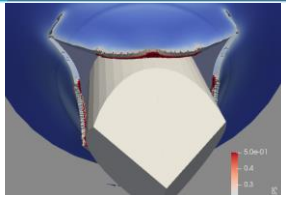
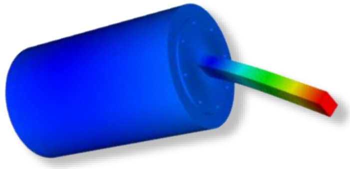
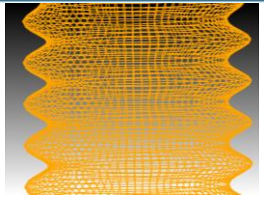
Verification and Convergence of the Method of Harmonic Balance within Sierra/SD FEA Code



Dalton L. Stein, Nathan E. Schill, Oliver G. Khan

Mentors: David M. Day, Robert J. Kuether

August 5th, 2025

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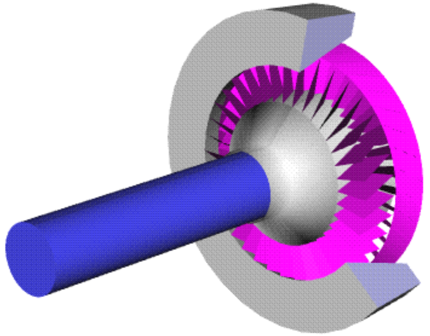
Background: Steady-State Response to Cyclic Loading

- Cyclic loading is prevalent in civil, aerospace, automotive, and defense industries
- Examples:
  - Rotating machinery [1]
  - Thermal cycling [2]
  - Dynamic Mechanical Analysis (DMA) [3]
- **Time Integration** (TI) is a typical approach for calculating steady-state response to periodic forcing
  - Often requires small time steps and long simulation times to reach steady-state



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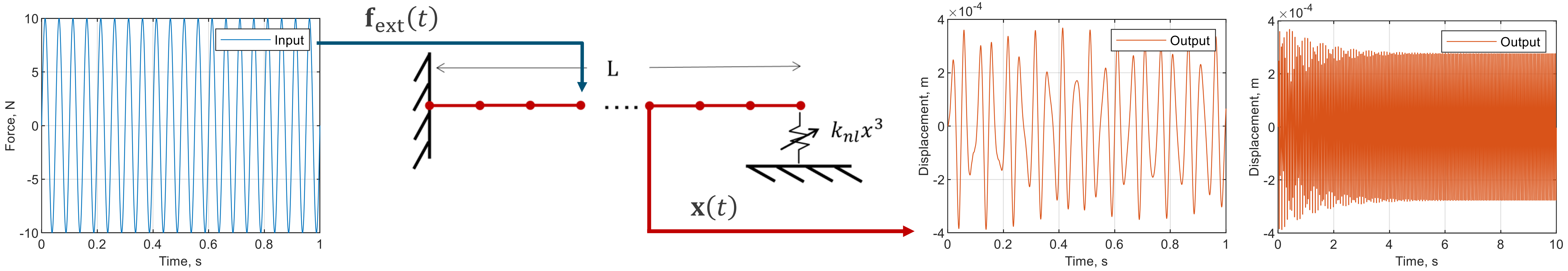


Opportunity: Multi-Harmonic Balance (MHB) [1,4]

- Approximates the steady-state response of a nonlinear system to periodic forcing

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} + \mathbf{f}_{\mathrm{nl}}(\mathbf{x}, \dot{\mathbf{x}}) = \mathbf{f}_{\mathrm{pre}} + \mathbf{f}_{\mathrm{ext}}(t)$$

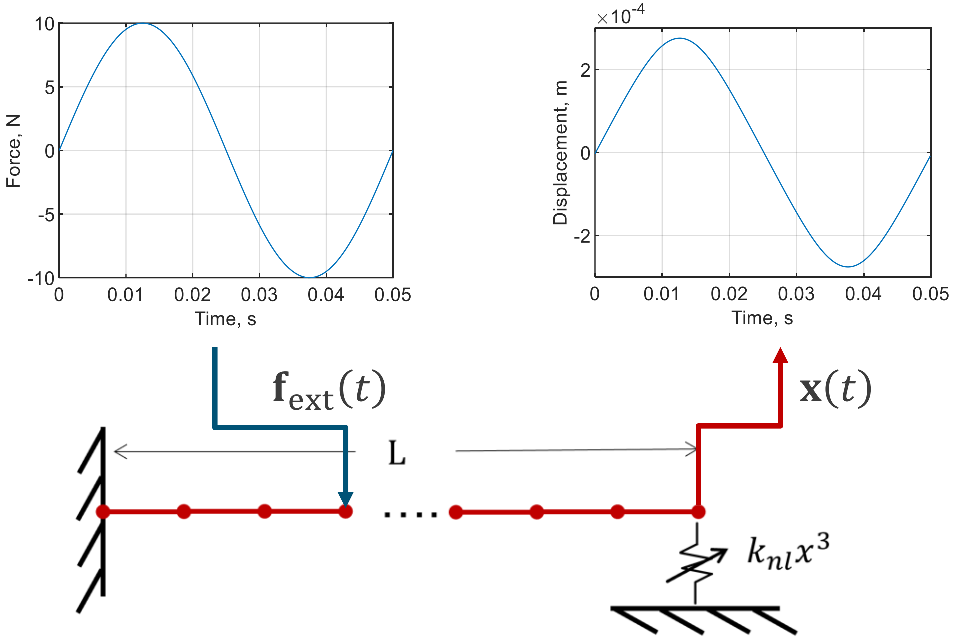
- Assumes a truncated Fourier series for the periodic response

$$\mathbf{x}(t) = \frac{\mathbf{c}_0^x}{\sqrt{2}} + \sum_{k=1}^{N_h} \left[\mathbf{s}_k^x \sin(k\omega t) + \mathbf{c}_k^x \cos(k\omega t)\right]$$

- Substitution and Fourier–Galerkin projection onto orthogonal periodic functions

$$\mathbf{r}(\mathbf{z}, \omega) = \mathbf{A}(\omega)\mathbf{z} + \mathbf{b}_{\mathrm{nl}}(\mathbf{z}) - \mathbf{b}_{\mathrm{pre}} - \mathbf{b}_{\mathrm{ext}} = \mathbf{0}$$

- Benefits of MHB compared to Time Integration (TI):
  - Orders of magnitude faster (minutes vs. hours)
  - Captures unstable periodic orbits



Example of harmonic forcing producing harmonic response

Opportunity: Multi-Harmonic Balance (MHB) [1,4]

- Approximates the steady-state response of a nonlinear system to periodic forcing

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} + \mathbf{f}_{\mathrm{nl}}(\mathbf{x}, \dot{\mathbf{x}}) = \mathbf{f}_{\mathrm{pre}} + \mathbf{f}_{\mathrm{ext}}(t)$$

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$$\mathbf{x}(t) = \frac{\mathbf{c}_0^x}{\sqrt{2}} + \sum_{k=1}^{N_h} \left[\mathbf{s}_k^x \sin(k\omega t) + \mathbf{c}_k^x \cos(k\omega t)\right]$$

