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Energy Balance for a Moving Defect in a Peridynamic Solid

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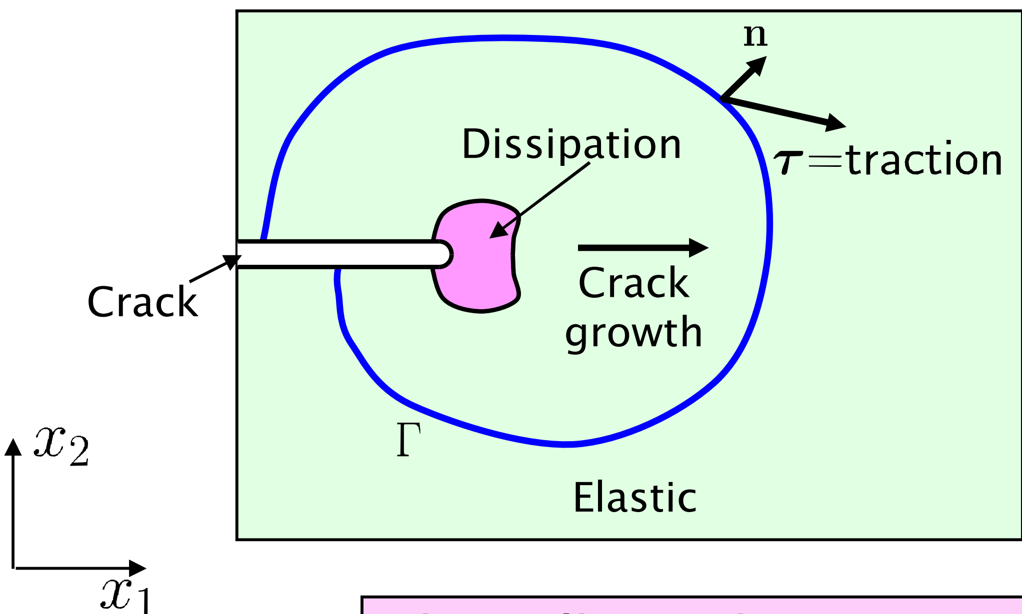
Energy flow into a defect: Results from the local theory

- The energy dissipated by a moving defect can be obtained from the elastic fields far from the defect.
  - Don't need to know all the small-scale details.
- **Eshelby** (1956): 3D energy-momentum tensor and its surface integral.
- **Rice** (1968): 2D J-integral and its relation to plastic flow.
- **Knowles & Sternberg** (1972): Obtained J-integral from Noether's theorem.



$$J = \int_\Gamma \left[W n_2 - \boldsymbol{\tau} \cdot \frac{\partial \mathbf{u}}{\partial x_1} \right] d\ell$$

= energy per unit crack area
= "driving force" on defect

This talk: Find a J-integral in peridynamics.

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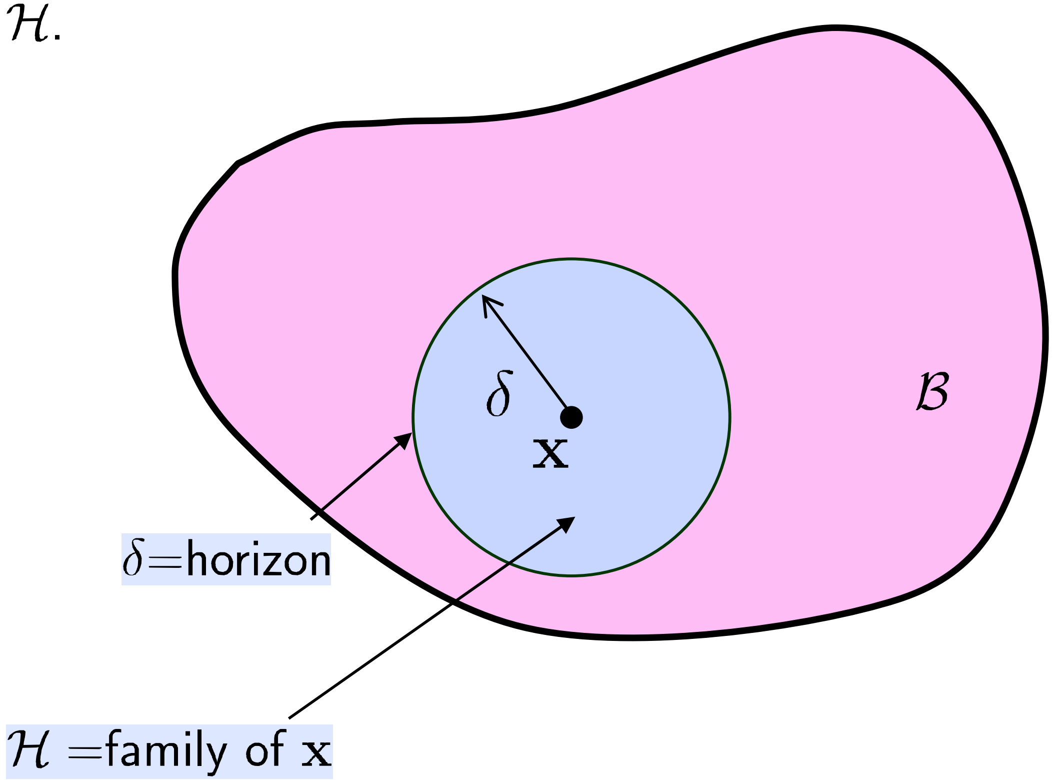
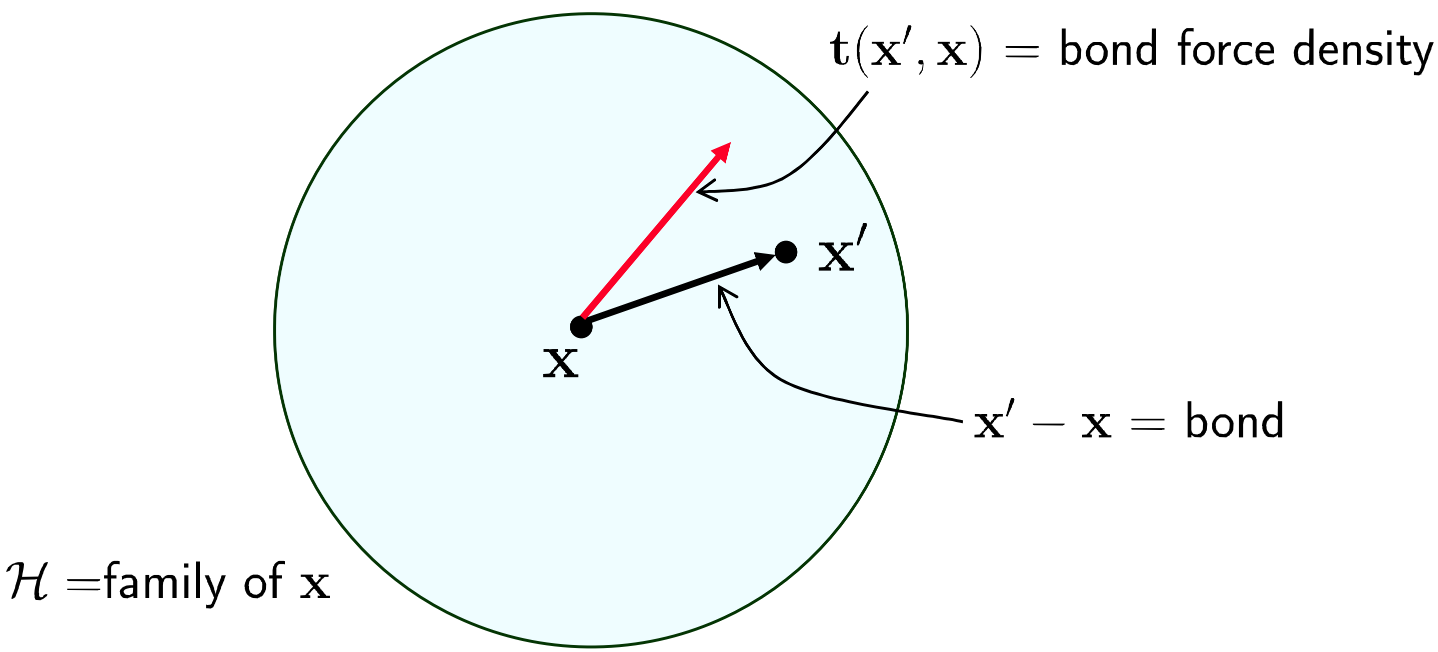
Peridynamics basics: Horizon and family

