A characterization of the quadrilateral meshes of a surface which admit a compatible hexahedral mesh of the enclosed volume

or

Which quadrilateral meshes admit a hexahedral mesh?

Scott Mitchell

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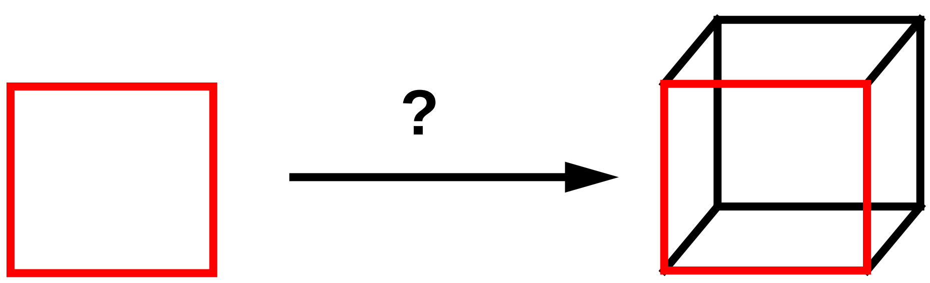

Problem

- start with a quadrilateral surface mesh
- fill the interior of volume with hexahedra
  - hexes must exactly match surface mesh

Folklore says hard/impossible to do.

We say, can almost always be done!

- ignore geometry/shape
- but, pathological connectivity ruled out

Problem

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Important industrial problem

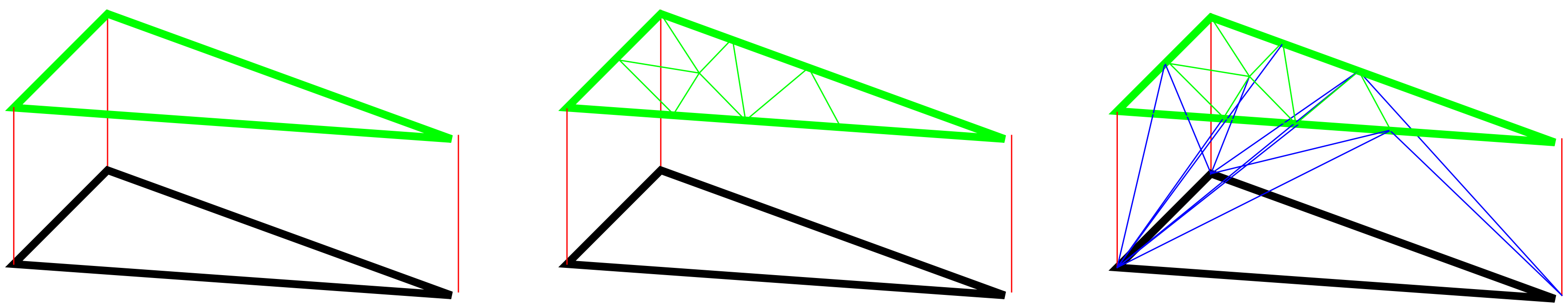
- new objects added touching meshed scene
- meshing large scenes/complicated parts
- meshing touching objects independently
  - e.g. partition domain into some mappable pieces, some not
  - e.g. parallel meshing of subdomains

No algorithms known

Previous work

For triangular/tetrahedral (Bern 1993)

- buffer surface mesh with one layer
- mesh interior
- match up (*hard part)



*No dice so far for quad/hex

Proof background

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<u>THE WAY</u> to think of quad/hex meshes:

Spatial Twist Continuum (STC)

-> higher level interpretation of duality

Dual of quad mesh:

- arrangement of curves, and vertices of intersection

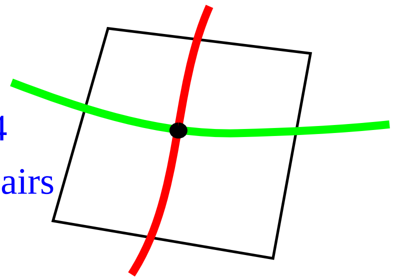
Dual of hex mesh:

