A Characterization of the Quadrilateral Meshes of a Surface Which Admit a Compatible Hexahedral Mesh of the Enclosed Volume

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Abstract. A popular three-dimensional mesh generation scheme is to start with a quadrilateral mesh of the surface of a volume, and then attempt to fill the interior of the volume with hexahedra, so that the hexahedra touch the surface in exactly the given quadrilaterals[24]. Folklore has maintained that there are many quadrilateral meshes for which no such *compatible* hexahedral mesh exists. In this paper we give an existence proof which contradicts this folklore: A quadrilateral mesh need only satisfy some very weak conditions for there to exist a compatible hexahedral mesh. For a volume that is topologically a ball, any quadrilateral mesh composed of an even number of quadrilaterals admits a compatible hexahedral mesh. We extend this to certain non-ball volumes: there is a construction to reduce to the ball case, and we give a necessary condition as well.

Keywords. Computational Geometry, hexahedral mesh generation, existence.

1. Introduction

For some applications, meshes composed of quadrilateral and hexahedral (i.e. cube-like) elements possess better numerical properties than meshes composed of triangular and tetrahedral elements. Hence a sizable fraction of the mesh generation research conducted in recent years has been devoted to hexahedral meshes[15]. In some large-scale applications, the surface of an object is meshed before its interior. This is also a requirement for meshing several adjoining objects independently. This requirement may also arise in parallel mesh generation, where the domain is first divided into many small regions, one for each processor. The problem is to produce a hexahedral mesh that fills the volume and touches the surface in exactly the given surface mesh. We say that such hexahedral and quadrilateral meshes are *compatible*, that the hexahedral mesh *respects* the quadrilateral mesh, and the quadrilateral mesh *admits* the hexahedral mesh.

For many years, meshing algorithm developers have tried to solve this problem, without even knowing if it could be done. The difficulty of this problem has lead many to

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speculate that the surface mesh must be highly constrained in order for it to admit a compatible hexahedral mesh. Our result is that this is not the case. We show that for an input that is topologically a ball, all that is required is that the quadrilateral mesh has an even number of quadrilaterals! This condition is also necessary. We extend this result to a necessary condition, and a slightly stronger sufficient condition, for the surface mesh of a sphere with *n*-handles to admit a compatible hexahedral mesh.

Bern[8] has previously given an algorithm that generates a linear-size, straight-sided compatible tetrahedral mesh of a polyhedron given any triangular mesh of its surface. settling the existence question. Schneiders has posed as an open problem the characterization of which quadrilateral surface meshes admit a compatible hexahedral mesh[21]. We provide this characterization (ignoring the straight-edges constraint), but do not immediately give an algorithm. Our techniques are novel and entirely different from Bern's[8]. Our approach relies on the *spatial twist continuum*, or STC[17], a global interpretation of the connectivity of the dual of a hexahedral mesh, and some theorems of topology concerning regular curves and manifolds[22].

Our result is entirely topological and combinatorial. We define the quadrilateral mesh, and the hexahedral mesh, purely in terms of their topology. We ignore the geometric embedding of the mesh. In particular, we have no guarantees on the shape of the elements of the mesh, except that certain combinatorial pathologies that a priori require bad elements are ruled out. For the most part, our techniques are elementary. The outline of the proof is as follows: First, we map the object to a ball, mapping the given surface mesh to a mesh of the sphere. Second, we form the two-dimensional STC of the surface mesh. This is an arrangement of regular curves on the sphere, whose induced cell complex is the combinatorial dual[18] of the quadrilateral mesh. Third, we use theorems of topology[22] to show that the curve arrangement can be extended into an arrangement of regular manifolds through the ball. Fourth, we add additional manifolds, entirely interior to the ball, so that the cell complex induced by the arrangement satisfies certain combinatorial constraints. Fifth, the induced cell complex of the manifold arrangement is dualized to create a hexahedral mesh, and mapped from the ball back to the original object. The combinatorial constraints on the manifold arrangement ensure that the hexahedral mesh is well defined. With the exception of showing the existence of the initial manifolds, the third part, the proof is constructive. For handle-bodies, we reduce to the ball case.