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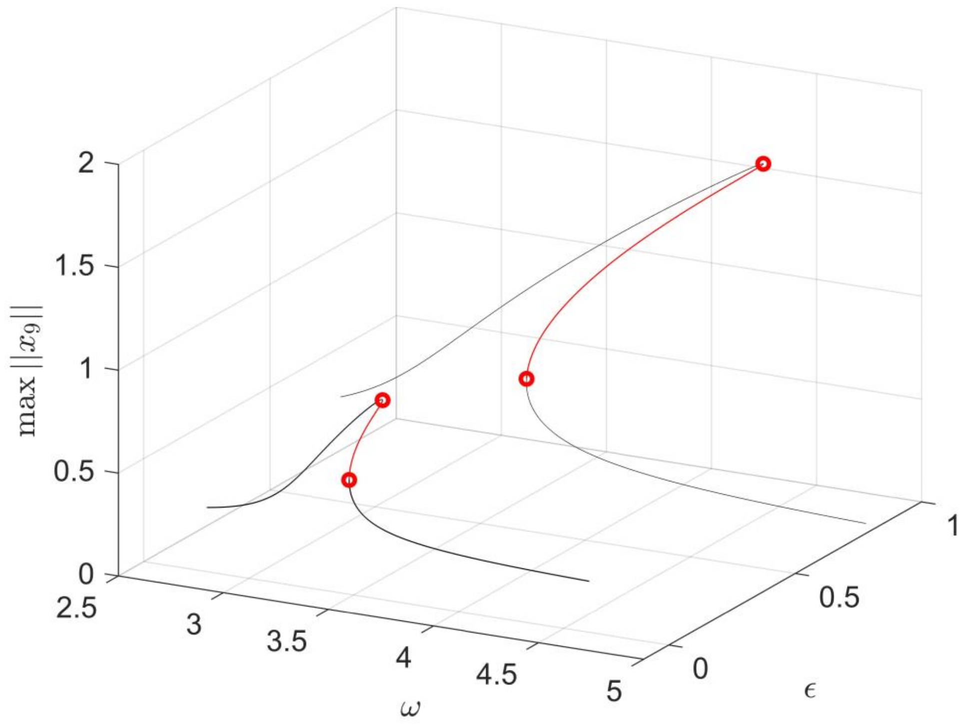
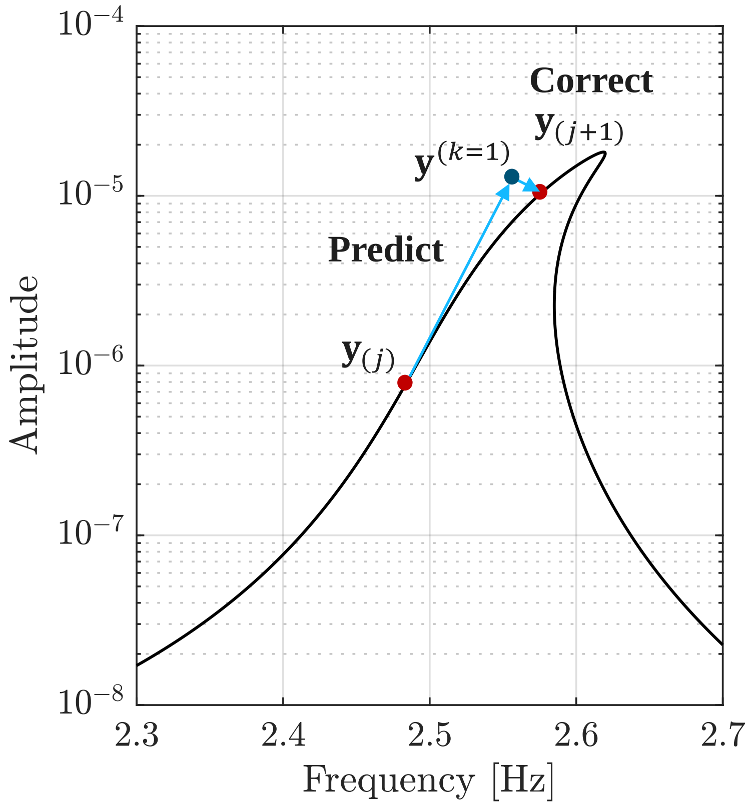


Red curves represent unstable periodic orbits computed with Multi-Harmonic Balance

Numerical Continuation [5]

- MHB can be combined with numerical continuation to compute a curve of periodic solutions as a parameter is varied
  - **Nonlinear Frequency Response (NLFR)** curves show the amplitude of a steady-state response as the forcing frequency changes

- Numerical continuation computes subsequent points on a solution branch in two main steps:
  1. **Predict** where the next point will be
  2. **Correct** the prediction until the point is on the solution curve



Nonlinear Frequency Response (NLFR) curve illustrating predictor-corrector numerical continuation scheme [5]

Objectives

1. **Implement** a MHB + Pseudo-Arclength Continuation scheme in C++ to be integrated into Sierra/SD, Sandia’s Finite Element Analysis (FEA) software for structural dynamics (SD)
2. **Verify** C++ and MATLAB MHB algorithms by comparison with Time Integration
3. Assess **convergence** of C++ and MATLAB MHB algorithms for varying selections of harmonics
4. Experiment with **adaptive algorithm** to automatically adjust harmonics included in response

Overview of Harmonic Balance in Sierra/SD

C++ Sierra Implementation

Objective: Implement a draft MHB code in C++ with the following requirements:

- Path-following capabilities – pass through fold bifurcations
- Robust to large, ill-conditioned systems
- Iterative solver for large, sparse linear systems

Householder Transformation

1. Find a $\mathbf{P}$ such that, $\mathbf{PV} = \pm\mathbf{e}_i$
2. Set $\mathbf{Q} = \mathbf{PI}_i$, where $\mathbf{I}_i$ obtained by eliminating the i^{th} column of $\mathbf{I}$
3. Transformation: $\mathbf{F}'(\mathbf{x}_k)\mathbf{Q}\mathbf{y}_k = -\mathbf{F}(\mathbf{x}_k)$ for $\mathbf{y}_k \in \mathbb{R}^n$

Sierra Implementation

- Path-following: Pseudo-Arclength Continuation
  - Construct augmented system via the constraint:

$$\mathbf{V}^{\mathrm{T}}(\mathbf{x}_k - \mathbf{x}_{k-1}) = 0$$

- Linear transformation of the augmented Jacobian [1]
- Solve using biconjugate gradient stabilized method (BiCGSTAB)

Solving Linear Systems

Prediction Vector:

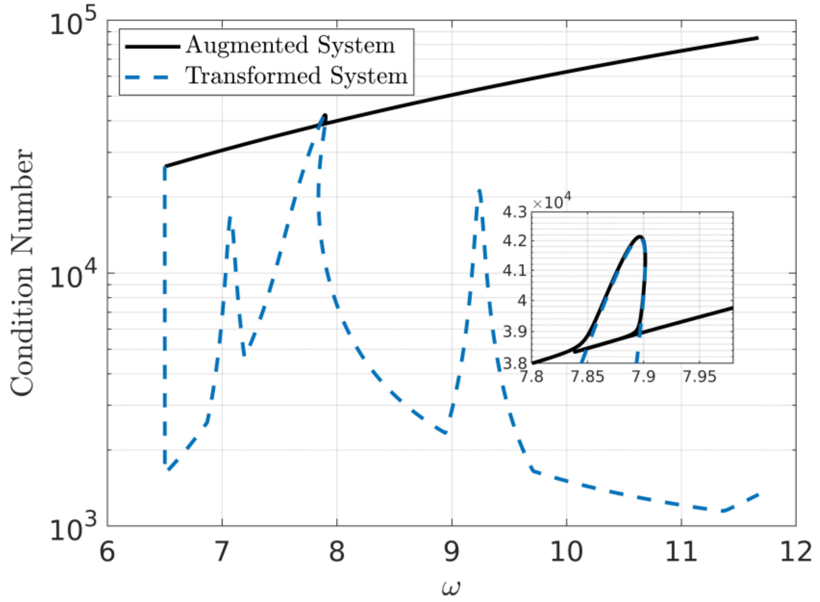
1. Solve transformed system
2. Map solutions back: $\mathbf{x}_k = \mathbf{Q}\mathbf{y}_k$
3. Update Prediction Vector: $\mathbf{V}_{\text{new}} = \frac{(\mathbf{V}_0 + \mathbf{x}_k)}{\|\mathbf{V}_0 + \mathbf{x}_k\|_2}$

Corrections:

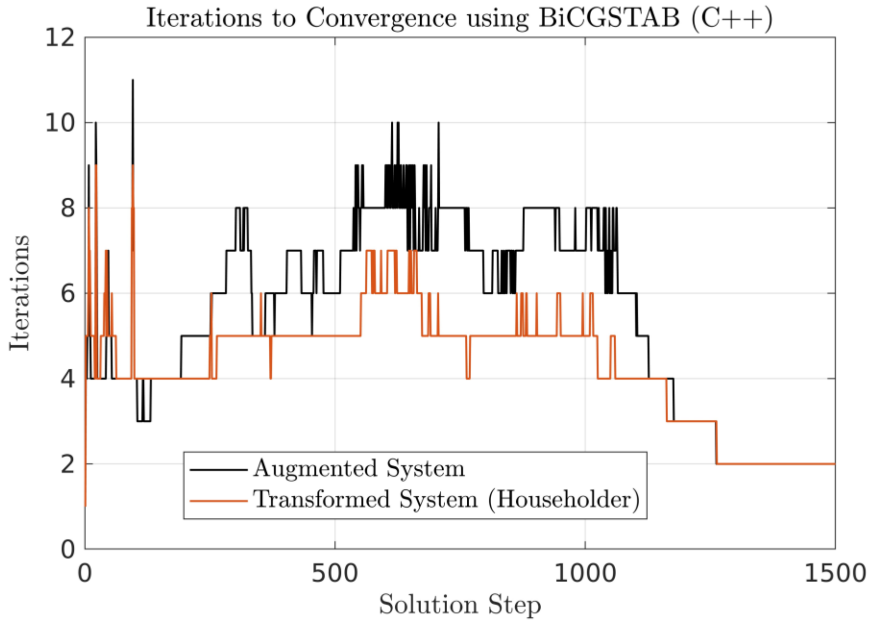
1. Solve transformed system
2. Map solutions back: $\mathbf{x}_k = \mathbf{Q}\mathbf{y}_k$
3. Update $\mathbf{z}_0$, ω_0

C++ Sierra Implementation – Motivation for Transformation

- Past work [5] explored the use of iterative solvers on the augmented MHB equations to improve computational performance over direct solvers
  - Current Sierra implementation will offer additional improvements
- For full rank Jacobian, $\mathbf{F}'(\mathbf{x}_k)$, and exact prediction vector, $\mathbf{P}_k$, the singular values of $\mathbf{F}'(\mathbf{x}_k)\mathbf{Q}$ are identical to the singulars values of $\mathbf{F}'(\mathbf{x}_k)$ [6].



Condition number of the Jacobians for transformed and augmented system



Number of iterations needed for solver to converge to specified tolerance

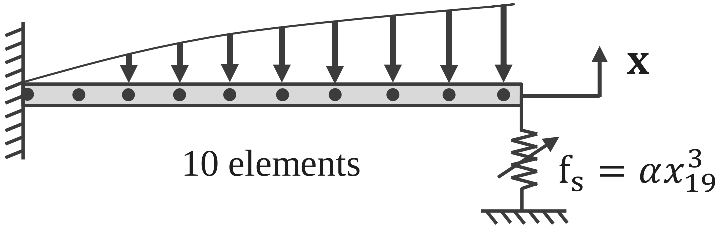
Verification of Harmonic Balance in C++ and MATLAB

Benchmark Systems

- Multi-Harmonic Balance was verified on two unified benchmark systems

Cantilever Beam with Cubic Spring

$\mathbf{f}_{\text{ext}}(t) = \varepsilon \boldsymbol{\phi}_{\mathbf{1}} \cos(\omega t)$



Length [m]	1
Width [mm]	6.35
Thickness [mm]	3.175
Young's Modulus [GPa]	70
Density [kg/m^3]	2880
Nonlinear Spring Stiffness, α [N/m^3]	518.7

Linear Change of Coordinates

Projection onto first normal mode

$$\mathbf{P} = \boldsymbol{\phi}_1$$

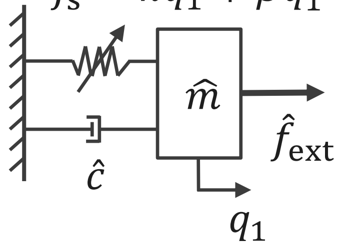
$$q_1 = \mathbf{P}^\dagger \mathbf{x},\ \dot{q}_1 = \mathbf{P}^\dagger \dot{\mathbf{x}},\ \ddot{q}_1 = \mathbf{P}^\dagger \ddot{\mathbf{x}}$$

$$\hat{m}\ddot{q}_1 + \hat{c}\dot{q}_1 + \hat{k}q_1 + \mathbf{P}^\dagger \mathbf{f}_{\text{nl}}(\mathbf{P}q_1, \mathbf{P}\dot{q}_1) = \mathbf{P}^\dagger \mathbf{f}_{\text{pre}} + \mathbf{P}^\dagger \mathbf{f}_{\text{ext}}(t)$$

$$\hat{m} = \mathbf{P}^\dagger \mathbf{M}\mathbf{P},\ \hat{c} = \mathbf{P}^\dagger \mathbf{C}\mathbf{P},\ \hat{k} = \mathbf{P}^\dagger \mathbf{K}\mathbf{P}$$

Duffing Oscillator

$\hat{f}_s = \hat{k}q_1 + \beta q_1^3$



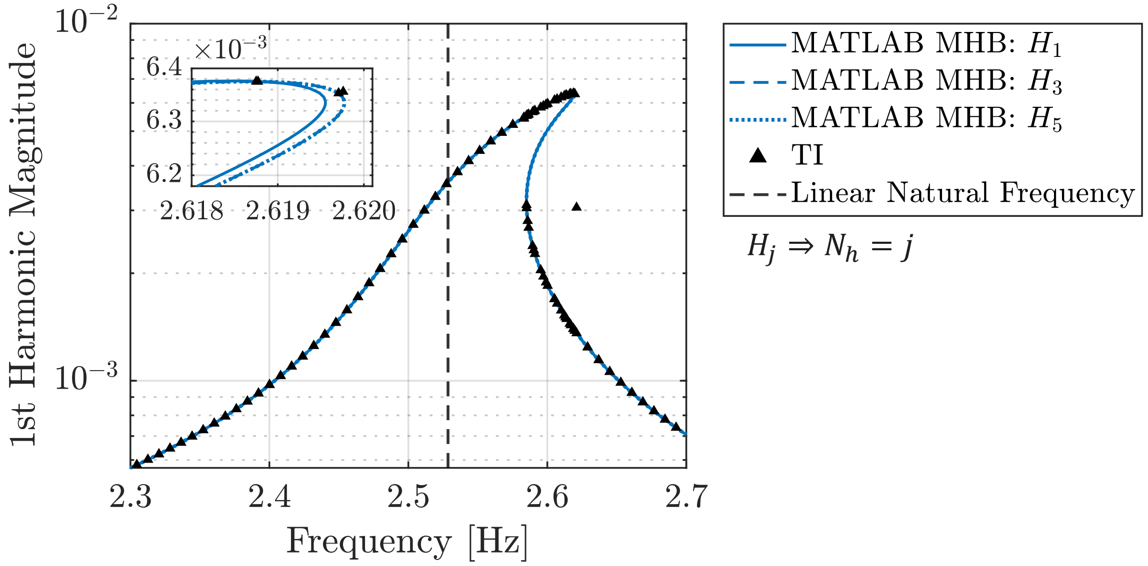
$\hat{f}_{\text{ext}}(t) = \varepsilon \cos(\omega t)$

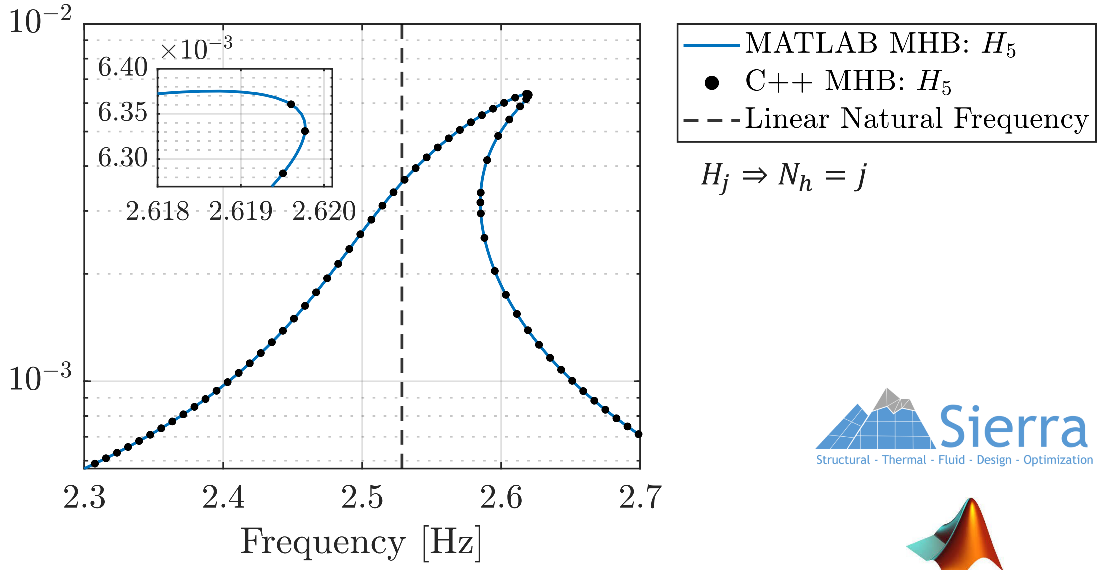
Damping Ratio, ζ [%]	0.75
Natural Frequency, ω_n [rad/s]	15.9
Nonlinear Spring Stiffness, β	10^6
Forcing Amplitude, ε	0.025

$$\boxed{\ddot{q}_1 + 2\zeta\omega_n\dot{q}_1 + \omega_n^2 q_1 + \beta q_1^3 = \varepsilon \cos(\omega t)}$$

