Combinatorial Optimization of Matrix-Vector Multiplication

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Abstract: Work by Kirby, et al. (2006) showed that combinatorial optimization of matrix/vector multiplication could lead to faster evaluation of finite element stiffness matrices. Using relationships between rows, an efficient set of operations can be generated to perform matrix-vector multiplication. My improved graph model of this problem solves this combinatorial optimization problem optimally for binary row relationships. I extend the representation by using hypergraphs to model more complicated row relationships, expressing an n-row relationship with an n-vertex hyperedge. My initial greedy algorithm for this hypergraph model has yielded significantly better results than the graph model for many matrices.

Motivation

- Reducing redundant operations in building stiffness matrices
  - Generate code to optimize construction of local stiffness matrices
  - Many local stiffness matrices built
  - Reuse optimized code when problem is rerun
- Finite element "Compilers"
  - FIAT (automates generation of FEs)
  - FFC (variational forms -> code for evaluation)
- Following on work by Kirby, et al., Texas Tech, University of Chicago
  - Optimization of FFC generated code
  - FErari

Matrix-Vector Multiplication

Kirby, et al. showed that each element of FE local stiffness matrix can be calculated by the following Frobenius product:

$$S^e_{i,j} = K_{i,j} : G^e = \sum_m^2 \sum_n^2 G^e_{m,n} K_{i,j,m,n}$$

Flattening the stiffness matrix into a vector, we get a matrix-vector multiplication operation (y=Ax) with the following inner products for each row:

$$y_{ni+j} = S^e_{i,j} = A_{(ni+j,*)}x$$

For the 2D Laplace equation, for example, we get the inner products of the following vectors:

$$A^T_{(ni+j,*)} = \begin{bmatrix} \left(\frac{\partial \phi_i}{\partial r}, \frac{\partial \phi_j}{\partial r}\right)_{\hat{e}} \\ \left(\frac{\partial \phi_i}{\partial r}, \frac{\partial \phi_j}{\partial s}\right)_{\hat{e}} \\ \left(\frac{\partial \phi_i}{\partial s}, \frac{\partial \phi_j}{\partial r}\right)_{\hat{e}} \\ \left(\frac{\partial \phi_i}{\partial s}, \frac{\partial \phi_j}{\partial s}\right)_{\hat{e}} \end{bmatrix} \qquad x = \det(J) \begin{bmatrix} \frac{\partial r}{\partial x}\frac{\partial r}{\partial x} + \frac{\partial r}{\partial y}\frac{\partial r}{\partial y} \\ \frac{\partial r}{\partial x}\frac{\partial s}{\partial x} + \frac{\partial r}{\partial y}\frac{\partial s}{\partial y} \\ \frac{\partial s}{\partial x}\frac{\partial r}{\partial x} + \frac{\partial s}{\partial y}\frac{\partial r}{\partial y} \\ \frac{\partial s}{\partial x}\frac{\partial s}{\partial x} + \frac{\partial s}{\partial y}\frac{\partial s}{\partial y} \end{bmatrix}$$

Element independent

Element dependent

Optimization Problem

Objective: To minimize the number of multiply add pairs (MAPs) in matrix/vector multiplication

e.g. $r_2 = \alpha r_1 \Rightarrow y_2 = \alpha y_1$

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} r_1^T \\ \alpha r_1^T \\ \vdots \\ r_m^T \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} r_1^T x \\ \alpha y_1 \\ \vdots \\ r_m^T x \end{bmatrix}$$

