Manipulation of Geographic Locations on an Ellipsoidal Earth

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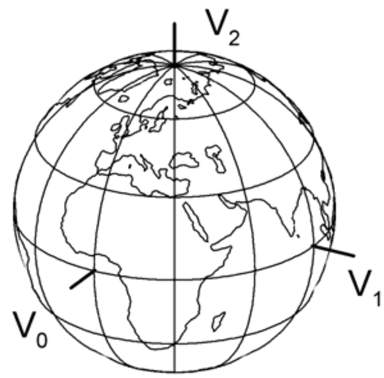
February 17, 2009

In order to manipulate points on or near the surface of the Earth, it is convenient to work in a Cartesian coordinate system where points are defined by a unit vector, **v**, with its origin at the center of the Earth, and a radial distance from the center of the Earth, *r*, measured in km. As illustrated in Figure 1, our coordinate system is oriented such that v_0 points from the center of the Earth towards the point on the surface with latitude and longitude 0°, 0°; v_1 points toward latitude, longitude 0°, 90° and v_2 points toward the north pole.



The parameters that define the GRS80 ellipsoid are

$$a = 6378.137 \text{ km};$$

$$\mathrm{b} = 6356.7523 \text{ km}$$

$$\mathrm{f} = 1 - \frac{b}{a} = 1/298.257222101$$

$$\mathrm{e}^2 = \frac{a^2 - b^2}{a^2} = 2f - f^2$$

where *a* and *b* are the equatorial and polar radii of the Earth, respectively, *f* is the flattening parameter, and *e* is the eccentricity (Snyder, 1987).

Geographic data, such as station locations, seismic event locations, etc., are generally given in geographic latitude, ϕ', longitude, θ', and depth, *z*. Geographic latitude is the acute angle between the equatorial plane and a line drawn perpendicular to the tangent of the reference ellipsoid at the point of interest (Figure 1). Geodesic latitude is another term for geographic latitude. Geocentric latitude is the acute angle between the equatorial plane and a line from the

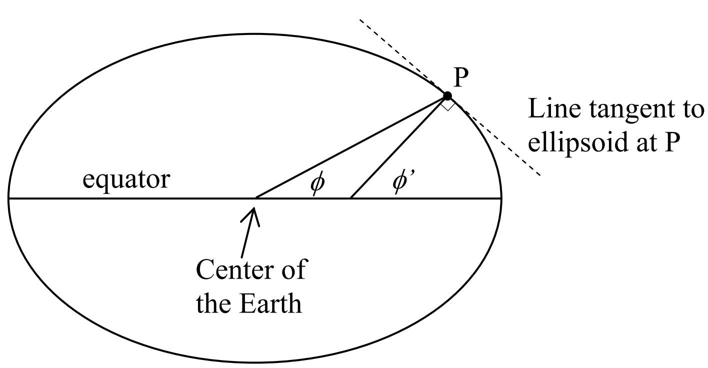
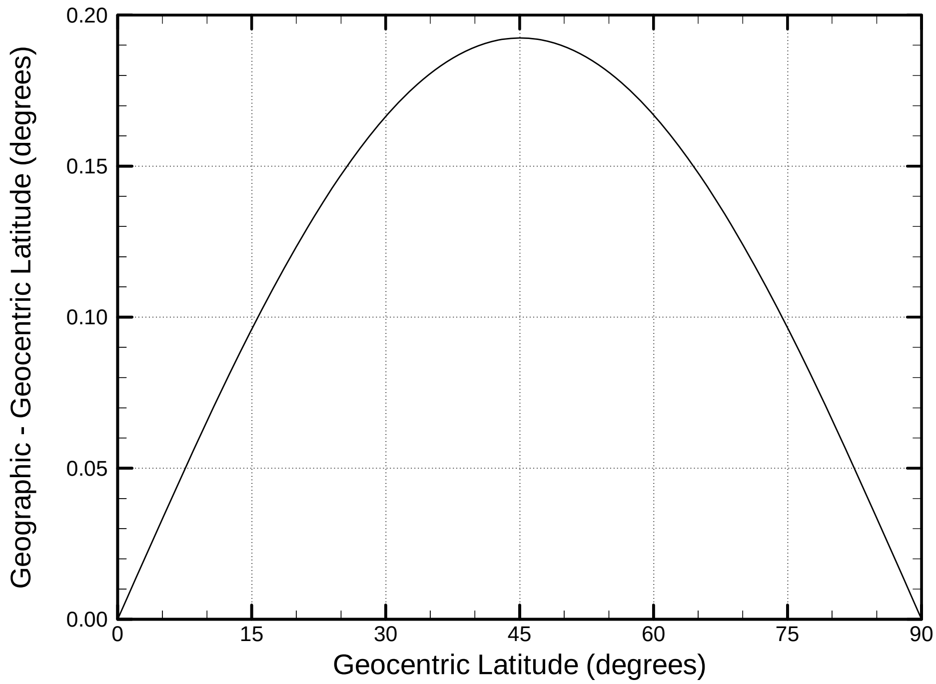


