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Variable Horizon in a Peridynamic Medium

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Abstract

A notion of material homogeneity is proposed for peridynamic bodies with variable horizon but constant bulk properties. A relation is derived that scales the force state according to the position-dependent horizon while keeping the bulk properties unchanged. Using this scaling relation, if the horizon depends on position, artifacts called ghost forces may arise in a body under homogeneous deformation. These artifacts depend on the second derivative of horizon and can be reduced by use of a modified equilibrium equation using a new quantity called the *partial stress*. Bodies with piecewise constant horizon can be modeled without ghost forces by using a technique called a *splice* between the regions. As a limiting case of zero horizon, both partial stress and splice techniques can be used to achieve local-nonlocal coupling. Computational examples, including dynamic fracture in a one-dimensional model with local-nonlocal coupling, illustrate the methods.

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Contents

1 Introduction

The peridynamic theory is a strongly nonlocal formulation of solid mechanics that is adapted to the study of continuous bodies with evolving discontinuities, including cracks, and long-range forces [16, 20]. Each material point $\mathbf{x}$ in the reference configuration of a body $\mathcal{B}$ interacts through the material model with other material points within a distance δ of itself. The maximum interaction distance δ is called the *horizon*, and the material within the horizon of $\mathbf{x}$ is called the *family* of $\mathbf{x}$, which is denoted $\mathcal{H}_\mathbf{x}$. The vector from $\mathbf{x}$ to any neighboring material point $\mathbf{q} \in \mathcal{H}_\mathbf{x}$ is called a *bond*, $\boldsymbol{\xi} = \mathbf{q} - \mathbf{x}$.

In an elastic peridynamic solid, the strain energy density $W(\mathbf{x})$ is determined by the collective deformation of $\mathcal{H}_\mathbf{x}$. To express this collective deformation, it is convenient to define the function $\underline{\mathbf{Y}}[\mathbf{x}, t]\langle \cdot \rangle : \mathcal{H}_\mathbf{x} \to \mathbb{R}^3$ that maps bonds into their images under the deformation $\mathbf{y}$. For any bond $\boldsymbol{\xi} \in \mathcal{H}_\mathbf{x}$, let

$$\underline{\mathbf{Y}}[\mathbf{x}]\langle \mathbf{q} - \mathbf{x} \rangle = \mathbf{y}(\mathbf{q}) - \mathbf{y}(\mathbf{x}).$$

The function $\underline{\mathbf{Y}}[\mathbf{x}, t]\langle \cdot \rangle$ is an example of a *state*, which is a mapping with domain $\mathcal{H}_\mathbf{x}$. By convention, the bond in $\mathcal{H}_\mathbf{x}$ that a state operates on is written in angle brackets, $\langle \boldsymbol{\xi} \rangle$. $\underline{\mathbf{Y}}[\mathbf{x}, t]$ is called the *deformation state* at $\mathbf{x}$. The deformation state is the basic kinematical quantity for purposes of material modeling and in this role is analogous to the deformation gradient $\mathbf{F} = \partial \mathbf{y} / \partial \mathbf{x}$ in the standard theory.

In an elastic material, the strain energy density $W(\mathbf{x})$ depends through the material model on the deformation state, and this dependence is written

$$W(\mathbf{x}) = \hat{W}(\underline{\mathbf{Y}}[\mathbf{x}]).$$

Let $\Phi_\mathbf{y}$ denote the total potential energy in a bounded elastic body under external body force density field $\mathbf{b}$, subjected to the deformation $\mathbf{y}$:

$$\Phi_\mathbf{y} = \int_\mathcal{B} (W(\mathbf{x}) - \mathbf{b}(\mathbf{x}) \cdot \mathbf{y}(\mathbf{x}))\, dV_\mathbf{x}.$$

Requiring the first variation of $\Phi_\mathbf{y}$ to vanish leads to the following Euler-Lagrange equation:

$$\mathbf{L}^{\text{pd}}(\mathbf{x}) + \mathbf{b}(\mathbf{x}) = \mathbf{0}$$

for all $\mathbf{x} \in \mathcal{B}$, where the *peridynamic internal force density* at $\mathbf{x}$ is given by

$$\mathbf{L}^{\text{pd}}(\mathbf{x}) = \int_\mathcal{B} \left\{ \underline{\mathbf{T}}[\mathbf{x}]\langle \mathbf{q} - \mathbf{x} \rangle - \underline{\mathbf{T}}[\mathbf{q}]\langle \mathbf{x} - \mathbf{q} \rangle \right\} dV_\mathbf{q}. \tag{1}$$

The pairwise bond force $\mathbf{f}$ on a point $\mathbf{p}$ due to interaction with any point $\mathbf{q}$ is given by

$$\mathbf{f}(\mathbf{q}, \mathbf{p}) = \underline{\mathbf{T}}[\mathbf{p}]\langle \mathbf{q} - \mathbf{p} \rangle - \underline{\mathbf{T}}[\mathbf{q}]\langle \mathbf{p} - \mathbf{q} \rangle. \tag{2}$$

In general,

$$\underline{\mathbf{T}}[\mathbf{p}] = \hat{\underline{\mathbf{T}}}(\underline{\mathbf{Y}}[\mathbf{p}]), \qquad \underline{\mathbf{T}}[\mathbf{q}] = \hat{\underline{\mathbf{T}}}(\underline{\mathbf{Y}}[\mathbf{q}]). \tag{3}$$

Here, $\underline{\mathbf{T}}[\mathbf{x}]$ is the *force state* at $\mathbf{x}$, which is related to the strain energy density by

$$\underline{\mathbf{T}}[\mathbf{x}] = \hat{W}_{\underline{\mathbf{Y}}}(\underline{\mathbf{Y}}[\mathbf{x}]) \tag{4}$$

where $\hat{W}_{\underline{\mathbf{Y}}}$ is the Fréchet derivative of $\hat{W}$ with respect to the deformation state $\underline{\mathbf{Y}}$. The Fréchet is a functional derivative with the property that if $d\underline{\mathbf{Y}}$ is a small increment in the deformation state,

$$\hat{W}(\underline{\mathbf{Y}} + d\underline{\mathbf{Y}}) = \hat{W}(\underline{\mathbf{Y}}) + \hat{W}_{\underline{\mathbf{Y}}}(\underline{\mathbf{Y}}) \bullet d\underline{\mathbf{Y}} \tag{5}$$

plus higher order terms. This expression uses the inner product of two states, defined for any states $\underline{\mathbf{A}}$ and $\underline{\mathbf{B}}$ by

$$\underline{\mathbf{A}} \bullet \underline{\mathbf{B}} = \int_{\mathcal{H}} \underline{\mathbf{A}}\langle\boldsymbol{\xi}\rangle \cdot \underline{\mathbf{B}}\langle\boldsymbol{\xi}\rangle \, dV_{\boldsymbol{\xi}}. \tag{6}$$

The mechanical interpretation of the force state is that $\underline{\mathbf{T}}\langle\mathbf{q} - \mathbf{x}\rangle$ represents a bond force density on $\mathbf{x}$ due to its interaction with $\mathbf{q}$. More general material models, whether elastic or not, may be written in the form

$$\underline{\mathbf{T}}[\mathbf{x}] = \underline{\hat{\mathbf{T}}}(\underline{\mathbf{Y}}[\mathbf{x}], \ldots, \mathbf{x})$$

where $\underline{\hat{\mathbf{T}}}$ is the material model, which may depend on additional variables besides $\underline{\mathbf{Y}}$.

If at any $\mathbf{x}$, $\underline{\mathbf{T}}$ depends *only* on $\underline{\mathbf{Y}}$ only through its current value (but not additional variables such as loading history), then the material model is *simple*, and we write

$$\underline{\mathbf{T}}[\mathbf{x}] = \underline{\hat{\mathbf{T}}}(\underline{\mathbf{Y}}[\mathbf{x}], \mathbf{x}).$$

All elastic materials are simple.