Most known hexahedral mesh generation codes don't address the problem of respecting a given quadrilateral surface mesh. Of those that do, all allow the user to chose between either changing the surface mesh, or having at least some non-hexahedral elements. Plastering[9], the current version of Whisker Weaving[24][10], and Algor's Hexagen coupled with Houdini[7] are all in this category. Either this is unnecessary, or there are some unknown conditions which a mesh must further satisfy in order to be embeddable with straight edges. In particular, Whisker Weaving[24] is a heuristic algorithm that attempts to create a valid STC, which is then dualized to a hexahedral mesh: The fix-up rules given in Section 6 should allow it to produce all-hexahedral elements without changing the surface mesh.

The remainder of this paper is organized as follows. In Section 2 we describe our assumptions and requirements about quadrilateral and hexahedral meshes. In Section 3 we define the STC. In Section 4 we present the necessary conditions, and in Section 5 show that the two-dimensional STC of a surface mesh satisfying these conditions can be extended to a three-dimensional STC. In Section 6 we show how to add to the arrangement so that the induced cell complex dualizes to a valid hexahedral mesh. In Section 7 we extend our results to non-ball input. In Section 8 we present conclusions.

2. Mesh Definitions and Assumptions

We define a quadrilateral mesh as a geometric *cell complex*[23] composed of 0-dimensional nodes, 1-dimensional edges, and 2-dimensional quadrilaterals. We require:

- **a.** Each edge contains two distinct nodes.
- **b.** Each face is contained in at least one higher-dimensional face. I.e. each node is in an edge, each edge is in a quadrilateral.
- **c.** Every edge is in exactly two distinct quadrilaterals.
- **d.** Each quadrilateral is bounded by a cycle of four distinct edges.
- **e.** Two nodes have at most one edge between them.
- **f.** Two quadrilaterals share at most one edge[16].

We define a hexahedral mesh as a three-dimensional geometric cell complex, adding 3-dimensional hexahedra to the above. This definition rules out some combinatorial pathologies, but may be too weak for certain numerical applications. For example, it may not be possible to embed the mesh with non-obtuse dihedral angles. We require:

- **A.** Each edge contains two distinct nodes.
- **B.** Each face is contained in at least one higher-dimensional face. I.e. each node is in an edge, each edge is in a quadrilateral, and each quadrilateral is in a hexahedron.
- **C.** Every quadrilateral is contained in exactly two distinct hexes, except that those of the surface mesh are contained in exactly one hex. The edges of the quadrilateral have the opposite ordering in the two hexes (i.e. the hexes are on "opposite sides" of the quadrilateral).
- **D.** Surface nodes, edges and quadrilaterals are distinct. That is, two surface faces cannot be treated as the same face by the internal mesh.
- **E.** Each quadrilateral is bounded by a cycle of four distinct edges.
- **F.** A hex is bounded by six distinct quadrilaterals. Furthermore, these quadrilaterals pairwise share edges in the following way. Quadrilateral 0's ordered edge cycle *={abcd}*, 1=*{aiel}*, 2=*{bjfi}*, 3=*{efgh}*, 4=*{cjgk}*, 5=*{dlhk}*. Distinct letters represent distinct edges.
- **G.** Two nodes have at most one edge between them.
- **H.** Any two quadrilaterals share at most one edge. This also implies that any two hexes share at most one quadrilateral.

A mesh satisfying **A**-**G** can be refined to satisfy **H**. Mitchell et al.[16] gives an algorithm for this that heuristically generates elements of good quality. The surface mesh does not change if it satisfies **f**. Below, we use constructive techniques similar to Mitchell et al.[16] to ensure that a cell complex satisfies **A**-**G**.

3. STC Definition

The *spatial twist continuum* (STC)[17] is a special structure superimposed on the combinatorial dual[18] of a quadrilateral or hexahedral mesh. Any quadrilateral and any hexahedral mesh induces an STC. By duality, it is also possible to derive a mesh from a given STC.

3.1 Two-Dimensional STC

The STC[17] of a given quadrilateral mesh is any one of the of arrangements of *chords* whose induced cell complex is the combinatorial dual of the mesh[18], disregarding the geometric embedding. The chords are *regular* curves, meaning the curve's tangent vector is continuously turning and is non-trivial. Any *non-degenerate* arrangement of chords will be called an STC. By non-degenerate, we mean that the chords are nowhere tangent and intersect a finite number of times, and at most two chords meet at a point.

To construct the STC for a quadrilateral mesh, first form its dual, second form a chain of dual edges defining the structure of each chord, and third replace each chain by a nearby regular curve. Find the chains by grouping the edges that are dual to the opposite sides of a quadrilateral into the same chord. See Figure 1. Chords that are closed curves are called *loops*.