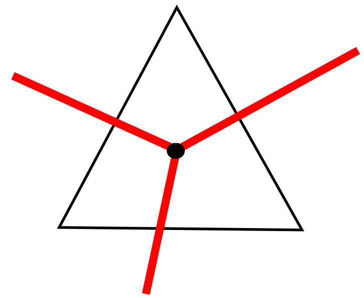
- arrangement of surfaces, and curves and vertices of intersection

Captures difficult global connectivity constraints inherent to hexahedral meshing

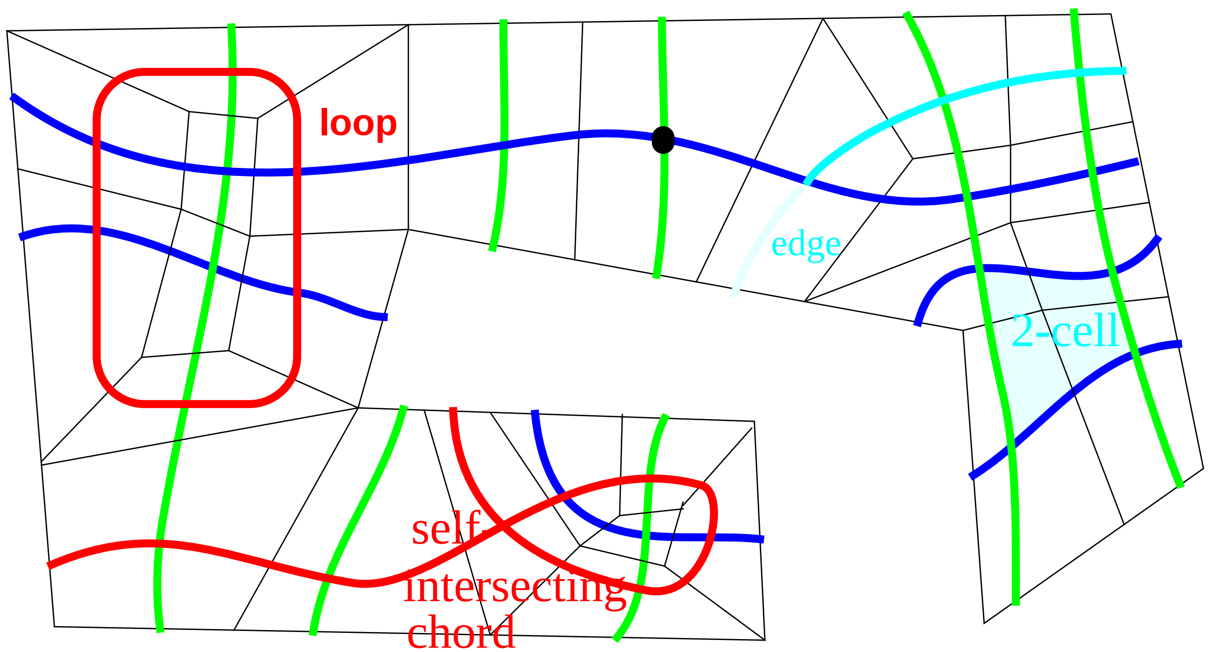
2d STC

Quad:
dual vertices degree 4
edges form 2 opposite pairs





Triangle:
dual vertices degree 3
no “opposite” edge.



A quadrilateral mesh and the corresponding STC.