If $\underline{\hat{\mathbf{T}}}$ has no explicit dependence on $\mathbf{x}$, then the body is *homogeneous*, and we write

$$\underline{\mathbf{T}}[\mathbf{x}] = \underline{\hat{\mathbf{T}}}(\underline{\mathbf{Y}}[\mathbf{x}]).$$

Since δ is in effect a material property, any homogeneous body has constant δ. This is typically the case in applications. If the body is homogeneous, then the region of integration in (1) may be changed from $\mathcal{B}$ to $\mathcal{H}$, which is the usual statement in the literature.

2 Peridynamic stress tensor

As shown in [6], the peridynamic internal force density can be expressed without approximation as

$$\mathbf{L}^{\mathrm{pd}} = \nabla \cdot \boldsymbol{\nu}^{\mathrm{pd}}$$

where $\boldsymbol{\nu}^{\mathrm{pd}}$ is the *peridynamic stress tensor* defined by

$$\boldsymbol{\nu}^{\mathrm{pd}}(\mathbf{x}) = \frac{1}{2}\int_{\mathcal{S}}\int_0^\infty\int_0^\infty (v+w)^2 \mathbf{f}(\mathbf{x}+v\mathbf{m}, \mathbf{x}-w\mathbf{m}) \otimes \mathbf{m}\; dw\; dv\; d\Omega_{\mathbf{m}} \tag{7}$$

where $\mathcal{S}$ is the unit sphere, $d\Omega_{\mathbf{m}}$ is a differential solid angle in the direction of unit vector $\mathbf{m}$, and $\mathbf{f}$ is given by (2) and (3).

Suppose the deformation is such that there is a tensor $\mathbf{F}$ such that

$$\mathbf{y}(\mathbf{x}+\boldsymbol{\xi}) - \mathbf{y}(\mathbf{x}) = \mathbf{F}\boldsymbol{\xi} \qquad \forall \mathbf{x} \in \mathcal{B},\ \forall \boldsymbol{\xi} \in \mathcal{H}_{\mathbf{x}}.$$

Then the deformation is called *uniform*. In this case, the deformation state at any $\mathbf{x}$ is given by $\underline{\mathbf{Y}}[\mathbf{x}] = \underline{\mathbf{F}}$, where $\underline{\mathbf{F}}$ is the state defined by

$$\underline{\mathbf{F}}\langle\boldsymbol{\xi}\rangle = \mathbf{F}\boldsymbol{\xi} \qquad \forall \boldsymbol{\xi} \in \mathcal{H}_{\mathbf{x}},$$

or, more succinctly,

$$\underline{\mathbf{F}} = \mathbf{F}\underline{\mathbf{X}}$$

where $\underline{\mathbf{X}}$ is the *identity* state defined by

$$\underline{\mathbf{X}}\langle\boldsymbol{\xi}\rangle = \boldsymbol{\xi} \qquad \forall \boldsymbol{\xi} \in \mathcal{H}_{\mathbf{x}}.$$

If the body is homogeneous, then the peridynamic stress tensor is easily computed making use of the change of variables $z = v + w$:

$$\begin{aligned}
\boldsymbol{\nu}^{\mathrm{pd}} &= \int_{\mathcal{S}}\int_0^\infty\int_v^\infty z^2 \hat{\underline{\mathbf{T}}}(\underline{\mathbf{F}})\langle z\mathbf{m}\rangle \otimes \mathbf{m}\; dz\; dv\; d\Omega_{\mathbf{m}} \\
&= \int_{\mathcal{S}}\int_0^\infty\int_0^z z^2 \hat{\underline{\mathbf{T}}}(\underline{\mathbf{F}})\langle z\mathbf{m}\rangle \otimes \mathbf{m}\; dv\; dz\; d\Omega_{\mathbf{m}} \\
&= \int_{\mathcal{S}}\int_0^\infty z^3 \hat{\underline{\mathbf{T}}}(\underline{\mathbf{F}})\langle z\mathbf{m}\rangle \otimes \mathbf{m}\; dz\; d\Omega_{\mathbf{m}} \\
&= \int_{\mathcal{S}}\int_0^\infty \hat{\underline{\mathbf{T}}}(\underline{\mathbf{F}})\langle z\mathbf{m}\rangle \otimes (z\mathbf{m})(z^2\; dz\; d\Omega_{\mathbf{m}}) \\
&= \boldsymbol{\nu}^0
\end{aligned}$$

where $\boldsymbol{\nu}^0$ is the *collapsed stress tensor* defined by

$$\boldsymbol{\nu}^0 = \int_{\mathcal{H}} \hat{\underline{\mathbf{T}}}(\underline{\mathbf{F}})\langle\boldsymbol{\xi}\rangle \otimes \boldsymbol{\xi}\; dV_{\boldsymbol{\xi}}. \tag{8}$$

As discussed in [19], the collapsed stress tensor is a legitimate Piola stress tensor whose constitutive model depends on the local deformation gradient tensor through (8). The collapsed stress tensor field provides the "local limit of peridynamics" in the sense that as $\delta \to 0$,

$$\mathbf{L}^{\mathrm{pd}} \to \nabla \cdot \boldsymbol{\nu}^0$$

provided $\mathbf{u}^0$ is twice continuously differentiable and $\underline{\hat{\mathbf{T}}}$ obeys the scaling relation derived in the next section.

3 Rescaling a material model at a point

The remainder of this paper concerns methods for allowing changes in horizon as a function of position. The first step is to define a notion of changing the horizon in a material model such that the "bulk properties" are invariant to this change.

Suppose an elastic material model is given for a particular value of horizon (without loss of generality, we will assume that this horizon is 1), and call the strain energy density function $\hat{W}_1$. Now consider a different value of horizon δ, and call the modified strain energy density function $\hat{W}$. It is required that for any deformation state $\underline{\mathbf{Y}}$,

$$\hat{W}(\underline{\mathbf{Y}}) = \hat{W}_1(\underline{\mathbf{Y}}_1) \tag{9}$$

where $\underline{\mathbf{Y}}_1$ is the *reference deformation state* defined by

$$\underline{\mathbf{Y}}_1\langle \mathbf{n} \rangle = \delta^{-1}\underline{\mathbf{Y}}\langle \delta\mathbf{n} \rangle \qquad \mathbf{n} \in \mathcal{H}_1 \tag{10}$$

where $\mathcal{H}_1$ is the family of $\mathbf{x}$ with horizon 1. To derive the force state $\underline{\mathbf{Y}}$, observe that (9) implies

$$\hat{W}_{\underline{\mathbf{Y}}} \bullet d\underline{\mathbf{Y}} = (\hat{W}_1)_{\underline{\mathbf{Y}}_1} \bullet d\underline{\mathbf{Y}}_1.$$

Hence, using (4),

$$\underline{\mathbf{T}} \bullet d\underline{\mathbf{Y}} = \underline{\mathbf{T}}_1 \bullet d\underline{\mathbf{Y}}_1.$$

Combining this with (5), (6) and (10) leads to the following scaling relation for peridynamic material models in three dimensions:

$$\hat{\underline{\mathbf{T}}}(\underline{\mathbf{Y}})\langle \boldsymbol{\xi} \rangle = \delta^{-4}\hat{\underline{\mathbf{T}}}_1(\underline{\mathbf{Y}}_1)\langle \delta^{-1}\boldsymbol{\xi} \rangle \qquad \forall \boldsymbol{\xi} \in \mathcal{H}$$

for any deformation state $\underline{\mathbf{Y}}$ on $\mathcal{H}$, where $\underline{\mathbf{Y}}_1$ is given by (10). Repeating the above derivation for 1- or 2-dimensional models leads to

$$\hat{\underline{\mathbf{T}}}(\underline{\mathbf{Y}})\langle \boldsymbol{\xi} \rangle = \delta^{-(1+D)}\hat{\underline{\mathbf{T}}}_1(\underline{\mathbf{Y}}_1)\langle \delta^{-1}\boldsymbol{\xi} \rangle \qquad \forall \boldsymbol{\xi} \in \mathcal{H} \tag{11}$$

where D is the number of dimensions and $\underline{\mathbf{Y}}_1$ is given by (10). $\hat{\underline{\mathbf{T}}}_1$ is called the *reference material model*.