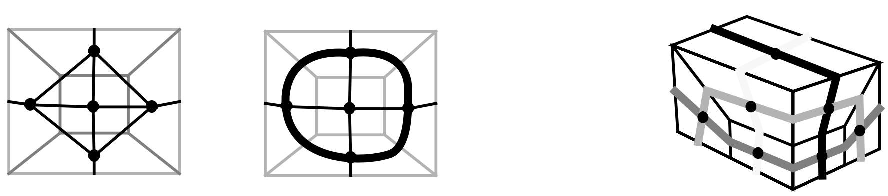

Figure 1. Left, the dual of a portion of a quadrilateral mesh. Center, a chord of the STC drawn with a wide line width. Left shows a perspective view of the loops of a closed-surface mesh.

3.2 Three-Dimensional STC

A three-dimensional STC is a non-degenerate arrangement of regular manifolds called *sheets*[17]. At most three sheets meet at a point. Figure 2 shows the STC for a mesh composed of four hexahedra. The curves of intersection are called *chords* and the points of intersection are called *centroids*. The STC of a hexahedral mesh is any one of the of arrangements whose induced cell complex is the combinatorial dual of the mesh. The intersection of the arrangement with the surface of the object is the two-dimensional

STC of the surface mesh. (The loops are closed curves but may be non-regular if e.g. the object surface is planar facetted. This is unimportant.)

The STC definition may be extended to meshes composed of *d*-cubes in any dimension, but similar structures have not been found for other types of meshes. Murdoch et al.[17] independently discovered the STC for the engineering community, but the concept of dualizing an arrangement to create a mesh has appeared before. For one thread see e.g. Rosenstiehl[19] and Grünbaum[14]. Topologists Aitchison and Rubinstein[1, 2, 3, 4, 5, 6] define and study something very much like the STC. See also de Bruijn[11]. Physicists invented the idea of partitioning space into cubes for the study of general relativity; Regge calculus. See for example Zamalodchikov[27]. A combinatorial description of arrangements of curves called Gauss codes[14, 20] has been studied for some time, but is not directly related.

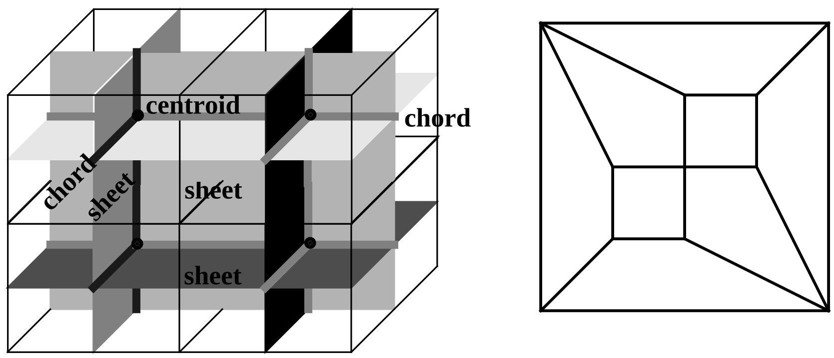

Figure 2. Right, the spatial twist continuum (STC) for a mesh of four hexahedra. Left, a quadrilateral surface mesh of a convex, planar-facetted polyhedron that satisfies conditions **a-f** above but has an odd (9) number of quadrilaterals. Hence no hexahedral mesh respects it. The ninth face forms the base of the figure and is bounded by the large outer square.

4. Necessary Condition

Theorem 1 *Any hexahedral mesh has an even number of quadrilaterals on its boundary.*

Proof. Let h be the number of hexes in the mesh, q the number of interior quads, and b the number of boundary quads. Then b quads are in one hex, and q quads are in two hexes. Each hex contains six quadrilaterals. Hence $6h = 2q + b$, so b is even. ∎

5. Constructing an Initial STC

For simplicity, we first take the given input solid and map it to the ball. Provided the surface mesh is even, we now show that there exists an arrangement of sheets inside the ball that meets the sphere in exactly the two-dimensional STC of the surface mesh. Note that since the sphere is closed every mesh edge is in exactly two quadrilaterals, and so every surface chord is a closed loop.

The necessary even-ness condition is first translated into a condition on the parity of the number of loop self-intersections.

