Compressively sensed complex networks

J. Ray, A. Pinar and D. Dunlavy
Sandia National Laboratories

Acknowledgements: ASCR, Office of Science, Dept. of Energy.

Introduction

- **Aim:** **To develop low dimension parametric (deterministic) models of complex networks**
  - **Use compressive sensing (CS) and multiscale analysis to do so**
    - **Exploit the structure of complex networks (some are self-similar under coarsening)**
- **Motivation:**
  - **Graphs often form the understructure over which dynamics occur**
    - **e.g., chemical reactions, epidemiological processes, cascading failures, etc**
  - **Dynamics are easy to observe, but the graphical structure unknown**
  - **A low-dimension model of a graph allows its “discovery” by fitting to data**
    - **Inverse problem, network discovery, estimation of graphs etc**

Today's talk

- Can compressive sensing actually work?
  - Under what circumstances?
- Some assumptions/characteristics of the networks
  - Networks will be small – O(100) nodes
    - In inverse problems, not enough info to fit detailed models
  - Networks will be assumed to have densely connected "cliques" of different sizes
    - Leads to "blocky" adjacency matrices
- Outline
  - Compressive sensing – what is it?
    - Sampling technique
    - Reconstruction technique
  - Test cases
    - 2 synthetic networks
    - Results – different levels of sampling/order reduction and reconstruction fidelity

What is compressive sensing?

- A technique to efficiently encode/decode a random vector *x* of length *N*
  - *x* can be a signal, a time-series
  - The process is lossy – the decoding is approximate
- Efficiency of representation rests on the presumption that the information content of the signal is small
  - i.e., a sparse representation exists for x
  - In conjunction with efficient sampling, only a few samples are needed
- "Sampling" a signal means projection on a *sampling basis set* ψ_i

$$y = \Psi x$$

  - *y* is the "signature"; y_i are the projections of *x*
  - Under certain conditions size(*y*) ~ *log(N)*

When and why is Y compressive?

- If *x* can be described sparsely in an orthogonal basis set (e.g., wavelets), Φ, then the basis weights can be sampled directly

$$x = \Phi s, \quad y = \Psi s$$

  – Where only *K* elements of *s* are non-zero. K << size(*s*) = *N*

- An efficient sampling of *x* collects information on all elements of *s* per projection
  – Can be done if ψ_i are random vectors
    - e.g., chosen from a high dimensional sphere (uniform spherical ensemble)
  – Ψ is then an orthogonal matrix
- Under these conditions

$$M = size(y) \geq cK \log(N / K) << N$$

Decoding - reconstructing *x* from *Y*

- Canonically, decoding is performed via l_1 minimization

$$\min \|s\|_1 \quad \text{subject to} \; \|y - \Psi s\|_2 < \varepsilon$$

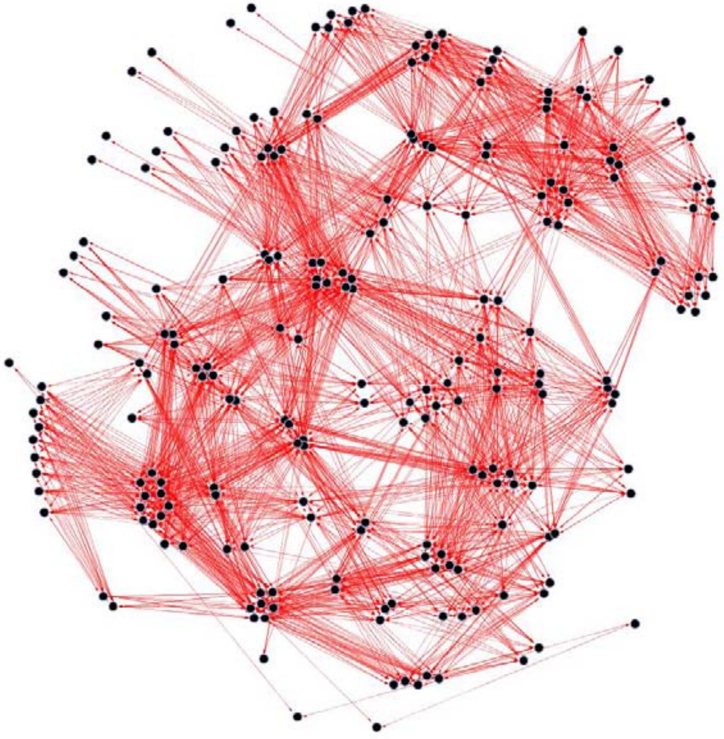
  – Exact solutions (e.g., basis pursuit) too computationally expensive, so usually an approximate form is solved in practice
- Our algorithm, called StOMP (Stagewise Orthogonal Matching Pursuit)
  – Donoho et al, 2006 (preprint)
  – Iteratively finds the non-zero elements of *s*
  – Number of iterations are bounded
    - But assumes that *s* is sparse
  – Computational cost comes from the pseudoinverse of Ψ
  – Suitable for large problems

Extending Compressive Sensing to networks

- Based on the CS of adjacency matrices
- We expect that the rows/columns of an adjacency matrix can be re-ordered to create a "blocky" adjacency matrix
  - Alternatively, node in a clique should have similar node-ids
- Networks showing self-similarity will show structure within the blocks
- Exploiting multi-resolution:
  - Decompose the adjacency matrix on a wavelet basis
    - In our case, Haar wavelets
  - Haar wavelet coefficients stored and treated hierarchically
    - Resolution by resolution, with no inter-resolution dependence modeled or exploited
  - Non-zero wavelet coefficients at each resolution will be sparse
  - Sampling and reconstruction too are performed hierarchically