3d STC

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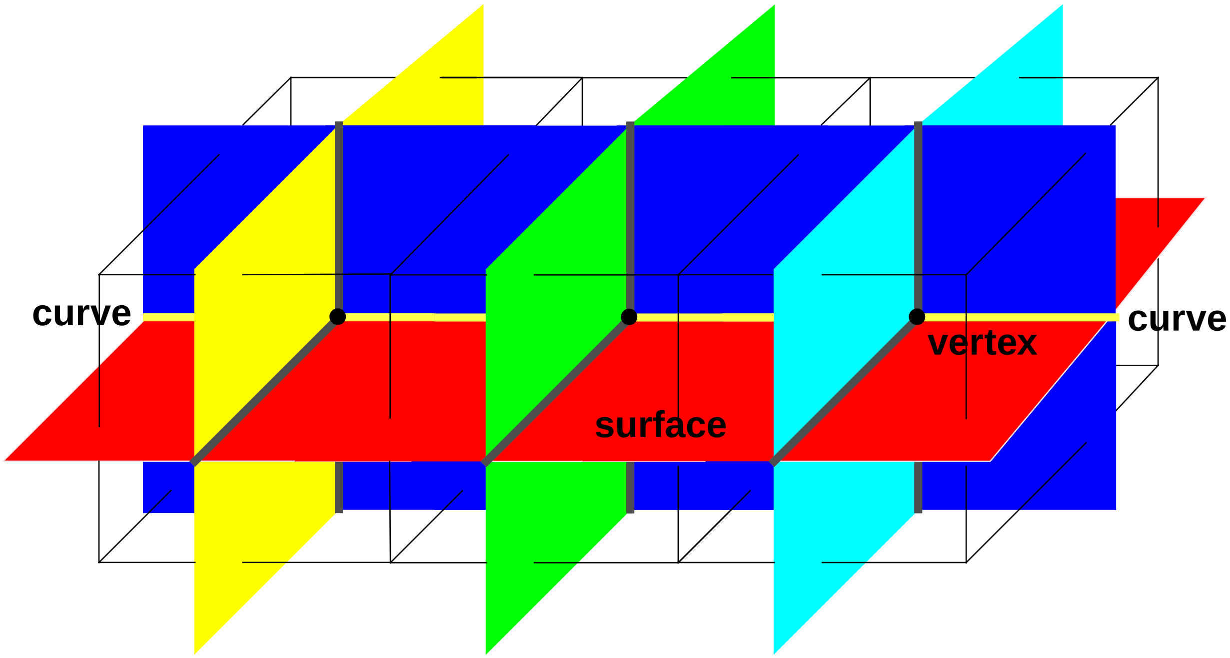


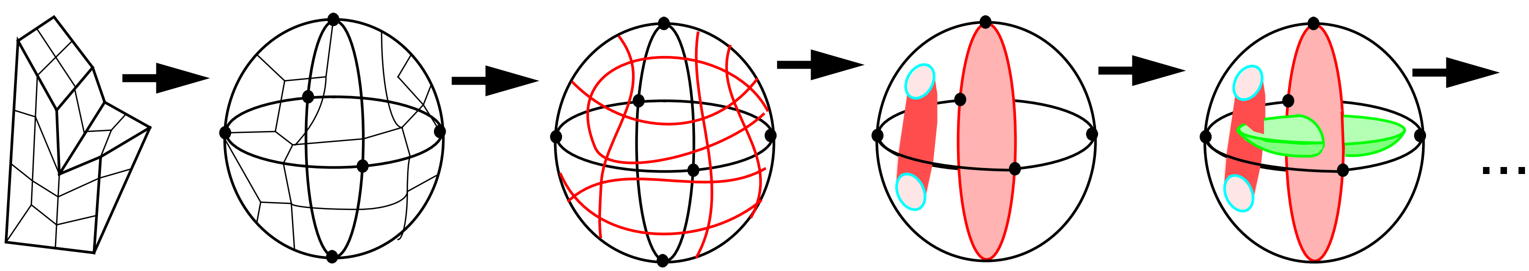
Surface represents a layer of hexes

Curve of intersection represents a line of hexes

Vertex of intersection represents a single hex

Outline similar to Whisker Weaving

- Map surface/mesh to a sphere (smooth)
- Form STC loops - smooth closed curves
- Extend loops into STC surfaces
- Fix surface arrangement to avoid pathologies
- Dualize back to form hexes
- Inverse map back to original object



Necessary condition

Necessary: surface mesh must have even #quads

Pf: Every hex mesh has an even #quads on surface:

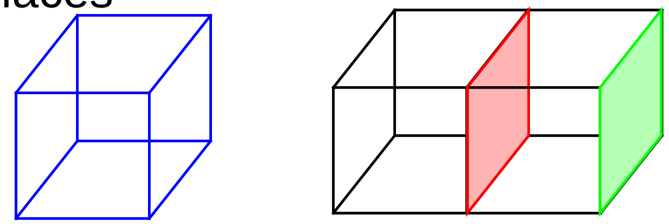
h = #hexes

f = total #faces

b = #surface faces

$$6h = 2f - b$$

count faces



Sufficient, too!