4 Variable scale homogeneous bodies

Recall from Section 1 that any homogeneous body necessarily has constant horizon. So, it is necessary to define a relaxed concept of homogeneity that captures what it means for a peridynamic body to have constant "bulk properties" without adhering to the strict definition of homogeneity. The reference material model defined in the previous section provides a way to do this.

Suppose a reference material model $\hat{\underline{\mathbf{T}}}_1$ is given in $\mathcal{B}$, independent of position. Also let the horizon $\delta(\mathbf{x})$ be prescribed as a function of position. Suppose that, at any $\mathbf{x} \in \mathcal{B}$,

$$\hat{\underline{\mathbf{T}}}(\underline{\mathbf{Y}}[\mathbf{x}], \mathbf{x})\langle \boldsymbol{\xi} \rangle = \frac{1}{(\delta(\mathbf{x}))^{1+D}} \hat{\underline{\mathbf{T}}}_1(\underline{\mathbf{Y}}_1[\mathbf{x}]) \left\langle \frac{\boldsymbol{\xi}}{\delta(\mathbf{x})} \right\rangle \qquad \boldsymbol{\xi} \in \mathcal{H}_{\mathbf{x}}. \tag{12}$$

Then $\mathcal{B}$ is a *variable scale homogeneous* (VSH) body. (A homogeneous body is a VSH body with constant horizon.)

By the results of Section 3, an elastic VSH body under uniform deformation has constant W. However, it does *not* necessarily have constant $\boldsymbol{\nu}^{\text{pd}}$. As shown in the next section, this leads to *ghost forces* at points where the horizon is changing.

5 Ghost forces

Here we demonstrate that in the absence of body forces, a uniform deformation of a VSH body is not necessarily in equilibrium. To see this, assume that δ is twice continuously differentiable, and compute the net internal force density $\mathbf{L}^{\mathrm{pd}}(\mathbf{x})$. From (1) and (12), for any $\mathbf{x}$,

$$\begin{aligned}\mathbf{L}^{\mathrm{pd}}(\mathbf{x}) &= \int \{\underline{\mathbf{T}}[\mathbf{x}]\langle\mathbf{q}-\mathbf{x}\rangle - \underline{\mathbf{T}}[\mathbf{q}]\langle\mathbf{x}-\mathbf{q}\rangle\}\ dV_{\mathbf{q}} \\ &= \int \left\{\delta^{-(1+D)}(\mathbf{x})\underline{\mathbf{T}}_1\langle\mathbf{m}\rangle - \delta^{-(1+D)}(\mathbf{q})\underline{\mathbf{T}}_1\langle\mathbf{n}\rangle\right\}\ dV_{\mathbf{q}}\end{aligned} \tag{13}$$

where

$$\mathbf{m} = \frac{\mathbf{q}-\mathbf{x}}{\delta(\mathbf{x})}, \qquad \mathbf{n} = \frac{\mathbf{x}-\mathbf{q}}{\delta(\mathbf{q})}. \tag{14}$$

Holding $\mathbf{x}$ fixed,

$$dV_{\mathbf{q}} = \left|\det\left(\frac{\partial\mathbf{m}}{\partial\mathbf{q}}\right)\right|^{-1} = \delta^D(\mathbf{q})dV_{\mathbf{m}}, \tag{15}$$

$$dV_{\mathbf{q}} = \left|\det\left(\frac{\partial\mathbf{n}}{\partial\mathbf{q}}\right)\right|^{-1} = \frac{\delta^D(\mathbf{q})}{1+\mathbf{n}\cdot\nabla\delta(\mathbf{q})+O(|\nabla\delta|^2)}dV_{\mathbf{n}}. \tag{16}$$

From (13), (14), (15), and (16), it follows that

$$\begin{aligned}\mathbf{L}^{\mathrm{pd}}(\mathbf{x}) &= \int \frac{\underline{\mathbf{T}}_1\langle\mathbf{m}\rangle}{\delta(\mathbf{x})}\ dV_{\mathbf{m}} - \int \frac{\underline{\mathbf{T}}_1\langle\mathbf{n}\rangle}{\delta(\mathbf{q})}\left(\frac{1}{1+\mathbf{n}\cdot\nabla\delta(\mathbf{q})+O(|\nabla\delta|^2)}\right)\ dV_{\mathbf{n}} \\ &= \int \epsilon(\mathbf{x},\mathbf{q})\underline{\mathbf{T}}_1\langle\mathbf{m}\rangle\ dV_{\mathbf{m}}\end{aligned} \tag{17}$$

where

$$\begin{aligned}\epsilon(\mathbf{x},\mathbf{q}) &= \frac{1}{\delta(\mathbf{x})} - \frac{1}{\delta(\mathbf{q})}\left(\frac{1}{1+\mathbf{n}\cdot\nabla\delta(\mathbf{q})+O(|\nabla\delta|^2)}\right) \\ &= \frac{1}{\delta(\mathbf{x})} - \frac{1}{\delta(\mathbf{q})+\nabla\delta(\mathbf{q})\cdot(\mathbf{x}-\mathbf{q})+O(\delta|\nabla\delta|^2)}.\end{aligned} \tag{18}$$

Now approximate $\delta(\mathbf{x})$ by the first three terms of a Taylor expansion centered at $\mathbf{q}$, that is,

$$\delta(\mathbf{x}) = \delta(\mathbf{q}) + \nabla\delta(\mathbf{q})\cdot(\mathbf{x}-\mathbf{q}) + \nabla\nabla\delta(\mathbf{q})(\boldsymbol{\xi}\otimes\boldsymbol{\xi})/2 + \dots$$

where $\boldsymbol{\xi} = \mathbf{q}-\mathbf{x}$. Using this to eliminate the term $\nabla\delta(\mathbf{q})\cdot(\mathbf{x}-\mathbf{q})$ in (18) leads, after further straightforward manipulations, to

$$\begin{aligned}\epsilon(\mathbf{x},\mathbf{q}) &= \frac{1}{\delta(\mathbf{x})} - \frac{1}{\delta(\mathbf{x})-\nabla\nabla\delta(\mathbf{q})(\boldsymbol{\xi}\otimes\boldsymbol{\xi})/2+\dots} \\ &= -\frac{1}{2}\nabla\nabla\delta(\mathbf{x})\mathbf{m}\otimes\mathbf{m}+\dots\end{aligned}$$

where $\mathbf{m}$ is again given by (14). From this result and (17), the internal force density at $\mathbf{x}$ is estimated from

$$
\begin{aligned}
\mathbf{L}^{\rm pd}(\mathbf{x}) &= -\frac{1}{2}\int \nabla\nabla\delta(\mathbf{x})(\mathbf{m}\otimes\mathbf{m})\underline{\mathbf{T}}_1\langle\mathbf{m}\rangle\, dV_{\mathbf{m}} + \dots \\
&= O(|\nabla\nabla\delta|)O(|\underline{\mathbf{T}}_1|). \qquad (19)
\end{aligned}
$$

The departure from equilibrium represented by nonzero $\mathbf{L}^{\rm pd}$ is called *ghost force* and is an artifact of the position dependence of horizon. Observe that the leading term in the ghost force depends on the *second* derivative of δ. In fact, it can be shown that if δ is a linear function of position, then the ghost force vanishes.

An illustration of the effect of ghost force in a VSH bar in equilibrium is shown in Figure 1. The peridynamic reference material model $\hat{\underline{\mathbf{T}}}_1$ is a bond-based model [18] with a nominal Young's modulus of 1. The horizon in the bar depends on position as shown in the top figure. Two cases are considered for dependence of horizon: piecewise linear ("not smoothed") and cubic spline ("smoothed"). The ends of the bar have prescribed displacements corresponding to a nominal strain in the bar of 1. The strain (du/dx) in equilibrium for the two cases is shown in the lower figure. If there were no ghost forces, the strain would be constant and equal to 1. Because of ghost forces, anomalies in strain ("ghost strains") appear that equilibrate the ghost forces. The smoothed $\delta(x)$ has lower ghost strains than the non-smoothed case. This result is consistent with (19), which predicts ghost forces proportional to the second derivative of $\delta(x)$.