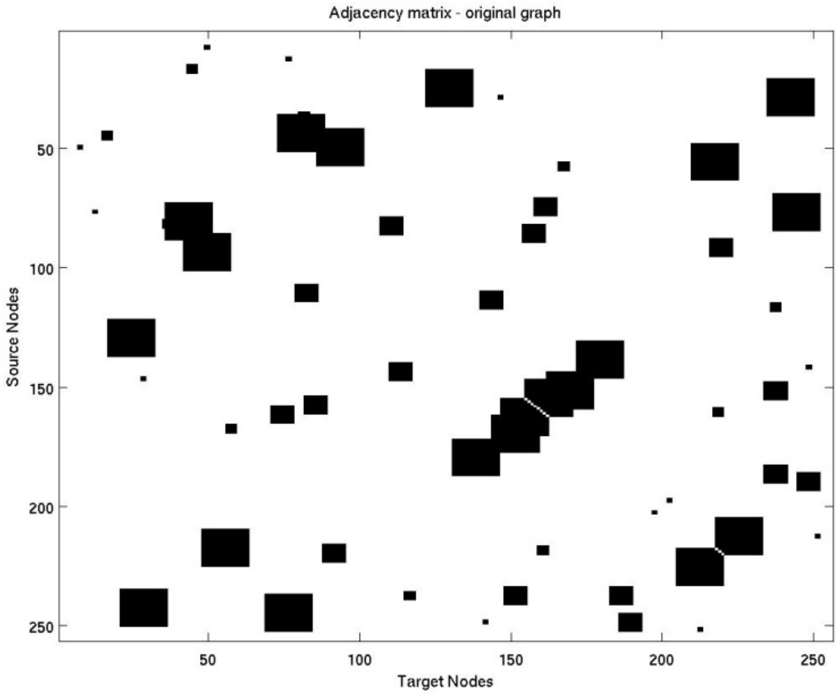
Sandia National Laboratories

A graph and its multiscale decomposition - I



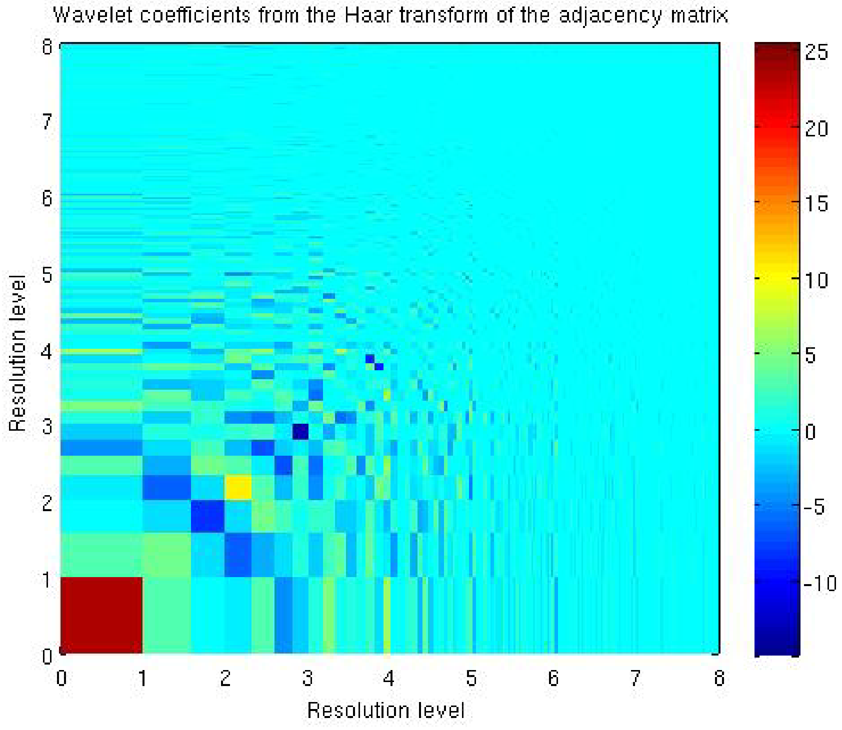
The network

Adjacency matrix



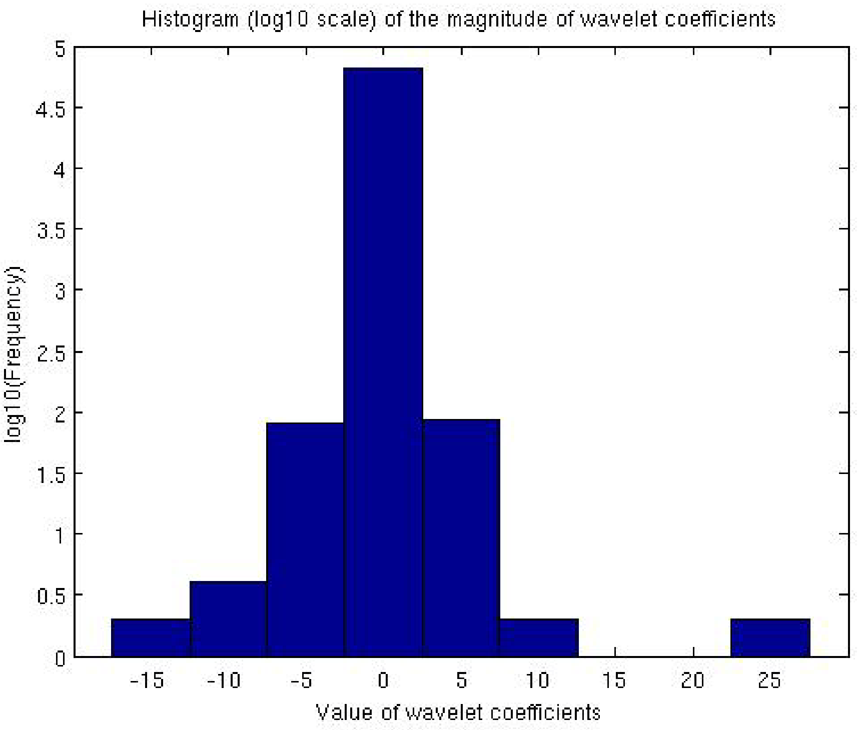
256 nodes, 256^2 wavelet coefficients

A graph and its multiscale decomposition - II



Wavelet coefficients at different resolutions

Histogram of coefficient values
(log10(Frequency))

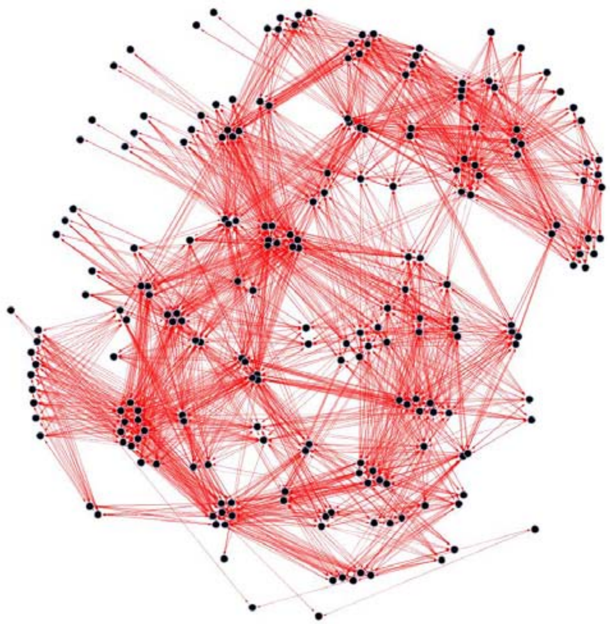


256 nodes, 256^2 wavelet coefficients

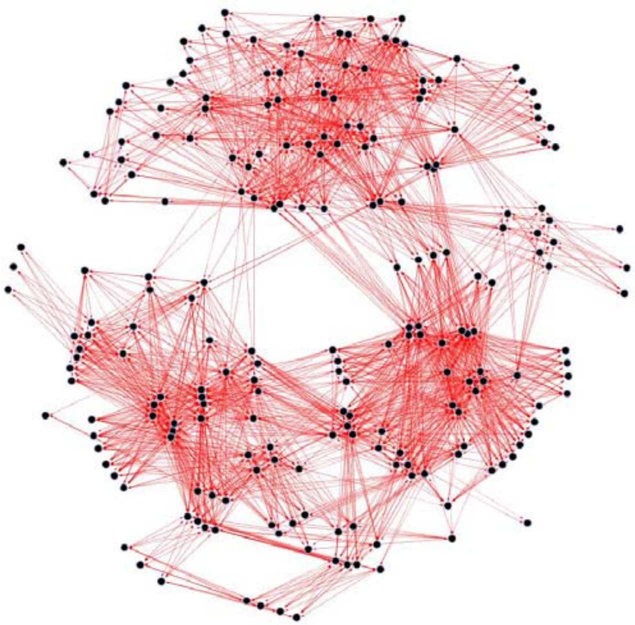
Sampling issues

- The number of wavelet coefficients at level l is 4^l
- How many samples per level?
  - Generally expressed as a fraction of 4^l