Initial STC

Def. Non-degenerate arrangement of surfaces

- nowhere tangent
- at most three surfaces meet at any point
- surfaces *regular* (C^1 continuous, parameterization)

Theorem [Smale, 1957]. On a sphere:

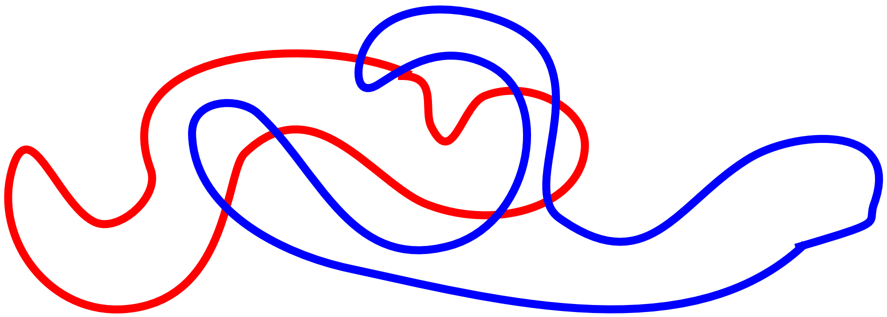
- loop with an even #self-intersections has a regular map to a circle
- two loops with odd #self-intersections have a regular map between them

Can use map to sweep out a surface

Maps to surfaces

Even #non-self intersections

- two closed curves on sphere intersect even times



Even #quads <=> even #self-intersections

- every pair of odd loops -> one surface
- every even loop -> one surface

Every even quad mesh (of sphere) admits an STC

Not good enough yet!

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STC may not dualize to a reasonable mesh, need

Table 1:

	Hex	STC
A	edge has two distinct nodes	2-cell in two distinct 3-cells
B	facet in a higher dimensional facet	facet contains lower dim facet
C	face in two distinct hexes	edge contains two distinct centroids
D	surface facets distinct from one another	only one surface cell in an internal cell
E	face has four distinct edges	edge contained in four distinct 2-cells
F	hex has six faces, ordered, etc.	centroid has six edges, ordered, etc.
G	two nodes are in only one common edge	two 3-cells share at most one 2-cell
H	two faces share one edge	skip- fix in pillowing talk later

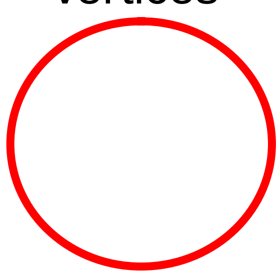
- distinct -> STC can’t be too coarse
- mostly satisfied automatically, except distinct

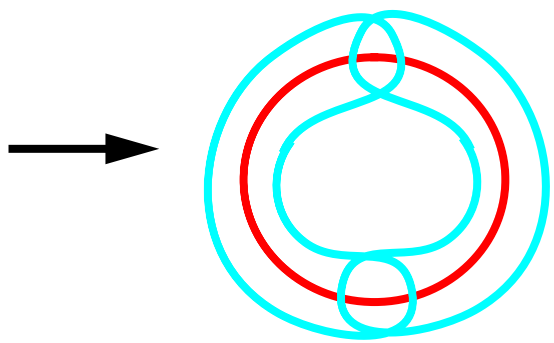
Idea: Not distinct? Put a sphere around it!