Duffing Oscillator – Frequency Response Curves for 1st Harmonic

- MATLAB MHB agreed with Time Integration (TI) when 3 or more harmonics were included
- C++ MHB agreed with MATLAB MHB when it included the same number of harmonics



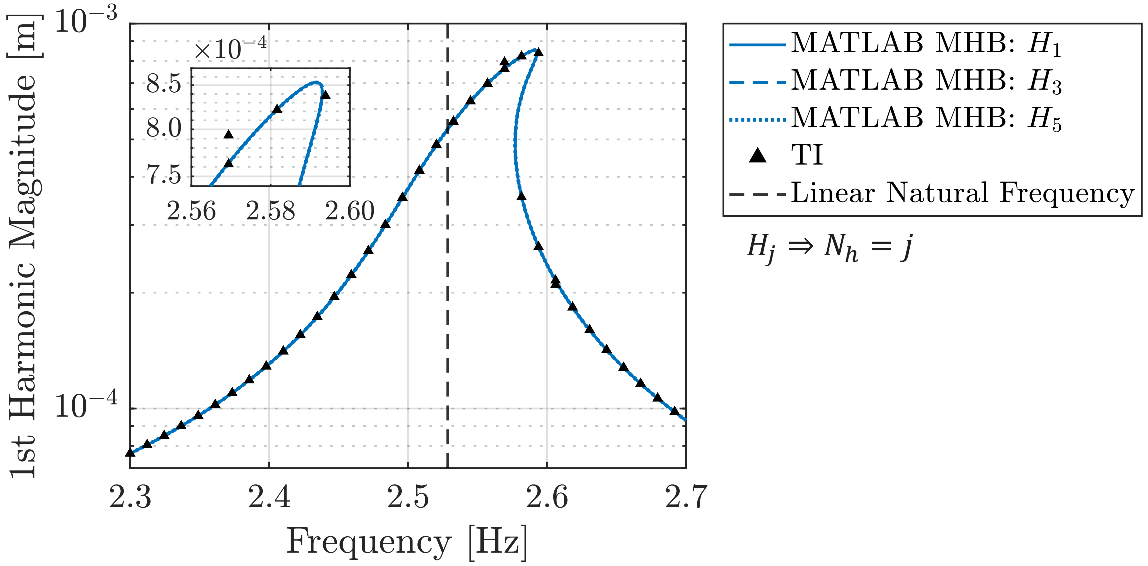


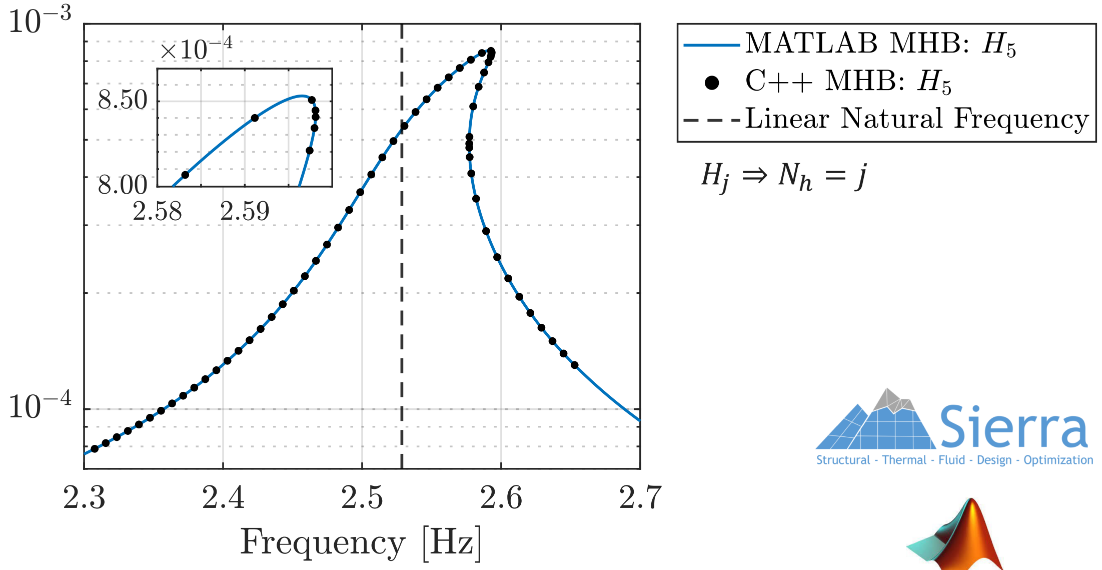
Sierra
Structural - Thermal - Fluid - Design - Optimization

Displacement of the 1st harmonic of q_1 vs. forcing frequency, for Time Integration and MATLAB MHB algorithm (left) and C++ and MATLAB MHB algorithms (right)

Cantilever Beam – Frequency Response Curves for 1st Harmonic

- MATLAB MHB agreed with Time Integration (TI) for all harmonic orders
- C++ and MATLAB MHB agreed when they included the same number of harmonics





1st harmonic of the beam tip displacement, x_{19}, vs. forcing frequency, for Time Integration and MATLAB MHB algorithm (left) and C++ and MATLAB MHB algorithms (right)

Convergence of Harmonic Balance in C++ and MATLAB

Convergence Study Overview

- Studying convergence revealed:
  - Which harmonics were needed to achieve desired level of accuracy
  - Maximum achievable accuracy, given the tolerance constraints of nonlinear equation solver
- Application of convergence study:
  - Adaptive algorithm that automatically selects harmonics to include in the response for each forcing frequency and degree of freedom

Convergence in Time Domain

Evaluate how well MHB reproduced the entire response using L^2 norm of time domain residual

$$\mathbf{R}(t) = \mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} + \mathbf{f}_{\text{nl}}(\mathbf{x}, \dot{\mathbf{x}}) - \mathbf{f}_{\text{pre}} - \mathbf{f}_{\text{ext}}(t)$$

$$\|\mathbf{R}\|_{L^2}^2 = \int_0^{2\pi/\omega} \mathbf{R}^{\mathrm{T}}(t)\mathbf{R}(t)\ dt$$

$$\text{Error metric} = \|\mathbf{R}\|_{L^2} / \left\|\mathbf{f}_{\text{pre}} + \mathbf{f}_{\text{ext}}\right\|_{L^2}$$

Convergence in Frequency Domain

Evaluate how well MHB reproduced certain harmonics using frequency domain residual [5]