  - Should this fraction vary with levels
- Some observations
  - Sampling has to be minimal at high resolutions
    - i.e. fine detail cannot be captured
    - Emphasis on “blockiness”, structure should be evident at a certain resolution
  - All wavelet coefficients at coarser resolutions can be retained
- So CS can be expected to work for graphs that are somewhat dense
  - Will not work for very sparse graphs

Tests



Network I



Network II

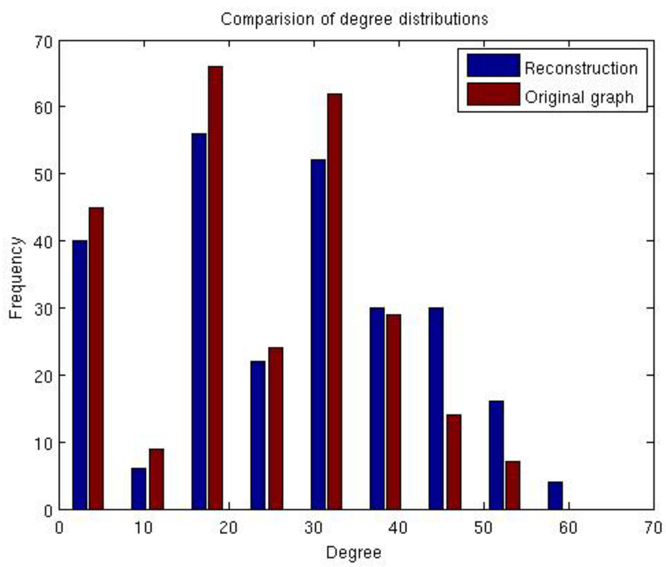
- **2 synthetic directed graphs, of 256 nodes. Average degree of 23 & 17**
- **Sample (to various degrees) and reconstruct**
- **Performance wrt number of samples and sparsity of graph**