Fix-ups may guide fix-ups in algorithms

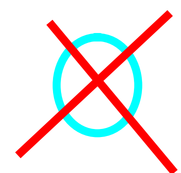
- A. 2-cell in two 3-cells
  - surfaces orientable, divides sphere, sides distinct
- B. facet contains one lower dim. facet

curve w/out vertices



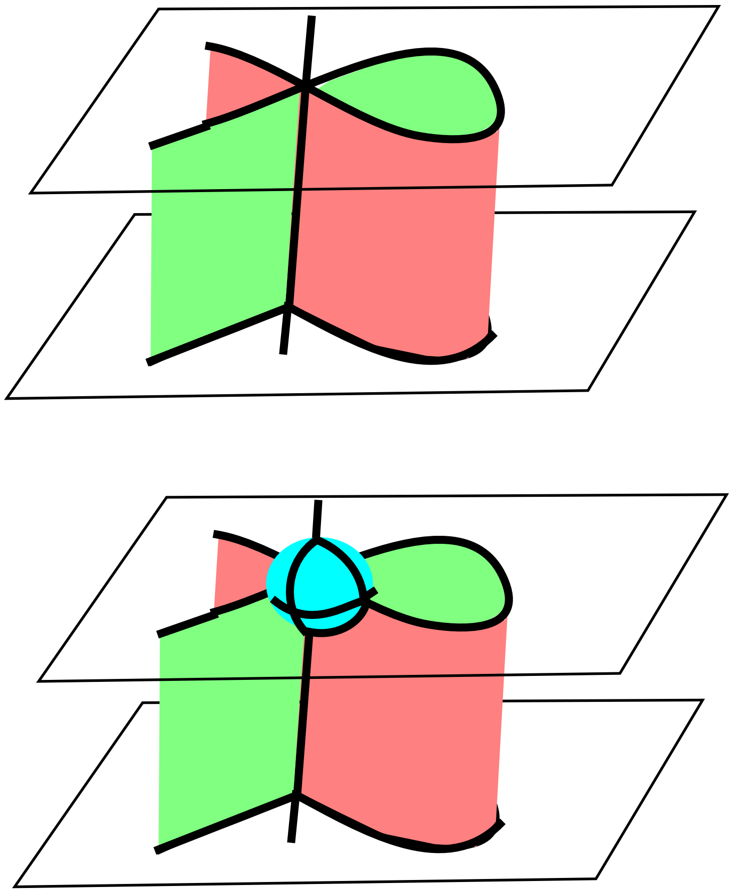


surround curve with two tubes

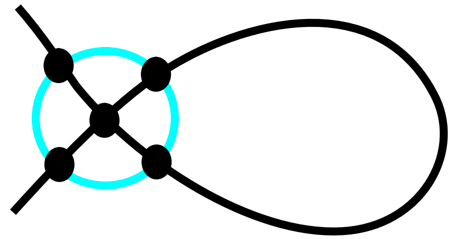


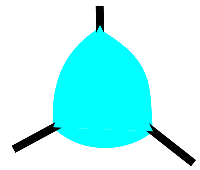
Fix STC

- C. edge has two distinct centroids