Figure 1 – An exaggerated ellipsoid illustrating the difference between geographic latitude, ϕ', and geocentric latitude, ϕ.

center of the Earth to the point in question. Geographic, geodesic and geocentric longitudes are all equivalent. Geographic and geocentric latitudes are compared in Figure 2.



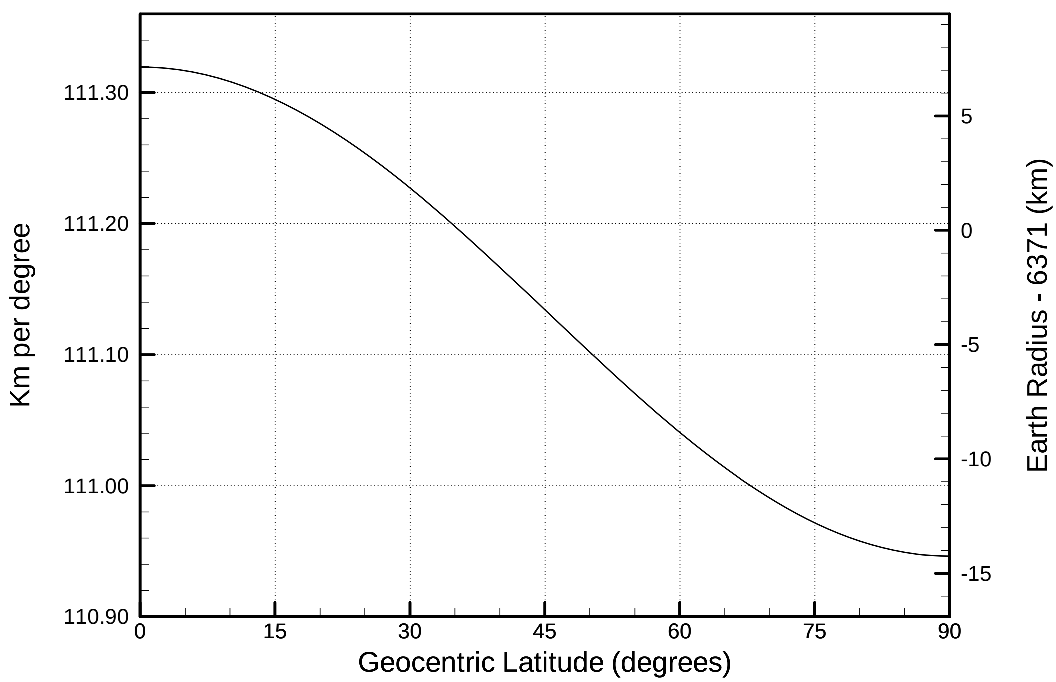


Figure 2 – a) Comparison of geographic and geocentric latitudes for the GRS80 ellipsoid. b) Km per degree and Earth radius as function of geocentric latitude.

To convert the position of a point in space from geographic to Cartesian coordinates, we must first convert from geographic to geocentric coordinates. Given the geographic latitude ϕ', and geographic longitude θ', of a point, the geocentric latitude, ϕ, and geocentric longitude, θ, are (Snyder, 1987)

$$\begin{aligned} \phi &= \arctan((1-e^2)\tan\phi') \\ \theta &= \theta' \end{aligned} \tag{1}$$

Then we convert from geocentric to Cartesian coordinates (Zwillinger, 2003)

$$\begin{aligned} v_0 &= \cos\phi\cos\theta \\ v_1 &= \cos\phi\sin\theta \\ v_2 &= \sin\phi \end{aligned} \tag{2}$$

We must also convert depth, z, to radius

$$r = R(\phi) - z \tag{3}$$

where $R(\phi)$, the radius of the Earth at geocentric latitude ϕ, is given by (Zwillinger, 2003)

$$R(\phi) = \frac{ab}{\sqrt{a^2\sin^2\phi + b^2\cos^2\phi}} = a\left(1 + \frac{e^2}{1-e^2}v_2^2\right)^{-1/2} \tag{4}$$

To convert a unit vector, $\mathbf{v}$, and radius r back to geographic latitude, longitude and depth

$$\begin{aligned} \phi' &= \arctan\left(\frac{\tan(\arcsin v_2)}{1-e^2}\right) \\ \theta' &= \arctan\left({}^{v_1}\!/_{v_0}\right) \\ z &= R(\phi) - r \end{aligned} \tag{5}$$

Once geographic information has been converted to unit vectors, a variety of useful calculations can be performed.