- Any point $\mathbf{x}$ interacts directly with other points within a finite distance δ called the "horizon."
- The material within a distance δ of $\mathbf{x}$ is called the "family" of $\mathbf{x}$, $\mathcal{H}$.



Bond forces

- A force density $\mathbf{t}(\mathbf{x}', \mathbf{x})$ is associated with each bond in the family of $\mathbf{x}$.
- Dimensions of $\mathbf{t}$ are force/volume^2.
- $\mathbf{t}$ is not necessarily parallel to the deformed bond.



Bond forces are determined by the collective deformation of the family

- The *deformation state* $\underline{\mathbf{Y}}$ is the function that maps bonds to their deformed images.

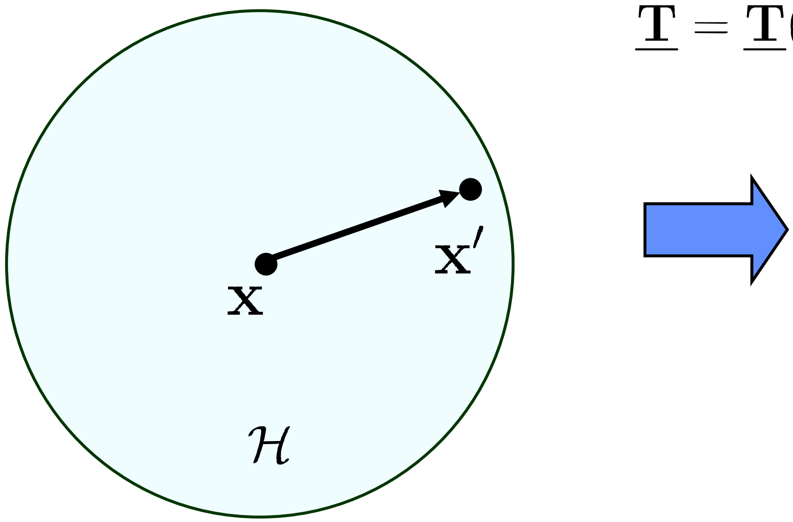
$$\mathbf{y}' - \mathbf{y} = \underline{\mathbf{Y}}\langle \mathbf{x}' - \mathbf{x} \rangle.$$

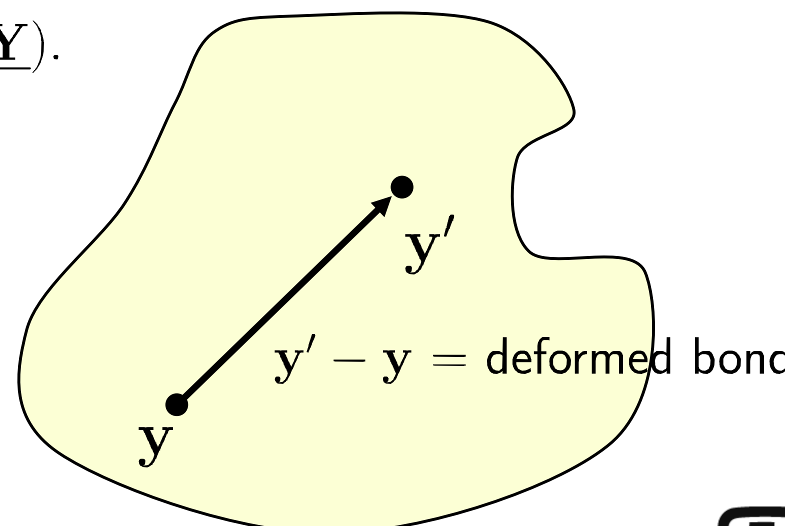
- The *force state* $\underline{\mathbf{T}}$ is the function that maps bonds to bond forces.

$$\mathbf{t}(\mathbf{x}', \mathbf{x}) = \underline{\mathbf{T}}\langle \mathbf{x}' - \mathbf{x} \rangle.$$

- The *constitutive model* $\underline{\hat{\mathbf{T}}}$ relates $\underline{\mathbf{T}}$ and $\underline{\mathbf{Y}}$:

$$\underline{\mathbf{T}} = \underline{\hat{\mathbf{T}}}(\underline{\mathbf{Y}}).$$