Lemma 1 *In the arrangement of loops of a two-dimensional STC of a quadrilateral mesh of the sphere, the parity of the number of loop self-intersections is equal to the parity of the number of surface quadrilaterals.*

Proof. Each quadrilateral of loop A is the intersection of loop A with some other loop, or A with itself. In Gauss[13] and Rosenstiehl[19], and by the Jordan curve theorem, we note that two nowhere-tangent closed curves on the sphere (plane) intersect an even number of times. Hence the number of intersections of A with all other loops is even. ∎

We now introduce a theorem of topology. Suppose loops A and B are parameterized by $I = [0, 1]$. A *homotopy* h between A and B is a continuous mapping such that $h(s, 0) = A(s)$ and $h(s, 1) = B(s)$ and $h(0, t) = h(1, t)$. A regular homotopy is a homotopy which at every stage is a regular curve, keeps end points and directions fixed and such that the tangent vector moves continuously with the homotopy[22].

Theorem 2 [Smale] *Let x_0 be a point in the unit tangent bundle T of a Riemannian manifold M. Then there is a 1-1 correspondence between the set π_0 of classes (under regular homotopy) of regular curves on M which start and end at the point and direction determined by x_0 and $\pi_1(T, x_0)$* [22].

For M equal to the sphere, this says that there is a regular homotopy between any loop with an even number of self-intersections and a regular curve with no self-intersections. Similarly, there is a regular homotopy between any loop with an odd number of self-intersections and a regular curve with one self-intersection; hence between any two loops with an odd number of self intersections. This fact can also be derived from Whitney[26]. If there are an even number of surface quadrilaterals then the loops with odd numbers of self-intersection can be taken in pairs.

Theorem 3 *A two-dimensional STC of an even mesh of the sphere admits a compatible arrangement of regular manifolds through the ball.*

Proof. We construct a manifold homeomorphic to a disk for each even loop L. By Theorem 2 there exists a regular homotopy h with $h(I, 0) = L$ and $h(I, 1)$ a circle. This homotopy can be used to define a sheet for the loop; for example, the manifold $(1 - t/2, h(I, t))$, where (r,s) denotes the curve s at radius r from the center of the ball. We can extend this to close the circle with a disk inside the ball.

For each *pair* of odd loops K and L, there is a regular homotopy h from K to L. Take the manifold $(\frac{3}{4}+\left(t-\frac{1}{2}\right)^2, h(I,t))$. This is a sheet having exactly K and L as its boundary. If necessary, the manifolds are perturbed so that they are regular and non-degenerate. ∎

6. Refining the STC to Satisfy the Constraints

The dual of the cell complex induced by the above arrangement (together with the sphere) is a cell complex that respects the surface mesh. By adding extra sheets conditions **A** through **G** may be satisfied. These additional sheets are topologically spheres, and lie inside the ball so that the arrangement still respects the surface mesh.

The dual conditions **A+** through **G+** for an arrangement's cell complex corresponding to **A** through **G** are the following. For simplicity, the sphere itself is considered to be part of the arrangement. *Internal* denotes cells that are not on the sphere.

- **A+.** Each internal 2-cell is contained in exactly two distinct 3-cells.
- **B+.** Each face contains at least one lower-dimensional face (excepting centroids).
- **C+.** Each STC edge has two distinct centroids.
- **D+.** Every internal cell contains at most one surface cell of one lower dimension.
- **E+.** Each internal STC edge is contained in exactly four distinct 2-cells.
- **F+.** Each internal STC centroid is contained in six STC edges. The edges are in twelve common 2-cells as follows. Edge 0 is in 2-cells *{abcd}*, 1=*{aiel}*, 2=*{bjfi}*, 3=*{efgh}*, 4=*{cjgk}*, 5=*{dlhk}*. Distinct letters are distinct 2-cells.
- **G+.** Two 3-cells have at most one 2-cell in common.

Theorem 4 *For any non-degenerate arrangement of regular manifolds in a ball, whose boundary is a given two-dimensional STC on a sphere, there exists additional sheets interior to the ball, such that the combined arrangement satisfies conditions* ***A+*** *to* ***G+****.*

Proof. Many of these conditions are satisfied by the initial arrangement without modification. The exception is that some faces are not distinct because the arrangement is locally too coarse. Adding additional sheets alleviates this problem. Each added sheet surrounds some arrangement face at a small distance, so that no other part of the arrangement or surface mesh is intersected. The order in which the conditions are considered is important: When adding sheets to satisfy a condition, it is assumed that all previous conditions are already satisfied.