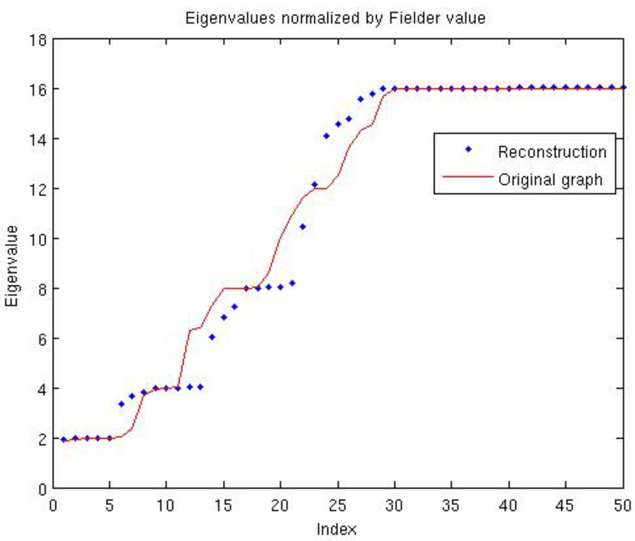
Test I(a) – fine sampling (60%)

- **Network I, with 256 nodes and 5914 edges**
  - **Needs $2 \cdot N_{edges}$ to store**
- **Reconstruction with ~60% sampling**

	Original	Recons-truction
No. of samples	11,824	7463 (~60%)
Average degree	23.1	27.2
Fiedler Value	1.85	1.91
No. edges	5914	6924