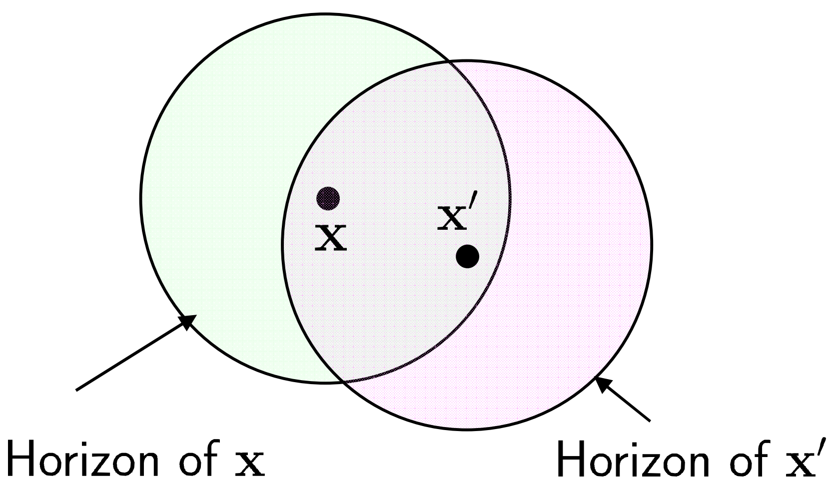


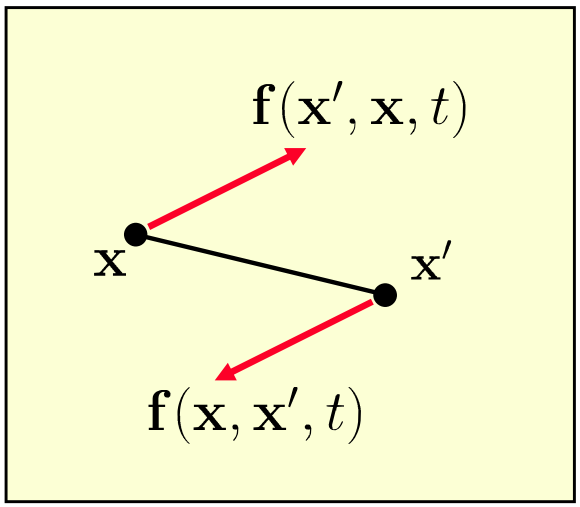
Peridynamic equation of motion

- At any point $\mathbf{x}$ in the reference configuration of the body $\mathcal{B}$:

$$\rho(\mathbf{x})\ddot{\mathbf{y}}(\mathbf{x},t) = \int_{\mathcal{H}} \big(\mathbf{t}(\mathbf{x}',\mathbf{x},t) - \mathbf{t}(\mathbf{x},\mathbf{x}',t)\big)\, dV' + \mathbf{b}(\mathbf{x},t)$$

Dual force density $\mathbf{f}(\mathbf{x}',\mathbf{x},t)$





Energy balance at a point

- First law expression for peridynamics:

$$\dot{\varepsilon} = \underline{\mathbf{T}} \bullet \dot{\underline{\mathbf{Y}}} + h + r$$

where ε=internal energy density, h=heat transport rate, r=energy source rate.

- Dot product represents, at any $\mathbf{x}$,

$$\underline{\mathbf{T}} \bullet \dot{\underline{\mathbf{Y}}} = \int_{\mathcal{H}} \mathbf{t}(\mathbf{x}', \mathbf{x}, t) \cdot \big(\mathbf{v}(\mathbf{x}', t) - \mathbf{v}(\mathbf{x}, t)\big) \, dV'$$

- Assume an adiabatic, source-free process: $h = r = 0$.

- The term $\underline{\mathbf{T}} \bullet \dot{\underline{\mathbf{Y}}}$ is called the *absorbed power density* (peridynamic version of stress power $\boldsymbol{\sigma} \cdot \dot{\mathbf{F}}$).

Energy balance for a subregion

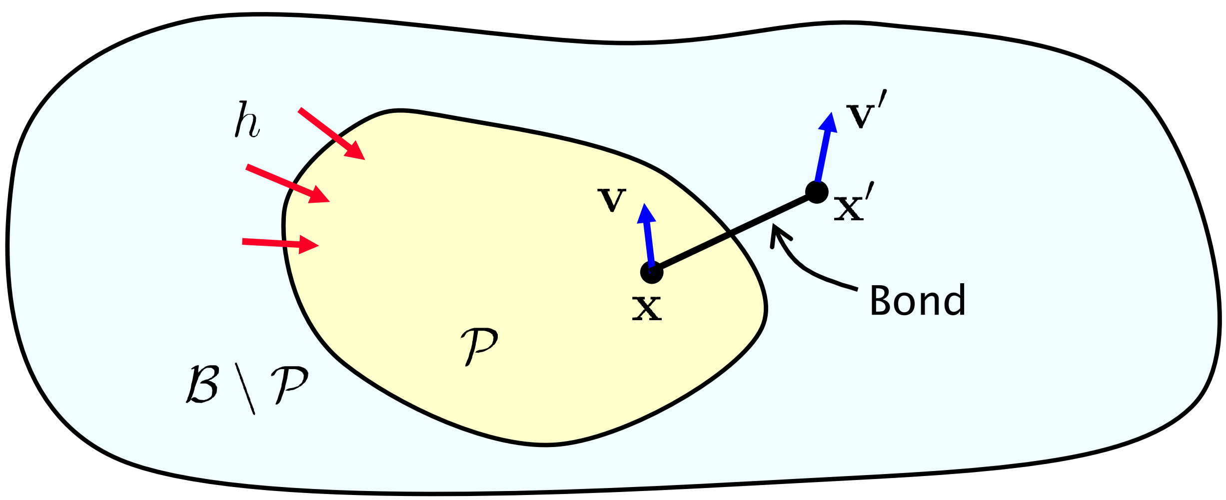
- Global first law expression in peridynamics for a subregion $\mathcal{P} \subset \mathcal{B}$:

$$\frac{d}{dt}\int_{\mathcal{P}}\left(\varepsilon + \frac{\rho \mathbf{v}\cdot\mathbf{v}}{2}\right)\, dV = \int_{\mathcal{P}}\int_{\mathcal{B}\setminus\mathcal{P}} (\mathbf{t}\cdot\mathbf{v}' - \mathbf{t}'\cdot\mathbf{v})\, dV'\, dV + \int_{\mathcal{P}} (h+r)\, dV + \int_{\mathcal{P}} \mathbf{b}\cdot\mathbf{v}\, dV$$

where following short notation is used:

$$\mathbf{t} = \mathbf{t}(\mathbf{x}', \mathbf{x}, t), \qquad \mathbf{v} = \mathbf{v}(\mathbf{x}, t)$$

$$\mathbf{t}' = \mathbf{t}(\mathbf{x}, \mathbf{x}', t), \qquad \mathbf{v}' = \mathbf{v}(\mathbf{x}', t).$$



Free energy and the force state

- Free energy is defined by

$$\psi = \varepsilon - \theta\eta$$

where θ=temperature, η=entropy density.