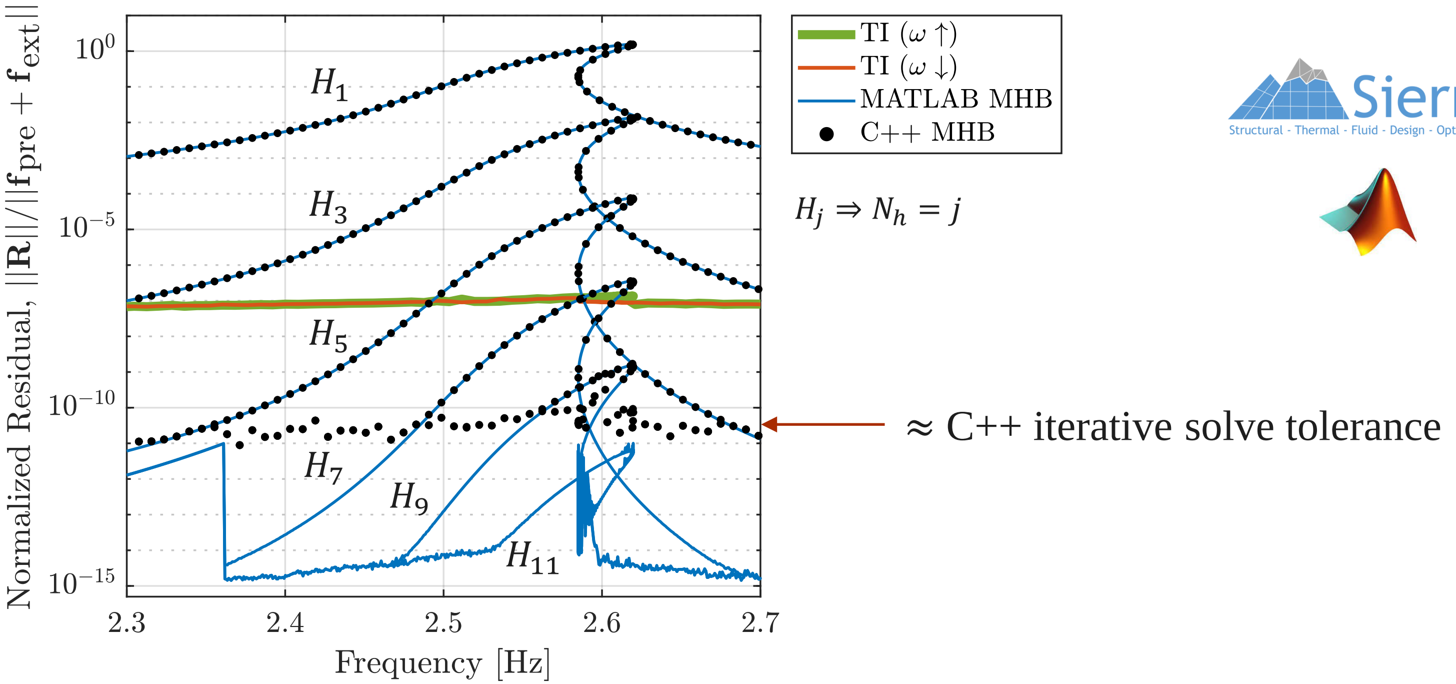
$$\mathbf{r}(\mathbf{z}, \omega) = \mathbf{A}(\omega)\mathbf{z} + \mathbf{b}_{\text{nl}}(\mathbf{z}) - \mathbf{b}_{\text{pre}} - \mathbf{b}_{\text{ext}}$$

$$\|\mathbf{r}(\mathbf{z}, \omega)\|_2^2 = \mathbf{r}^{\mathrm{T}}(\mathbf{z}, \omega)\mathbf{r}(\mathbf{z}, \omega)$$

$$\text{Error metric} = \|\mathbf{r}\|_2 / \left\|\mathbf{b}_{\text{pre}} + \mathbf{b}_{\text{ext}}\right\|_2$$

Duffing Oscillator – Convergence of Time Domain Residual

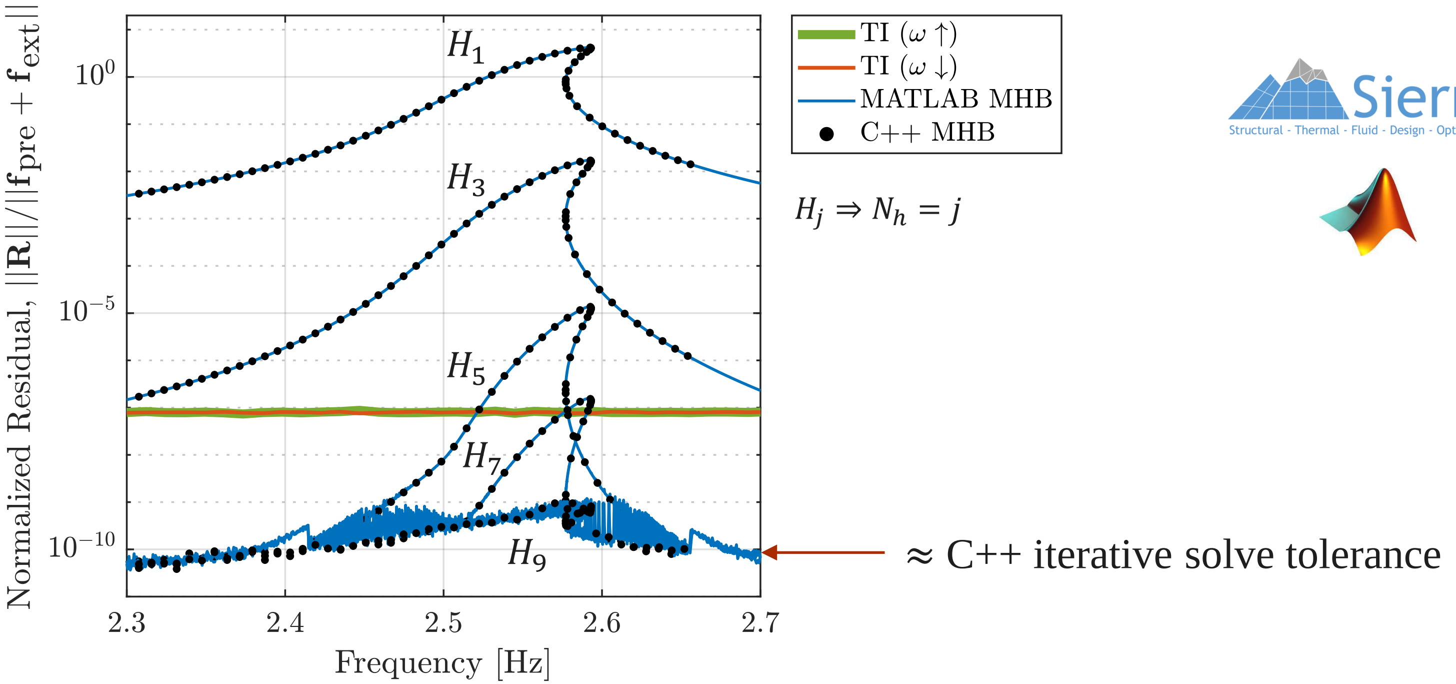
- C++ MHB and MATLAB MHB outperformed TI when more than 7 harmonics were included
- C++ MHB and MATLAB MHB performed similarly until the C++ algorithm’s higher linear-solve tolerance



Normalized time domain residual vs. forcing frequency for Time Integration and C++ and MATLAB implementations of Multi-Harmonic Balance

Cantilever Beam – Convergence of Time Domain Residual

- C++ MHB and MATLAB MHB outperformed TI when more than 7 harmonics were included
- MATLAB MHB's residual tolerance matched C++ MHB's linear solve tolerance, leading both of their residuals to saturate at a similar value



Normalized time domain residual vs. forcing frequency for Time Integration and C++ and MATLAB implementations of Multi-Harmonic Balance

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  - Maximum achievable accuracy, given the tolerance constraints of nonlinear equation solver
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Convergence in Time Domain

Evaluate how well MHB reproduced the entire response using L^2 norm of time domain residual

$$\mathbf{R}(t) = \mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} + \mathbf{f}_{\text{nl}}(\mathbf{x}, \dot{\mathbf{x}}) - \mathbf{f}_{\text{pre}} - \mathbf{f}_{\text{ext}}(t)$$

$$\|\mathbf{R}\|_{L^2}^2 = \int_0^{2\pi/\omega} \mathbf{R}^{\mathrm{T}}(t)\mathbf{R}(t)\ dt$$

$$\text{Error metric} = \|\mathbf{R}\|_{L^2} / \left\|\mathbf{f}_{\text{pre}} + \mathbf{f}_{\text{ext}}\right\|_{L^2}$$

