a union of surface balls using power crust poles for sizing estimation. However, their goal was to produce a coarse homeomorphic surface reconstruction. As in [6], the use of balls and α -shapes for surface reconstruction was explored earlier, e.g., ball-pivoting [19, 54], but the connection to Voronoi meshing has been absent. In contrast, VoroCrust aims at a decomposition of the enclosed volume into fat Voronoi cells conforming to an isotopic surface reconstruction with quality guarantees.

In a previous paper [4], we explored the related problem of generating a Voronoi mesh that conforms to restricted classes of piecewise-linear complexes, with more challenging inputs left for future work. The approach adopted in [4] does not use a union of balls and relies instead on similar ideas to those proposed for conforming Delaunay meshing [29, 45, 49].

In this paper, we present a theoretical analysis of an abstract version of the VoroCrust algorithm. This establishes the quality and approximation guarantees of its output for volumes bounded by smooth surfaces. A description of the algorithm we analyze is given next; see Figure 1 for an illustration in 2D, and also our accompanying multimedia contribution [2].

The abstract VoroCrust algorithm

1. Take as input a sample \mathcal{P} on the surface \mathcal{M} bounding the volume \mathcal{O} .
2. Define a ball B_i centered at each sample p_i , with a suitable radius r_i , and let $\mathcal{U} = \cup_i B_i$.
3. Initialize the set of sites \mathcal{S} with the corner points of $\partial\mathcal{U}$, \mathcal{S}^\uparrow and \mathcal{S}^\downarrow , on both sides of \mathcal{M} .
4. Optionally, generate additional sites $\mathcal{S}^{\downarrow\downarrow}$ in the interior of \mathcal{O} , and include $\mathcal{S}^{\downarrow\downarrow}$ into \mathcal{S} .
5. Compute the Voronoi diagram $\text{Vor}(\mathcal{S})$ and retain the cells with sites in $\mathcal{S}^\downarrow \cup \mathcal{S}^{\downarrow\downarrow}$ as the volume mesh $\hat{\mathcal{O}}$, where the facets between \mathcal{S}^\uparrow and \mathcal{S}^\downarrow yield a surface approximation $\hat{\mathcal{M}}$.



■ **Figure 1** VoroCrust reconstruction, demonstrated on a planar curve.

In this paper, we assume \mathcal{O} is a bounded open subset of \mathbb{R}^3 , whose boundary \mathcal{M} is a closed, bounded and smooth surface. We further assume that \mathcal{P} is an ϵ -sample, with a weak σ -sparsity condition, and r_i is set to δ times the local feature size at p_i . For appropriate values of ϵ , σ and δ , we prove that $\hat{\mathcal{O}}$ and $\hat{\mathcal{M}}$ are isotopic to \mathcal{O} and \mathcal{M} , respectively. We also show that simple techniques for sampling within \mathcal{O} , e.g., octree refinement, guarantee an upper bound on the fatness of all cells in $\hat{\mathcal{O}}$, as well as the number of samples.

Ultimately, we seek a conforming Voronoi mesher that can handle realistic inputs possibly containing sharp features, can estimate a sizing function and generate samples, and can guarantee the quality of the output mesh. This is the subject of a forthcoming paper [3] which describes the design and implementation of the complete VoroCrust algorithm.

The rest of the paper is organized as follows. Section 2 introduces the key definitions and notation used throughout the paper. Section 3 describes the placement of Voronoi seeds and basic properties of our construction assuming the union of surface balls satisfies a structural property. Section 4 proves this property holds and establishes the desired approximation guarantees under certain conditions on the input sample. Section 5 considers the generation of interior samples and bounds the fatness of all cells in the output mesh. Section 6 concludes the paper with pointers for future work. A number of proofs are deferred to the appendices; see also the accompanying multimedia contribution [2].

2 Preliminaries

Throughout, standard general position assumptions [38] are made implicitly to simplify the presentation. We use $\mathbf{d}(p, q)$ to denote the Euclidean distance between two points $p, q \in \mathbb{R}^3$, and $\mathbb{B}(c, r)$ to denote the Euclidean ball centered at $c \in \mathbb{R}^3$ with radius r . We proceed to introduce the notation and recall the key definitions used throughout, following those in [13, 26, 37].

2.1 Sampling and approximation

We take as input a set of sample points $\mathcal{P} \subset \mathcal{M}$. A local scale or *sizing* is used to vary the sample density. Recall that the *medial axis* [13] of \mathcal{M} , denoted by \mathcal{A} , is the closure of the set of points in \mathbb{R}^3 with more than one closest point on \mathcal{M} . Hence, \mathcal{A} has one component inside \mathcal{O} and another outside. Each point of \mathcal{A} is the center of a *medial ball* tangent to \mathcal{M} at multiple points. Likewise, each point on \mathcal{M} has two tangent medial balls, not necessarily of the same size. The *local feature size* at $x \in \mathcal{M}$ is defined as $\text{lfs}(x) = \inf_{a \in \mathcal{A}} \mathbf{d}(x, a)$. The set \mathcal{P} is an ϵ -sample [9] if for all $x \in \mathcal{M}$ there exists $p \in \mathcal{P}$ such that $\mathbf{d}(x, p) \leq \epsilon \cdot \text{lfs}(x)$.

We desire an approximation of \mathcal{O} by a Voronoi mesh $\hat{\mathcal{O}}$, where the boundary $\hat{\mathcal{M}}$ of $\hat{\mathcal{O}}$ approximates \mathcal{M} . Recall that two topological spaces are *homotopy-equivalent* [26] if they have the same topology type. A stronger notion of topological equivalence is *homeomorphism*, which holds when there exists a continuous bijection with a continuous inverse from \mathcal{M} to $\hat{\mathcal{M}}$. The notion of isotopy captures an even stronger type of equivalence for surfaces *embedded* in Euclidean space. Two surfaces $\mathcal{M}, \hat{\mathcal{M}} \subset \mathbb{R}^3$ are *isotopic* [16, 25] if there is a continuous mapping $F : \mathcal{M} \times [0, 1] \rightarrow \mathbb{R}^3$ such that for each $t \in [0, 1]$, $F(\cdot, t)$ is a homeomorphism from \mathcal{M} to $\hat{\mathcal{M}}$, where $F(\cdot, 0)$ is the identity of \mathcal{M} and $F(\mathcal{M}, 1) = \hat{\mathcal{M}}$. To establish that two surfaces are *geometrically close*, the distance between each point on one surface and its closest point on the other surface is required. Such a bound is usually obtained in the course of proving isotopy.

2.2 Diagrams and triangulations

The set of points defining a Voronoi diagram are traditionally referred to as *sites* or *seeds*. When approximating a manifold by a set of sample points of varying density, it is helpful to assign weights to the points reflective of their density. In particular, a point p_i with weight w_i , can be regarded as a ball B_i with center p_i and radius $r_i = \sqrt{w_i}$.

Recall that the *power distance* [37] between two points p_i, p_j with weights w_i, w_j is $\pi(p_i, p_j) = \mathbf{d}(p_i, p_j)^2 - w_i - w_j$. Unless otherwise noted, points are *unweighted*, having weight equal to zero. There is a natural geometric interpretation of the weight: all points q on the boundary of B_i have $\pi(p_i, q) = 0$, inside $\pi(p_i, q) < 0$ and outside $\pi(p_i, q) > 0$. Given a set of weighted points \mathcal{P} , this metric gives rise to a natural decomposition of \mathbb{R}^3 into the *power cells* $V_i = \{q \in \mathbb{R}^3 \mid \pi(p_i, q) \leq \pi(p_j, q) \forall p_j \in \mathcal{P}\}$. The *power diagram* $\text{wVor}(\mathcal{P})$ is the cell complex defined by collection of cells V_i for all $p_i \in \mathcal{P}$.

The nerve [37] of a collection \mathcal{C} of sets is defined as $\mathcal{N}(\mathcal{C}) = \{X \subseteq \mathcal{C} \mid \cap X \neq \emptyset\}$. Observe that $\mathcal{N}(\mathcal{C})$ is an abstract simplicial complex because $X \in \mathcal{N}(\mathcal{C})$ and $Y \subseteq X$ imply $Y \in \mathcal{N}(\mathcal{C})$. With that, we obtain the *weighted Delaunay triangulation*, or *regular triangulation*, as $\text{wDel}(\mathcal{P}) = \mathcal{N}(\text{wVor}(\mathcal{P}))$. Alternatively, $\text{wDel}(\mathcal{P})$ can be defined directly as follows. A subset $T \subset \mathbb{R}^d$, with $d \leq 3$ and $|T| \leq d + 1$ defines a d -simplex σ_T . Recall that the *orthocenter* [27] of σ_T , denoted by z_T , is the unique point $q \in \mathbb{R}^d$ such that $\pi(p_i, z_T) = \pi(p_j, z_T)$ for all $p_i, p_j \in T$; the *orthoradius* of σ_T is equal to $\pi(p, z_T)$ for any $p \in T$. The *Delaunay condition* defines $\text{wDel}(\mathcal{P})$ as the set of tetrahedra σ_T with an *empty orthosphere*, meaning $\pi(p_i, z_T) \leq \pi(p_j, z_T)$ for all $p_i \in T$ and $p_j \in \mathcal{P} \setminus T$, where $\text{wDel}(\mathcal{P})$ includes all faces of σ_T .

There is a natural duality between $\text{wDel}(\mathcal{P})$ and $\text{wVor}(\mathcal{P})$. For a tetrahedron σ_T , the definition of z_T immediately implies z_T is a *power vertex* in $\text{wVor}(\mathcal{P})$. Similarly, for each k -face σ_S of $\sigma_T \in \text{wDel}(\mathcal{P})$ with $S \subseteq T$ and $k + 1 = |S|$, there exists a dual $(3 - k)$ -face σ'_S in $\text{wVor}(\mathcal{P})$ realized as $\cap_{p \in S} V_p$. When \mathcal{P} is unweighted, the same definitions yield the standard (unweighted) Voronoi diagram $\text{Vor}(\mathcal{P})$ and its dual Delaunay triangulation $\text{Del}(\mathcal{P})$.

2.3 Unions of balls

Let \mathcal{B} denote the set of balls corresponding to a set of weighted points \mathcal{P} and define the *union of balls* \mathcal{U} as $\cup \mathcal{B}$. It is quite useful to capture the structure of \mathcal{U} using a combinatorial representation like a simplicial complex [36, 37]. Let f_i denote $V_i \cap \partial B_i$ and \mathcal{F} the collection of all such f_i . Observing that $V_i \cap B_j \subseteq V_i \cap B_i \forall B_i, B_j \in \mathcal{B}$, f_i is equivalently defined as the spherical part of $\partial(V_i \cap B_i)$. Consider also the decomposition of \mathcal{U} by the cells of $\text{wVor}(\mathcal{P})$ into $\mathcal{C}(\mathcal{B}) = \{V_i \cap B_i \mid B_i \in \mathcal{B}\}$. The *weighted α -complex* $\mathcal{W}(\mathcal{P})$ is defined as the *geometric realization* of $\mathcal{N}(\mathcal{C}(\mathcal{B}))$ [37], i.e., $\sigma_T \in \mathcal{W}$ if $\{V_i \cap B_i \mid p_i \in T\} \in \mathcal{N}(\mathcal{C}(\mathcal{B}))$. It is not hard to see that \mathcal{W} is a subcomplex of $\text{wDel}(\mathcal{P})$.

To see why \mathcal{W} is relevant, consider its *underlying space*; we create a collection containing the convex hull of each simplex in \mathcal{W} and define the *weighted α -shape* $\mathcal{J}(\mathcal{P})$ as the union of this collection. It turns out that the simplices $\sigma_T \in \mathcal{W}$ contained in $\partial \mathcal{J}$ are dual to the faces of $\partial \mathcal{U}$ defined as $\cap_{i \in T} f_i$. Every point $q \in \partial \mathcal{U}$ defined by $\cap_{i \in T_q} f_i$, for $T_q \in \mathcal{B}$ and $k + 1 = |T_q|$, witnesses the existence of σ_{T_q} in \mathcal{W} ; the k -simplex σ_{T_q} is said to be *exposed* and $\partial \mathcal{J}$ can be defined directly as the collection of all exposed simplices [36]. In particular, the *corners* of $\partial \mathcal{U}$ correspond to the facets of $\partial \mathcal{J}$. Moreover, \mathcal{J} is homotopy-equivalent to \mathcal{U} [37].