- Assume ψ has the following dependencies:

$$\psi(\underline{\mathbf{Y}}, \theta, \underline{\phi})$$

where $\underline{\phi}$ is the *damage state* (next slide).

- Can show by Coleman-Noll procedure that

$$\underline{\mathbf{T}} = \psi_{\underline{\mathbf{Y}}}$$

where $\psi_{\underline{\mathbf{Y}}}$ is the Fréchet derivative of ψ with respect to $\underline{\mathbf{Y}}$.

Damage state

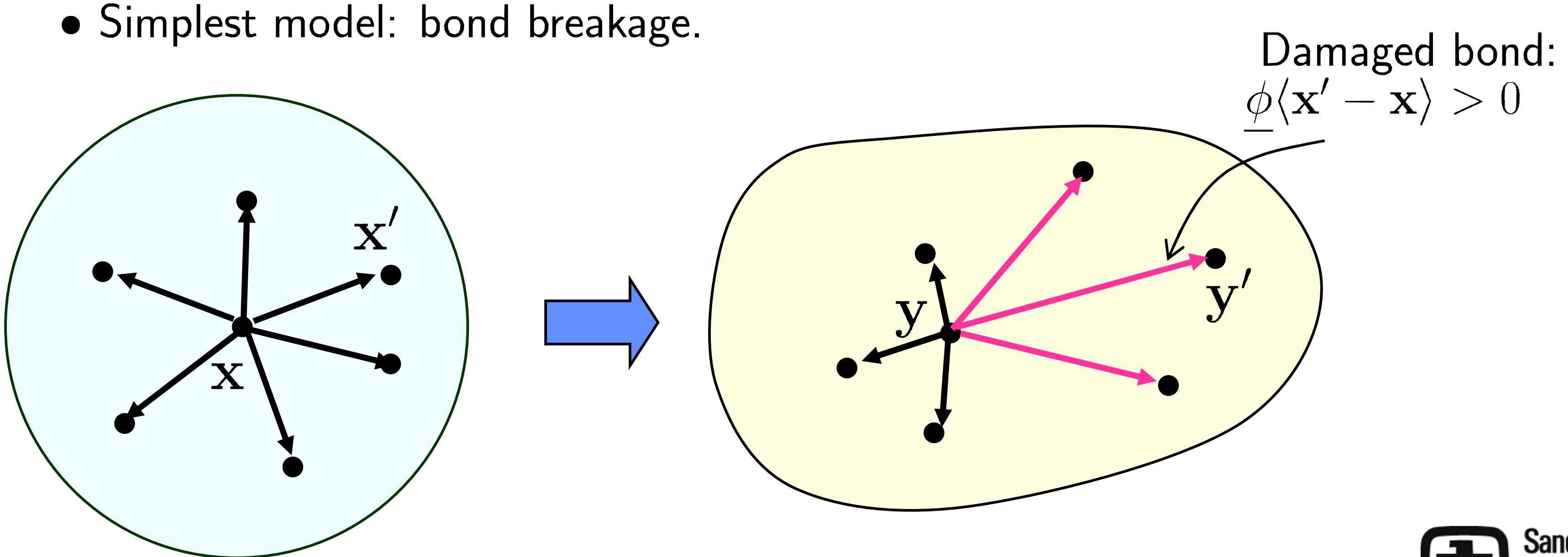
- The damage state increases monotonically for each bond $\mathbf{x}' - \mathbf{x}$:

$$\dot{\underline{\phi}}\langle \mathbf{x}' - \mathbf{x} \rangle \geq 0, \qquad 0 \leq \underline{\phi}\langle \mathbf{x}' - \mathbf{x} \rangle \leq 1.$$

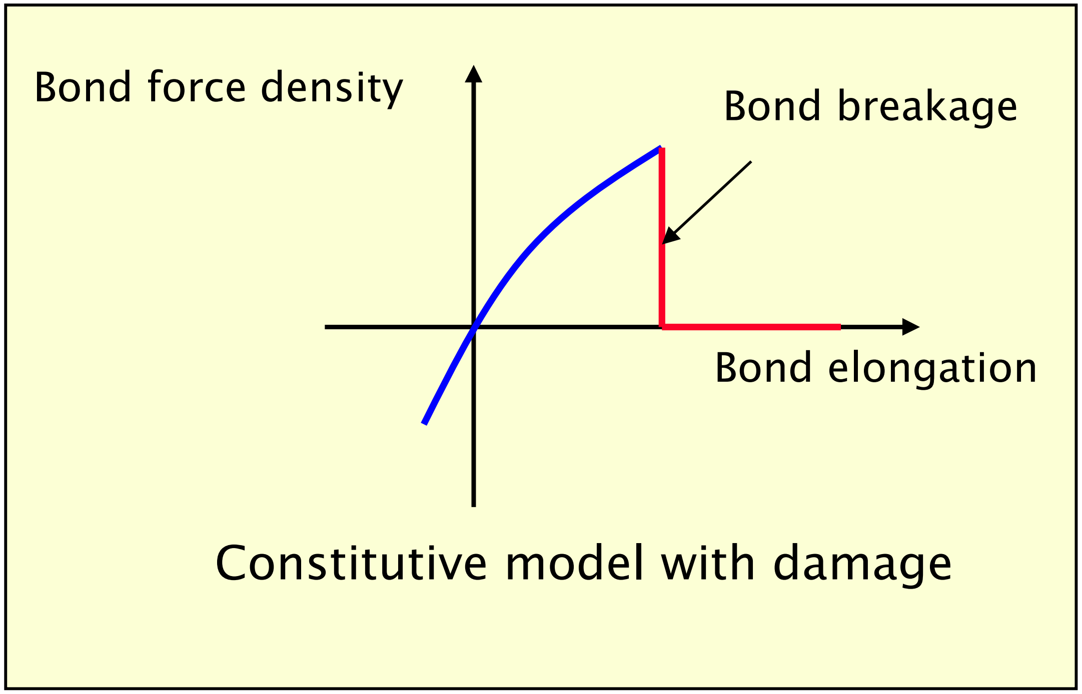
  according to some *damage evolution law*:

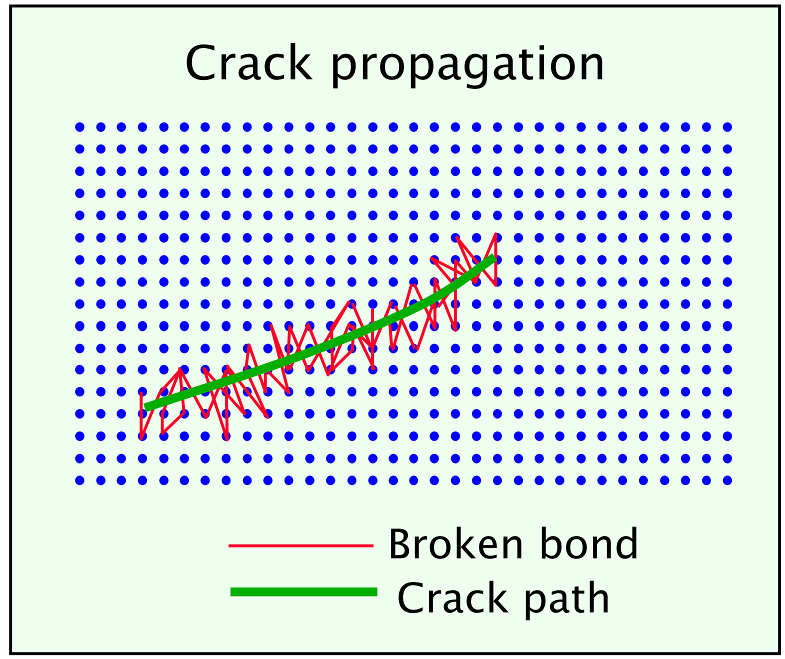
$$\underline{\phi} = \underline{D}(\underline{\mathbf{Y}}, \dot{\underline{\mathbf{Y}}}, \dots).$$

- Details of this are not important for present purposes.
- Simplest model: bond breakage.

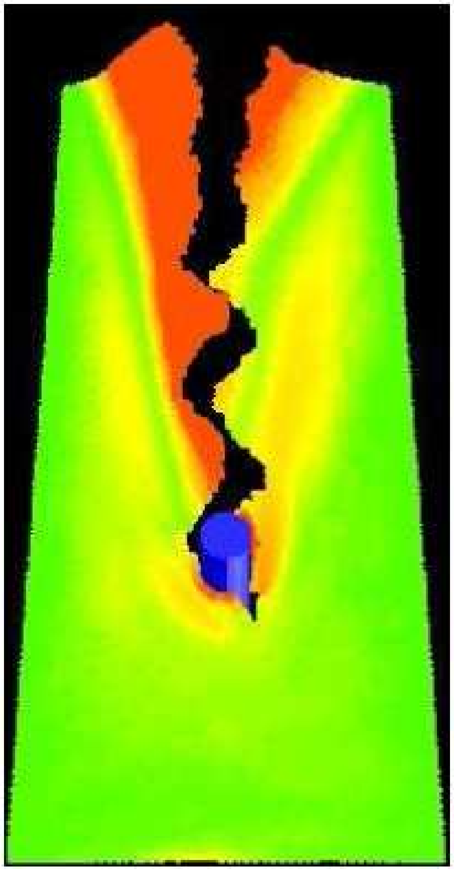


Bond breakage and progressive failure

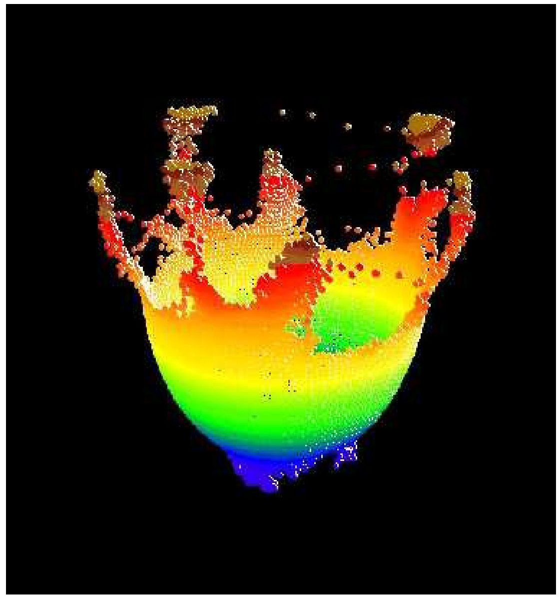




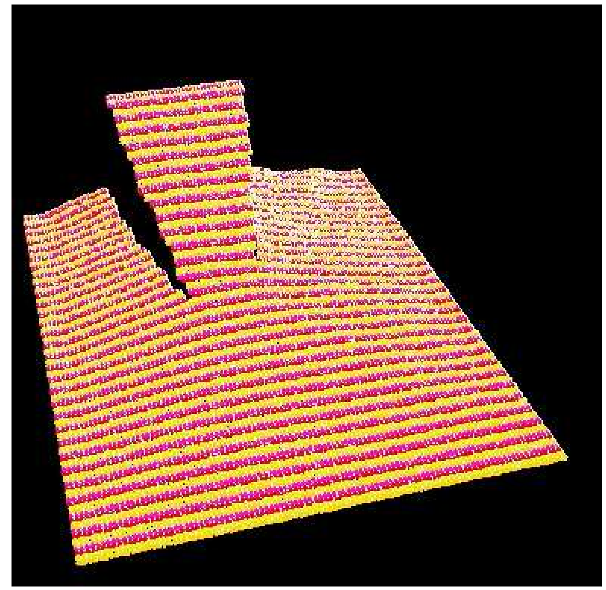
Damage leads to fracture



Tearing instability



Balloon pop



Peeling

Main advantage of peridynamics for crack modeling

Crack growth is autonomous: same field equations apply on or off of a discontinuity.

Entropy production and energy dissipation

- Again using Coleman-Noll, the rate of entropy production is

$$\dot{\eta} = \frac{\dot{\psi}^{\mathrm{d}}}{\theta}$$

where the *rate of energy dissipation* is given by

$$\begin{aligned} \dot{\psi}^{\mathrm{d}} &= -\psi_{\underline{\phi}} \bullet \underline{\dot{\phi}} \\ &:= -\int_{\mathcal{H}} \psi_{\underline{\phi}}\langle \mathbf{x}' - \mathbf{x} \rangle \, \underline{\dot{\phi}}\langle \mathbf{x}' - \mathbf{x} \rangle \, dV' \end{aligned}$$

where $\underline{\phi}$ is the Fréchet derivative of ψ with respect to $\underline{\phi}$.