Degree distribution



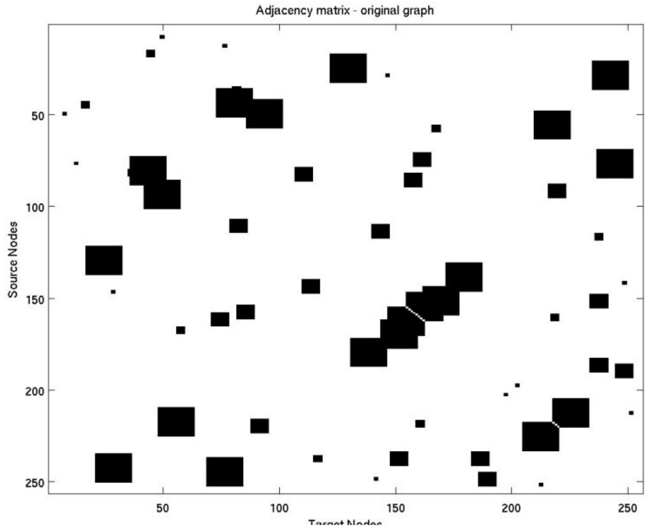


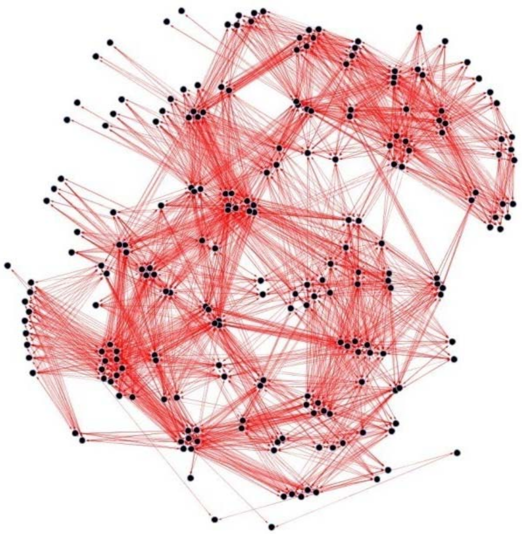
Eigenvalues of the Laplacian

Sandia National Laboratories

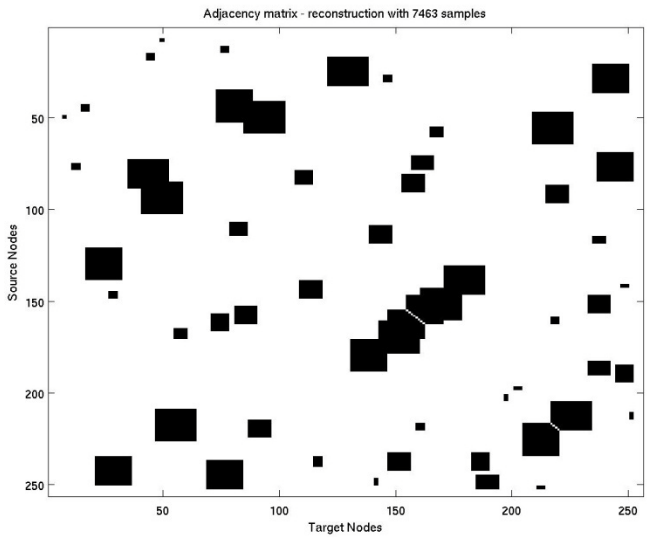
Test I(a) – comparison of nets

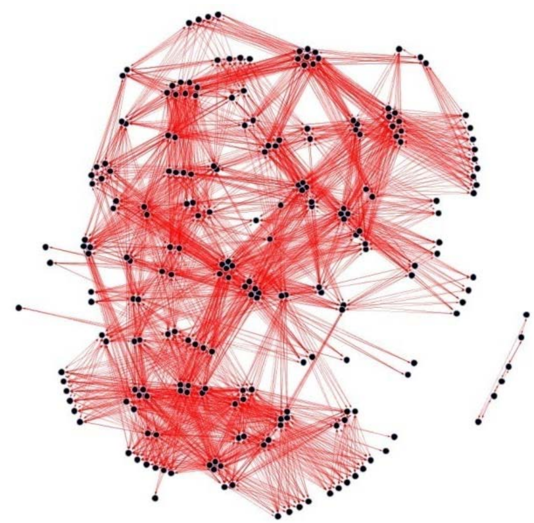
Original



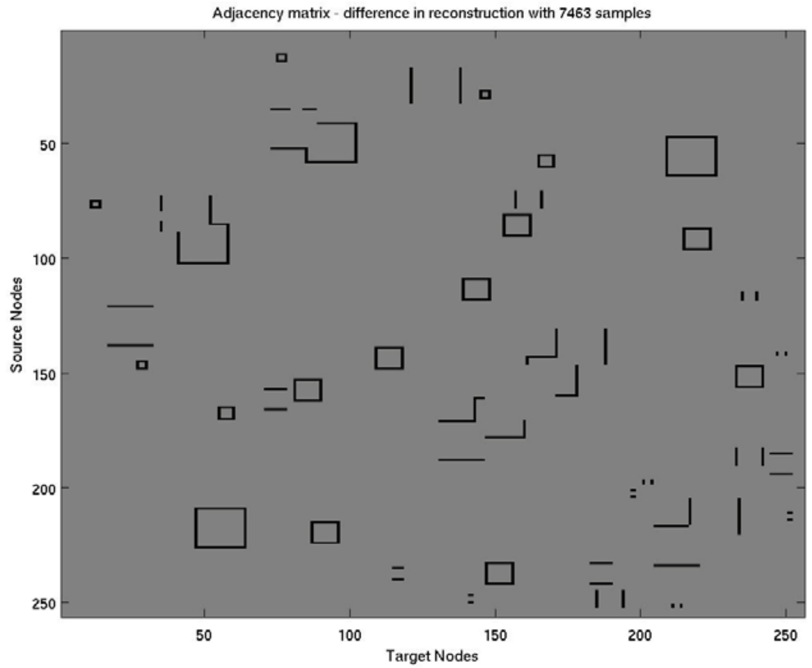


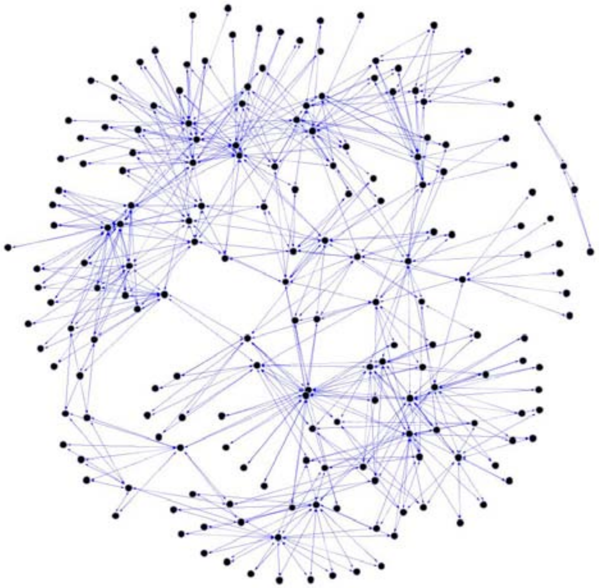
Reconstruction





Test I(a) – analysis of differences





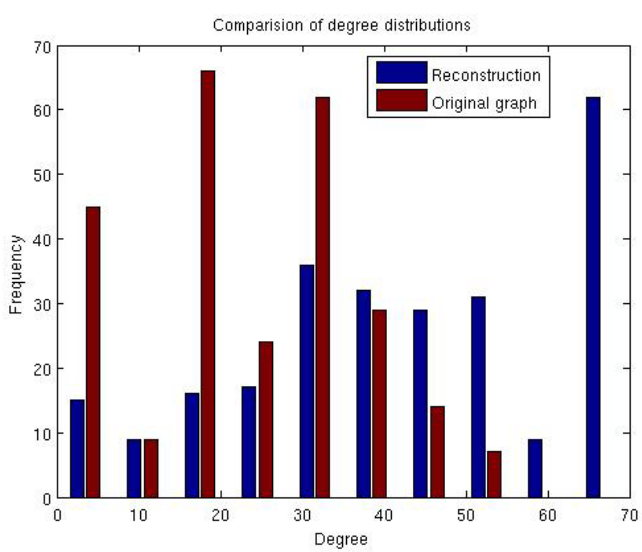
- **Difference of adjacency matrices**
  - **Small differences around the blocks**
  - **Normalized error (Frobenius norm) ~ 42%**
  - **Red edge: false negative; Blue edge: false positive**