Convergence in Frequency Domain

Evaluate how well MHB reproduced certain harmonics using frequency domain residual [5]

$$\mathbf{r}(\mathbf{z}, \omega) = \mathbf{A}(\omega)\mathbf{z} + \mathbf{b}_{\text{nl}}(\mathbf{z}) - \mathbf{b}_{\text{pre}} - \mathbf{b}_{\text{ext}}$$

$$\|\mathbf{r}(\mathbf{z}, \omega)\|_2^2 = \mathbf{r}^{\mathrm{T}}(\mathbf{z}, \omega)\mathbf{r}(\mathbf{z}, \omega)$$

$$\text{Error metric} = \|\mathbf{r}\|_2 / \left\|\mathbf{b}_{\text{pre}} + \mathbf{b}_{\text{ext}}\right\|_2$$

Frequency Domain Residual [5]

$$\mathbf{r}(\mathbf{z}, \omega) = \mathbf{A}(\omega)\mathbf{z} + \mathbf{b}_{\mathrm{nl}}(\mathbf{z}) - \mathbf{b}_{\mathrm{pre}} - \mathbf{b}_{\mathrm{ext}}$$

$$\mathbf{z} = \mathrm{DFT}(\mathbf{x})$$

Fourier coefficients of **solution**

$$\boxed{\mathbf{b}_{\mathrm{nl}}(\mathbf{z}) = \mathrm{DFT}\left[\mathbf{f}_{\mathrm{nl}}\left(\mathrm{IDFT}(\mathbf{z})\right)\right]}$$

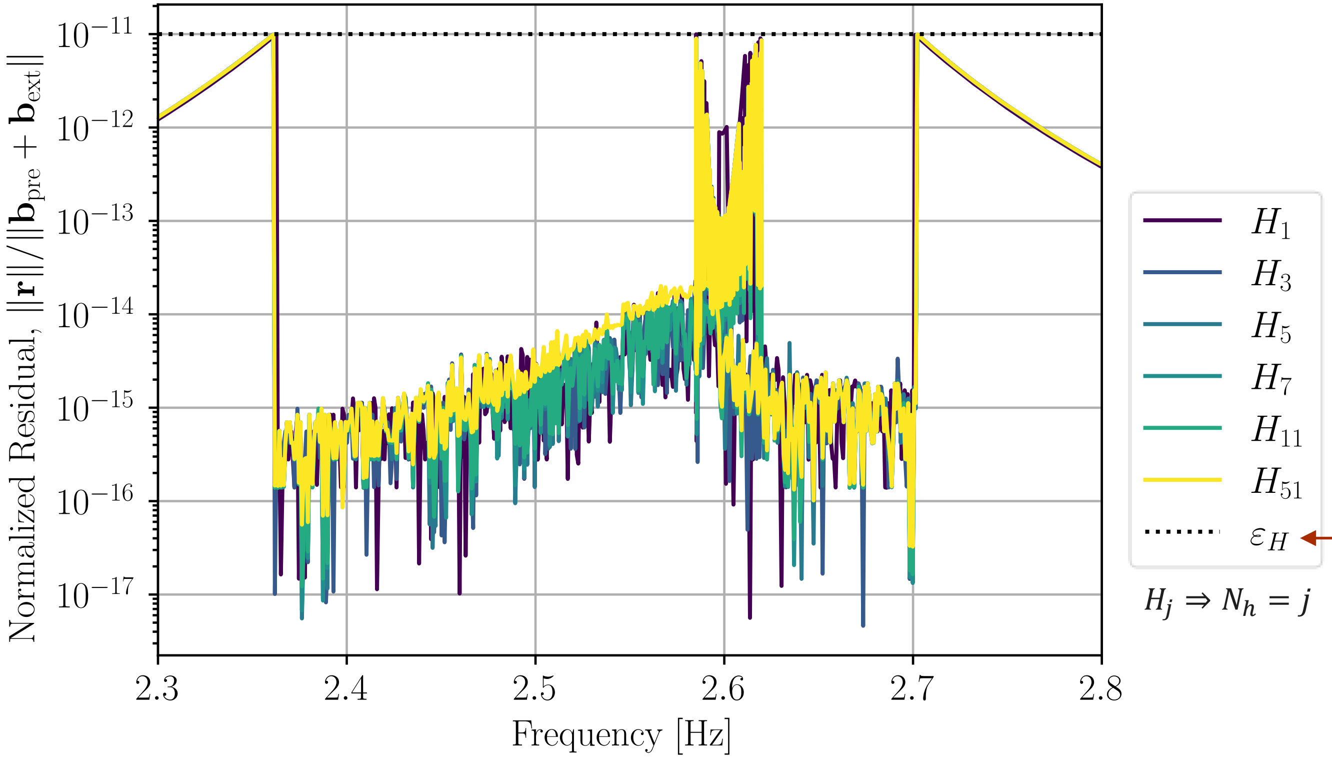
Fourier coefficients of **nonlinear force**

$$\mathbf{A}(\omega) = \begin{bmatrix} \mathbf{K} & & & & & \\ & \mathbf{K} - (\omega)^2\mathbf{M} & -\omega\mathbf{C} & & & \\ & \omega\mathbf{C} & \mathbf{K} - (\omega)^2\mathbf{M} & & & \\ & & & \ddots & & \\ & & & & \mathbf{K} - (N_h\omega)^2\mathbf{M} & -N_h\omega\mathbf{C} \\ & & & & N_h\omega\mathbf{C} & \mathbf{K} - (N_h\omega)^2\mathbf{M} \end{bmatrix}$$

Matrix describing **linear dynamics**

Duffing Oscillator – Convergence of Frequency Domain Residual

- Harmonic Balance required the normalized residual magnitude to be smaller than a specified tolerance, ε_H



$H_j \Rightarrow N_h = j$

normalized residual tolerance

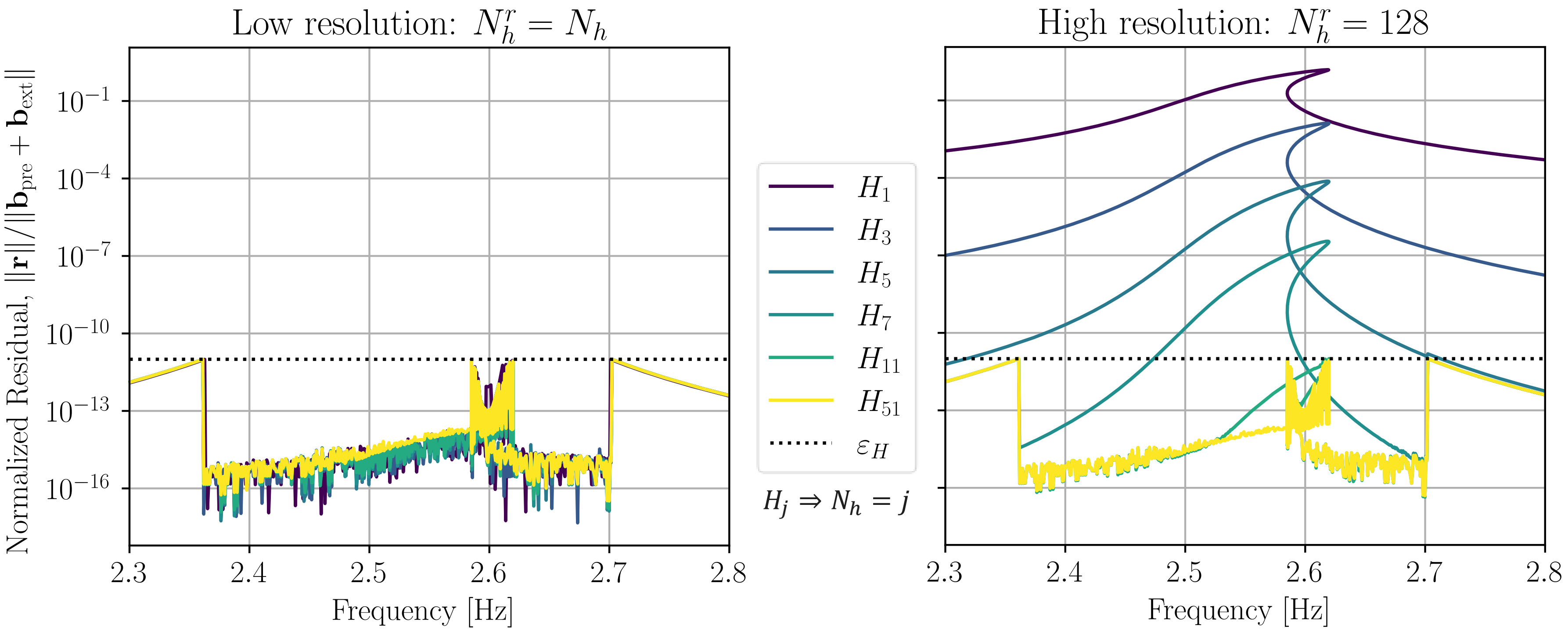
Normalized magnitude of frequency domain residual from MATLAB vs. forcing frequency; residual was computed using N_h harmonics