Possible Optimizations

Number of Nonzeros

$y_3 = 0$ 0 MAPs

$y_1 = 2x_1$ 1 MAP

$y_2 = 2x_1 + 2x_2$ 2 MAPs

$y_4 = 2x_1 + 2x_2 + 2x_3$ 3 MAPs

Scalar Multiples

$y_4 = y_2$ 0 MAPs

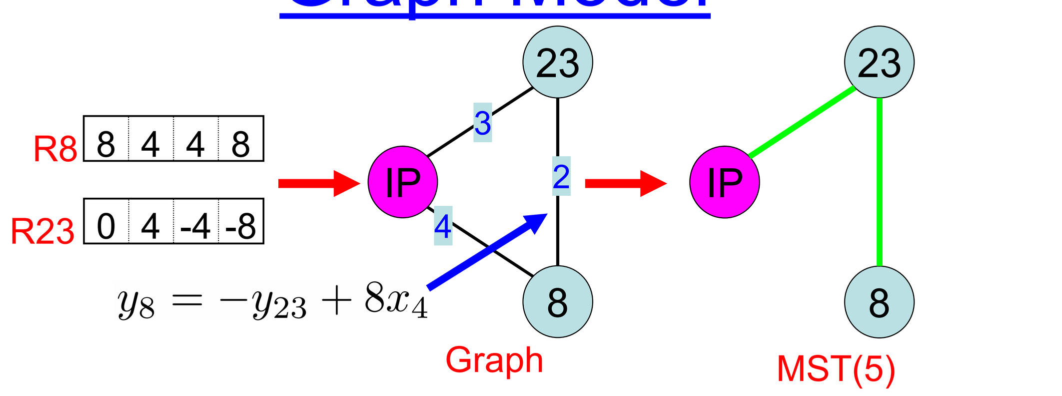
$y_2 = 4y_1$ 1 MAP

Partial Scalar Multiple $y_2 = 4y_4 + 4x_4$ 2 MAPs

Linear Combinations

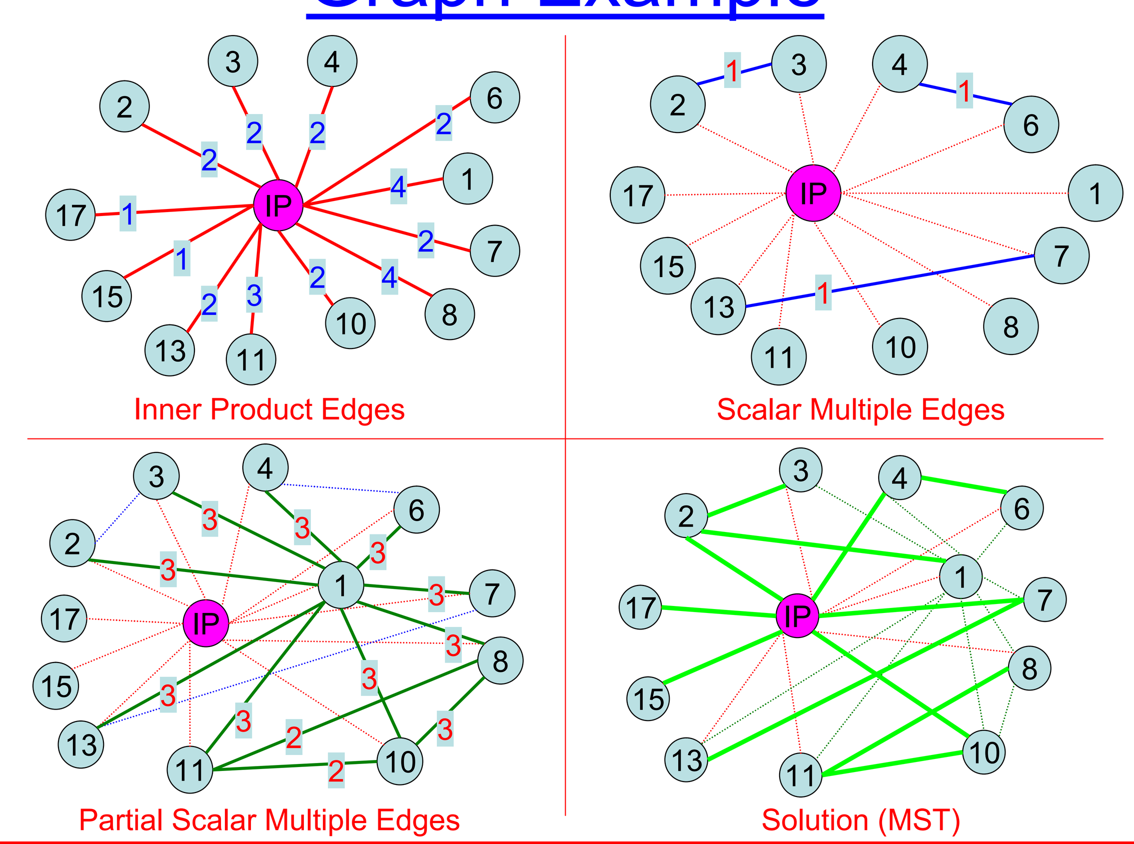
$y_2 = 4y_3 + 4y_4$ 2 MAPs

Graph Model



- Relationships between rows represented by edges between corresponding vertices
- Additional edges connect each vertex to IP vertex, representing inner product
  - FErari does not include this relationship in graph model
- Edge weights are MAP cost for particular operation