- For an isothermal process, therefore

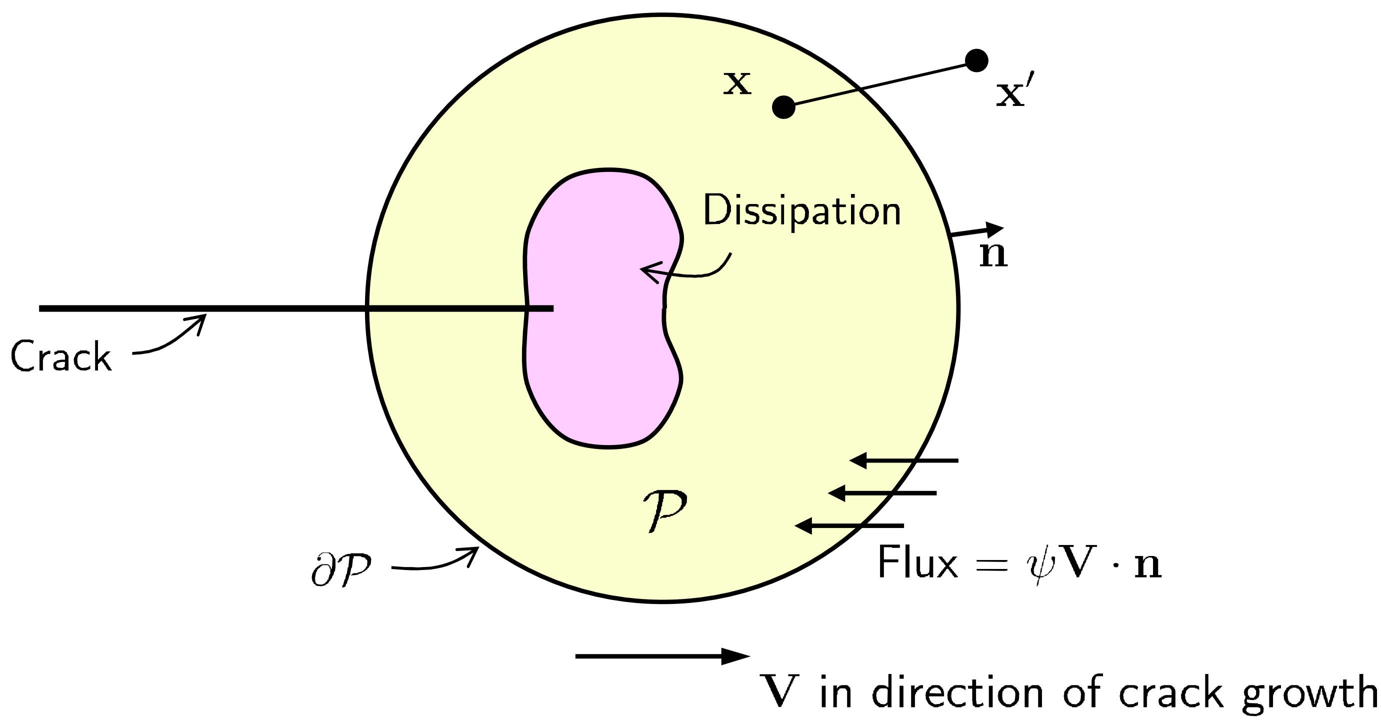
$$\dot{\psi} = \dot{\varepsilon} - \theta\dot{\eta}$$

so that

$$\dot{\psi} = \dot{\varepsilon} - \dot{\psi}^{\mathrm{d}}.$$

Analyze dissipation of energy near a defect

- Assume a homogeneous body.
- Assume a constant defect velocity .
- $\mathcal{P}$ moves with the defect through the reference configuration $\mathcal{B}$.