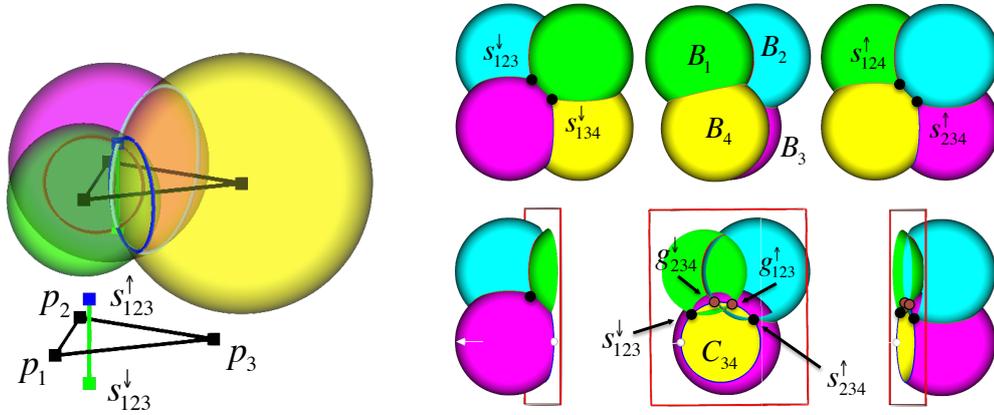
The union of balls defined using an ϵ -sampling guarantees the approximation of the manifold under suitable conditions on the sampling. Following earlier results on uniform sampling [46], an extension to non-uniform sampling establishes sampling conditions for the isotopic approximation of hypersurfaces and medial axis reconstruction [26].

3 Seed placement and surface reconstruction

We determine the location of Voronoi seeds using the union of balls \mathcal{U} . The correctness of our reconstruction depends crucially on how sample balls \mathcal{B} overlap. Assuming a certain structural property on \mathcal{U} , the surface reconstruction is embedded in the dual shape \mathcal{J} .

3.1 Seeds and guides

Central to the method and analysis are triplets of sample spheres, i.e., boundaries of sample balls, corresponding to a *guide triangle* in $\text{wDel}(\mathcal{P})$. The sample spheres associated with the vertices of a guide triangle intersect contributing a pair of *guide points*. The reconstruction consists of Voronoi facets, most of which are guide triangles.



(a) Overlapping balls and guide circles. (b) Pattern resulting in four half-covered seed pairs.

■ **Figure 2** (a) Guide triangle and its dual seed pair. (b) Cutaway view in the plane of circle C_{34} .

When a triplet of spheres $\partial B_i, \partial B_j, \partial B_k$ intersect at exactly two points, the intersection points are denoted by $g_{ijk}^\uparrow = \{g_{ijk}^\uparrow, g_{ijk}^\downarrow\}$ and called a pair of *guide points* or *guides*; see Figure 2a. The associated *guide triangle* t_{ijk} is dual to g_{ijk}^\uparrow . We use arrows to distinguish guides on different sides of the manifold with the *upper* guide g^\uparrow lying outside \mathcal{O} and the *lower* guide g^\downarrow lying inside. We refer to the edges of guide triangles as *guide edges* $e_{ij} = \overline{p_i p_j}$. A guide edge e_{ij} is associated with a dual *guide circle* $C_{ij} = \partial B_i \cap \partial B_j$, as in Figure 2a.

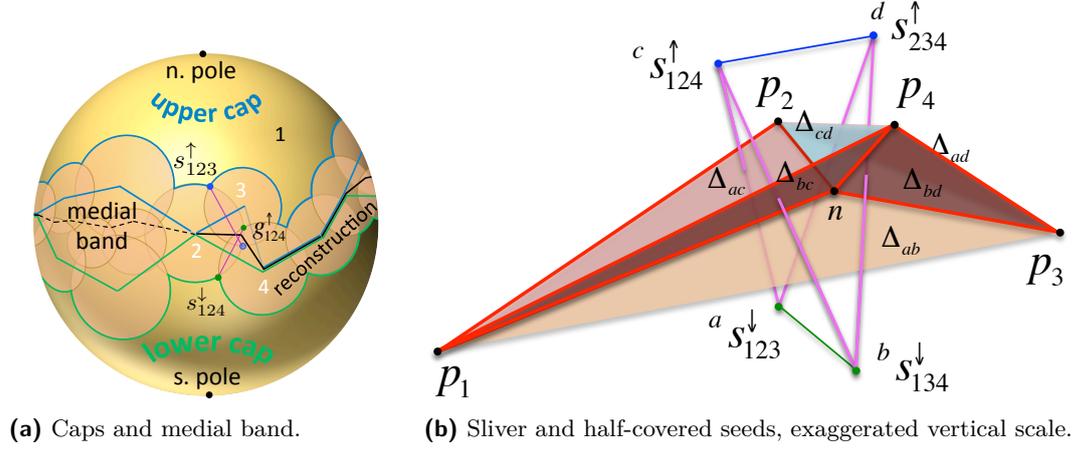
The Voronoi seeds in $\mathcal{S}^\uparrow \cup \mathcal{S}^\downarrow$ are chosen as the subset of guide points that lie on $\partial \mathcal{U}$. A guide point g which is not interior to any sample ball is *uncovered* and included as a *seed* s into \mathcal{S} ; covered guides are not. We denote *uncovered guides* by s and *covered guides* by g , whenever coverage is known and important. If only one guide point in a pair is covered, then we say the guide pair is *half-covered*. If both guides in a pair are covered, they are ignored. Let $\mathcal{S}_i = \mathcal{S} \cap \partial B_i$ denote the seeds on sample sphere ∂B_i .

As each guide triangle t_{ijk} is associated with at least one dual seed s_{ijk} , the seed witnesses its inclusion in \mathcal{W} and t_{ijk} is exposed. Hence, t_{ijk} belongs to $\partial \mathcal{J}$ as well. When such t_{ijk} is dual to a single seeds s_{ijk} it bounds the interior of \mathcal{J} , i.e., it is a face of a *regular component* of \mathcal{J} ; in the simplest and most common case, t_{ijk} is a facet of a tetrahedron as shown in Figure 3b. When t_{ijk} is dual to a pair of seeds $s_{ijk}^\uparrow, s_{ijk}^\downarrow$, it does not bound the interior of \mathcal{J} and is called a *singular face* of $\partial \mathcal{J}$. All singular faces of $\partial \mathcal{J}$ appear in the reconstructed surface.

3.2 Disk caps

We describe the structural property required on \mathcal{U} along with the consequences exploited by Vorocrust for surface reconstruction. This is partially motivated by the requirement that all sample points on the surface appear as vertices in the output Voronoi mesh.

We define the subset of ∂B_i inside other balls as the *medial band* and say it is *covered*. Let the caps K_i^\uparrow and K_i^\downarrow be the complement of the medial band in the interior and exterior of \mathcal{O} , respectively. Letting n_{p_i} be the normal line through p_i perpendicular to \mathcal{M} , the two intersection points $n_{p_i} \cap \partial B_i$ are called the *poles* of B_i . See Figure 3a.



(a) Caps and medial band.

(b) Sliver and half-covered seeds, exaggerated vertical scale.

■ **Figure 3** (a) Decomposing the sample sphere ∂B_1 . (b) Uncovered seeds and reconstruction facets. Let $\tau_p \in \mathcal{W}(\mathcal{P}) \subseteq \text{wDel}(\mathcal{P})$ and $\tau_s \in \text{Del}(\mathcal{S})$ denote the tetrahedra connecting the four samples and the four seeds shown, respectively. s_{123}^\downarrow and s_{134}^\downarrow are the uncovered lower guide seeds, with g_{123}^\uparrow and g_{134}^\uparrow covered. The uncovered upper guide seeds are s_{124}^\uparrow and s_{234}^\uparrow , with g_{124}^\downarrow and g_{234}^\downarrow covered. Δ_{ac} is the Voronoi facet dual to the Delaunay edge between $^a s_{123}^\downarrow$ and $^c s_{124}^\uparrow$, etc. Voronoi facets dual to magenta edges are in the reconstructed surface; those dual to green and blue edges are not. n is the circumcenter of τ_s and appears as a Voronoi vertex in $\text{Vor}(\mathcal{S})$ and a *Steiner vertex* in the surface reconstruction. In general, n is not the orthocenter of the sliver τ_p .

We require that \mathcal{U} satisfies the following structural property: each ∂B_i has *disk caps*, meaning the medial band is a *topological annulus* and the two caps contain the poles and are *topological disks*. In other words, each B_i contributes one connected component to each side of $\partial \mathcal{U}$. As shown in Figure 3a, all seeds in \mathcal{S}_i^\uparrow and \mathcal{S}_i^\downarrow lie on ∂K_i^\uparrow and ∂K_i^\downarrow , respectively, along the arcs where other sample balls intersect ∂B_i . In Section 4, we establish sufficient sampling conditions to ensure \mathcal{U} satisfies this property. In particular, we will show that both poles of each B_i lie on $\partial \mathcal{U}$.

The importance of disk caps is made clear by the following observation. The requirement that all sample points appear as Voronoi vertices in $\hat{\mathcal{M}}$ follows as a corollary.

► **Observation 1** (Three upper/lower seeds). *If ∂B_i has disk caps, then each of ∂K_i^\uparrow and ∂K_i^\downarrow has at least three seeds and the seeds on ∂B_i are not all coplanar.*

Proof. Every sphere $S_{j \neq i}$ covers strictly less than one hemisphere of ∂B_i because the poles are uncovered. Hence, each cap is composed of at least three arcs connecting at least three upper seeds $\mathcal{S}_i^\uparrow \subset \partial K_i^\uparrow$ and three lower seeds $\mathcal{S}_i^\downarrow \subset \partial K_i^\downarrow$. Further, any hemisphere through the poles contains at least one upper and one lower seed. It follows that the set of seeds $\mathcal{S}_i = \mathcal{S}_i^\uparrow \cup \mathcal{S}_i^\downarrow$ is not coplanar. ◀

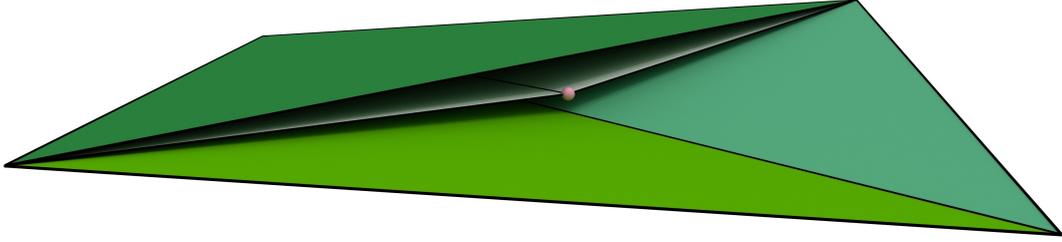
► **Corollary 2** (Sample reconstruction). *If ∂B_i has disk caps, then p_i is a vertex in $\hat{\mathcal{M}}$.*

Proof. By Observation 1, the sample is equidistant to at least four seeds which are not all coplanar. It follows that the sample appears as a vertex in the Voronoi diagram and not in the relative interior of a facet or an edge. Being a common vertex to at least one interior and one exterior Voronoi seed, VoroCrust retains this vertex in its output reconstruction. ◀

3.3 Sandwiching the reconstruction in the dual shape of \mathcal{U}

Triangulations of smooth surfaces embedded in \mathbb{R}^3 can have half-covered guides pairs, with one guide covered by the ball of a fourth sample not in the guide triangle dual to the guide pair. The tetrahedron formed by the three samples of the guide triangle plus the fourth covering sample is a *sliver*, i.e., the four samples lie almost uniformly around the equator of a sphere. In this case we do not reconstruct the guide triangle, and also do not reconstruct some guide edges. We show that the reconstructed surface $\hat{\mathcal{M}}$ lies entirely within the region of space bounded by guide triangles, i.e., the α -shape of \mathcal{P} , as stated in the following theorem.

► **Theorem 3.** *If all sample balls have disk caps, then $\hat{\mathcal{M}} \subseteq \mathcal{J}(\mathcal{P})$.*



■ **Figure 4** Cutaway view of a sliver tetrahedron $\tau_p \in \mathcal{W}(\mathcal{P}) \subseteq \text{wDel}(\mathcal{P})$, drawn to scale. Half-covered guides give rise to the Steiner vertex (pink), which results in a surface reconstruction using four facets (only two are shown) sandwiched within τ_p . In contrast, filtering $\text{wDel}(\mathcal{P})$ chooses two of the four facets of τ_p , either the bottom two, or the top two (only one is shown).