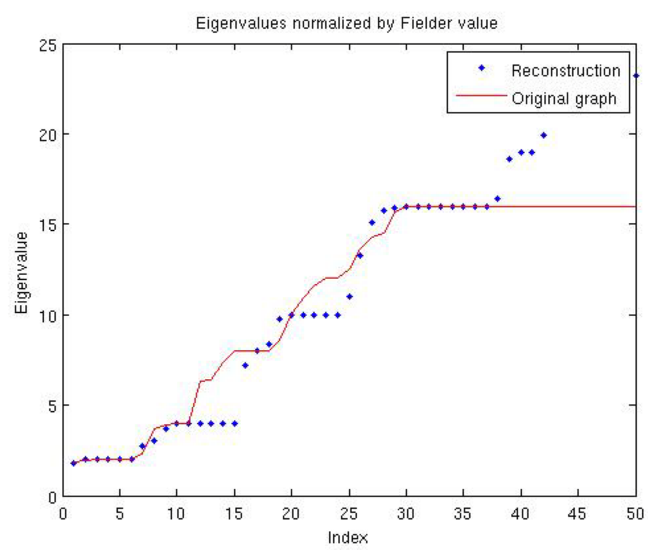
Test I(b) – coarse sampling (25%)

- **Network I, with 256 nodes and 5914 edges**
  - **Needs $2*N_{edges}$ to store**
- **Reconstruction with ~ 25% sampling**

	Original	Recons-truction
No. of samples	11,824	2478 (~25%)
Average degree	23.1	44.6
Fiedler Value	1.85	1.81
No. edges	5914	11424

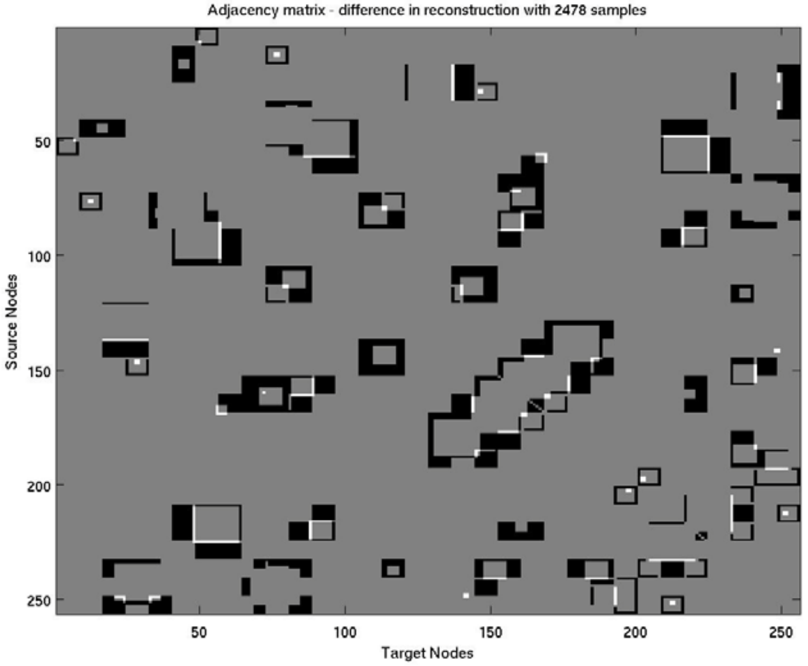
Degree distribution

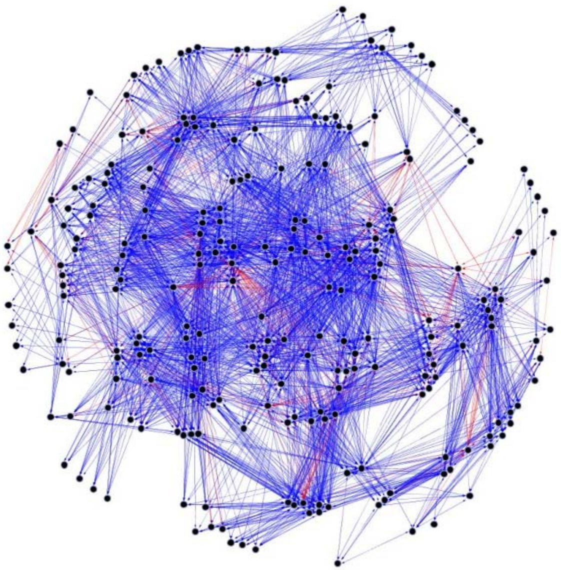




Eigenvalues of the Laplacian

Test I(b) – analysis of differences





- **Difference of adjacency matrices**
  - **Significant structural differences**
  - **Normalized error (Frobenius norm) ~ 100%**
  - **Red edge: false negative; Blue edge: false positive**