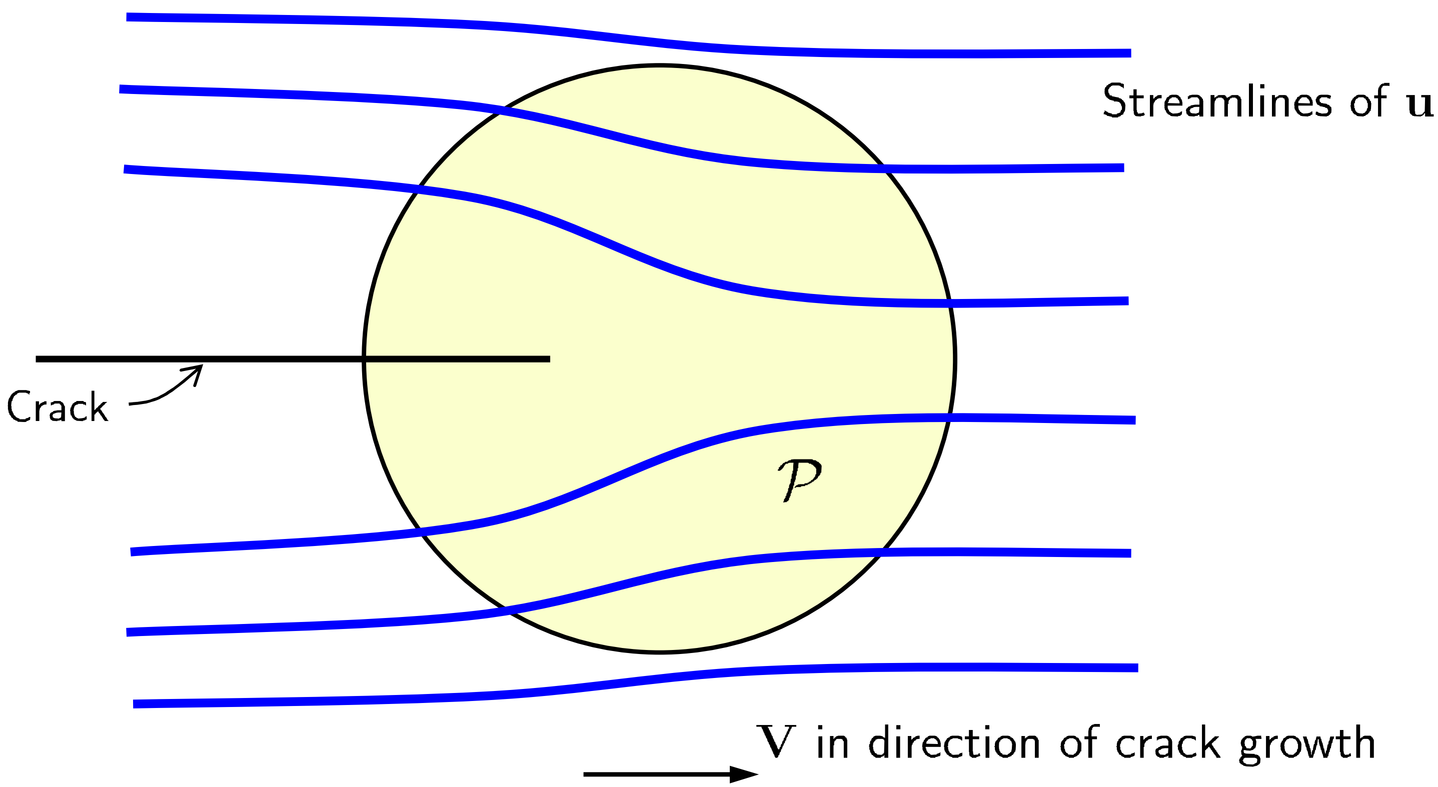


Steady–state motion

- Assumed motion is

$$\mathbf{y}(\mathbf{x}, t) = \mathbf{x} + \mathbf{u}(\mathbf{x} - \mathbf{V}t)$$

where $\mathbf{u}$ is a given function and $|\mathbf{V}|$ is small.



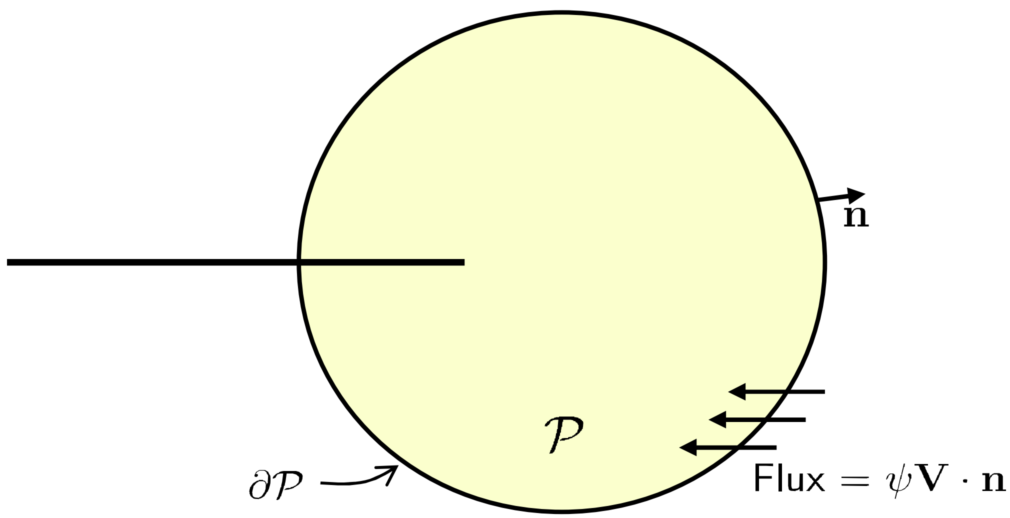
Free energy balance

- Reynolds transport theorem implies

$$\frac{d}{dt}\int_{\mathcal{P}} \psi \, dV = \int_{\mathcal{P}} \dot{\psi} \, dV + \int_{\partial\mathcal{P}} \psi \mathbf{V} \cdot \mathbf{n} \, dA$$

but steady-state implies

$$\frac{d}{dt}\int_{\mathcal{P}} \psi \, dV = 0.$$



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Use first law to compute nonlocal work done across the boundary

- Recall

$$\dot{\psi} = \dot{\varepsilon} - \dot{\psi}^{\mathrm{d}}.$$

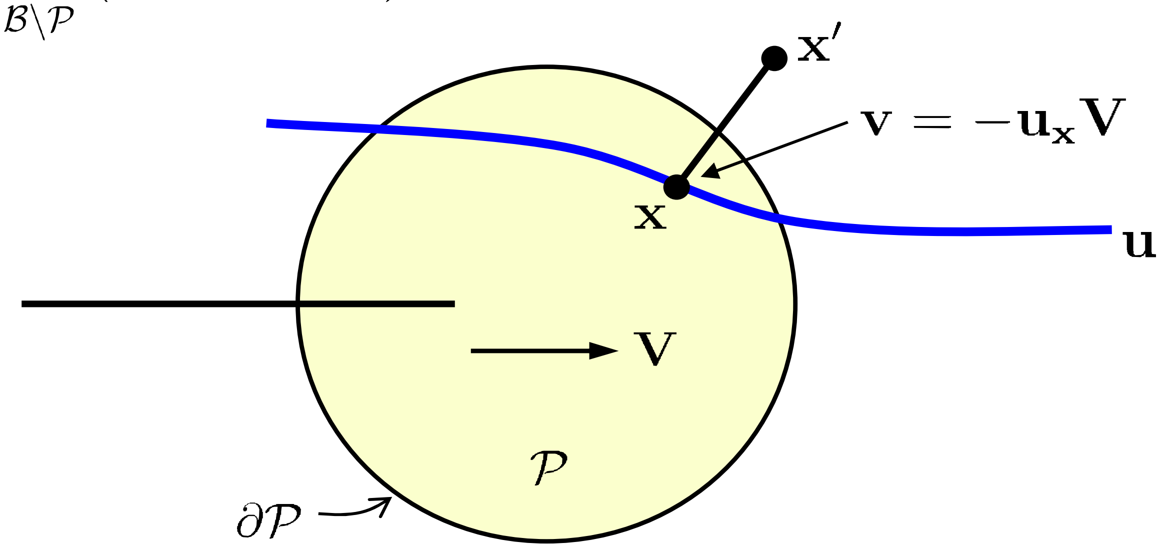
hence

$$\int_{\mathcal{P}} \left(\dot{\varepsilon} - \dot{\psi}^{\mathrm{d}}\right) dV + \int_{\partial\mathcal{P}} \psi \mathbf{V}\cdot\mathbf{n}\, dA = 0$$

- Global first law under present assumptions reduces to

$$\int_{\mathcal{P}} \dot{\varepsilon}\, dV = \int_{\mathcal{P}} \int_{\mathcal{B}\setminus\mathcal{P}} \left(\mathbf{t}\cdot\mathbf{v}' - \mathbf{t}'\cdot\mathbf{v}\right) dV'\, dV$$

$$\boxed{\begin{aligned} \mathbf{u}_{\mathbf{x}} &= \mathrm{grad}\; \mathbf{u}(\mathbf{x}) \\ \mathbf{u}'_{\mathbf{x}} &= \mathrm{grad}\; \mathbf{u}(\mathbf{x}') \end{aligned}}$$