The simple case of a single isolated sliver tetrahedron is illustrated in Figures 3b, 4 and 2b. A sliver has a pair of lower guide triangles and a pair of upper guide triangles. For instance, t_{124} and t_{234} are the pair of upper triangles in Figure 3b. In such a tetrahedron, there is an edge between each pair of samples corresponding to a non-empty circle of intersection between sample balls, like the circles in Figure 2a. For this circle, the arcs covered by the two other sample balls of the sliver overlap, so each of these balls contributes exactly one uncovered seed, rather than two. In this way the upper guides for the upper triangles are uncovered, but their lower guides are covered; also only the lower guides of the lower triangles are uncovered. The proof of Theorem 3 follows by analyzing the Voronoi cells of the seed points located on the overlapping sample balls and is deferred to Appendix A. Alternatively, Theorem 3 can be seen as a consequence of Theorem 2 in [15].

4 Sampling conditions and approximation guarantees

We take as input a set of points \mathcal{P} sampled from the bounding surface \mathcal{M} such that \mathcal{P} is an ϵ -sample, with $\epsilon \leq 1/500$. We require that \mathcal{P} satisfies the following sparsity condition: for any two points $p_i, p_j \in \mathcal{P}$, $\text{lfs}(p_i) \geq \text{lfs}(p_j) \implies \mathbf{d}(p_i, p_j) \geq \sigma \epsilon \text{lfs}(p_j)$, with $\sigma \geq 3/4$.

Such a sampling \mathcal{P} can be obtained by known algorithms. Given a suitable representation of \mathcal{M} , the algorithm in [21] computes a loose ϵ' -sample E which is a $\epsilon'(1+8.5\epsilon')$ -sample. More specifically, whenever the algorithm inserts a new sample p into the set E , $\mathbf{d}(p, E) \geq \epsilon' \text{lfs}(p)$. To obtain E as an ϵ -sample, we set $\epsilon'(\epsilon) = (\sqrt{34\epsilon + 1} - 1)/17$. Observing that $3\epsilon/4 \leq \epsilon'(\epsilon)$ for $\epsilon \leq 1/500$, the returned ϵ -sample satisfies our required sparsity condition with $\sigma \geq 3/4$.

We start by adapting Theorem 6.2 and Lemma 6.4 from [26] to the setting just described. For $x \in \mathbb{R}^3 \setminus M$, let $\Gamma(x) = \mathbf{d}(x, \tilde{x})/\text{lfs}(\tilde{x})$, where \tilde{x} is the closest point to x on \mathcal{M} .

► **Corollary 4.** *For an ϵ -sample \mathcal{P} , with $\epsilon \leq 1/20$, the union of balls \mathcal{U} with $\delta = 2\epsilon$ satisfies:*

1. \mathcal{M} is a deformation retract of \mathcal{U} ,
2. $\partial\mathcal{U}$ contains two connected components, each isotopic to \mathcal{M} ,
3. $\Gamma^{-1}([0, a']) \subset U \subset \Gamma^{-1}([0, b'])$, where $a' = \epsilon - 2\epsilon^2$ and $b' \leq 2.5\epsilon$.

Proof. Theorem 6.2 from [26] is stated for balls with radii within $[a, b]$ times the lfs. We set $a = b = \delta$ and use $\epsilon \leq 1/20$ to simplify fractions. This yields the above expressions for $a' = (1 - \epsilon)\delta - \epsilon$ and $b' = \delta/(1 - 2\delta)$. The general condition requires $(1 - a')^2 + (b' - a' + \delta(1 + 2b' - a')/(1 - \delta))^2 < 1$, as we assume no noise. Plugging in the values of a' and b' , we verify that the inequality holds for the chosen range of ϵ . ◀

Furthermore, we require that each ball $B_i \in \mathcal{B}$ contributes one facet to each side of $\partial\mathcal{U}$. Our sampling conditions ensure that both poles are outside any ball $B_j \in \mathcal{B}$.

► **Lemma 5 (Disk caps).** All balls in \mathcal{B} have disk caps for $\epsilon \leq 0.066$, $\delta = 2\epsilon$ and $\sigma \geq 3/2$.

Proof. Fix a sample p_i and let x be one of the poles of B_i and $B_x = \mathbb{B}(c, \text{lfs}(p_i))$ the tangent ball at p_i with $x \in B_x$. Letting p_j be the closest sample to x in $P \setminus \{p_i\}$, we assume the worst case where $\text{lfs}(p_j) \geq \text{lfs}(p_i)$ and p_j lies on ∂B_x . To simplify the calculations, take $\text{lfs}(p_i) = 1$ and let ℓ denote $\mathbf{d}(p_i, p_j)$. As lfs is 1-Lipschitz, we get $\text{lfs}(p_j) \leq 1 + \ell$. By the law of cosines, $\mathbf{d}(p_j, x)^2 = \mathbf{d}(p_i, p_j)^2 + \mathbf{d}(p_i, x)^2 - 2\mathbf{d}(p_i, p_j)\mathbf{d}(p_i, x)\cos(\phi)$, where $\phi = \angle p_j p_i c$. Letting $\theta = \angle p_i c p_j$, observe that $\cos(\phi) = \sin(\theta/2) = \ell/2$. To enforce $x \notin B_j$, we require $\mathbf{d}(p_j, x) > \delta \text{lfs}(p_j)$, which is equivalent to $\ell^2 + \delta^2 - \delta\ell^2 > \delta^2(1 + \ell)^2$. Simplifying, we get $\ell > 2\delta^2/(1 - \delta - \delta^2)$ where sparsity guarantees $\ell > \sigma\epsilon$. Setting $\sigma\epsilon > 2\delta^2/(1 - \delta - \delta^2)$ we obtain $4\sigma\epsilon^2 + (8 + 2\sigma)\epsilon - \sigma < 0$, which requires $\epsilon < 0.066$ when $\sigma \geq 3/4$. ◀

Corollary 4 together with Lemma 5 imply that each ∂B_i is decomposed into a covered region $\partial B_i \cap \cup_{j \neq i} B_j$, the *medial band*, and two uncovered caps $\partial B_i \setminus \cup_{j \neq i} B_j$, each containing one pole. Recalling that seeds arise as pairs of intersection points between the boundaries of such balls, we show that seeds can be classified correctly as either inside or outside \mathcal{M} .

► **Corollary 6.** *If a seed pair lies on the same side of \mathcal{M} , then at least one seed is covered.*

Proof. Fix such a seed pair $\partial B_i \cap \partial B_j \cap \partial B_k$ and recall that $\mathcal{M} \cap \partial B_i$ is contained in the medial band on ∂B_i . Now, assume for contradiction that both seeds are uncovered and lie on the same side of \mathcal{M} . It follows that $B_j \cap B_k$ intersects B_i away from its medial band, a contradiction to Corollary 4. ◀

Corollary 4 guarantees that the medial band of B_i is a superset of $\Gamma^{-1}([0, a']) \cap \partial B_i$, which means that all seeds s_{ijk} are at least $a' \text{lfs}(\tilde{s}_{ijk})$ away from \mathcal{M} . It will be useful to bound the elevation of such seeds above T_{p_i} , the *tangent plane* to \mathcal{M} at p_i .

► **Lemma 7.** For a seed $s \in \partial B_i$, $\theta_s = \angle s p_i s' \geq 29.34^\circ$ and $\theta_s > \frac{1}{2} - 5\epsilon$, where s' is the projection of s on T_{p_i} , implying $\mathbf{d}(s, s') \geq h_s^\perp \delta \text{lfs}(p_i)$, with $h_s^\perp > 0.46$ and $h_s^\perp > \frac{1}{2} - 5\epsilon$.

Proof. Let $\text{lfs}(p_i) = 1$ and $B_s = \mathbb{B}(c, 1)$ be the tangent ball at p_i with $s \notin B_s$; see Figure 5a. Observe that $\mathbf{d}(s, \mathcal{M}) \leq \mathbf{d}(s, x)$, where $x = \overline{sc} \cap \partial B_s$. By the law of cosines, $\mathbf{d}(s, c)^2 = \mathbf{d}(p_i, c)^2 + \mathbf{d}(p_i, s)^2 - 2\mathbf{d}(p_i, c)\mathbf{d}(p_i, s)\cos(\pi/2 + \theta_s) = 1 + \delta^2 + 2\delta\sin(\theta_s)$. We may write² $\mathbf{d}(s, c) \leq 1 + \delta^2/2 + \delta\sin(\theta_s)$. It follows that $\mathbf{d}(s, x) \leq \delta^2/2 + \delta\sin(\theta_s)$. As lfs is 1-Lipschitz and $\mathbf{d}(p_i, x) \leq \delta$, we get $1 - \delta \leq \text{lfs}(x) \leq 1 + \delta$. There must exist a sample p_j such that $\mathbf{d}(x, p_j) \leq \epsilon \text{lfs}(x) \leq \epsilon(1 + \delta)$. Similarly, $\text{lfs}(p_j) \geq (1 - \epsilon(1 + \delta))(1 - \delta)$. By the triangle inequality, $\mathbf{d}(s, p_j) \leq \mathbf{d}(s, x) + \mathbf{d}(x, p_j) \leq \delta^2/2 + \delta\sin(\theta_s) + \epsilon(1 + \delta)$. Setting $\mathbf{d}(s, p_j) < \delta(1 - \delta)(1 - \epsilon(1 + \delta))$ implies $\mathbf{d}(s, p_j) < \delta \text{lfs}(p_j)$, which shows that for small values of θ_s , s cannot be a seed and $p_j \neq p_i$. Substituting $\delta = 2\epsilon$, we get $\theta_s \geq \sin^{-1}(2\epsilon^3 - 5\epsilon + 1/2) \geq 29.34^\circ$ and $\theta_s > 1/2 - 5\epsilon$. \blacktriangleleft

We make frequent use of the following bound on the distance between related samples.

► Claim 8. *If $B_i \cap B_j \neq \emptyset$, then $\mathbf{d}(p_i, p_j) \in [\kappa\epsilon, \kappa\delta] \cdot \text{lfs}(p_i)$, with $\kappa = 2/(1 - \delta)$ and $\kappa\epsilon = \sigma\epsilon/(1 + \sigma\epsilon)$.*

Proof. The upper bound comes from $\mathbf{d}(p_i, p_j) \leq r_i + r_j$ and $\text{lfs}(p_j) \leq \text{lfs}(p_i) + \mathbf{d}(p_i, p_j)$ by 1-Lipschitz, and the lower bound from $\text{lfs}(p_i) - \mathbf{d}(p_i, p_j) \leq \text{lfs}(p_j)$ and the sparsity. \blacktriangleleft

Bounding the circumradii is the culprit behind why we need such small values of ϵ .

► Lemma 9. *The circumradius of a guide triangle t_{ijk} is at most $\varrho_f \cdot \delta \text{lfs}(p_i)$, where $\varrho_f < 1.38$, and at most $\bar{\varrho}_f \cdot \mathbf{d}(p_i, p_j)$ where $\bar{\varrho}_f < 3.68$.*

Proof. Let p_i and p_j be the triangle vertices with the smallest and largest lfs values, respectively. From Claim 8, we get $\mathbf{d}(p_i, p_j) \leq \kappa\delta \text{lfs}(p_i)$. It follows that $\text{lfs}(p_j) \leq (1 + \kappa\delta)\text{lfs}(p_i)$. As t_{ijk} is a guide triangle, we know that it has a pair of intersection points $\partial B_i \cap \partial B_j \cap \partial B_k$. Clearly, the seed is no farther than $\delta \text{lfs}(p_j)$ from any vertex of t_{ijk} and the orthoradius of t_{ijk} cannot be bigger than this distance.