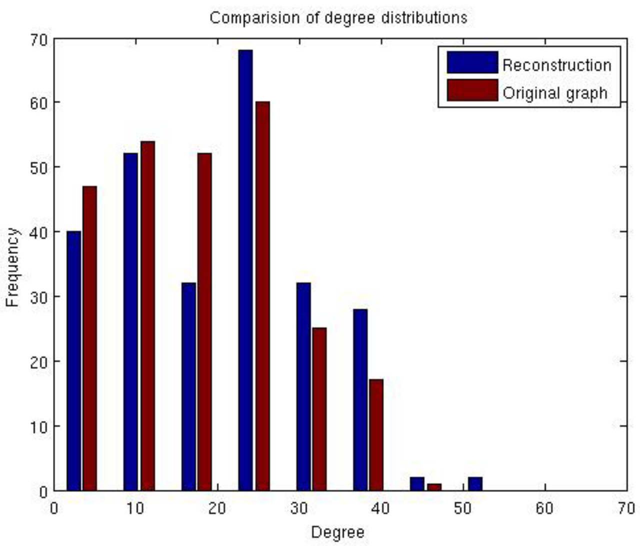
Sandia National Laboratories

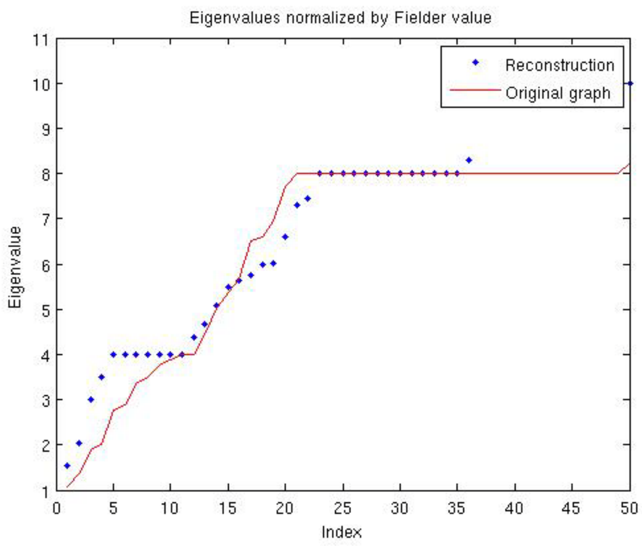
Test II - a sparser net

- **Network II, with 256 nodes and 4426 edges**
- **Reconstruction with ~ 85% sampling**

	Original	Recons-truction
No. of samples	8852	7589 (~85%)
Average degree	17.29	20.1
Fiedler Value	1.08	1.54
No. edges	4426	5144