In this example, the anomalies in strain are less than 2%, even for the non-smoothed case. This departure from constant strain could be acceptable in many applications. Ghost forces in a VSH peridynamic body are always self-equilibrated: they do not exert a net force on the body. This follows from the fact that the peridynamic equilibrium equation always conserves linear momentum, even if the material model depends on position. To address applications in which these ghost forces are not acceptable, or it is desired to have a discontinuous jump in $\delta(x)$, we will introduce two methods, *partial stress* and *splice* that exhibit zero ghost forces in a uniform deformation of a VSH body.

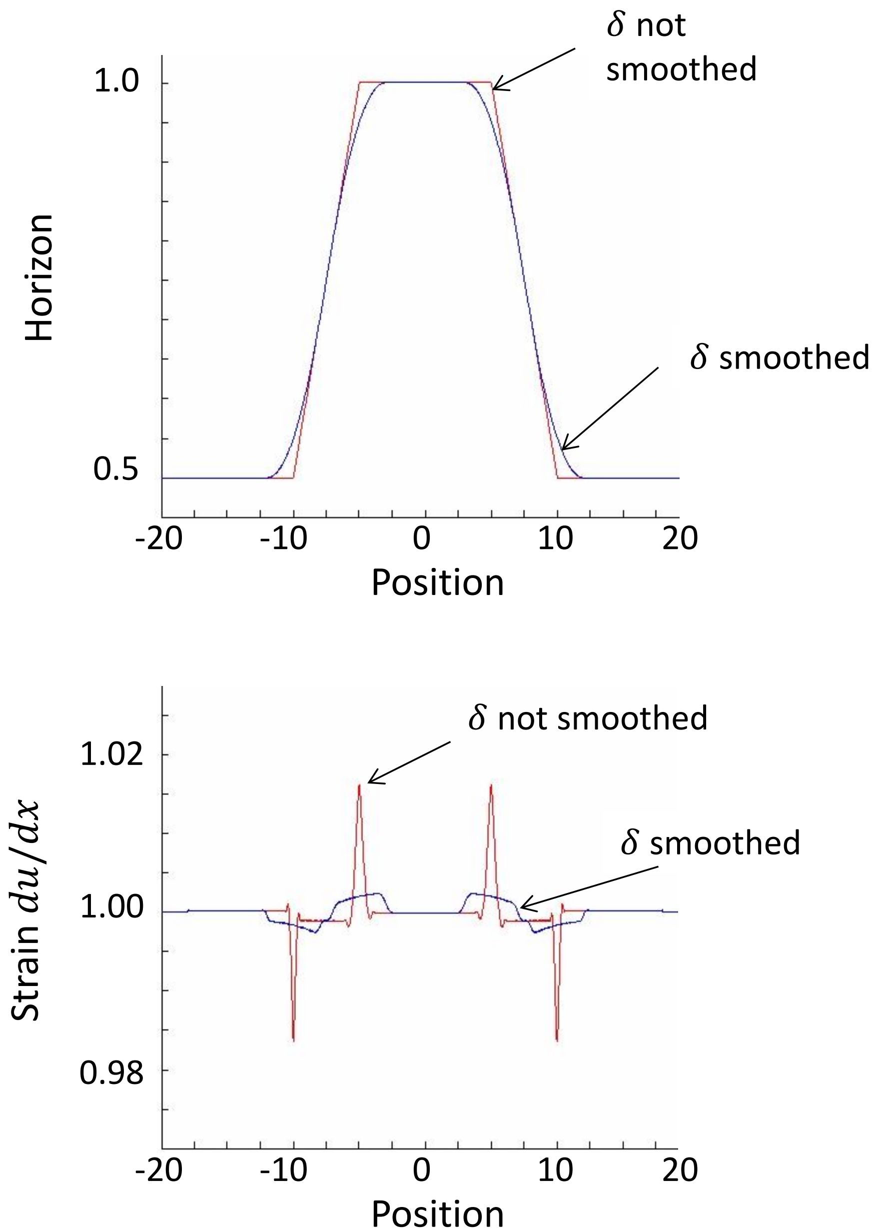

Figure 1. Ghost strain in a VSH body in equilibrium.

6 Partial stress field

Here we investigate a modified form of the momentum balance that eliminates ghost forces in a VSH body under uniform deformation. The momentum balance is expressed in terms of a new field called the "partial stress tensor" field.

Consider a peridynamic body $\mathcal{B}$ and let its force state field $\underline{\mathbf{T}}$ be given. Let the *partial stress* field $\boldsymbol{\nu}^{\rm ps}$ be the tensor field defined by

$$\boldsymbol{\nu}^{\rm ps}(\mathbf{x}) = \int_{\mathcal{H}_{\mathbf{x}}} \underline{\mathbf{T}}[\mathbf{x}]\langle\boldsymbol{\xi}\rangle \otimes \boldsymbol{\xi}\, dV_{\boldsymbol{\xi}} \qquad \forall \mathbf{x} \in \mathcal{B}. \tag{20}$$

Also define the *partial internal force* by

$$\mathbf{L}^{\rm ps}(\mathbf{x}) = \nabla \cdot \boldsymbol{\nu}^{\rm ps}(\mathbf{x}) \qquad \forall \mathbf{x} \in \mathcal{B}. \tag{21}$$

If the material model is simple, the partial stress can be expressed in the form

$$\boldsymbol{\nu}^{\rm ps}(\mathbf{x}) = \hat{\boldsymbol{\nu}}^{\rm ps}(\underline{\mathbf{Y}}[\mathbf{x}]) \qquad \forall \mathbf{x} \in \mathcal{B}.$$

where

$$\hat{\boldsymbol{\nu}}^{\rm ps}(\underline{\mathbf{Y}}) = \int_{\mathcal{H}} \hat{\underline{\mathbf{T}}}(\underline{\mathbf{Y}})\langle\boldsymbol{\xi}\rangle \otimes \boldsymbol{\xi}\, dV_{\boldsymbol{\xi}}. \tag{22}$$

Note the similarity between $\boldsymbol{\nu}^{\rm ps}$ and $\boldsymbol{\nu}^0$ defined by (8). The difference is that $\boldsymbol{\nu}^{\rm ps}$ depends on the (nonlocal) deformation state $\underline{\mathbf{Y}}$, while $\boldsymbol{\nu}^0$ depends on the (local) deformation gradient tensor. In the special case of uniform deformation of a homogeneous body (which implies $\delta =$ constant and $\underline{\mathbf{Y}} = \underline{\mathbf{F}}$), clearly $\boldsymbol{\nu}^{\rm ps} \equiv \boldsymbol{\nu}^0$.

By repeating the manipulations leading to (12), it is easily shown that in a VSH body with reference material model $\hat{\underline{\mathbf{T}}}_1$,

$$\boldsymbol{\nu}^{\rm ps}(\mathbf{x}) = \int_{\mathcal{H}_1} \hat{\underline{\mathbf{T}}}_1(\underline{\mathbf{Y}}_1[\mathbf{x}])\langle\mathbf{n}\rangle \otimes \mathbf{n}\, dV_{\mathbf{n}} \tag{23}$$

where $\underline{\mathbf{Y}}_1$ is given by (10). Since, in a uniform deformation, $\underline{\mathbf{Y}}_1$ is constant (and equal to $\underline{\mathbf{F}}$), it follows from (21) and (23) that in a VSH body under uniform deformation,

$$\mathbf{L}^{\rm ps} \equiv \mathbf{0}.$$

This establishes that ghost forces are absent in the partial stress formulation of the momentum balance. This observation motivates the use of this modified momentum balance in the transition region of horizon.

7 Partial stress as an approximation to peridynamics

With the intent of modeling some parts of a body using partial stress and other parts using the full peridynamic integral equations, the relation between the two will now be investigated. Since the plan is to transition between partial stress and full peridynamic equations where the horizon is constant, we investigate the way these fields approximate each other under this assumption.