Duffing Oscillator – Convergence of Frequency Domain Residual

- Using more DFT points allowed access to higher harmonics present in nonlinear force

$$\mathbf{b}_{\mathrm{nl},\,N_h^r}(\mathbf{z}) = \mathrm{DFT}_{N_h^r}\left[\mathbf{f}_{\mathrm{nl}}\left(\mathrm{IDFT}_{N_h}(\mathbf{z})\right)\right]$$

$$\mathbf{r}_{N_h^r}(\mathbf{z},\omega) = \mathbf{A}(\omega)\mathbf{z} + \mathbf{b}_{\mathrm{nl},\,N_h^r}(\mathbf{z}) - \mathbf{b}_{\mathrm{pre}} - \mathbf{b}_{\mathrm{ext}}$$



Normalized magnitude of frequency domain residual from MATLAB vs. forcing frequency; residual was computed using N_h harmonics

Implementation of Adaptive Harmonic Selection

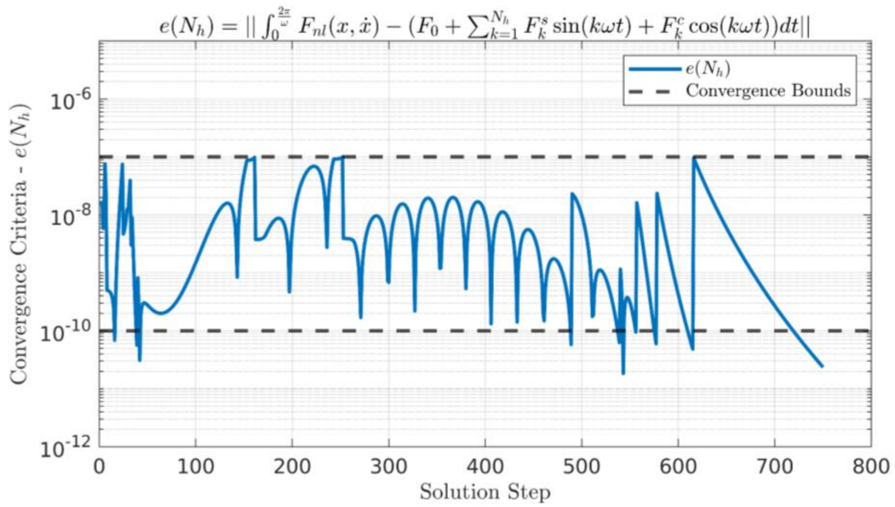
Adaptive Algorithm Overview

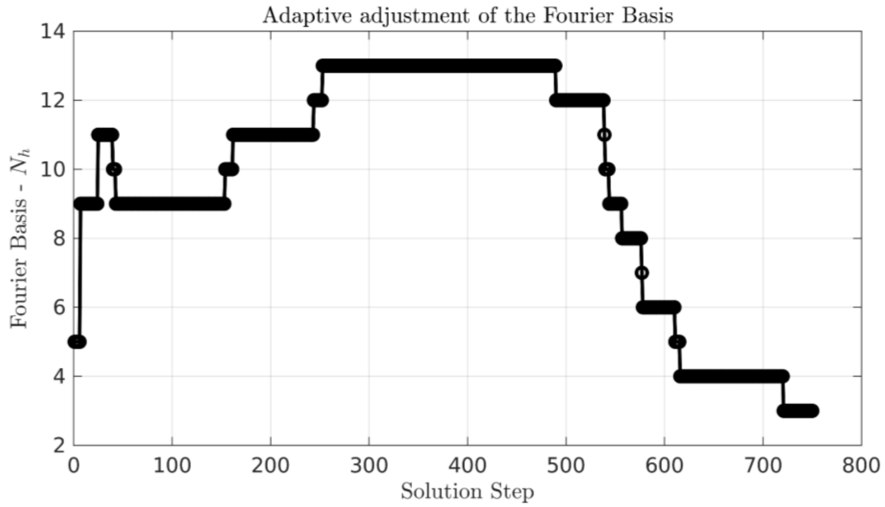
Objective:

- Incorporate adaptive harmonic selection while continuing along solution branch in MATLAB MHB code

General Idea:

- New Inputs:
  1. Tolerance bounds (lower/upper)
  2. Maximum harmonic order, $N_{\max}$ (safety net)
  3. Error metric selection
- Initialization:
  1. Compute FFT operators for $N_{\max}$
  2. Allocate storage containers of size $N_{\max}$
- Continuation:
  1. Pass in relevant continuation variables and blocks of FFT operators to the subroutines
  2. Evaluate error metric after converged solution reached
  3. Test convergence bounds
     - If greater than upper bound, add harmonics and repeat until less than upper bound or $N_h = N_{\max}$
     - If within convergence bounds, begin next iteration and maintain order
     - If less than lower bound, keep solution, subtract harmonics, and begin next iteration