Degree distribution

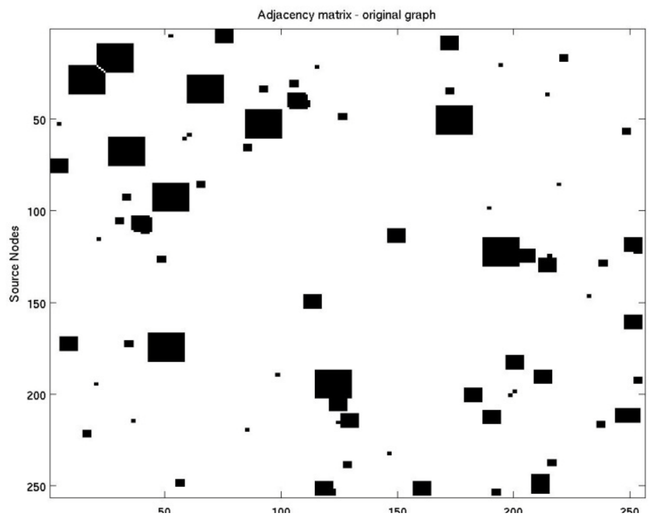


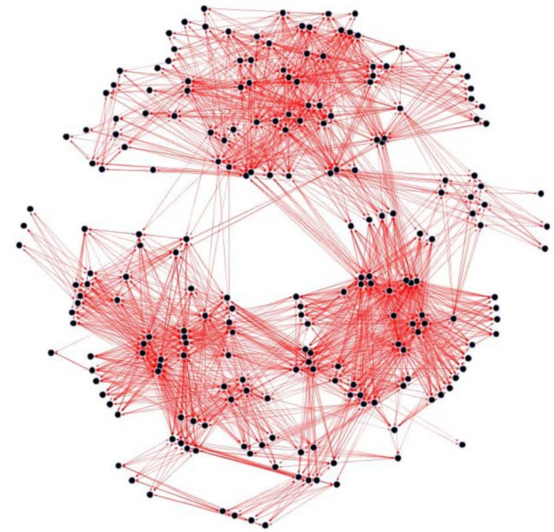


Eigenvalues of the Laplacian

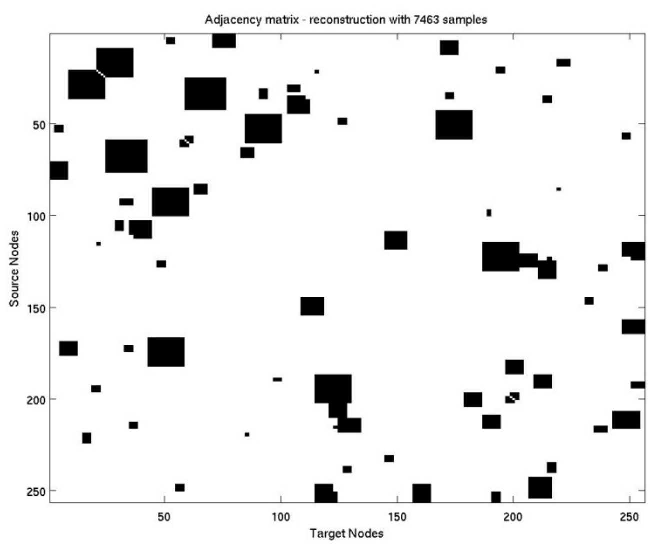
Test II – comparison of nets

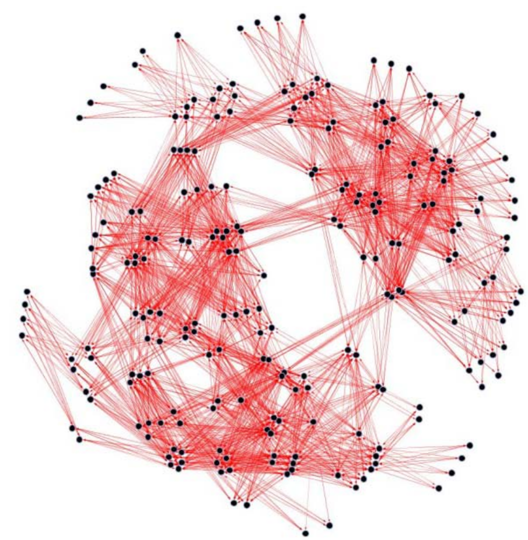
Original



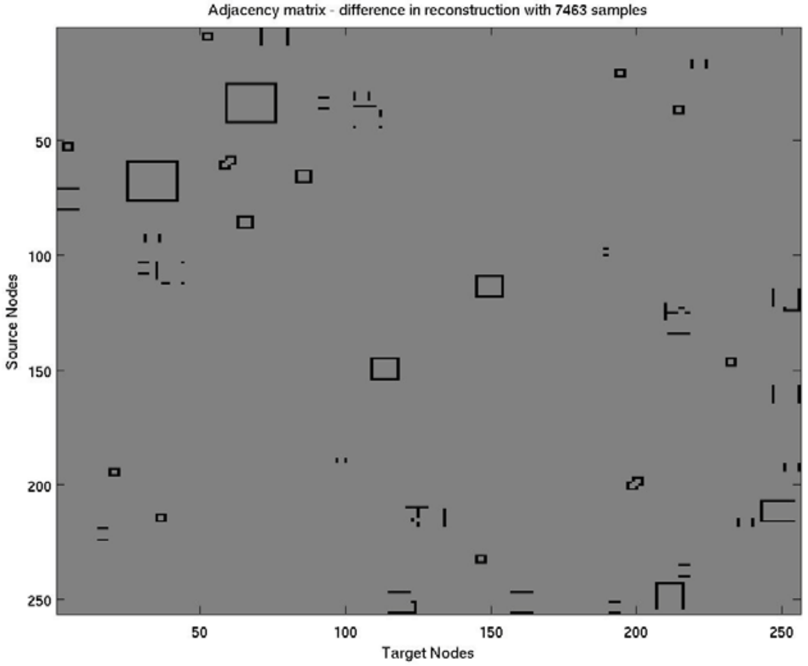


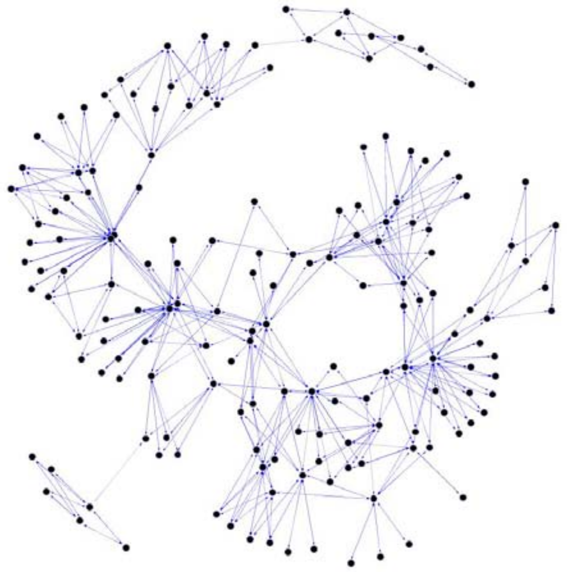
Reconstruction





Test II – analysis of differences





- **Difference of adjacency matrices**
  - **Small differences around the blocks**
  - **Normalized error (Frobenius norm) ~ 40%**
  - **Red edge: false negative; Blue edge: false positive**

Sandia National Laboratories

Summary of the tests

- Define: Average link probability = average degree / # of nodes
- For an average link probability of around 10%:
  - 60% sampling gives excellent reconstruction
  - 25% sampling leads to over estimation of average degree
    - i.e., the reconstructed graph is very coarse & lacks detail
- For an average link probability of around 7%:
  - The technique requires too many samples (~85%) and is not competitive
- In general, matching the eigenvalue spectrum is easy
  - Fiedler value less so, but getting to +/- 10% is possible
- Matching the degree distribution is harder
  - 25% sampling does not do it
  - 60% or higher does it, depending upon the average link probability

Sandia National Laboratories

Summary and Conclusions

- CS provides a new way of sampling and reconstructing networks
- Approach based on multiresolution decomposition of the adjacency matrix and its efficient sampling
- Requires preprocessing of the adjacency matrix to make it "blocky"
  - Biggest (combinatorial) algorithm challenge.
- Current CS reconstruction algorithm makes no use of the structure of a graph – very general (and so not very efficient/customized)
  - Other model-based CS techniques exist, but not yet adapted to networks
  - Obvious starting point for future work to increase the efficiency of reconstruction