Proposition. Let $\mathcal{B}$ be a homogeneous body (which implies constant δ). Let a twice continuously differentiable deformation $\mathbf{y}$ of $\mathcal{B}$ be prescribed. Then

(i)

$$\boldsymbol{\nu}^{\mathrm{pd}} - \boldsymbol{\nu}^{\mathrm{ps}} = O(\delta)O(|\nabla\underline{\mathbf{T}}_1|) \quad \text{on } \mathcal{B} \tag{24}$$

(ii)

$$\mathbf{L}^{\mathrm{pd}} - \mathbf{L}^{\mathrm{ps}} = O(\delta)O(|\nabla\nabla\underline{\mathbf{T}}_1|) \quad \text{on } \mathcal{B}. \tag{25}$$

Proof of (i). Combining (3), (7), and (12), and setting $v = \delta v_1$, $w = \delta w_1$ leads to

$$\begin{aligned} \boldsymbol{\nu}^{\mathrm{pd}}(\mathbf{x}) &= \int_{\mathcal{H}_1}\int_0^\infty\int_0^\infty (v+w)^2\underline{\mathbf{T}}[\mathbf{x}-w\mathbf{m}]\langle(v+w)\mathbf{m}\rangle \otimes \mathbf{m}\, dw\, dv\, d\Omega_{\mathbf{m}} \\ &= \int_{\mathcal{H}_1}\int_0^\infty\int_0^\infty (v_1+w_1)^2\underline{\mathbf{T}}_1[\mathbf{x}-\delta w_1\mathbf{m}]\langle(v_1+w_1)\mathbf{m}\rangle \otimes \mathbf{m}\, dw_1\, dv_1\, d\Omega_{\mathbf{m}}. \end{aligned}$$

The first two terms of a Taylor expansion for $\underline{\mathbf{T}}_1$ provide

$$\underline{\mathbf{T}}_1[\mathbf{x}-\delta w_1\mathbf{m}] = \underline{\mathbf{T}}_1[\mathbf{x}] - \delta w_1 \nabla\underline{\mathbf{T}}_1[\mathbf{x}]\mathbf{m} + \dots.$$

Repeating the manipulations leading up to (8), the first term in this Taylor expansion, after evaluating the triple integral, is found to be the partial stress defined by (20). Thus

$$\boldsymbol{\nu}^{\mathrm{pd}} = \boldsymbol{\nu}^{\mathrm{ps}} + O(\delta)O(|\nabla\underline{\mathbf{T}}_1|),$$

proving (24).

Proof of (ii). Applying a Taylor expansion to $\underline{\mathbf{T}}$ near $\mathbf{x}$ and setting $\boldsymbol{\xi} = \mathbf{q} - \mathbf{x}$ leads to

$$\underline{\mathbf{T}}[\mathbf{q}] = \underline{\mathbf{T}}[\mathbf{x}] - \nabla\underline{\mathbf{T}}[\mathbf{x}]\boldsymbol{\xi} + \frac{1}{2}\nabla\nabla\underline{\mathbf{T}}[\mathbf{x}] \cdot (\boldsymbol{\xi}\otimes\boldsymbol{\xi}) + \dots.$$

Combining this with (1),

$$
\begin{aligned}
\mathbf{L}^{\rm pd}(\mathbf{x}) &= \int_{\mathcal{H}} \Big\{ \underline{\mathbf{T}}[\mathbf{x}]\langle\boldsymbol{\xi}\rangle - \underline{\mathbf{T}}[\mathbf{q}]\langle -\boldsymbol{\xi}\rangle \Big\}\, dV_{\boldsymbol{\xi}} \\
&= \int_{\mathcal{H}} \Big\{ \underline{\mathbf{T}}[\mathbf{x}]\langle\boldsymbol{\xi}\rangle - \big(\underline{\mathbf{T}}[\mathbf{x}]\langle -\boldsymbol{\xi}\rangle - \nabla\underline{\mathbf{T}}[\mathbf{x}]\langle -\boldsymbol{\xi}\rangle\boldsymbol{\xi} \\
&\qquad\qquad + \frac{1}{2}\nabla\nabla\underline{\mathbf{T}}[\mathbf{x}]\langle -\boldsymbol{\xi}\rangle \cdot (\boldsymbol{\xi}\otimes\boldsymbol{\xi}) + \dots \big) \Big\}\, dV_{\boldsymbol{\xi}}
\end{aligned}
$$

Replacing $\boldsymbol{\xi}$ by $-\boldsymbol{\xi}$, the first two terms in the integrand cancel. Bringing the gradient operator outside of the integral yields

$$
\begin{aligned}
\mathbf{L}^{\rm pd}(\mathbf{x}) &= \nabla\cdot\int_{\mathcal{H}} \underline{\mathbf{T}}[\boldsymbol{\xi}]\langle\boldsymbol{\xi}\rangle\otimes\boldsymbol{\xi}\, dV_{\boldsymbol{\xi}} - \frac{1}{2}\int_{\mathcal{H}} \nabla\nabla\underline{\mathbf{T}}[\mathbf{x}]\langle\boldsymbol{\xi}\rangle\cdot(\boldsymbol{\xi}\otimes\boldsymbol{\xi})\, dV_{\boldsymbol{\xi}} + \dots \\
&= \mathbf{L}^{\rm ps}(\mathbf{x}) - \frac{1}{2}\int_{\mathcal{H}} \nabla\nabla\underline{\mathbf{T}}[\mathbf{x}]\langle\boldsymbol{\xi}\rangle\cdot(\boldsymbol{\xi}\otimes\boldsymbol{\xi})\, dV_{\boldsymbol{\xi}} + \dots
\end{aligned}
$$

Using (11) to express the remainder in terms of the reference force state, and setting $\boldsymbol{\xi} = \delta\mathbf{m}$, this implies

$$
\mathbf{L}^{\rm pd}(\mathbf{x}) = \mathbf{L}^{\rm ps}(\mathbf{x}) - \frac{\delta}{2}\int_{\mathcal{H}_1} \nabla\nabla\underline{\mathbf{T}}_1[\mathbf{x}]\langle\mathbf{m}\rangle\cdot(\mathbf{m}\otimes\mathbf{m})\, dV_{\mathbf{m}} + \dots
$$

Since $\mathbf{m}$ is a unit vector, this proves (25). $\square$

Because of (25), it follows that at the interface between subregions where $\mathbf{L}^{\rm ps}$ and $\mathbf{L}^{\rm pd}$ are used in the momentum balance, there are no ghost forces if the deformation is uniform.

Figure 2 shows an example in which the horizon is a function of position in a VSH bar. The horizon increase from 0.1 on the left to 1.0 on the right through a transition region of thickness 0.1. An incident wave pulse is applied on the left boundary. The total thickness of the pulse is 4.0, which is thin enough that we expect to see some effect of nonlocality as it crosses into the high-δ region. Two cases are considered: (1) the full peridynamic equations applied throughout, and (2) the partial stress model is applied in a region surrounding the transition. In case (2), the total width of the partial stress region is 10, and the full peridynamic equations are applied everywhere else. Figure 3 compares the time history of displacement at the point $x = -10$ for the two cases, showing both the incident and reflected pulse. Evidently, the use of partial stress in the transition region reduces the reflections. Figure 4 shows the comparison at the point $x = 10$, showing the transmitted pulses, which are essentially the same for both models. The change in shape of the transmitted pulse compared with the incident pulse is primarily due to nonlocality: as δ increases, short wavelength components of the pulse experience a lower wave speed. This