Recall that the weight w_i associated with p_i is $\delta^2 \text{lfs}(p_i)^2$. We shift the weights of all the vertices of t_{ijk} by the lowest weight w_i , which does not change the orthocenter. With that $w_j - w_i = \delta^2(\text{lfs}(p_j)^2 - \text{lfs}(p_i)^2) \leq \delta^2 \text{lfs}(p_i)^2((1 + \kappa\delta)^2 - 1) = \kappa\delta^3 \text{lfs}(p_i)^2(\kappa\delta + 2)$. On the other hand, sparsity ensures that the closest vertex in t_{ijk} to p_j is at distance at least $N(p_j) \geq \sigma\epsilon \text{lfs}(p_j) \geq \sigma\epsilon(1 + \kappa\delta)\text{lfs}(p_i)$. Ensuring $\alpha^2 \leq (w_j - w_i)/N(p_i)^2 \leq \kappa\delta^3(2 + \kappa\delta)/(\sigma^2\epsilon^2(1 - \kappa\delta)^2) \leq 1/4$ suffices to bound the circumradius of t_{ijk} by $c_{rad} = 1/\sqrt{1 - 4\alpha^2}$ times its orthoradius, as required by Claim 4 in [27]. Substituting $\delta = 2\epsilon$ and $\sigma \geq 3/4$ we get $\alpha^2 \leq 78.97\epsilon$, which corresponds to $c_{rad} < 1.37$. It follows that the circumradius is at most $c_{rad}\delta \text{lfs}(p_j) \leq c_{rad}(1 + \kappa\delta)\delta \text{lfs}(p_i) < 1.38\delta \text{lfs}(p_i)$.

For the second statement, observe that $\text{lfs}(p_i) \geq (1 - \kappa\delta)\text{lfs}(p_j)$ and the sparsity condition ensures that the shortest edge length is at least $\sigma\epsilon \text{lfs}(p_i) \geq \sigma\epsilon(1 - \kappa\delta)\text{lfs}(p_j)$. It follows that the circumradius is at most $\frac{\delta c_{rad}}{\sigma\epsilon(1 - \kappa\delta)} < 3.68$ times the length of any edge of t_{ijk} . \blacktriangleleft

Given the bound on the circumradii, we are able to bound the deviation of normals.

► Lemma 10. *If t_{ijk} is a guide triangle, then (1) $\angle_a(n_{p_i}, n_{p_j}) \leq \eta_s \delta < 0.47^\circ$, with $\eta_s < 2.03$, and (2) $\angle_a(n_t, n_{p_i}) \leq \eta_t \delta < 1.52^\circ$, with $\eta_t < 6.6$, where n_{p_i} is the line normal to \mathcal{M} at p_i and n_t is the normal to t_{ijk} . In particular, t_{ijk} makes an angle at most $\eta_t \delta$ with T_{p_i} .*

² Define $f(u, v) = \sqrt{1 + u^2 + 2uv} - (1 + u^2/2 + uv)$ and observe that $f(u, -u/2) = 0$ is the only critical value of $f(u, \cdot)$. As $\partial^2 f / \partial v^2 \leq 0$ for $(u, v) \in \mathbb{R} \times [-1, 1]$, we get that $f(u, v) \leq 0$ in this range.

Proof. Claim 8 implies $\mathbf{d}(p_i, p_j) \leq \kappa \delta \text{lfs}(p_i)$ and (1) follows from the Normal Variation Lemma [14] with $\rho = \kappa \delta < 1/3$ yielding $\angle_a(n_{p_i}, n_{p_j}) \leq \kappa \delta / (1 - \kappa \delta)$. Letting R_t denote the circumradius of t , Lemma 9 implies that the $R_t \leq \varrho_f \cdot \delta \text{lfs}(p_i) \leq \text{lfs}(p_i) / \sqrt{2}$ and the Triangle Normal Lemma [31] implies $\angle_a(n_{p^*}, n_t) < 4.57\delta < 1.05^\circ$, where p^* is the vertex of t subtending a maximal angle in t . Hence, $\angle_a(n_{p_i}, n_t) \leq \angle_a(n_{p_i}, n_{p^*}) + \angle_a(n_{p^*}, n_t)$. ◀

Towards establishing homeomorphism, the next lemma on the monotonicity of distance to the nearest seed is critical. First, we show that the nearest seeds to any surface point $x \in \mathcal{M}$ are generated by nearby samples.

► **Lemma 11.** The nearest seed to $x \in \mathcal{M}$ lies on some ∂B_i where $\mathbf{d}(x, p_i) \leq 5.03 \cdot \epsilon \text{lfs}(x)$. Consequently, $\mathbf{d}(x, p_i) \leq 5.08 \cdot \epsilon \text{lfs}(p_i)$.

Proof. In an ϵ -sampling, there exists a p_a such that $\mathbf{d}(x, p_a) \leq \epsilon \text{lfs}(x)$, where $\text{lfs}(p_a) \leq (1 + \epsilon) \text{lfs}(x)$. The sampling conditions also guarantee that there exists at least one seed s_a on ∂B_a . By the triangle inequality, we get that $\mathbf{d}(x, s_a) \leq \mathbf{d}(x, p_a) + \mathbf{d}(p_a, s_a) \leq \epsilon \text{lfs}(x) + \delta \text{lfs}(p_a) \leq \epsilon(1 + 2(1 + \epsilon)) \text{lfs}(x) = \epsilon(2\epsilon + 3) \text{lfs}(x)$.

We aim to bound ℓ to ensure $\forall p_i$ s.t. $\mathbf{d}(x, p_i) = \ell \cdot \epsilon \text{lfs}(x)$, the nearest seed to x cannot lie on B_i . Note that in this case, $(1 - \ell\epsilon) \text{lfs}(x) \leq \text{lfs}(p_i) \leq (1 + \ell\epsilon) \text{lfs}(x)$. Let s_i be any seed on B_i . It follows that $\mathbf{d}(x, s_i) \geq \mathbf{d}(x, p_i) - \mathbf{d}(p_i, s_i) \geq \ell \cdot \epsilon \text{lfs}(x) - 2\epsilon \text{lfs}(p_i) \geq \epsilon((1 - 2\epsilon)\ell - 2) \text{lfs}(x)$.

Setting $\epsilon((1 - 2\epsilon)\ell - 2) \text{lfs}(x) \geq \epsilon(2\epsilon + 3) \text{lfs}(x)$ suffices to ensure $\mathbf{d}(x, s_i) \geq \mathbf{d}(x, s_a)$, and we get $\ell \geq (2\epsilon + 5)/(1 - 2\epsilon)$. Conversely, if the nearest seed to x lies on B_i , it must be the case that $\mathbf{d}(x, p_i) \leq \ell \epsilon \text{lfs}(x)$. We verify that $\ell\epsilon = \epsilon(2\epsilon + 5)/(1 - 2\epsilon) < 1$ for any $\epsilon < 0.13$. It follows that $\mathbf{d}(x, p_j) \leq \ell\epsilon/(1 - \ell\epsilon) \text{lfs}(p_i)$. ◀

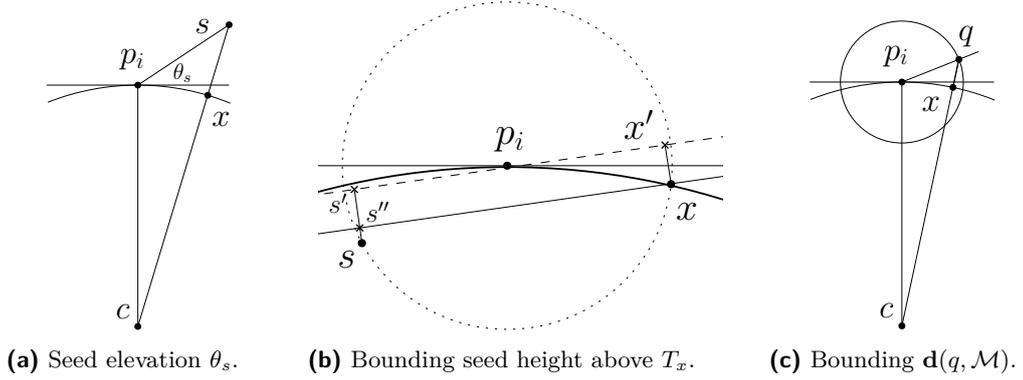
► **Lemma 12.** For any normal segment N_x issued from $x \in \mathcal{M}$, the distance to \mathcal{S}^\uparrow is either strictly increasing or strictly decreasing along $\Gamma^{-1}([0, 0.96\epsilon]) \cap N_x$. The same holds for \mathcal{S}^\downarrow .

Proof. Let n_x be the outward normal and T_x be the tangent plane to \mathcal{M} at x . By Lemma 11, the nearest seeds to x are generated by nearby samples. Fix one such nearby sample p_i . For all possible locations of a seed $s \in \mathcal{S}^\uparrow \cap \partial B_i$, we will show a sufficiently large lower bound on $\langle s - s'', n_x \rangle$, where s'' the projection of s onto T_x .

Take $\text{lfs}(p_i) = 1$ and let $B_s = \mathbb{B}(c, 1)$ be the tangent ball to \mathcal{M} at p_i with $s \in B_s$. Let A be the plane containing $\{p_i, s, x\}$. Assume in the worst case that $A \perp T_{p_i}$ and x is as far as possible from p_i on $\partial B_s \cap T_{p_i}$. By Lemma 11, $\mathbf{d}(p_i, x) \leq 5.08\epsilon$ and it follows that $\theta_x = \angle(n_x, n_{p_i}) \leq 5.08\epsilon/(1 - 5.08\epsilon) \leq 5.14\epsilon$. This means that T_x is confined within a $(\pi/2 - \theta_x)$ -cocone centered at x . Assume in the worst case that n_x is parallel to A and T_x is tilted to minimize $\mathbf{d}(s, s'')$; see Figure 5b.

Let T'_x be a translation of T_x such that $p_i \in T'_x$ and denote by x' and s' the projections of x and s , respectively, onto T'_x . Observe that T'_x makes an angle θ_x with T_{p_i} . From the isosceles triangle $\Delta p_i c x$, we get that $\theta'_x \leq 1/2 \angle p_i c x = \sin^{-1} 5.08\epsilon/2 \leq 2.54\epsilon$. Now, consider $\Delta p_i x x'$ and let $\phi = \angle x p_i x'$. We have that $\phi = \theta_x + \theta'_x \leq 2.54\epsilon + \delta/(1 - \delta) \leq 4.55\epsilon$. Hence, $\sin(\phi) \leq 4.55\epsilon$ and $\mathbf{d}(x, x') \leq 5.08\epsilon \sin(\phi) \leq 0.05\epsilon$. On the other hand, we have that $\angle s p_i s' = \psi \geq \theta_s - \theta_x$ and $\mathbf{d}(s, s') \geq \delta \sin \psi$, where $\theta_s \geq 1/2 - 5\epsilon$ by Lemma 7. Simplifying we get $\sin(\psi) \geq 1/2 - 10.08\epsilon$. The proof follows by evaluating $\mathbf{d}(s, s'') = \mathbf{d}(s, s') - \mathbf{d}(x, x')$. ◀

► **Theorem 13.** For every $x \in \mathcal{M}$ with closest point $q \in \hat{\mathcal{M}}$, and for every $q \in \hat{\mathcal{M}}$ with closest point $x \in \mathcal{M}$, we have $\|xq\| < h_t \cdot \epsilon^2 \text{lfs}(x)$, where $h_t < 30.52$. For $\epsilon < 1/500$, $h_t \cdot \epsilon^2 < 0.0002$. Moreover, the restriction of the mapping π to $\hat{\mathcal{M}}$ is a homeomorphism and $\hat{\mathcal{M}}$ and \mathcal{M} are ambient isotopic. Consequently, $\hat{\mathcal{O}}$ is ambient isotopic to \mathcal{O} as well.



■ **Figure 5** Constructions used for (a) Lemma 7, (b) Lemma 12 and (c) Theorem 13.

Proof. Fix a sample $p_i \in \mathcal{P}$ and a surface point $x \in \mathcal{M} \cap B_i$. We consider two cocones centered at x : a p -cocone contains all nearby surface points and a q -cocone contains all guide triangles incident at p_i . By Theorem 3, all reconstruction facets generated by seeds on B_i are sandwiched in the q -cocone.

Lemma 10 readily provides a bound on the q -cocone angle as $\gamma \leq \eta_t \delta$. In addition, since $\mathbf{d}(p_i, x) \leq \delta \text{lfs}(p_i)$, we can bound the p -cocone angle as $\theta \leq 2 \sin^{-1}(\delta/2)$ by Lemma 2 in [7]. We utilize a mixed pq -cocone with angle $\omega = \gamma/2 + \theta/2$, obtained by gluing the lower half of the p -cocone with the upper half of the q -cocone.