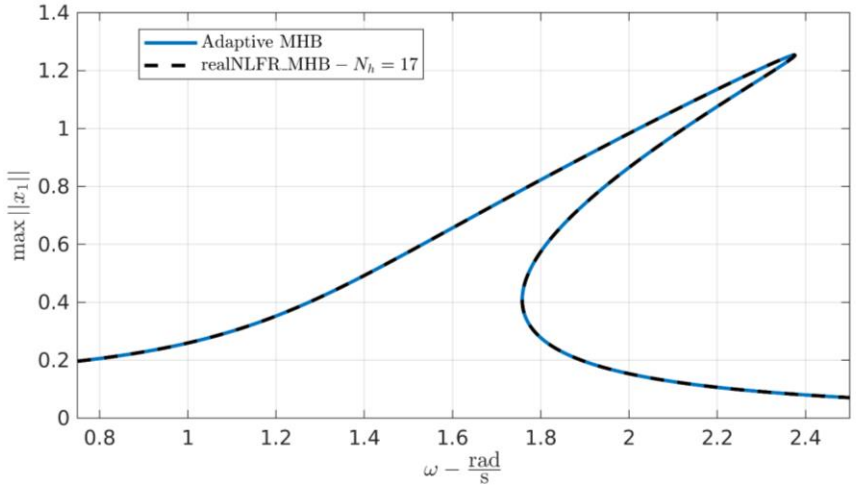


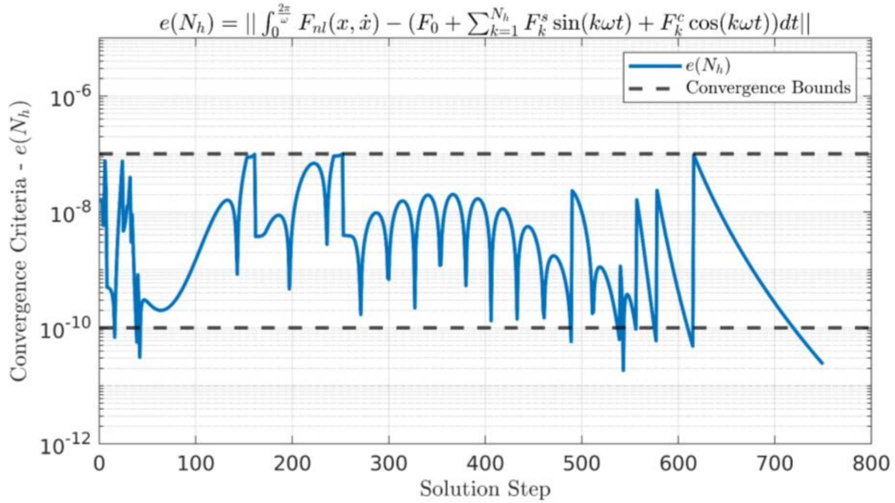


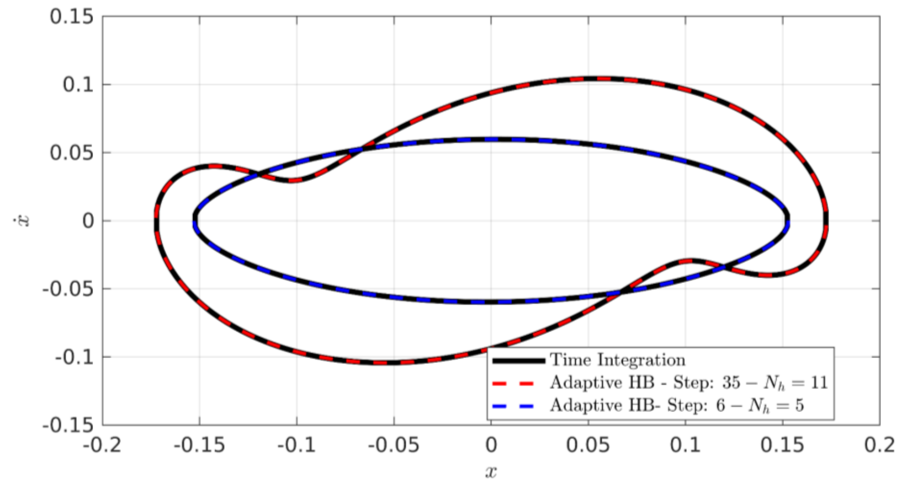
Adaptive Algorithm Overview

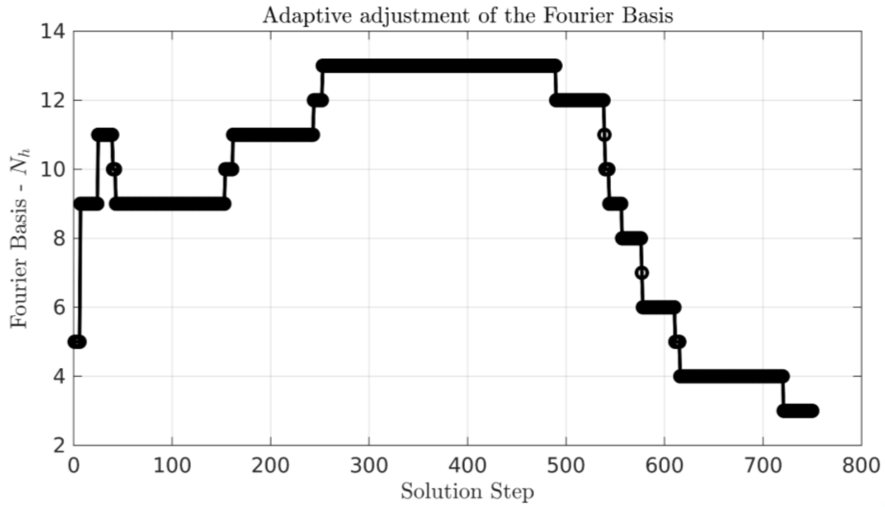
- Error function taken from literature [7]
- Convergence bounds chosen heuristically; goal is to validate implemented adaptive scheme

- Lower bound allows algorithm to discard unneeded harmonics along the solution branch





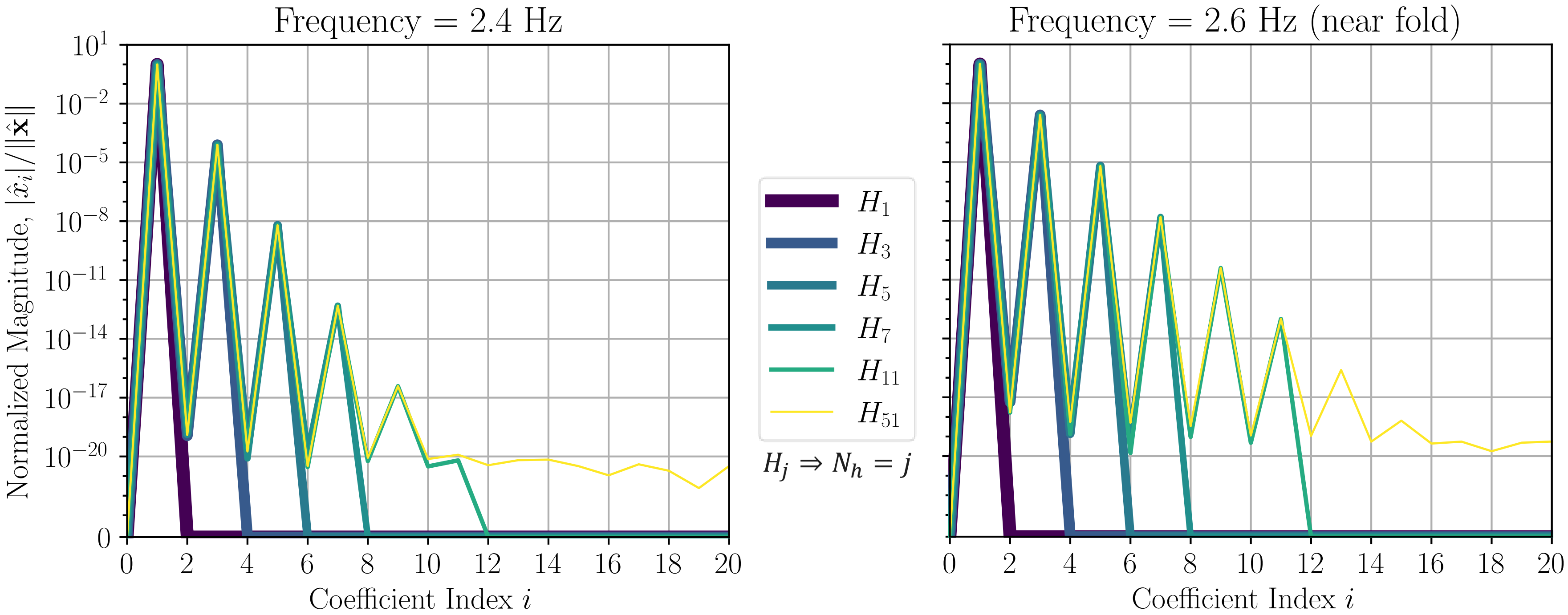


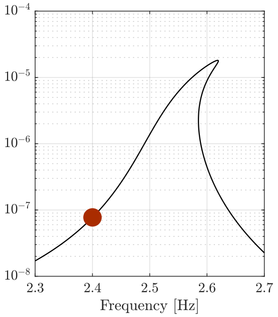


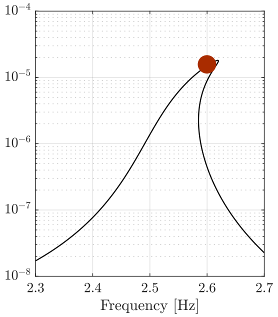
Adaptive Algorithm Applied to Duffing Oscillator [7]

- Near the fold of the frequency response curve, higher harmonics have larger magnitudes

$|\hat{x}_i| = \sqrt{(s_i^x)^2 + (c_i^x)^2}$ — magnitude of i-th harmonic of solution $\mathbf{z}$







Magnitude of the *i*-th harmonic of the displacement, normalized by the total magnitude of all Fourier coefficients (generated in MATLAB)

Conclusion

Conclusion

1. **Completed preliminary C++ implementation** of MHB + Pseudo-Arclength Continuation to integrate into Sierra/SD

2. **Verified** C++ and MATLAB MHB algorithms by comparison with Time Integration on two benchmark systems:
    - Duffing oscillator
    - Cantilever beam

3. Assessed **convergence** of C++ and MATLAB MHB for varying selections of harmonics via
    - Time domain residuals
    - Frequency domain residuals
    - Fourier coefficient magnitudes

4. Experimented with **adaptive algorithm** to automatically adjust harmonics included in response
    - Future developments will select harmonics for each DOF and isolate odd/even harmonics

Thank You — Questions?

Acknowledgements

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Sandia National Laboratories is a multimission laboratory managed and operated by National Technology and Engineering Solutions of Sandia, LLC, a wholly owned subsidiary of Honeywell International, Inc., for the U.S. Department of Energy’s National Nuclear Security Administration under contract DE-NA-0003525.

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