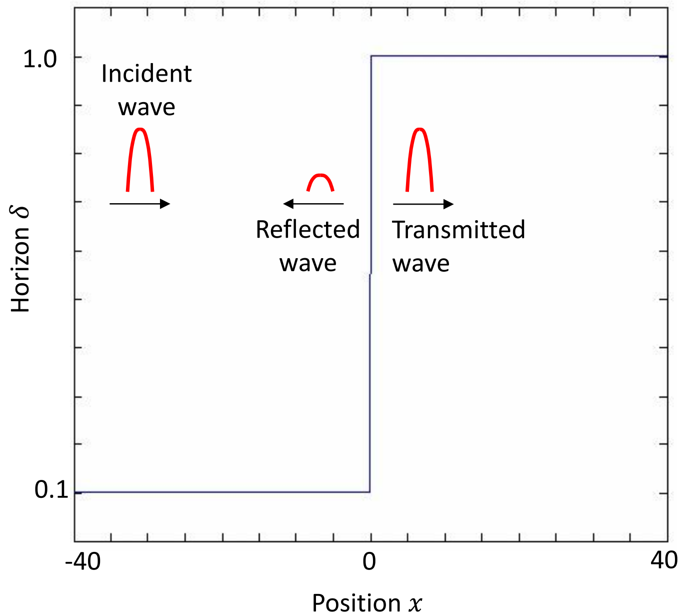

Figure 2. Stress pulse in a VSH bar: horizon as a function of position.

effect of nonlocality on wave speed is reflected in dispersion curves for linear waves depend on the horizon [20]. Since the partial stress method has no ghost forces under uniform deformation, this example illustrates that the effect of ghost forces in the fully peridynamic model is small compared to the effect of nonlocality on dispersion.

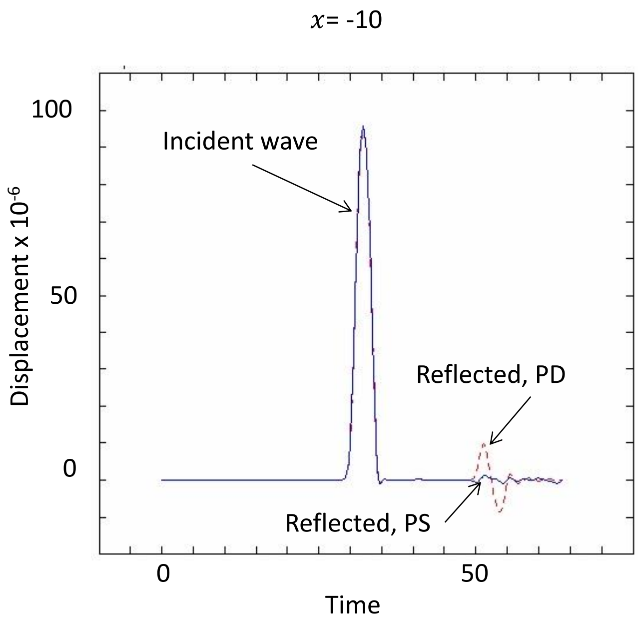

Figure 3. Stress pulse in a VSH bar: time history of displacement at $x = -10$. The two curves are for a fully peridynamic (PD) model, and for peridynamic with a partial stress (PS) region surrounding the transition in δ. The PS curve has a different reflection than the PD model.

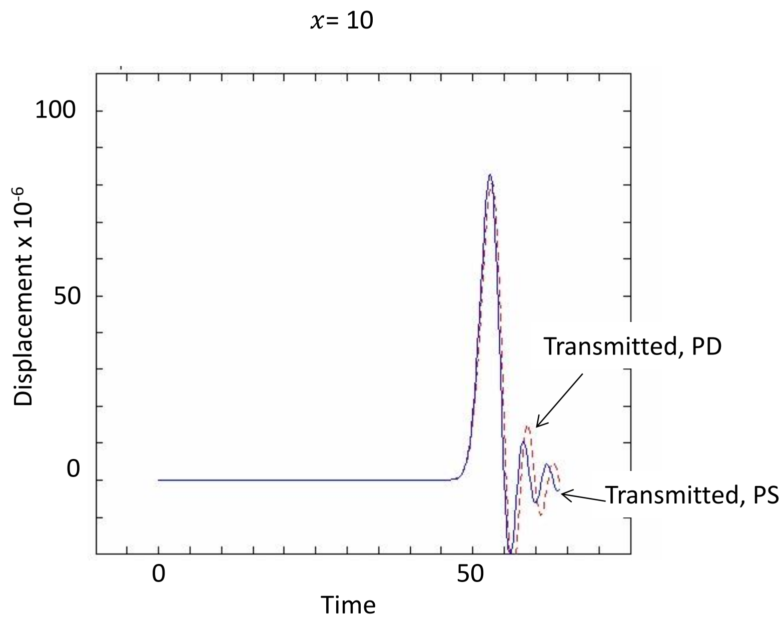

Figure 4. Stress pulse in a VSH bar: time history of displacement at $x = 10$. The transmitted pulses are similar between the two models, with the primary effect being the nonlocality of the material model.

8 Splice between two peridynamic subregions

Let two values of horizon be denoted δ_+ and δ_-. Let a reference material model $\hat{\underline{\mathbf{T}}}_1$ be given. Suppose, for a given deformation, two force state fields are computed everywhere using (11):

$$\underline{\mathbf{T}}_+[\mathbf{x}]\langle\boldsymbol{\xi}\rangle = \frac{1}{\delta_+^{1+D}}\hat{\underline{\mathbf{T}}}_1(\underline{\mathbf{Y}}_1[\mathbf{x}])\langle\boldsymbol{\xi}/\delta_+\rangle, \qquad \underline{\mathbf{T}}_-[\mathbf{x}]\langle\boldsymbol{\xi}\rangle = \frac{1}{\delta_-^{1+D}}\hat{\underline{\mathbf{T}}}_1(\underline{\mathbf{Y}}_1[\mathbf{x}])\langle\boldsymbol{\xi}/\delta_-\rangle.$$

Further suppose that $\mathcal{B}$ is divided into two subregions $\mathcal{B}_+$ and $\mathcal{B}_-$ and that the internal force density at any $\mathbf{x} \in \mathcal{B}$ is given by

$$\mathbf{L}(\mathbf{x}) = \mathbf{L}^{\text{splice}}(\mathbf{x}) := \begin{cases} \int_{\mathcal{B}} \left\{\underline{\mathbf{T}}_+[\mathbf{x}]\langle\mathbf{q}-\mathbf{x}\rangle - \underline{\mathbf{T}}_+[\mathbf{q}]\langle\mathbf{x}-\mathbf{q}\rangle\right\} dV_{\mathbf{q}}, & \text{if } \mathbf{x} \in \mathcal{B}_+ \\ \int_{\mathcal{B}} \left\{\underline{\mathbf{T}}_-[\mathbf{x}]\langle\mathbf{q}-\mathbf{x}\rangle - \underline{\mathbf{T}}_-[\mathbf{q}]\langle\mathbf{x}-\mathbf{q}\rangle\right\} dV_{\mathbf{q}}, & \text{if } \mathbf{x} \in \mathcal{B}_-. \end{cases}$$

The resulting model of $\mathcal{B}$ is called a *splice* of the subregions $\mathcal{B}_+$ and $\mathcal{B}_-$.

A splice is not the same as a VSH with $\delta(\mathbf{x})$ prescribed as a step function. The difference is that in a splice, a point $\mathbf{x}$ near the interface "sees" the force states on the other side of the interface corresponding to the same value of horizon as itself, $\delta(\mathbf{x})$. In contrast, in a VSH, each point is assigned a unique value of horizon, and the force state at each point is uniquely computed according to this horizon.

In many applications, a splice provides a viable and convenient way to model a VSH body that has piecewise constant values of horizon. It is immediate that a splice model has zero internal force density under homogeneous deformation, since the values of $\underline{\mathbf{T}}_+$ and $\underline{\mathbf{T}}_-$ are constant throughout. This implies that a splice model produces no ghost forces under uniform deformation. This is the main advantange of a splice over a VSH body.