put a ball around the non-distinct vertex

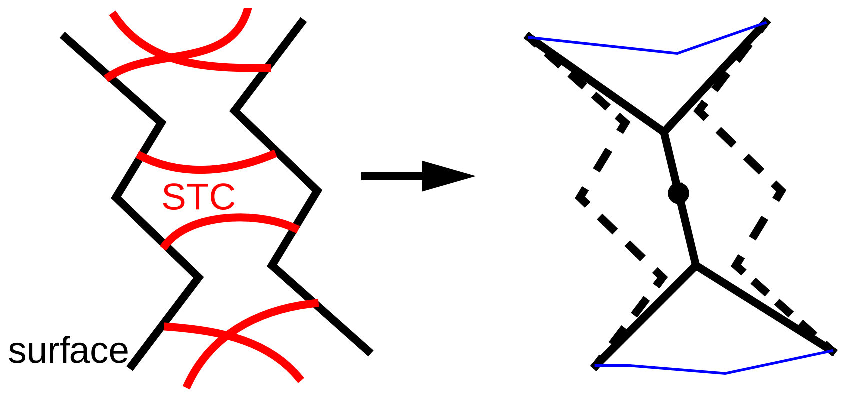




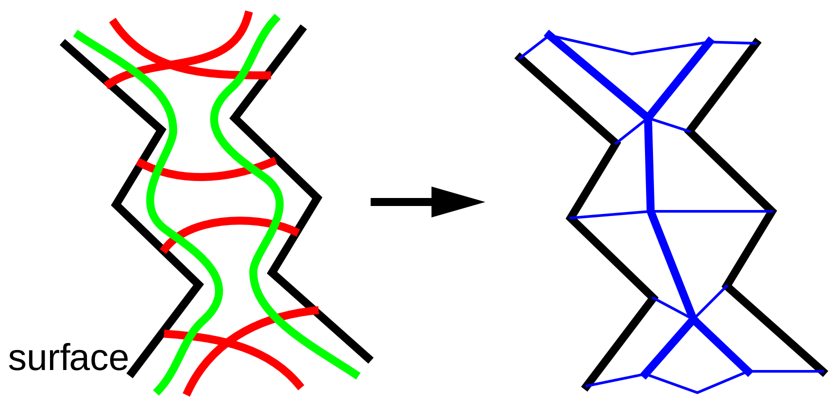
Fix STC

Dual must keep surface entities distinct

- D. surface cell in at most one internal cell



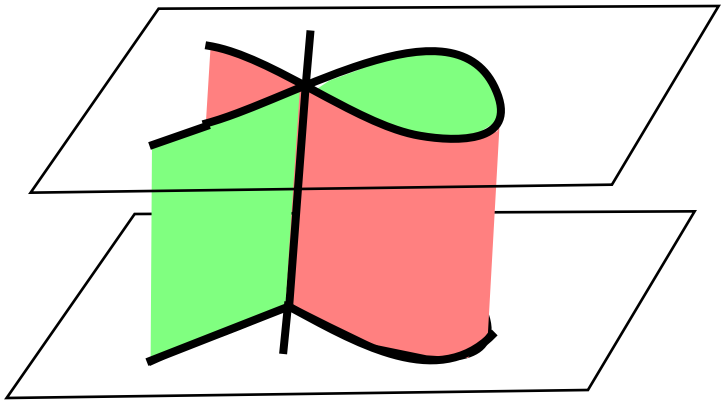
- add a ball close to surface

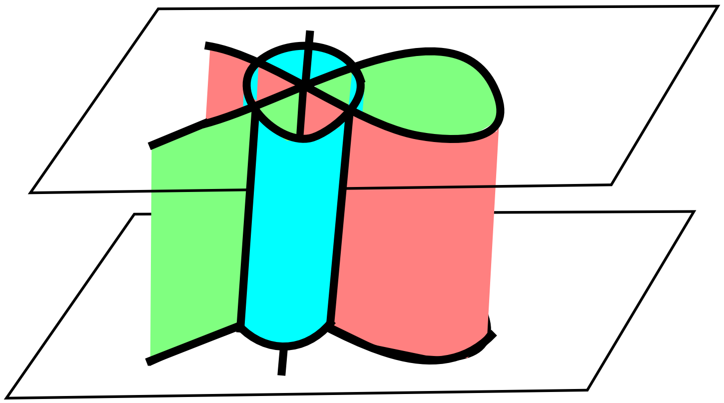


boundary layer
preserves surface

Fix up

- E. Edge in four distinct 2-cells
  - fix like C.