Let $q \in \hat{\mathcal{M}}$ and consider its closest point $x \in \mathcal{M}$. Again, fix $p_i \in \mathcal{P}$ such that $x \in B_i$; see Figure 5c. By sandwiching, we know that any ray through q intersects at least one guide triangle, in some point y , after passing through x . Let us assume the worst case that y lies on the upper boundary of the pq -cocone. Then, $\mathbf{d}(q, x) \leq \mathbf{d}(y, y') = h = \delta \sin(\omega) \text{lfs}(p_i)$, where y' is the closest point on the lower boundary of the pq -cocone point to q . We also have that, $\mathbf{d}(p_i, x) \leq \cos(\omega) \delta \text{lfs}(p_i) \leq \delta \text{lfs}(p_i)$, and since lfs is 1-Lipschitz, $\text{lfs}(p_i) \leq \text{lfs}(x)/(1 - \delta)$. Simplifying, we write $\mathbf{d}(q, x) < \delta \omega / (1 - \delta) \cdot \text{lfs}(x) < h_t \epsilon^2 \text{lfs}(x)$.

With $\mathbf{d}(q, x) \leq 0.55 \epsilon \text{lfs}(x)$, Lemma 12 shows that the normal line from any $p \in \mathcal{M}$ intersects $\hat{\mathcal{M}}$ exactly once close to the surface. It follows that for every point $x \in \mathcal{M}$ with closest point $q \in \hat{\mathcal{M}}$, we have $\mathbf{d}(x, q) \leq \mathbf{d}(x, q')$ where $q' \in \hat{\mathcal{M}}$ with x its closest point in \mathcal{M} . Hence, $\mathbf{d}(x, q) \leq h_t \epsilon^2 \text{lfs}(x)$ as well.

Building upon Lemma 12, as a point moves along the normal line at x , it is either the case that the distance to \mathcal{S}^\uparrow is decreasing while the distance to \mathcal{S}^\downarrow is increasing or the other way around. It follows that these two distances become equal at exactly one point on the Voronoi facet above or below x separating some seed $s^\uparrow \in \mathcal{S}^\uparrow$ from another seed $s^\downarrow \in \mathcal{S}^\downarrow$. Hence, the restriction of the mapping π to $\hat{\mathcal{M}}$ is a homeomorphism.

This shows that $\hat{\mathcal{M}}$ and \mathcal{M} homeomorphic. Recall that Corollary 4(3) implies \mathcal{U} is a *topological thickening* [25] of \mathcal{M} . In addition, Theorem 3 guarantees that $\hat{\mathcal{M}}$ is embedded in the interior of \mathcal{U} , such that it separates the two surfaces comprising $\partial \mathcal{U}$. These three properties imply $\hat{\mathcal{M}}$ is isotopic to \mathcal{M} in \mathcal{U} by virtue of Theorem 2.1 in [25]. Finally, as $\hat{\mathcal{M}}$ is the boundary of $\hat{\mathcal{O}}$ by definition, it follows that $\hat{\mathcal{O}}$ is isotopic to \mathcal{O} as well. ◀

5 Quality guarantees and output size

We establish a number of quality guarantees on the output mesh. The main result is an upper bound on the *fatness* of all Voronoi cell. See Appendix B for the proofs.

Recall that fatness is the outradius to inradius ratio, where the outradius is the radius of the smallest enclosing ball, and the inradius is the radius of the largest enclosed ball. The good quality of guide triangles allows us to bound the inradius of Voronoi cells.

► **Lemma 14.** For all guide triangles t_{ijk} : (1) Edge length ratios are bounded: $\ell_k/\ell_j \leq \kappa_\ell = \frac{2\delta}{1-\delta} \frac{\sigma\epsilon}{1+\sigma\epsilon}$. (2) Angles are bounded: $\sin(\theta_i) \geq 1/(2\bar{\rho}_f)$ implying $\theta_i \in (7.8^\circ, 165^\circ)$. (3) Altitudes are bounded: the altitude above e_{ij} is at least $\alpha_t|e_{ij}|$, where $\alpha_t = 1/4\bar{\rho}_f > 0.067$.

Observe that a guide triangle is contained in the Voronoi cell of its seed, even when one of the guides is covered. Hence, the tetrahedron formed by the triangle together with its seed lies inside the cell, and the cell inradius is at least the tetrahedron inradius.

► **Lemma 15.** For seeds $s_{ijk} \in \mathcal{S}^\uparrow \cup \mathcal{S}^\downarrow$, the inradius of the Voronoi cell is at least $\rho_v \delta \cdot \text{lfs}(p_i)$ with $\rho_v = \hat{h}_s / (1 + \frac{3}{2\sigma\bar{\rho}_f}) > 0.3$ and $\hat{h}_s \geq \frac{1}{2} - (5 + 2\eta_t)\epsilon$.

To get an upper bound on cell outradii, we must first generate seeds interior to \mathcal{O} . We consider a simple algorithm for generating \mathcal{S}^\downarrow based on a standard octree over \mathcal{O} . For sizing, we extend lfs beyond \mathcal{M} , using the point-wise maximal 1-Lipschitz extension $\text{lfs}(x) = \inf_{p \in \mathcal{M}} (\text{lfs}(p) + \mathbf{d}(x, p))$ [44]. An octree box \square is refined if the length of its diagonal is greater than $2\delta \cdot \text{lfs}(c)$, where c is the center of \square . After refinement terminates, we add an interior seed at the center of each empty box, and do nothing with boxes containing one or more guide seeds. Applying this scheme, we obtain the following.

► **Lemma 16.** The fatness of interior cells is at most $\frac{8\sqrt{3}(1+\delta)}{1-3\delta} < 14.1$.

► **Lemma 17.** The fatness of boundary cells is at most $\frac{4(1+\delta)}{(1-3\delta)(1-\delta)^2\rho_v} < 13.65$.

As the integral of lfs^{-3} is bounded over a single cell, it effectively counts the seeds.

► **Lemma 18.** $|\mathcal{S}^\downarrow| \leq 18\sqrt{3}/\pi \cdot \epsilon^{-3} \int_{\mathcal{O}} \text{lfs}^{-3}$.

6 Conclusions

We have analyzed an abstract version of the VoroCrust algorithm for volumes bounded by smooth surfaces. We established several guarantees on its output, provided the input samples satisfy certain conditions. In particular, the reconstruction is isotopic to the underlying surface and all 3D Voronoi cells have bounded fatness, i.e., outradius to inradius ratio. The triangular faces of the reconstruction have bounded angles and edge-length ratios, except perhaps in the presence of slivers. In a forthcoming paper [3], we describe the design and implementation of the complete VoroCrust algorithm, which generates conforming Voronoi meshes of realistic models, possibly containing sharp features, and produces samples that follow a natural sizing function and ensure output quality.

For future work, it would be interesting to ensure both guides are uncovered, or both covered. The significance would be that no tetrahedral slivers arise and no Steiner points are introduced. Further, the surface reconstruction would be composed entirely of guide triangles, so it would be easy to show that triangle normals converge to surface normals as sample density increases. Alternatively, where Steiner points are introduced on the surface, it would be helpful to have conditions that guaranteed the triangles containing Steiner points have good quality. In addition, the minimum edge length in a Voronoi cell can be a limiting factor in certain numerical solvers. Post-processing by mesh optimization techniques [5, 53] can help eliminate short Voronoi edges away from the surface. Finally, we expect that the abstract algorithm analyzed in this paper can be extended to higher dimensions.

References

- 1 A. Abdelkader, C. Bajaj, M. Ebeida, A. Mahmoud, S. Mitchell, J. Owens, and A. Rushdi. Sampling conditions for conforming Voronoi meshing by the VoroCrust algorithm. In *34th International Symposium on Computational Geometry (SoCG 2018)*, pages 1:1–1:16, 2018.
- 2 A. Abdelkader, C. Bajaj, M. Ebeida, A. Mahmoud, S. Mitchell, J. Owens, and A. Rushdi. VoroCrust Illustrated: Theory and Challenges (Multimedia Contribution). In *34th International Symposium on Computational Geometry (SoCG 2018)*, pages 77:1–77:4, 2018.
- 3 A. Abdelkader, C. Bajaj, M. Ebeida, A. Mahmoud, S. Mitchell, J. Owens, and A. Rushdi. VoroCrust: Voronoi meshing without clipping. *Manuscript*, In preparation.
- 4 A. Abdelkader, C. Bajaj, M. Ebeida, and S. Mitchell. A Seed Placement Strategy for Conforming Voronoi Meshing. In *Canadian Conference on Computational Geometry*, 2017.
- 5 A. Abdelkader, A. Mahmoud, A. Rushdi, S. Mitchell, J. Owens, and M. Ebeida. A constrained resampling strategy for mesh improvement. *Computer Graphics Forum*, 36(5):189–201, 2017.
- 6 O. Aichholzer, F. Aurenhammer, B. Kornberger, S. Plantinga, G. Rote, A. Sturm, and G. Vegter. Recovering structure from r-sampled objects. *Computer Graphics Forum*, 28(5):1349–1360, 2009.
- 7 N. Amenta and M. Bern. Surface reconstruction by Voronoi filtering. *Discrete & Computational Geometry*, 22(4):481–504, December 1999.
- 8 N. Amenta, M. Bern, and D. Eppstein. The crust and the β -skeleton: Combinatorial curve reconstruction. *Graphical models and image processing*, 60(2):125–135, 1998.
- 9 N. Amenta, M. Bern, and D. Eppstein. Optimal point placement for mesh smoothing. *Journal of Algorithms*, 30(2):302–322, 1999.
- 10 N. Amenta, M. Bern, and M. Kamvysselis. A new Voronoi-based surface reconstruction algorithm. In *Proceedings of the 25th Annual Conference on Computer Graphics and Interactive Techniques*, pages 415–421, 1998.
- 11 N. Amenta, S. Choi, T. Dey, and N. Leekha. A simple algorithm for homeomorphic surface reconstruction. In *16th Annual Symposium on Computational Geometry*, pages 213–222, 2000.
- 12 N. Amenta, S. Choi, and R.-K. Kolluri. The power crust. In *Proceedings of the Sixth ACM Symp. on Solid Modeling and Applications*, pages 249–266, 2001.
- 13 N. Amenta, S. Choi, and R.-K. Kolluri. The power crust, unions of balls, and the medial axis transform. *Computational Geometry*, 19(2):127–153, July 2001.
- 14 N. Amenta and T. Dey. Normal variation for adaptive feature size. *CoRR*, abs/1408.0314, 2014.
- 15 N. Amenta and R.-K. Kolluri. The medial axis of a union of balls. *Computational Geometry*, 20(1):25 – 37, 2001. Selected papers from the 12th Annual Canadian Conference.
- 16 N. Amenta, T. Peters, and A. Russell. Computational topology: ambient isotopic approximation of 2-manifolds. *Theoretical Computer Science*, 305(1):3 – 15, 2003. Topology in Computer Science.
- 17 L. Beirão da Veiga, F. Brezzi, L.-D. Marini, and A. Russo. The hitchhiker’s guide to the virtual element method. *Mathematical Models and Methods in Applied Sciences*, 24(08):1541–1573, 2014.
- 18 N. Bellomo, F. Brezzi, and G. Manzini. Recent techniques for PDE discretizations on polyhedral meshes. *Mathematical Models and Methods in Applied Sciences*, 24(08):1453–1455, 2014.
- 19 F. Bernardini, J. Mittleman, H. Rushmeier, C. Silva, and G. Taubin. The ball-pivoting algorithm for surface reconstruction. *IEEE Transactions on Visualization and Computer Graphics*, 5(4):349–359, Oct 1999.