9 Local-nonlocal coupling

The situation frequently arises that we wish to model most of a body using the standard local theory, and only a small portion (such as in the vicinity of a growing crack) with peridynamics. The method for achieving the transition between the two is called *local-nonlocal coupling*. Methods that have been proposed for local-nonlocal coupling include Arlequin [5], morphing [9, 1, 2] and blending [14, 15]. Liu and Hong proposed a coupling method using interface elements [8]. Methods that achieve coupling by treating peridynamic bonds as finite elements are described in Macek and Silling [10] and the book chapter by Oterkus and Madenci [11] which contains additional references.

Here, we first explore local-nonlocal coupling using the partial stress field. Formally, replacing $\underline{\mathbf{Y}}$ by $\underline{\mathbf{F}}$ in the definition of $\boldsymbol{\nu}^{\rm ps}$ (22) results in the collapsed stress $\boldsymbol{\nu}^0$ defined by (8). After analyzing the relation between the two stress tensors in more detail using a Taylor expansion of $\underline{\mathbf{Y}}_1$ near $\underline{\mathbf{F}}$, assuming the deformation is twice continuously differentiable, one finds that

$$\boldsymbol{\nu}^{\rm ps} = \boldsymbol{\nu}^0 + O(\delta)O(|\nabla\underline{\mathbf{T}}_1|) \qquad \text{as } \delta \to 0$$

$$\mathbf{L}^{\rm ps} = \mathbf{L}^0 + O(\delta)O(|\nabla\nabla\underline{\mathbf{T}}_1|) \qquad \text{as } \delta \to 0.$$

Thus, by making δ sufficiently small in the computation of the partial stress, a local model is obtained in which the material model is supplied by the collapsed stress $\boldsymbol{\nu}^0$.

Using the idea of a splice described in the previous section, a body can be divided into subregions $\mathcal{B}_0$ and $\mathcal{B}_{\rm pd}$ that use the local model ($\delta = 0$) and the full peridynamic model ($\delta > 0$) respectively. The internal force density in the splice model is given by

$$\mathbf{L}(\mathbf{x}) = \begin{cases} \int_{\mathcal{B}} \left\{ \underline{\mathbf{T}}[\mathbf{x}]\langle \mathbf{q}-\mathbf{x}\rangle - \underline{\mathbf{T}}[\mathbf{q}]\langle \mathbf{x}-\mathbf{q}\rangle \right\} dV_{\mathbf{q}}, & \text{if } \mathbf{x} \in \mathcal{B}_{\rm pd} \\ \nabla \cdot \boldsymbol{\nu}^0 & \text{if } \mathbf{x} \in \mathcal{B}_0. \end{cases}$$

As an example, we apply two methods for local-nonlocal coupling to the problem of spall initiated by the impact of two brittle elastic plates. The impactor has half the thickness of the target and strikes the target from the left side. As shown in the wave diagram in Figure 6, the compressive waves that issue from the contact surface eventually intersect each other at the midplane of the target plate. When this happens, the waves, which by that time are both tensile, reinforce to create a thin region where the stress is strongly tensile. Within this tensile region, the strength of the material is exceeded and a crack forms. The formation of this crack creates a relief pulse on the free surface of the target bar. In experiments, this pulse can be measured using VISAR or other techniques [4] as it reflects from the free surface of the target. With the help of analysis or computational modeling, the exact characteristics of the crack release (or "pullback") pulse can be interpreted using suitable data processing techniques to reveal the dynamic stength properties of materials under strong tension (spall).

In the computational model of this spall experiment, the impactor and target plates have thicknesses of 20 and 40 respectively. The impact velocity is 0.1. The elastic modulus and density of both plates are 1. The reference material model $\underline{\mathbf{T}}_1$ is bond-based microelastic with a critical bond strain for failure of 0.04. The entire region is discretized into 1000 nodes. The objective is to model the small part of the body where damage can occur using the full peridynamic equations. This peridynamic region is coupled to local regions using either of two methods:

- Partial stress: a peridynamic region of thickness 10, centered at the midpoint, is enclosed by layers of thickness 4 where the partial stress method is applied. Beyond this, the local equations are used. In the peridynamic and partial stress regions, the horizon is $\delta = 0.13$.

- Splice: a peridynamic region of thickness 10 and horizon $\delta = 0.13$, centered at the midpoint of the target plate, is spliced to local regions.

For comparison, results with the entire model fully peridynamic ($\delta = 0.13$ throughout) are also computed.

The computed velocity profile using the splice method for local-nonlocal coupling is shown in Figure 5. The time of this snapshot is $t = 70$. Comparing this figure with the wave diagram in Figure 6, a number of salient features may be seen. The crack appears as a sharp jump in velocity as a function of position $x = 40$. The two release (pullback) pulses move away from the crack at the wave velocity, which is $c = 1.0$. The computed velocity history at the free surface is shown in Figure 7. The dips in velocity represent the crack release pulse created in the interior of the target due to spall.

As shown in the figures, the three methods give nearly the same results in this example. However, a fully peridynamic model in multiple dimensions would require a much higher computational cost due to the large number of nonlocal interactions required to discretize the material model. In other words, the splice or partial stress methods give essentially the same result in this problem as the fully peridynamic model at greatly reduced cost (and without ghost forces).

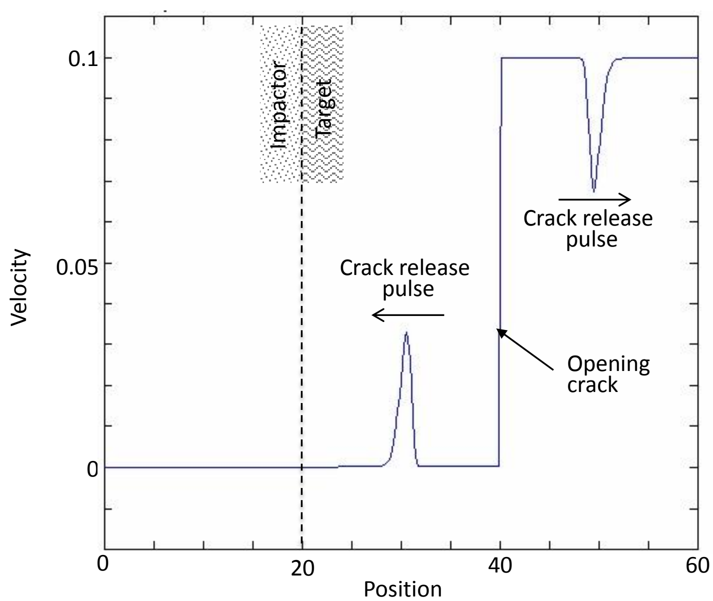

Figure 5. Velocity as a function of position at $t = 70$ in the spall example problem using the splice method for local-nonlocal coupling. There are no significant artifacts from the local-nonlocal transitions, which are located at $x = 37$ and $x = 43$.

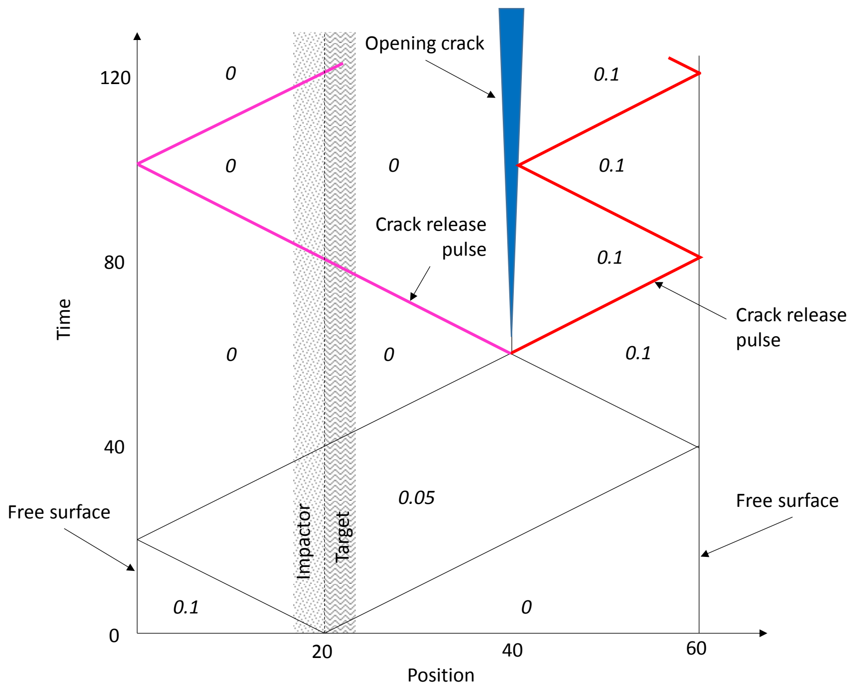

Figure 6. Wave diagram for the impact of a plate (from the left) on a target plate. Waves reinforce at the midplane of the target plate to cause fracture. Numbers in italics represent particle velocity.