<u>Distance between two points</u>

Given two points defined by unit vectors, $\mathbf{u}$ and $\mathbf{v}$, the angular separation of the two points is

$$\Delta = \arccos(\mathbf{u}\cdot\mathbf{v}) \tag{6}$$

To find the separation of **u** and **v** at the surface of the Earth in km, it is necessary to either perform the following integration numerically

$$d = \int_{\mathbf{u}}^{\mathbf{v}} R(\phi) d\delta \tag{7}$$

or consider the algorithm of Vincenty (1975).

Azimuth from one point to another

The azimuth, α, from **u** to **v**, measured clockwise from north, is

$$\begin{aligned} &\alpha = \arccos(|\mathbf{u}\times\mathbf{v}| \cdot |\mathbf{u}\times\mathbf{n}|) \\ &if\,(|\mathbf{u}\times\mathbf{v}| \cdot \mathbf{n} < 0) \qquad \alpha = 2\pi - \alpha \end{aligned} \tag{8}$$

where **n** is the vector pointing to the north pole, **n** = [0, 0, 1].

Points on a great circle

Given two points defined by unit vectors **u** and **v**, to find another point, **w,** that lies on the great circle defined by **u** and **v,** at some angular distance δ, measured from **u** in the direction of **v** (see Figure 3)

$$\begin{aligned} &\mathbf{t} = |(\mathbf{u}\times\mathbf{v})\times\mathbf{u}| \\ &\mathbf{w} = \mathbf{u}\cos\delta + \mathbf{t}\sin\delta \end{aligned} \tag{9}$$

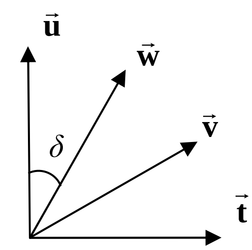


Figure 3 – Calculation of **w** given **u** and **v**. All vectors are of unit length and lie entirely in the plane of the figure.

Note that if many points, $\mathbf{w}_i$, are to be found along the same great circle defined by **u** and **v**, the normalized vector triple product, **t**, only needs to be computed once.

Finding a new point some distance and azimuth from another point

To find a point, **w**, that is some specified distance δ from **p** in direction φ, (see Figure 4) we first find an intermediate point **u**, distance δ north of **p** by applying Equation 9 with **v** = **n** = [0, 0, 1]. Then **w** is found by rotating **u** around **p** by angle α = -φ.

$$\mathbf{w} = \mathbf{u}\cos\alpha + \mathbf{p}(\mathbf{p}\cdot\mathbf{u})(1-\cos\alpha) + (\mathbf{p}\times\mathbf{u})\sin\alpha \tag{10}$$

α is equal to –φ because rotations defined by equation 10 are positive clockwise when viewed in the direction of the pole of rotation **p**.

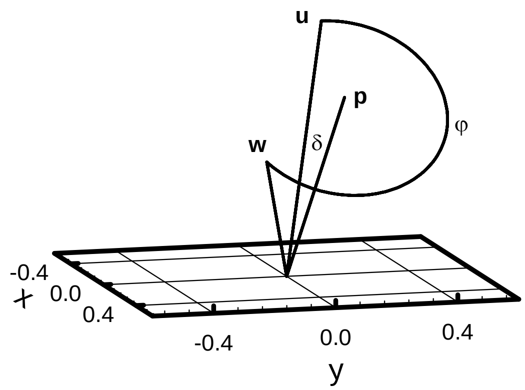


Figure 4 – Start at point **p**, located at latitude, longitude 45°, 0°. Find a new point **u** δ = 20° north of **p**. Then rotate **u** φ = 235° around **p** to position **w**. Note that ∠**pu** = ∠**pw** = δ = 20°.

Radius and Depth

It should be noted that equation 3 is only approximately true since depth, *z*, is normally measured in a direction perpendicular to the surface of the earth, not along a radial line emanating from the center of the earth (see Figure 1). Given the small amount of flattening however, the error is insignificant for most applications.

References

Snyder, J. P., Map Projections – A Working Manual, USGS Prof. Paper 1395, 1987.

Vincenty, T., Survey Review, 23, No 176, p 88-93, 1975

Zwillinger, D., CRC Standard Mathematical Tables and Fomulae, 31rst Edition, 2003.