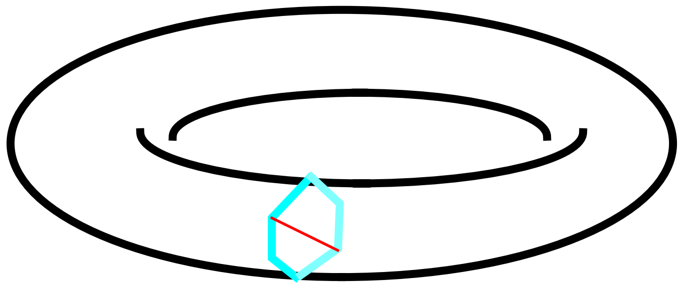




put an elongated ball around the edge
(caps not shown)

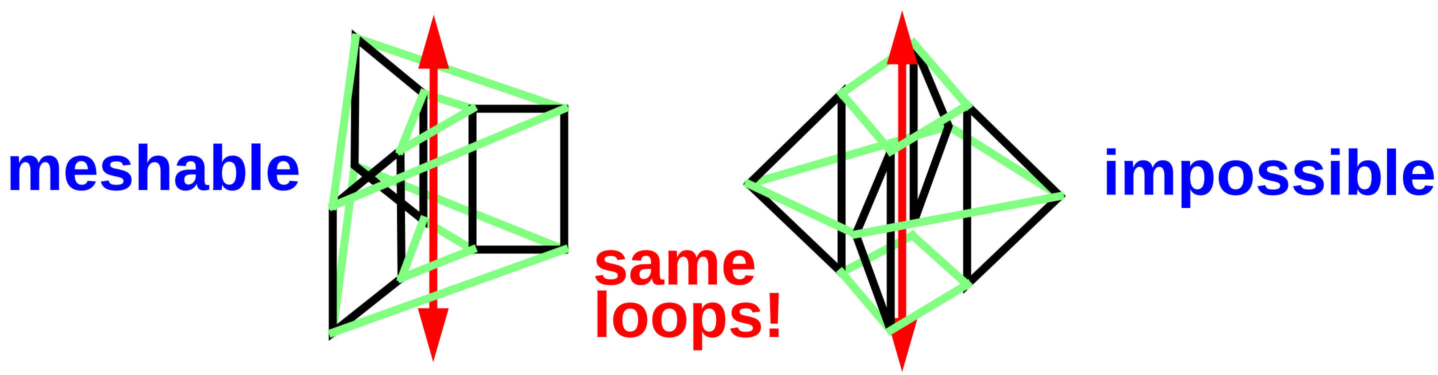
Extensions to non-ball

Idea: in a hex mesh, some edge-cycles bound a quad-rilateralized disk, hence the cycle is even



Necessary condition, mesh exists only if

- all cycles of edges contractible to a point in the volume are even
- even #quadrilaterals



Extensions to non-ball

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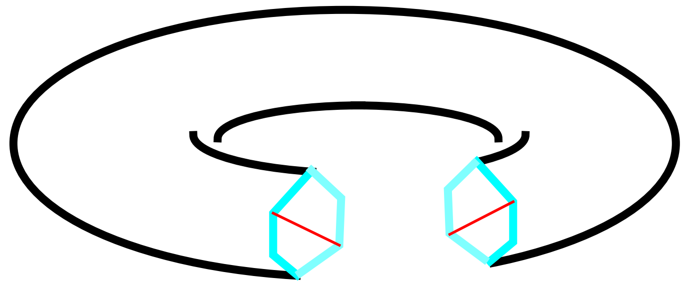
Idea: find a disk cutting each handle

- treat as two-sided to reduce to ball case



Sufficient condition, mesh exists if

- for each handle, a topological disk cutting it can be found, bounded by an even cycle of edges

Improvements

- tighter conditions (Bill Thurston), for wild topology/surface mesh
- V.A. Gasilov et al. collaboration for practical algorithm for finding disks

Conclusions

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Surface quad mesh satisfying mild conditions admits a compatible hexahedral mesh

- ball: even #quadrilaterals = necessary and sufficient
- non-ball: +null-homotopic curves crossed even #times by STC loops, sufficient to find such curves

Proof yields algorithm ideas

Experience applying proof:

small problems can have non-obvious solutions (Schneiders),

unlikely to be chosen by non-STC heuristics

STC slowly being used by developers (ICEM CFD, etc)

V.A. Gasilov developing non-ball reduction