- 20 J. Bishop. Simulating the pervasive fracture of materials and structures using randomly close packed Voronoi tessellations. *Computational Mechanics*, 44(4):455–471, September 2009.
- 21 J.-D. Boissonnat and S. Oudot. Provably good sampling and meshing of surfaces. *Graphical Models*, 67(5):405 – 451, 2005. Solid Modeling and Applications.
- 22 T. Brochu, C. Batty, and R. Bridson. Matching fluid simulation elements to surface geometry and topology. *ACM Trans. Graph.*, 29(4):47:1–47:9, 2010.
- 23 F. Cazals, T. Dreyfus, S. Sachdeva, and N. Shah. Greedy geometric algorithms for collection of balls, with applications to geometric approximation and molecular coarse-graining. *Computer Graphics Forum*, 33(6):1–17, 2014.
- 24 F. Cazals, H. Kanhere, and S. Lorient. Computing the volume of a union of balls: A certified algorithm. *ACM Trans. Math. Softw.*, 38(1):3:1–3:20, 2011.
- 25 F. Chazal and D. Cohen-Steiner. A condition for isotopic approximation. *Graphical Models*, 67(5):390 – 404, 2005. Solid Modeling and Applications.
- 26 F. Chazal and A. Lieutier. Smooth manifold reconstruction from noisy and non-uniform approximation with guarantees. *Computational Geometry*, 40(2):156 – 170, 2008.
- 27 S.-W. Cheng, T. Dey, H. Edelsbrunner, M. Facello, and S.-H. Teng. Silver exudation. *J. ACM*, 47(5):883–904, September 2000.
- 28 S.-W. Cheng, T. Dey, and J. Shewchuk. *Delaunay Mesh Generation*. CRC Press, 2012.
- 29 D. Cohen-Steiner, É.-C. de Verdière, and M. Yvinec. Conforming Delaunay triangulations in 3D. In *Proceedings of the Eighteenth Annual Symposium on Computational Geometry, SCG '02*, pages 199–208, 2002.
- 30 K. Sykes D. Letscher. On the stability of medial axis of a union of disks in the plane. In *28th Canadian Conference on Computational Geometry, CCCG 2016*, pages 29–33, 2016.
- 31 T. Dey. *Curve and Surface Reconstruction: Algorithms with Mathematical Analysis*. Cambridge University Press, New York, NY, USA, 2006.
- 32 T. Dey, K. Li, E. Ramos, and R. Wenger. Isotopic reconstruction of surfaces with boundaries. In *Computer Graphics Forum*, volume 28:5, pages 1371–1382, 2009.
- 33 T. Dey and L. Wang. Voronoi-based feature curves extraction for sampled singular surfaces. *Computers & Graphics*, 37(6):659–668, October 2013. Shape Modeling International (SMI) Conference 2013.
- 34 L. Duan and F. Lafarge. Image partitioning into convex polygons. In *2015 IEEE Conference on Computer Vision and Pattern Recognition (CVPR)*, pages 3119–3127, June 2015.
- 35 M. Ebeida and S. Mitchell. Uniform random Voronoi meshes. In *International Meshing Roundtable (IMR)*, pages 258–275, 2011.
- 36 H. Edelsbrunner. *Weighted alpha shapes*. University of Illinois at Urbana-Champaign, Department of Computer Science, 1992.
- 37 H. Edelsbrunner. The union of balls and its dual shape. *Discrete & Computational Geometry*, 13(3):415–440, Jun 1995.
- 38 H. Edelsbrunner and E.-P. Mücke. Simulation of simplicity: A technique to cope with degenerate cases in geometric algorithms. *ACM Trans. Graph.*, 9(1):66–104, January 1990.
- 39 R. Eymard, T. Gallouët, and R. Herbin. Finite volume methods. In *Techniques of Scientific Computing (Part 3)*, volume 7 of *Handbook of Numerical Analysis*, pages 713 – 1018. Elsevier, 2000.
- 40 Ø. Klemetsdal, R. Berge, K.-A. Lie, H. Nilsen, and O. Møyner. *SPE-182666-MS*, chapter Unstructured Gridding and Consistent Discretizations for Reservoirs with Faults and Complex Wells. Society of Petroleum Engineers, 2017.
- 41 D. Kuzmin. A guide to numerical methods for transport equations. *University Erlangen-Nuremberg*, 2010.

- 42 G. Manzini, A. Russo, and N. Sukumar. New perspectives on polygonal and polyhedral finite element methods. *Mathematical Models and Methods in Applied Sciences*, 24(08):1665–1699, 2014.
- 43 R. Merland, G. Caumon, B. Lévy, and P. Collon-Drouaillet. Voronoi grids conforming to 3D structural features. *Computational Geosciences*, 18(3):373–383, 2014.
- 44 G. Miller, D. Talmor, and S.-H. Teng. Data generation for geometric algorithms on non-uniform distributions. *International Journal of Computational Geometry and Applications*, 09(06):577–597, 1999.
- 45 M. Murphy, D. Mount, and C. Gable. A point-placement strategy for conforming Delaunay tetrahedralization. *International Journal of Computational Geometry & Applications*, 11(06):669–682, 2001.
- 46 P. Niyogi, S. Smale, and S. Weinberger. Finding the homology of submanifolds with high confidence from random samples. *Discrete & Computational Geometry*, 39(1-3):419–441, 2008.
- 47 A. Okabe, B. Boots, K. Sugihara, and S.-N. Chiu. *Spatial Tessellations: Concepts and Applications of Voronoi Diagrams*, volume 501. John Wiley & Sons, 2009.
- 48 M. Peric and S. Ferguson. The advantage of polyhedral meshes. *Dynamics - Issue 24*, page 4–5, Spring 2005. The customer magazine of the CD-adapco Group, currently maintained by Siemens at <http://siemens.com/mdx>. The issue is available at <http://mdx2.plm.automation.siemens.com/magazine/dynamics-24> (accessed March 29, 2018).
- 49 A. Rand and N. Walkington. Collars and intestines: Practical conforming Delaunay refinement. In *Proceedings of the 18th International Meshing Roundtable*, pages 481–497, 2009.
- 50 C. Rycroft. Voro++: A three-dimensional Voronoi cell library in C++. *Chaos*, 19(4):–, 2009. Software available online at <http://math.1bl.gov/voro++/>.
- 51 M. Sents and C. Gable. Coupling LaGrit Unstructured Mesh Generation and Model Setup with TOUGH2 Flow and Transport. *Comput. Geosci.*, 108(C):42–49, 2017.
- 52 H. Si, K. Gärtner, and J. Fuhrmann. Boundary conforming Delaunay mesh generation. *Computational Mathematics and Mathematical Physics*, 50(1):38–53, 2010.
- 53 D. Sieger, P. Alliez, and M. Botsch. Optimizing Voronoi diagrams for polygonal finite element computations. In *International Meshing Roundtable (IMR)*, pages 335–350. Springer, 2010.
- 54 P. Stelldinger. Topologically correct surface reconstruction using alpha shapes and relations to ball-pivoting. In *Pattern Recognition, 2008. ICPR 2008. 19th International Conference on*, pages 1–4. IEEE, 2008.
- 55 D.-M. Yan, W. Wang, B. Lévy, and Y. Liu. Efficient computation of clipped Voronoi diagram for mesh generation. *Computer-Aided Design*, 45(4):843–852, April 2013.
- 56 Dong-Ming Yan, Bruno Lévy, Yang Liu, Feng Sun, and Wenping Wang. Isotropic remeshing with fast and exact computation of restricted Voronoi diagram. *Computer Graphics Forum*, 28(5):1445–1454, July 2009.
- 57 M. Yip, J. Mohle, and J. Bolander. Automated modeling of three-dimensional structural components using irregular lattices. *Computer-Aided Civil and Infrastructure Engineering*, 20(6):393–407, 2005.

A Sandwich analysis

This appendix provides the proof of Theorem 3. In what follows, we assume all sample spheres have disk caps; see Section 3.2. By Corollary 2, each $p_i \in \mathcal{P}$ appears as a vertex in the surface reconstruction $\hat{\mathcal{M}}$. For an appropriate sampling, $\hat{\mathcal{M}}$ is watertight and the set of reconstruction facets containing p_i is a topological disk with p_i in its interior. We seek to characterize the angular orientation of the fan-facets around a sample p_i , namely that they lie sandwiched between upper and lower guide triangles. For orientation, it suffices to consider only the seeds on ∂B_i , because these are the seeds whose cells contain p_i . By suppressing other seeds, the fan-facets are all triangles extending radially from p_i to infinity. To argue about the actual reconstruction $\hat{\mathcal{M}}$, those other seeds are introduced later.

We make use of technical arguments in spherical geometry concerning the spherical arcs formed by the intersections of ∂B_i with extended reconstruction facets, and extended guide triangles. These arcs are great circle arcs between the two points where ∂B_i intersects the two rays from p_i that bound the extended triangle (facet).

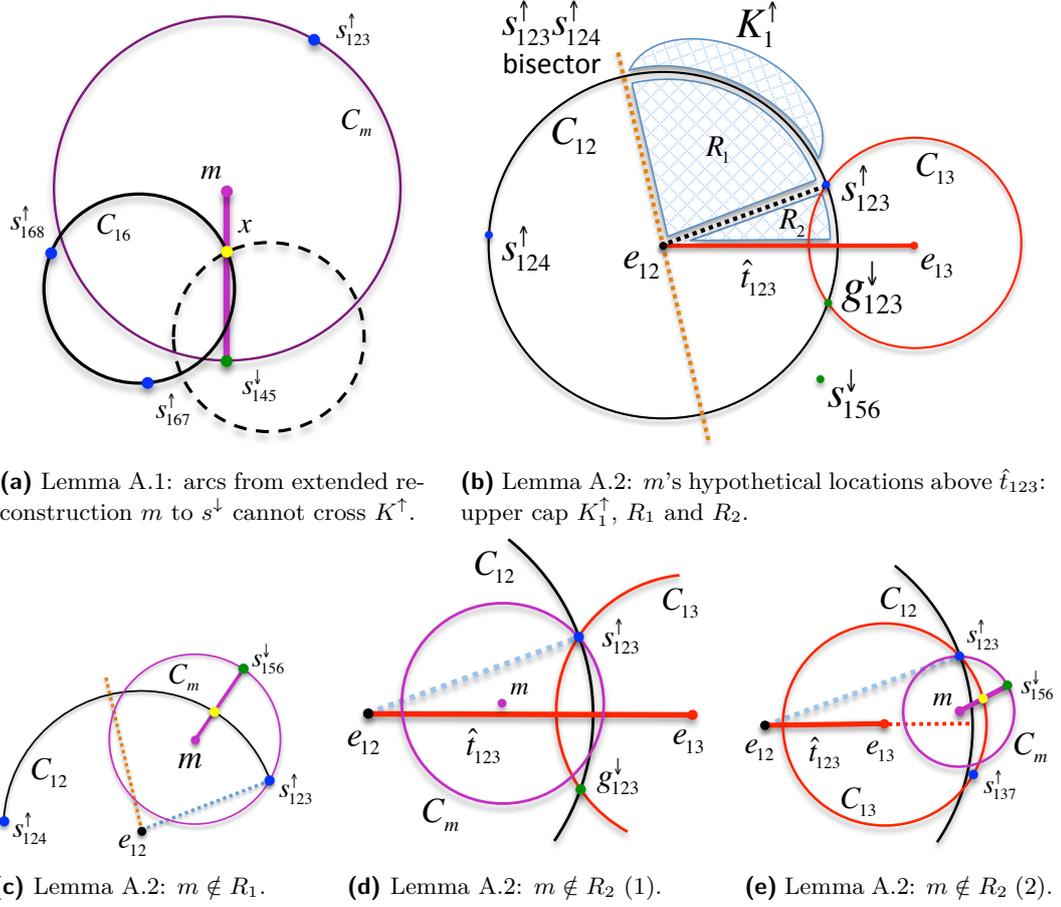
For a reconstruction facet $t \in \hat{\mathcal{M}}$ incident to $p_i \in \mathcal{P}$, let \hat{t} be its radial extension from p_i to infinity. Similarly, we denote by \hat{t}_{ijk} the radial extension of the guide triangle t_{ijk} . An *extended fan-facet point* is any point $m \in \hat{t} \cap \partial B_i$. We will show that such points m lie sandwiched between extended guide triangles incident to p_i . Since t is the intersection of the Voronoi cells of one upper and one lower seed, $m \in \hat{t}$ is the center of a *Voronoi ball* B_m having those two seeds on its boundary and no seed in its interior. The intersection of the empty ball B_m with ∂B_i is the *spherical disk* $D_m \subset \partial B_i$, and also has an interior \overline{D}_m empty of seeds. The following technical lemma is essential to the sandwiching arguments.

► **Lemma A.1** (Path m to s misses K). *If all sample balls have disk caps, then for any m on an extended reconstruction facet \hat{t} having $p_i \in \mathcal{P}$ as a vertex, the shorter great circle arc on ∂B_i from m to the lower (upper) seed of t does not pass through the upper (lower) cap.*

Proof. We use numerals to index the unusually large number of samples and seeds involved in the configurations at hand. Without loss of generality let $i = 1$ and fix one such facet t . We consider up to seven spheres $\partial B_{\{2, \dots, 8\}}$, not necessarily distinct, intersecting ∂B_1 .