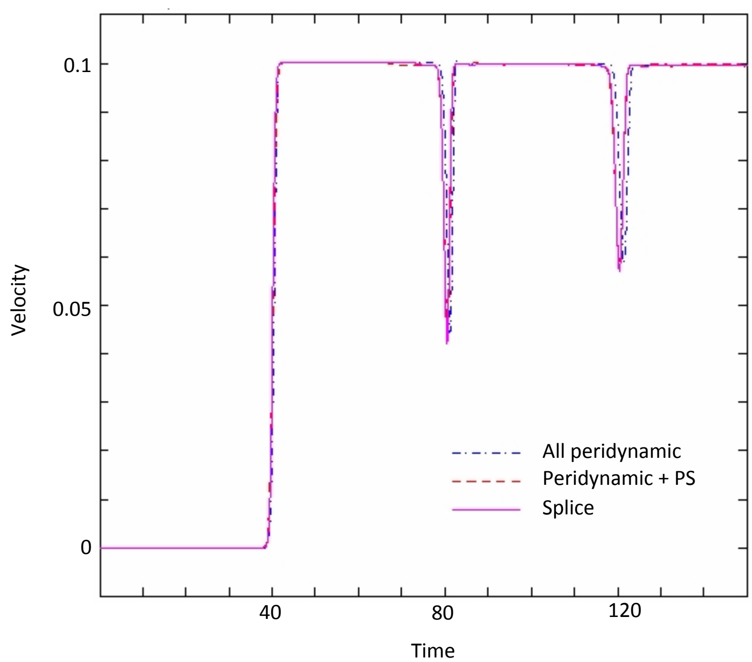

Figure 7. Velocity history at the free (right) surface of the target plate, showing the release pulse from the dynamic fracture occurring in the interior of the target bar. The three curves are for fully peridynamic (PD) local-nonlocal coupling using partial stress (PS), and local-nonlocal coupling using a splice.

10 Discussion

In this paper, we proposed a notion of a homogeneous body (a "variable scale homogeneous" body) that characterizes a peridynamic medium with variable horizon but with constant bulk material properties. We analyzed the origin and effect of ghost forces due to changes in horizon in the full peridynamic model. The importance of these ghost forces depends on the application; they may or may not be acceptable in a computational model. These anomalies can be reduced by making $\delta(\mathbf{x})$ a smoothly varying function.

If the ghost forces under uniform deformation are not acceptable, the partial stress field can be used achieve a transition between regions with different horizon. This field appears in a modified form of the equilibrium equation. The partial stress approaches the collapsed stress in the limit of zero horizon, if the deformation is continuously differentiable. The collapsed stress $\hat{\boldsymbol{\nu}}^0(\mathbf{F})$ provides a material model for Piola stress in the local formulation.

The partial stress formulation is not thermodynamically consistent in the sense that along a cyclic loading path in an elastic material, nonzero net work may be appear:

$$\oint \hat{\boldsymbol{\nu}}^{\mathrm{ps}}(\underline{\mathbf{Y}}) \cdot d\mathbf{F} \neq 0. \tag{26}$$

(An exception is a uniform deformation, in which case $\underline{\mathbf{Y}} = \underline{\mathbf{F}}$.) The root cause of this inconsistency, which is evident in (26), is that the partial stress concept mixes local and nonlocal kinematics. The partial stress is therefore not suitable as a basis for a general theory of continuum mechanics. The full peridynamic theory, like the local theory, *is* thermodynamically consistent [20, 12, 13], that is,

$$\oint \hat{\underline{\mathbf{T}}}(\underline{\mathbf{Y}}) \bullet d\underline{\mathbf{Y}} = 0, \qquad \text{and} \qquad \oint \hat{\boldsymbol{\nu}}^0(\mathbf{F}) \cdot d\mathbf{F} = 0.$$

To connect subregions with constant δ to each other, or to achieve local-nonlocal coupling, subregions can be connected using a splice. This method allows the peridynamic material model in each subregion to "see" displacements in the adjacent subregion for purposes of evaluating the nonlocal force state. The methods proposed in this paper may provide a means to reduce the cost of modeling a crack with peridynamics by connecting it to a larger surrounding region that is modeled with the local theory.

11 Appendix: Computational details

All the numerical examples in this paper involving nonlocal models used the method described in [17]. This method uses midpoint quadrature in a Lagrangian discretization of the body. The approximation in one dimension for the peridynamic internal force density is

$$L_i^{\mathrm{pd}} = \sum_{j \in \mathcal{H}_i} (T_{ij} - T_{ji}) V_j$$

where i and j are node numbers, V_j is the volume of node j, $\mathcal{H}_i$ is the set of nodes in the family of i, and

$$T_{ij} = \hat{\underline{T}}(\underline{Y}[x_i], x_i)\langle x_j - x_i \rangle, \qquad T_{ji} = \hat{\underline{T}}(\underline{Y}[x_j], x_j)\langle x_i - x_j \rangle,$$

$$\underline{Y}[x_i]\langle x_j - x_i \rangle = y_j - y_i, \qquad \underline{Y}[x_j]\langle x_i - x_j \rangle = y_i - y_j.$$

The partial stress is computed from

$$\nu_i^{\mathrm{ps}} = \sum_{j \in \mathcal{H}_i} (T_{ij} - T_{ji})(x_j - x_i) V_j.$$

The internal force density in the partial stress formulation is computed from

$$L_i^{\mathrm{ps}} = \frac{\nu_{i+1}^{\mathrm{ps}} - \nu_{i-1}^{\mathrm{ps}}}{2\Delta x}$$

where Δx is the mesh spacing. For computations involving the PDEs of the local theory, the Piola (collapsed) stress is approximated by the finite difference formula

$$\nu_{i+1/2}^0 = \hat{\nu}^0(F_{i+1/2}), \qquad F_{i+1/2} = \frac{y_{i+1} - y_i}{\Delta x}$$

where $\hat{\nu}^0$ is the material model for the Piola stress. The internal force density in the local model is approximated by

$$L_i^0 = \frac{\nu_{i+1/2}^0 - \nu_{i-1/2}^0}{\Delta x}.$$

For dynamics, time integration is performed using explicit central differencing:

$$v_i^{n+1/2} - v^{n-1/2} = a_i^n \Delta t,$$

$$y_i^{n+1} - y_i^n = v_i^{n+1/2} \Delta t$$

where Δt is the time step size. At time step n, the acceleration is computed from

$$\rho a_i^n = L_i^n + b_i^n + \eta_i^n,$$

where L is either L^{pd}, L^{ps}, or L^0, ρ is density, b is body force density, and η is linear artificial viscosity:

$$\eta_i^n = \alpha \rho c \Delta x (v_{i+1}^n - 2v_i^n + v_{i-1}^n)$$

where c is the wave speed and α is a dimensionless constant on the order of 1/4. The linear artificial viscosity is effective in reducing zero energy mode oscillations that can appear due to the non-invertibility of the partial stress tensor and, when a correspondence material model is used, of the peridynamic force state. Other methods for controlling zero energy mode oscillations are described by Breitenfeld *et al.* [3] and by Littlewood [7]. Artificial viscosity η_i^n is not applied if the bonds from i to $i+1$ or i to $i-1$ are damaged according to the material damage model at time step n. Otherwise, the artificial viscosity would create nonphysical forces across a crack surface.

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