- Edges in minimum spanning tree (MST) represent operations to optimally compute (with these operations) matrix-vector multiplication
- In my work, I used Prim's algorithm to calculate MST

Graph Example



Graph Model Results

2D Laplace FErari Matrices, Matvec Costs

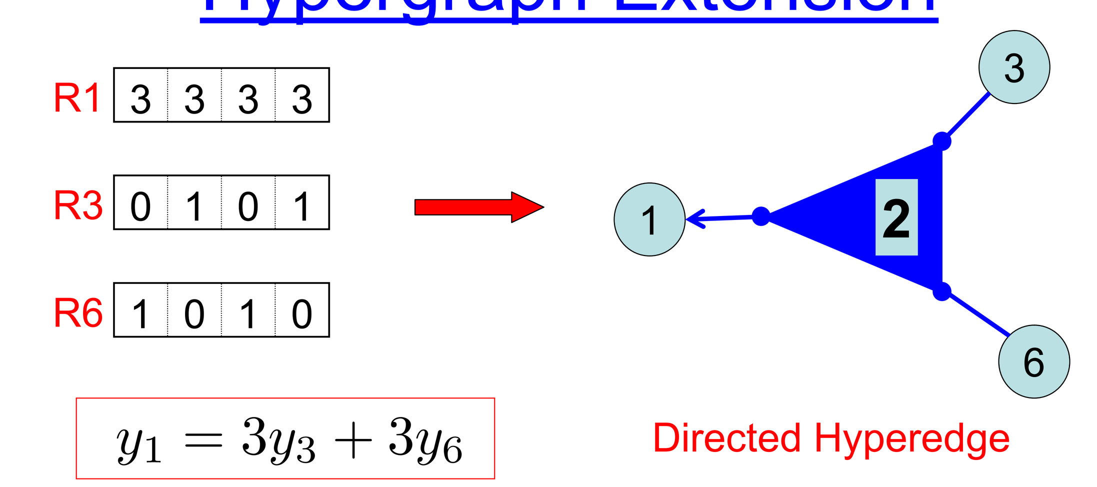
Order	Unoptimized MAPs	Nonzero MAPs	FErari MAPs	My MAPs
1	18	10	7	7
2	63	34	15	14*
3	165	108	45	43
4	360	292	176	152
5	693	589	443	366
6	1218	1070	867	678

3D Laplace FErari Matrices, Matvec Costs

Order	Unoptimized MAPs	Nonzero MAPs	FErari MAPs	My MAPs
1	60	21	-	17
2	330	177	94	79
3	1260	789	362	342
4	3780	2586	1118	1049
5	9576	7125	-	3567