Denote the upper and lower seeds associated with t by s_{123}^\uparrow and s_{145}^\downarrow . We show that $\overline{ms_{145}^\downarrow}$ does not pass through K_1^\uparrow ; the case of $\overline{ms_{123}^\uparrow}$ and K_1^\downarrow is similar. The proof follows by examining great circle arcs on ∂B_1 restricted within D_m . By the definition of disk caps, K_1^\uparrow is a topological disk and ∂K_1^\uparrow forms a closed path. In addition ∂K_1^\uparrow is disjoint from ∂K_1^\downarrow , as K_1^\uparrow and K_1^\downarrow are separated by a medial band. Being a lower seed, s_{145}^\downarrow lies on ∂K_1^\downarrow .

Suppose for contradiction that $\overline{ms_{145}^\downarrow}$ goes through K_1^\uparrow and let $x \in \overline{ms_{145}^\downarrow} \subset \partial D_m$ be the last point where $\overline{ms_{145}^\downarrow}$ crosses ∂K_1^\uparrow in the direction from the interior of K_1^\uparrow to its exterior. We have that x lies on an uncovered arc $\overline{s_{167}^\uparrow s_{168}^\uparrow} \subset \partial K_1^\uparrow$ on the guide circle C_{16} with seeds s_{167}^\uparrow and s_{168}^\uparrow as endpoints. Figure 6a depicts a hypothetical path $\overline{ms_{145}^\downarrow}$. Since D_m is empty, s_{167}^\uparrow and s_{168}^\uparrow both lie outside D_m . In addition, s_{145}^\downarrow cannot lie inside C_{16} , as s_{145}^\downarrow is an uncovered seed and C_{16} bounds the region on ∂B_1 which is covered by B_6 ; this impossible configuration is illustrated by the dashed circle in Figure 6a. It would follow that $\overline{ms_{145}^\downarrow}$ must cross $\overline{s_{167}^\uparrow s_{168}^\uparrow}$ (a second time to reach s_{145}^\downarrow , a contradiction to x being the last point where $\overline{ms_{145}^\downarrow}$ intersects ∂K_1^\uparrow . (Special cases include K_1^\uparrow and K_1^\downarrow meeting at a point, or x being a point of tangency between C_{16} and $\overline{ms_{145}^\downarrow}$.) ◀



■ **Figure 6** Fan-facet sandwich lemmas.

A reconstructed fan-facet is *sandwiched* at p_i if the intersection of its radial extension with ∂B_i lies in the region bounded by the extension of guide triangles incident to p_i .

► **Lemma A.2** (Sandwich fan-facets). *If all sample balls have disk caps, then all facets in $\hat{\mathcal{M}}$ with a sample $p_i \in \mathcal{P}$ as a vertex are sandwiched between guide triangles.*

Proof. Take $i = 1$ and consider the spherical arcs on ∂B_1 which arise from intersecting ∂B_1 with extended guide triangles having p_1 as a vertex. We partition the subset of ∂B_1 above these arcs, and show that no point m of an extended reconstruction facet can lie in any of these partitions; the argument for the subset of ∂B_1 below these arcs is analogous. By Lemma A.1, such m may only lie on the medial band.

Consider the spherical arcs bounding the upper cap K_1^\uparrow . As in the proof of the prior lemma, we label up to seven other spheres $\partial B_{\{2, \dots, 7\}}$ that intersect ∂B_1 . Each consecutive pair of such arcs intersect at an upper seed s_*^\uparrow and a lower guide g_*^\downarrow which may or may not be covered. Fix such an arc $\widehat{s_{123}^\uparrow s_{124}^\uparrow} \subset \partial K_1^\uparrow$ on the guide circle C_{12} . With a slight abuse of notation, we denote by e_{1j} the intersection of the ray $\overrightarrow{p_1 p_j}$ with ∂B_1 . Let D_{1j} denote the spherical disk bounded by C_{1j} and \bar{D}_{1j} its interior. We split D_{12} in half by the great circle arc through e_{12} bisecting $\widehat{s_{123}^\uparrow s_{124}^\uparrow}$. Without loss of generality, we consider the half disk containing s_{123}^\uparrow and argue that no point m on an extended reconstruction facet can lie above the extended upper guide triangle \hat{t}_{123} ; the case of lower guide triangles is similar.

We partition the half disk above the extended upper guide triangle \hat{t}_{123} into R_1 , above $\widehat{e_{12}s_{123}^\uparrow}$ (and R_2 , below $\widehat{e_{12}s_{123}^\uparrow}$ (and above the extended upper guide facet \hat{t}_{123} . See Figure 6b. Suppose there is some point m on an extended reconstruction facet whose two closest and equidistant seeds are s_{123}^\uparrow and some lower seed which we denote by s_{156}^\downarrow . We show that m cannot lie above \hat{t}_{123} , neither in R_1 nor R_2 . Observe that if $m = e_{12}$, then it clearly does not lie above \hat{t}_{123} . Hence, we assume $m \neq e_{12}$.

Suppose $m \in R_1$ and consider the spherical disk D_m on ∂B_1 , with center m and s_{123}^\uparrow on its boundary. As no seed is closer to m than s_{123}^\uparrow , D_m is empty of other seeds and cannot contain s_{124}^\uparrow in its interior. Hence, the only portion of D_m outside D_{12} is bounded by $\widehat{s_{123}^\uparrow s_{124}^\uparrow}$ (see Figure 6c). In addition, s_{156}^\downarrow cannot lie in $\overline{D_{12}}$. It follows that the shorter great circle arc from m to $s_{156}^\downarrow \in \partial D_m \setminus \overline{D_{12}}$ must cross $\widehat{s_{123}^\uparrow s_{124}^\uparrow} \subset \partial K_1^\uparrow$ into the upper cap K_1^\uparrow ; a contradiction to Lemma A.1. We conclude that $m \notin R_1$.

Suppose $m \in R_2$ and consider the spherical disk D_m centered at e_{13} with s_{123}^\uparrow on its boundary and no seed in its interior. We have two subcases: either $D_m \subset D_{12} \cup D_{13}$ or not. Suppose $D_m \subset D_{12} \cup D_{13}$; see Figure 6d. It would follow that the only possible position for s_{156}^\downarrow is g_{123}^\downarrow . But then, $m \in \widehat{e_{12}e_{13}} \subset \hat{t}_{123}$; a contradiction to $m \in R_2$. In the second subcase, let s_{137}^\uparrow be the other seed on C_{13} such that $\widehat{s_{123}^\uparrow s_{137}^\uparrow} \subset \partial K_1^\uparrow$; see Figure 6e. As in the case when m was assumed to lie in R_1 , the only portion of D_m outside $D_{12} \cup D_{13}$ is bounded by $\widehat{s_{123}^\uparrow s_{137}^\uparrow}$ (where s_{156}^\downarrow cannot lie in $\overline{D_{12}} \cup \overline{D_{13}}$). It follows that the shorter great circle arc from m to $s_{156}^\downarrow \in \partial D_m \setminus (\overline{D_{12}} \cup \overline{D_{13}})$, must cross $\widehat{s_{123}^\uparrow s_{137}^\uparrow} \subset \partial K_1^\uparrow$ into K_1^\uparrow ; a contradiction to Lemma A.1. As neither subcase can be true, we conclude $m \notin R_2$. ◀

► **Theorem 3.** *If all sample balls have disk caps, then $\hat{\mathcal{M}} \subseteq \mathcal{J}(\mathcal{P})$.*

Proof. Let V_{ijk} denote the Voronoi cell of seed s_{ijk} and $\mathcal{P}_{ijk} = \{p_i, p_j, p_k\}$. By Corollary 2, all three samples \mathcal{P}_{ijk} appear as Voronoi vertices of V_{ijk} and are retained as vertices in $\hat{\mathcal{M}}$. For an appropriate sampling, $\hat{\mathcal{M}}$ is watertight and each sample is surrounded by a fan of reconstruction facets. By Lemma A.2, the reconstruction facets of each such fan are sandwiched between the guide triangles incident on the corresponding sample. In particular, the subset of such reconstruction facets on the boundary of V_{ijk} and incident on samples in \mathcal{P}_{ijk} , denoted by \mathcal{F}_{ijk} , are sandwiched in this manner. We consider upper seeds; the case of lower seeds is similar. As V_{ijk}^\uparrow is convex, the planes containing each of the facets in $\mathcal{F}_{ijk}^\uparrow$ are tangents to V_{ijk}^\uparrow . By Lemma A.2, the planes containing the subset of guide triangles incident on each of the samples in \mathcal{P}_{ijk} , and bounding the reconstruction facets in $\mathcal{F}_{ijk}^\uparrow$ from below, are also tangents to V_{ijk}^\uparrow . Hence, all the Voronoi cells of upper seeds lie above the guide triangles incident on samples in \mathcal{P}_{ijk} . Similarly, all the Voronoi cells of lower seeds lie below the upper guide triangles incident on the associated triplet of sample points. As $\hat{\mathcal{M}}$ is the intersection of the Voronoi cells of upper and lower seeds, it follows that $\hat{\mathcal{M}}$ is sandwiched between upper and lower guide triangles, which constitute $\partial \mathcal{J}(\mathcal{P})$. ◀

B Quality bounds

This appendix provides the details of the proposed octree refinement for seeding the interior of \mathcal{O} along with the proofs of the statements in Section 5. Namely, we bound the fatness of all cells in the Voronoi mesh $\hat{\mathcal{O}}$ as well as a number of quality measures of guide triangles which constitute the majority of facets in the surface reconstruction $\hat{\mathcal{M}}$.

B.1 At the boundary

Recall that the seeds in $\mathcal{S}^\uparrow \cup \mathcal{S}^\downarrow \subset \partial\mathcal{U}$ are used to define the surface reconstruction $\hat{\mathcal{M}}$ as the set of Voronoi facets common to one seed from \mathcal{S}^\uparrow and another \mathcal{S}^\downarrow . Guide triangles contribute the majority of facets in $\hat{\mathcal{M}}$. Thanks to the sparsity of the sampling, we can derive several quality measures for guide triangles.

► **Lemma 14.** For all guide triangles t_{ijk} : (1) Edge length ratios are bounded: $\ell_k/\ell_j \leq \kappa_\ell = \frac{2\delta}{1-\delta} \frac{\sigma\epsilon}{1+\sigma\epsilon}$. (2) Angles are bounded: $\sin(\theta_i) \geq 1/(2\bar{\rho}_f)$ implying $\theta_i \in (7.8^\circ, 165^\circ)$. (3) Altitudes are bounded: the altitude above e_{ij} is at least $\alpha_t|e_{ij}|$, where $\alpha_t = 1/4\bar{\rho}_f > 0.067$.

Proof. The edge ratio bound is basically a restatement of Claim 8. Denote by ℓ_i and θ_i the length of the triangle edge opposite to p_i and the angle at vertex p_i , respectively. Claim 8 implies $\ell_k \leq \kappa \delta \text{lfs}(p_i)$ and the sparsity condition guarantees that $\ell_j \geq \kappa_\epsilon \text{lfs}(p_i)$, hence $\ell_i/\ell_k \leq \kappa_\ell$ for any pair of edges.

Let R_{ijk} denote t_{ijk} 's circumradius. By the Central Angle Theorem, $\sin(\theta_i) = \ell_i/(2R_{ijk})$, and we also have $R_{ijk} \leq \bar{\rho}_f \ell_i$ from Lemma 9. Hence $\sin(\theta_i) \geq 1/(2\bar{\rho}_f)$.

For the worst case altitude, let the edge under consideration be the longest, $e = \ell_k$, and the second longest edge ℓ_j , so $\ell_j \geq \ell_k/2$. The altitude is then $\sin(\theta_i)\ell_j \geq \ell_k/(4\bar{\rho}_f)$. ◀

Before proceeding to study the decomposition of the interior of \mathcal{O} , we establish a bound on the inradius of Voronoi cells with seeds in $\mathcal{S}^\uparrow \cup \mathcal{S}^\downarrow$.

► **Corollary B.1.** If t_{ijk} is a guide triangle with associated seed s , then $\angle sp_i s'' \geq \frac{1}{2} - \eta'_t \epsilon$, where s'' is the projection of s on the plane of t_{ijk} and $\eta'_t \leq 5 + 2\eta_t < 18.18$, implying $\mathbf{d}(s, s'') \geq \hat{h}_s \delta \text{lfs}(p_i)$ with $\hat{h}_s \geq \frac{1}{2} - \eta'_t \epsilon$.