Proof A+ Each internal 2-cell is contained in exactly two distinct 3-cells. A 2-cell is a subset of a sheet. The sheets are orientable with boundary on the sphere, so one cannot travel inside the ball from one "side" of the manifold to the other without crossing it. Alternatively, a non-orientable sheet can be subdivided by spheres into a collection of orientable sheets.

Proof B+ Each face contains at least one face. Every 3-cell contains a 2-cell, since the ball is bounded. Every 2-cell contains an edge, since every sheet introduced so far has a boundary. Every edge contains a centroid, except for the following case: A chord *C* may be composed of a single edge with no centroids when it is the circle of intersection between two sheets (or a sheet and itself). In that case we add two sheets *Y* and *Z* as in Figure 3, each one containing slightly more than half of *C*.

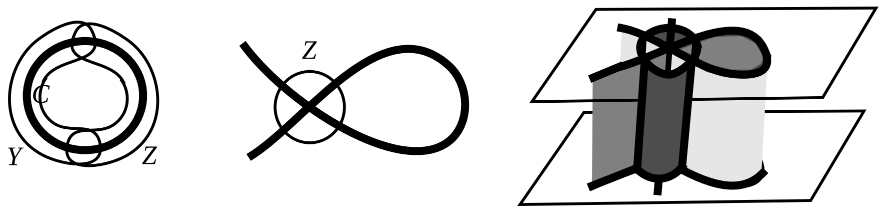

Figure 3. Right shows adding sheets *Y* and *Z* to refine *C* to satisfy **B+.** Center shows adding a sheet *Z* around a centroid to satisfy **C+**. Left shows a cut-away of adding a sheet around an edge.

Proof C+ Each STC edge has two distinct centroids. An edge may contain the same internal centroid twice only in the case that a sheet self-intersects at the centroid. We introduce a small ball sheet *Z* around that centroid as in Figure 3 center. The intersection of *Z* with the arrangement is topologically equivalent to the intersection of the sphere with the three planes through the coordinate axis.

Proof D+ Every internal face contains at most one surface face of one lower dimension. This condition requires some explanation. Suppose internal cell *A* contains surface cell *B* of one lower dimension. When we dualize to create a hexahedral mesh, the mesh face for *A* must be taken to be the surface mesh face that is the dual of *B*. Hence A must contain only one such *B*, else two surface faces are considered to be the same.

Construction: If this doesn't hold, then we introduce a spherical sheet *Z* just inside the surface sphere. Thus, every surface face is the base of a prismatic cell whose top is a face of *Z*. The prism's side-faces are all internal. Further constructions below may subdivide the faces containing surface faces, but surface faces themselves are not subdivided and the condition will continue to hold.

Proof E+ Each internal STC edge is contained in four distinct 2-cells. If not, place a sheet around the edge, analogous to C+. The centroids of the edge are distinct because C+ already holds. The details are omitted for brevity. See Figure 3 right.

Proof F+ Each STC centroid is in six STC edges... follows from **E+** and **C+**.

Proof G+ Two 3-cells have at most one 2-cell in common. If not, place a sheet surrounding the 2-cell, analogous to C+. This relies on the sub-cells of the 3-cells being distinct. The details are omitted for brevity. This concludes the proof of theorem 4. ∎

Dualizing the cell complex of the constructed arrangement we have:

Theorem 5 *Any even quadrilateral mesh (satisfying* ***a*** *through* ***f****) of a surface* *topologically equivalent to a sphere admits a compatible hexahedral mesh (satisfying* ***A*** *through* ***H)*** *of the enclosed volume.*

7. Extensions to Non-Ball Input

We now outline a strategy for reducing from the general case of a volume whose boundary is a sphere with *n*-handles, to the case of a volume whose boundary is a sphere. The basic idea is to reduce to the spherical case by cutting the handles before creating the initial STC. This strategy appears to be sound for the topologically uncomplicated examples likely to occur in CAD models, for example models with screw holes or oil conduits. Future work will give tight necessary and sufficient conditions that properly consider the topological complications that arise with knotted holes and very coarse surface meshes.