*optimal

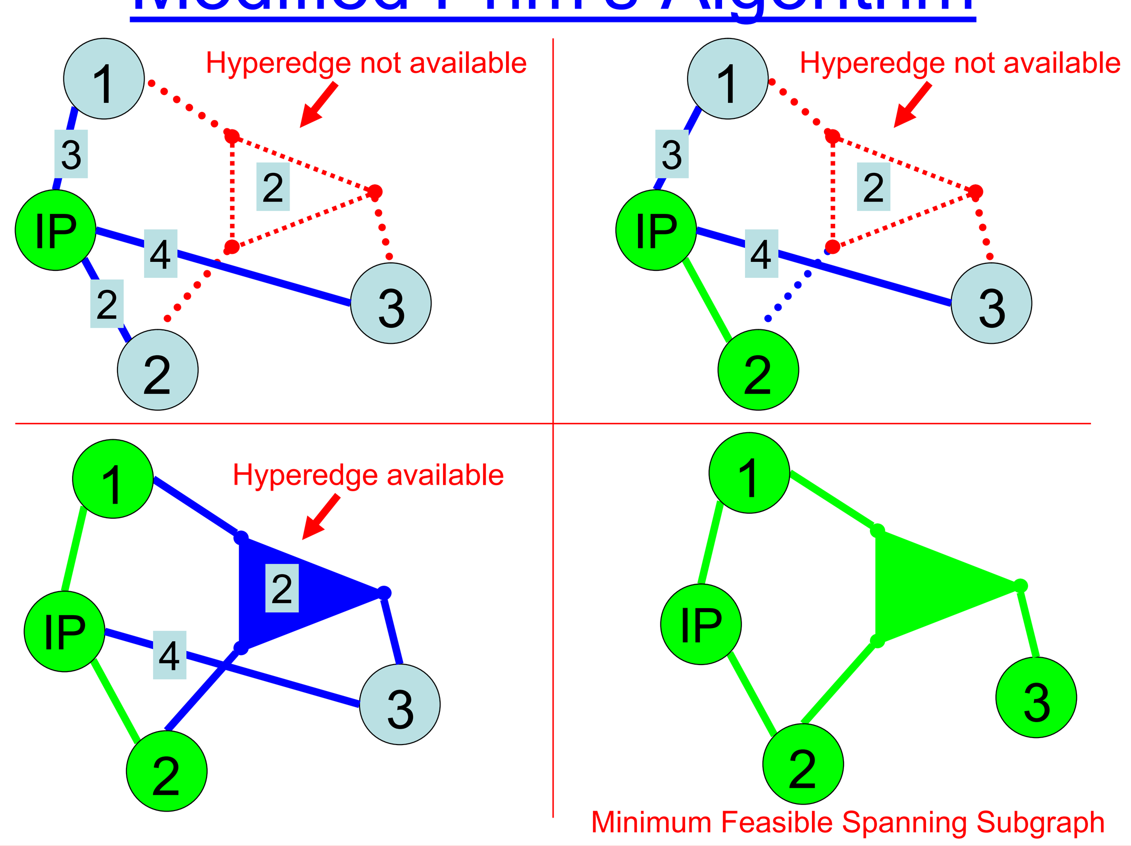
Hypergraph Extension



First Pass: Greedy Algorithm

- Modified Prim's algorithm to include hyperedges
- Polynomial time algorithm
- Solution no longer a tree
- No guarantee of optimum solution for hypergraph

Modified Prim's Algorithm



Hypergraph Model Results

2D Laplace FErari Matrices, Matvec Costs

Order	Unoptimized MAPs	Nonzero MAPs	Graph MAPs	HGraph MAPs
1	18	10	7	6*
2	63	34	14*	14*
3	165	108	43	43
4	360	292	152	150
5	693	589	366	363
6	1218	1070	678	678

3D Laplace FErari Matrices,Matvec Costs

Order	Unoptimized MAPs	Nonzero MAPs	FErari Geom. MAPs	My Graph MAPs	My HGraph MAPs
1	60	21	–	17	14*
2	330	177	105	79	68
3	1260	789	327	342	296
4	3780	2586	1045	1049	833
5	9576	7125	–	3567	3250

*optimal

Matrix-Vector Multiplication Timings

2D Laplace FErari Matrices

Order	Original Time (s)	Optimized Time (s)
1	2.16×10^{-3}	1.80×10^{-4}
2	7.74×10^{-3}	6.95×10^{-4}
3	1.94×10^{-2}	1.87×10^{-3}
4	4.23×10^{-2}	5.20×10^{-3}
5	8.09×10^{-2}	1.02×10^{-2}
6	1.53×10^{-1}	1.85×10^{-2}

3D Laplace FErari Matrices

Order	Original Time (s)	Optimized Time (s)
1	7.09×10^{-3}	3.65×10^{-4}
2	3.70×10^{-2}	1.68×10^{-3}
3	1.72×10^{-1}	9.62×10^{-3}
4	5.96×10^{-1}	3.15×10^{-2}

- 10,000 matrix vector multiplications per timing

Future Work

- Larger size hyperedges
  - Have implemented 3 and 4 vertex hyperedges
  - Higher degree perhaps useful for 3D FE problems
- More nearly optimal and optimal solution methods

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