Proof. Combining Lemma 7 with Lemma 10, we have $\angle sp_i s'' \geq \angle sp_i s' - \angle_a(n_{t_{ijk}}, n_{p_i})$. ◀

► **Lemma 15.** For seeds $s_{ijk} \in \mathcal{S}^\uparrow \cup \mathcal{S}^\downarrow$, the inradius of the Voronoi cell is at least $\rho_v \delta \cdot \text{lfs}(p_i)$ with $\rho_v = \hat{h}_s / (1 + \frac{3}{2\sigma\bar{\rho}_f}) > 0.3$ and $\hat{h}_s \geq \frac{1}{2} - (5 + 2\eta_t)\epsilon$.

Proof. Fix a seed s_{ijk} and observe that $\{p_i, p_j, p_k\}$ belong to its Voronoi cell. By the convexity of the cell, it follows that the tetrahedron $T = p_i p_j p_k s_{ijk}$ is contained inside it. We establish a lower bound on the cell's inradius by bounding the inradius of T . Let f_i denote the facet of T opposite to p_i and f_0 denote t_{ijk} . Let A_i be the area of f_i .

Observe that the incenter c_T divides T into four smaller tetrahedra, one for each facet of T , where the distance from c_T to the plane of each facet is equal to the inradius r . This allows us to express the volume of T as $V = \sum_{i=0}^3 r A_i / 3$. Hence, we have that $r = 3V / \sum_i A_i$. We may also express V as $H A_0 / 3$, where H is the distance from s_{ijk} to the plane of t_{ijk} . Substituting for V and factoring out A_0 , we get that $r = H / (1 + \sum_{i>0}^3 A_i / A_0)$.

Triangle area ratios A_i/A_0 are bounded because triangle angles are bounded, and edge lengths are bounded by the local feature size. Consider the edge $e_i = \overline{p_j p_k}$ common to f_i and t_{ijk} and let α_s and α_p be the altitudes of e_i in f_i and t_{ijk} , respectively. It follows that $A_i/A_0 = \alpha_s/\alpha_p$. Note α_s is less than the length of the longest edge of f_i .

Hence, assuming that $\text{lfs}(p_j) \geq \text{lfs}(p_k)$, we get that $\alpha_s \leq \delta \text{lfs}(p_j)$. On the other hand, the sparsity condition guarantees $\mathbf{d}(p_j, p_k) \geq \sigma \epsilon \text{lfs}(p_j)$, allowing us to rewrite $\alpha_s \leq \frac{\delta}{\sigma \epsilon} \mathbf{d}(p_j, p_k)$. From Lemma 14, we have that $\alpha_p \geq \mathbf{d}(p_j, p_k)/(4\bar{\rho}_f)$. It follows that $A_i/A_0 \leq \frac{1}{2\sigma\bar{\rho}_f}$. The proof follows by invoking Corollary B.1 to bound $H \geq \hat{h}_s \delta \text{lfs}(p_i)$. ◀

B.2 Octree refinement and outradii

Towards bounding the aspect ratio of Voronoi cells, we begin by proving some basic properties of the octree described in Section 5. Given an octree box \square_i , denote by c_i its center and r_i its radius (half its diagonal length). Assume that the input P has been scaled and shifted to fit into the unit cube $[0, 1]^3$. Starting with the unit cube as the box associated with the root node of the octree, the refinement process terminates with $r_i \leq \delta \text{lfs}(c_i)$ for all leaf boxes \square_i . Note that refinement depends only on lfs and is independent of the number of points in P , and the distances between them. We establish the following Lipschitz-like properties for the size of leaf boxes.

► **Claim B.2.** *If \square_i is a leaf box, then $\frac{\delta}{2+\delta} \text{lfs}(c_i) \leq r_i \leq \delta \text{lfs}(c_i)$.*

Proof. By definition the leaf box was not split, so $r_i \leq \delta \text{lfs}(c_i)$. Letting \square_j be the parent of \square_i , it is clear that \square_j had to be split. Hence, $r_j = 2r_i > \delta \text{lfs}(c_j)$. By Lipschitzness, $\text{lfs}(c_i) \leq \text{lfs}(c_j) + r_i \leq r_i(1 + 2/\delta)$. ◀

► **Claim B.3.** *For any $p \in \square_i$, where \square_i is a leaf box, $\frac{\delta}{2(1+\delta)} \leq r_i \leq \frac{\delta}{1-\delta} \text{lfs}(p)$.*

Proof. Observe that $\mathbf{d}(p, c_i) \leq r_i$, so $\text{lfs}(p)$ is bounded in terms of $\text{lfs}(c_i)$. Conveniently, Claim B.2 bounds $\text{lfs}(c_i)$ in terms of r_i . To get the lower bound, we write $\text{lfs}(p) \leq \text{lfs}(c_i) + r_i \leq (\frac{2+\delta}{\delta} + 1)r_i$. For the upper bound, we write $\text{lfs}(p) \geq \text{lfs}(c_i) - r_i \geq (1/\delta - 1)r_i$. ◀

► **Lemma B.4.** *If \square_i and \square_j are two leaf boxes sharing a corner, then $r_i/r_j \in [1/2, 2]$.*

Proof. Assume that $r_j \leq r_i$. From Claim B.2 we have $r_i \leq \delta \text{lfs}(c_i)$ and $r_j \geq \frac{\delta}{2+\delta} \text{lfs}(c_j)$. Together with lfs being 1-Lipschitz, this gives $r_j \geq \frac{\delta}{2+\delta} (\text{lfs}(c_i) - (r_i + r_j)) \geq \frac{\delta}{2+\delta} (r_i/\delta - r_i - r_j)$. Simplifying, we get $r_j \geq \frac{r_i}{2} \frac{1-\delta}{1+\delta}$. For $\delta < 1/3$, we obtain $r_j > r_i/4$. As the ratio of box radii is a power of two, $r_j \in \{r_i/2, r_i\}$. ◀

These properties of the octree may be used to bound the outradius of Voronoi cells.

► **Lemma B.5.** *The Voronoi cell of $s \in \mathcal{S}$ has outradius at most $\frac{2\delta}{1-3\delta} \text{lfs}(s) \leq \frac{4(1+\delta)}{1-3\delta} r_i$, where \square_i is the leaf box containing s .*

Proof. Let v be a vertex on the Voronoi cell of s . The octree construction guarantees $v \in \square_j$, for some leaf box \square_j . Claim B.3 gives $r_j \leq \delta/(1-\delta) \text{lfs}(v)$. Fixing some $s' \in \square_j \cap \mathcal{S} \neq \emptyset$, it follows that $\mathbf{d}(v, s) \leq \mathbf{d}(v, s') \leq 2r_j$. Hence, $\text{lfs}(v) \geq \frac{1-\delta}{2\delta} \mathbf{d}(v, s)$. By Lipschitzness, $\text{lfs}(s) \geq \text{lfs}(v) - \mathbf{d}(v, s) \geq \frac{1-3\delta}{2\delta} \mathbf{d}(v, s)$. As $s \in \square_i$, Claim B.3 gives $\text{lfs}(s) \leq \frac{2(1+\delta)}{\delta} r_i$. It follows that $\mathbf{d}(v, s) \leq \frac{2\delta}{1-3\delta} \text{lfs}(s) \leq \frac{4(1+\delta)}{1-3\delta} r_i$. ◀

B.3 Aspect ratio and size bounds

Any Voronoi vertex is in some box, and every box has at least one seed. This provides an upper bound on the distance between a Voronoi vertex and its closest seed, and an upper bound on the cell outradius, for both interior and guide seeds. Interior seeds are at the center of a box containing no other seeds, so interior cell inradius is at least a constant factor times r . Combining the outradius and inradius bounds provides the following results.

► **Lemma 16.** *The fatness of interior cells is at most $\frac{8\sqrt{3}(1+\delta)}{1-3\delta} < 14.1$.*

Proof. Let $s \in \mathcal{S}$ be an interior seed and recall that s was inserted at the center of some empty leaf box \square_i . By construction, s is the only seed in \square_i . It follows that the inradius of $\text{Vor}(s)$ is at least $\frac{1}{2\sqrt{3}} r_i$, which is half the distance from c_i to any of its sides. The proof follows from the bound on the outradius in terms of r_i as provided by Lemma B.5. ◀

► **Lemma 17.** The fatness of boundary cells is at most $\frac{4(1+\delta)}{(1-3\delta)(1-\delta)^2\varrho_v} < 13.65$.

Proof. Let $s \equiv s_{ijk} \in \mathcal{S}$ be a boundary seed and recall the lower bound of $\varrho_v \epsilon \text{lfs}(p_i)$ on the inradius of $\text{Vor}(s)$ from Lemma 15. By Lipschitzness, we may express this as $\varrho_v \delta(1-\delta) \text{lfs}(s)$. On the other hand, an upper bound of $\frac{4(1+\delta)}{1-3\delta} r_a$ on the circumradius of $\text{Vor}(s)$ is provided by Lemma B.5, where \square_a is the leaf box containing s . From Claim B.3, we have that $r_a \leq \frac{\delta}{1-\delta} \text{lfs}(s)$. With both bounds expressed in terms of $\text{lfs}(s)$, we evaluate their ratio. ◀

► **Lemma 18.** $|\mathcal{S}^{\downarrow\downarrow}| \leq 18\sqrt{3}/\pi \cdot \epsilon^{-3} \int_{\mathcal{O}} \text{lfs}^{-3}$.

Proof. Let $\mathcal{I} = \mathcal{S}^{\downarrow} \cup \mathcal{S}^{\downarrow\downarrow}$ and $V(s)$ denote the Voronoi cell of seed s . Since the Voronoi cells of interior seeds in \mathcal{I} partition the volume \mathcal{O} , $\int_{\mathcal{O}} \text{lfs}^{-3} = \sum_{s \in \mathcal{I}} \int_{V(s)} \text{lfs}^{-3}$. Bounded outradii and inradii will bound each integral by as follows.

Fix a seed s and let R_s and r_s be the circumradius and inradius of $V(s)$, respectively. From Lemma B.5, we have $R \leq \frac{2\delta}{1-3\delta} \text{lfs}(s)$. By Lipschitzness, for any $x \in \text{Vor}(s)$, $\text{lfs}(x) \geq \frac{1-5\delta}{1-3\delta} \text{lfs}(s)$. Thus, $\int_{\text{Vor}(s)} \text{lfs}^{-3} \geq f_1(\delta) \text{lfs}^{-3}(s) \text{vol}(\text{Vor}(s))$, where $f_1(\delta) = \left(\frac{1-3\delta}{1-5\delta}\right)^3$.

If $s \in \mathcal{S}^{\downarrow\downarrow}$, Claim B.3 yields $r_s \geq \frac{\delta}{4\sqrt{3}(1+\delta)} \text{lfs}(s)$. Hence, $\text{vol}(\text{Vor}(s)) \geq f_2(\delta) \text{lfs}^3(s)$, where $f_2(\delta) = \frac{4\pi}{3} \left(\frac{\delta}{4\sqrt{3}(1+\delta)}\right)^3$. If $s = s_{ijk} \in \mathcal{S}^{\downarrow}$, Lemma 15 gives $r_s \geq \varrho_v \epsilon \text{lfs}(p_i)$. Recalling $\mathbf{d}(p_i, s_{ijk}) = \delta \text{lfs}(p_i)$ and the extension of lfs to the interior of \mathcal{O} , we get $\text{lfs}(s) \leq (1+\delta) \text{lfs}(p_i)$. It follows that $r_s \geq \frac{\varrho_v \delta}{1+\delta} \text{lfs}(s)$ and $\text{vol}(\text{Vor}(s)) \geq f_3(\delta) \text{lfs}^3(s)$, where $f_3(\delta) = \frac{4\pi}{3} \left(\frac{\varrho_v \delta}{1+\delta}\right)^3$.

Letting $f_4(\delta) = f_1(\delta) \cdot \min(f_2(\delta), f_3(\delta))$, we established that $\text{vol}(\text{Vor}(s)) \geq f_4(\delta) \text{lfs}^3(s)$. Plugging that into the above bound, we get $\int_{\text{Vor}(s)} \text{lfs}^{-3} \geq f_4(\delta)$. Hence, $\int_{\mathcal{O}} \text{lfs}^{-3} \geq f_4(\delta) |\mathcal{I}| \geq f_4(\delta) |\mathcal{S}^{\downarrow\downarrow}|$. The proof follows by observing that $\frac{1}{f_4(\delta)} \leq 18\sqrt{3}/\pi \cdot \epsilon^{-3}$. ◀