Lemma 2 *A quadrilateral mesh is bounded by an even number of edges.*

Proof. Let q be the number of quadrilaterals, e be the number of interior edges, and b the number of boundary edges of the mesh. Then $4q = 2e + b$, so b is even. ∎

Lemma 3 *All simple cycles of edges homotopic to one another have the same parity.*

Proof. By definition, a cycle homotopic to 0 on the surface comprises the entire boundary of a portion of the surface mesh whose interior is topologically an open disk. By Lemma 2 the cycle is even. Otherwise, take two cycles in the same homotopy class. By definition, they bound a connected portion of the mesh, each cycle appearing once on the boundary. Hence the sum of the number of edges in the two cycles is even. ∎

We have the following sufficient condition.

Theorem 6 *A hexahedral mesh respecting the surface mesh exists if the surface mesh has an even number of quadrilaterals, and further a non-self-intersecting topological disk bounded by an even cycle of edges can be found that cuts each of the n-handles in turn.*

Proof. In the sphere with no handles case this degenerates to Theorem 5. Otherwise, for each handle in turn, we take some non-self-intersecting disk *M* cutting the handle and bounded by a cycle *R* of edges. If *R* has even parity, we can quadrilateralize *M*. If we conceptually separate *M* and *R* into two, one for each side of *M*, i.e. “slicing” along *M*, we can recursively treat *P* as *P′*, a sphere with *n-1* handles. Note that since we treated *M* as two-sided, each quadrilateral of *M* appears on the surface of *P′* twice, so the parity of the number of surface quadrilaterals doesn’t change. Note also that future cutting disks use the surface of *P′* rather than *P*. ∎

We also have the following necessary condition.

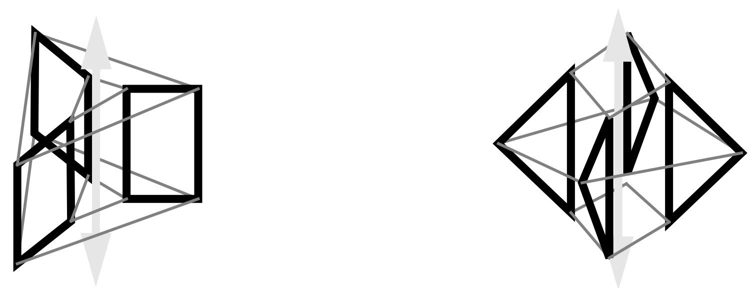

Figure 4. Two quadrilateral meshes of the surface of a torus. The axis of symmetry is vertical. Left, cycles in the non-trivial homotopy class have even parity and a hexahedral mesh of three elements is obvious. Right, cycles in the non-trivial homotopy class have odd parity, and no hexahedral mesh exists. Note that both surface meshes otherwise have the same combinatorial structure: the only difference is what is considered the inside and outside of the torus.

Theorem 7 *A hexahedral mesh respecting the surface mesh exists only if the surface mesh has an even number of quadrilaterals, and only if the cycles of edges that can be contracted to a point inside the volume are even.*

Proof. Consider a surface mesh with a contractible cycle *C*. Then there is a piecewise-linear manifold *M* whose boundary is exactly *C*. Suppose a compatible hex mesh exists for the surface mesh, and consider the intersection of the corresponding STC with *M* and *C*. This induces a two-dimensional STC on *M* respecting the dual of *C*, hence a quadrilateral mesh on *M* respecting *C*. But then, by Lemma 2, *C* must have an even number of edges.

8. Conclusions

We have shown that given mild conditions on a surface mesh, there exists a hexahedral mesh filling the interior of the volume. The fact that the sufficient conditions are minor runs counter to the experience of most mesh generation algorithm developers. This is probably due to the fact that previous work had no way to quantify the global connectivity constraints inherent in hexahedral meshes, while today we know that the STC captures these constraints beautifully and succinctly. Some steps of the proof are constructive, and may lead to practical algorithms.

CUBIT is a suite of mesh generation tools under development by Sandia National Laboratories and others under contract. Currently, Gasilov et al.[12] is developing a practical algorithm for CUBIT, along the lines of the proof of Theorem 6, to reduce the problem of constructing a compatible hexahedral mesh for a topological sphere with *n*-handles to the problem of constructing a compatible hexahedral mesh for a topological sphere. The sphere will then be meshed using the Whisker Weaving algorithm[24], or perhaps one of the other techniques available in CUBIT, such mapping identifiable subregions[25]. Whisker Weaving is based on the STC, and meshes a topological sphere by creating the arrangement of pseudo-manifolds in an advancing-front manner (in contrast to Section 5).

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