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Total rate of energy dissipation

- Eliminate $\dot{\varepsilon}$ term to find

$$\int_{\mathcal{P}} \dot{\psi}^{\mathrm{d}}\, dV = \int_{\mathcal{P}} \int_{\mathcal{B}\setminus\mathcal{P}} \big(\mathbf{t}\cdot(-\mathbf{u}'_{\mathbf{x}}\mathbf{V}) - \mathbf{t}'\cdot(-\mathbf{u}_{\mathbf{x}}\mathbf{V})\big)\, dV'\, dV + \int_{\partial\mathcal{P}} \psi \mathbf{V}\cdot\mathbf{n}\, dA$$

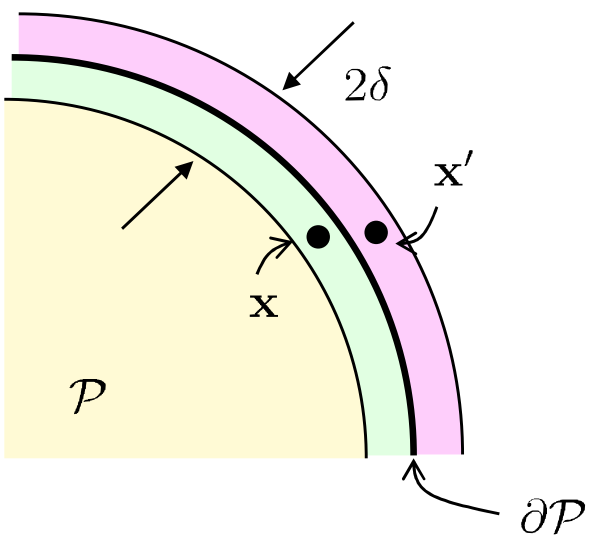
or

$$\int_{\mathcal{P}} \dot{\psi}^{\mathrm{d}}\, dV = \mathbf{J}\cdot\mathbf{V}$$

where

$$\boxed{\mathbf{J} = \int_{\mathcal{P}} \int_{\mathcal{B}\setminus\mathcal{P}} \big(\mathbf{u}_{\mathbf{x}}^T \mathbf{t}' - (\mathbf{u}'_{\mathbf{x}})^T \mathbf{t}\big)\, dV'\, dV + \int_{\partial\mathcal{P}} \psi \mathbf{n}\, dA}$$

Peridynamic J-integral (3D)

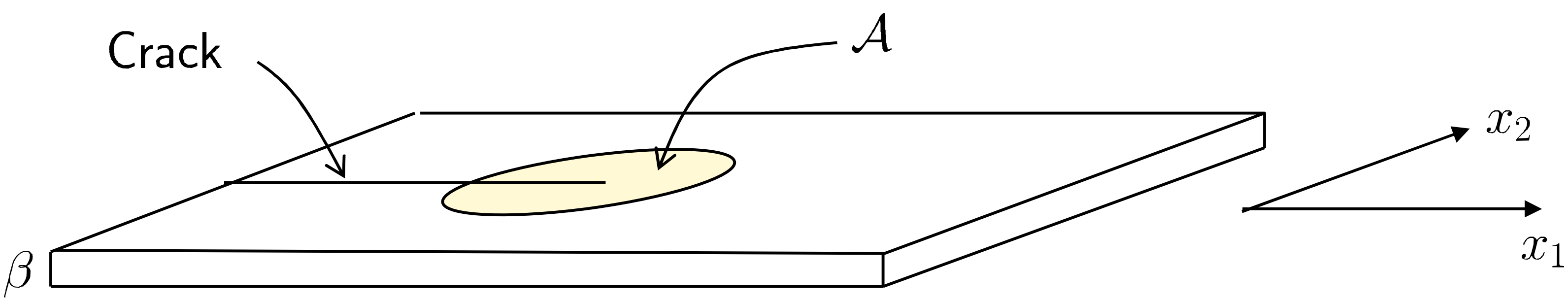


Integrand is nonzero only if $\mathbf{x}$ and $\mathbf{x}'$ are sufficiently close to $\partial\mathcal{P}$.

Crack in a plate

- Apply to a plate of thickness β. $\mathcal{A}$ is the interior of a curve in the plane.
- Assume crack grows in the x_1 direction.

$$J_1 = \beta^2 \int_{\mathcal{A}} \int_{\mathcal{B}\setminus\mathcal{A}} \left(\frac{\partial \mathbf{u}}{\partial x_1} \cdot \mathbf{t}' - \frac{\partial \mathbf{u}'}{\partial x_1} \cdot \mathbf{t} \right) \, dA' \, dA + \beta \int_{\partial \mathcal{A}} \psi n_2 \, ds$$



Crack in a plate: Limit of small horizon

- Small horizon:

$$\frac{\partial \mathbf{u}}{\partial x_1} \approx \frac{\partial \mathbf{u}'}{\partial x_1}$$

  hence

$$J_1 \approx \beta^2 \int_{\mathcal{A}} \frac{\partial \mathbf{u}}{\partial x_1} \cdot \left[\int_{\mathcal{B} \setminus \mathcal{A}} (\mathbf{t}' - \mathbf{t}) \, dA' \right] \, dA + \beta \int_{\partial \mathcal{A}} \psi n_2 \, ds$$

$$= \beta \int_{\partial \mathcal{A}} \left[-\frac{\partial \mathbf{u}}{\partial x_1} \cdot \boldsymbol{\tau} \, ds + \psi n_2 \right] \, ds$$

  where $\boldsymbol{\tau}$ is the traction vector on $\partial \mathcal{A}$.

- This is the same as Rice's J-integral in the standard theory (except for factor of β).

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Summary

$$\boxed{\mathbf{J} = \int_{\mathcal{P}} \int_{\mathcal{B}\setminus\mathcal{P}} \left(\mathbf{u}_{\mathbf{x}}^T \mathbf{t}' - (\mathbf{u}'_{\mathbf{x}})^T \mathbf{t}\right) dV' \, dV + \int_{\partial\mathcal{P}} \psi \mathbf{n} \, dA}$$

- Directly computed the free energy dissipated by a defect based on the first law and on steady-state assumptions.
- Did not need to assume anything about the physical dissipative mechanism.
- Did not assume that dissipation is confined to a small process zone.
- Defect may or may not involve a discontinuity in $\mathbf{u}$ (consistent with the "spirit of peridynamics").
- For more info: SS & RL, "Peridynamic Theory of Solid Mechanics," to appear in *Advances in Applied Mechanics*, vol. 44 (2010).