WATER RESOURCES RESEARCH, VOL. 27, NO. 8, PAGES 1947–1956, AUGUST 1991

Immiscible Displacement in Porous Media: Stability Analysis of Three-Dimensional, Axisymmetric Disturbances With Application to Gravity-Driven Wetting Front Instability

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As water infiltrates downward into an air-filled, water wettable porous medium, the wetting front which forms may become unstable and allow the formation of downward moving fingers within the vadose zone. In this paper we first review stability criteria and estimated finger widths determined from linear stability theory in two-dimensional systems. Two approaches reported in the literature which employ different formulations for the interfacial boundary conditions, yield different estimates of the finger width. We extend the analyses to investigate finger diameter in three-dimensional systems by considering axisymmetric disturbances. Results of the three-dimensional analyses are illustrated through comparison to previously reported experimental results in three-dimensional systems. Because either approach gives similar results for low system fluxes, in practice, it probably will not matter which formulation is used. However, one approach represents the data better and contains only traditionally measured porous media properties.

INTRODUCTION

The subsurface transport of water and contaminants through the vadose zone to an aquifer is extremely important in determining both groundwater recharge and contaminant loading. Many processes influence this transport, some a result of man, others a result of physical properties of the unsaturated zone, weather, and biological activity. While in many situations we have met with moderate success in predicting recharge, the prediction of contaminant loading has been far from successful. This failure is due in large part to the fact that path must be known in contaminant transport, and large-scale, averaged hydraulic properties, which are useful to model the gross transport of water in the field, are thus of less use.

The documented high variability of tracer and contaminant transport in the field raises the question of cause. Usually, the answer encompasses the spatial variability in hydraulic and transport properties, the presence of cracks/joints/fractures or "macropore flow," and the temporal or spatial variability in water and contaminant supply. In addition to these well-known causes, laboratory [*Tabuchi*, 1961; *Smith*, 1967; *Hill and Parlange*, 1972; *Diment and Watson*, 1985; *Tamai et al.*, 1987] and field [*Starr et al.*, 1978, 1986; *Glass et al.*, 1988*b*; *Hendrickx et al.*, 1988; *van Ommen and Dejksma*, 1988; *van Ommen et al.*, 1988] experiments have shown gravity-driven instability in the flow field itself to occur under some conditions. Figure 1 shows a sequence of two photographs of water infiltrating through an initially dry, water wettable, two-layer sand system with a fine-textured layer overlying a coarse-textured bottom layer. As can be seen in the photographs, water moves through "fingers" in the coarse bottom layer as a result of the phenomenon of wetting front instability. When wetting front instability occurs, the flat wetting front moving downward through the unsaturated zone becomes unstable and breaks into fingers which move vertically to the water table, bypassing a large portion of the vadose zone. Solute transport through such an unstable flow field is extremely nonuniform, and therefore contaminant loading at the water table will be quite different than if transport is assumed to be one-dimensional [*Glass et al.*, 1988*a*, 1989*a*].

The subject of gravity-driven wetting front instability has received increased attention in recent years with an emphasis on systematic experimentation [*Glass and Steenhuis*, 1984; *Glass et al.*, 1989*b*, *c*, *d*; *Baker and Hillel*, 1990]. Most of the research has been concerned with fingering in thin slab laboratory chambers where two-dimensional fingers are forced. In horizontally extensive systems, however, fingers are three-dimensional; therefore stability criteria and relationships for finger properties as a function of system parameters must be formulated for three-dimensional systems. *Glass et al.* [1990] have conducted a laboratory study in large cylinders (30-cm diameter) to study wetting front instability in initially dry sand (see Figure 2). Using the dimensional analysis of *Glass et al.* [1989*b*], they measured relationships for individual finger diameter and finger velocity as functions of system parameters. In this paper we use linear stability theory to derive stability criteria and formulations for the diameter of three-dimensional fingers and then compare the theoretical results to the experiments of *Glass et al.* [1990].

THEORY

The general theory of hydrodynamic stability addresses the question of whether a given fluid flow is stable relative to imposed disturbances [e.g., *Lin*, 1955; *Chandrasekhar*, 1961; *Drazin and Reid*, 1981]. If it is not stable, then the flow will

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Paper number 91WR00836.
0043-1397/91/91WR-00836$05.00

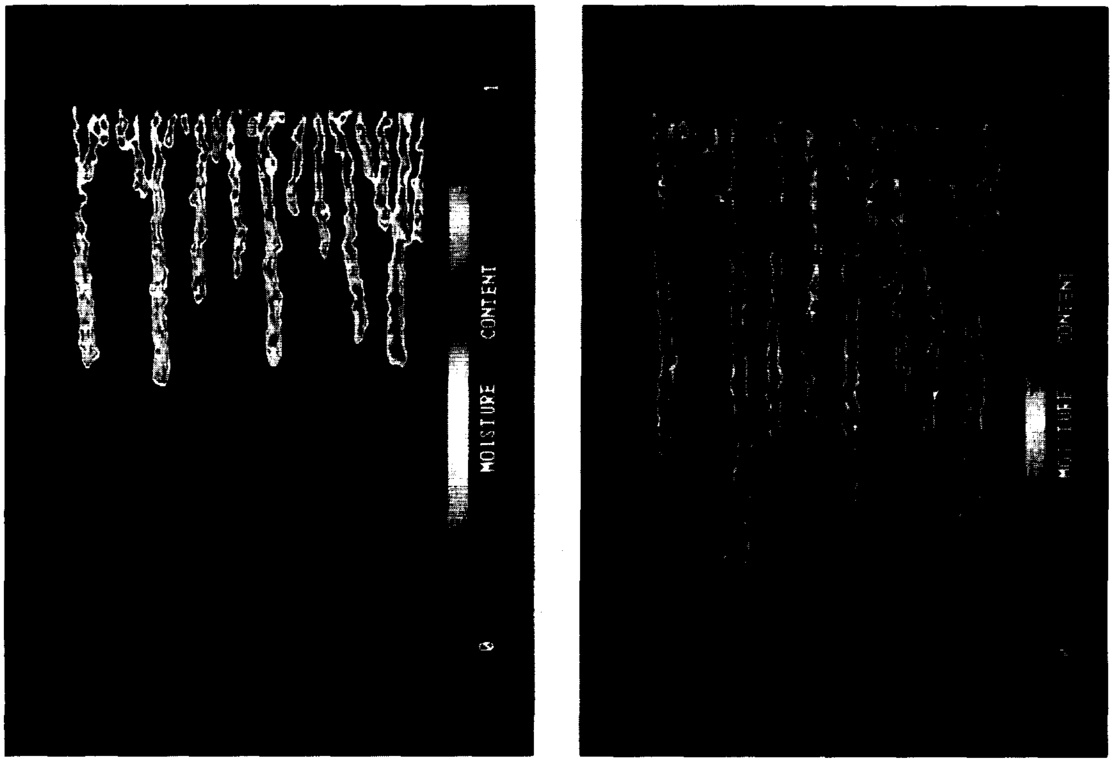

Fig. 1. The downward growth of fingers within an initially dry homogeneous porous medium is shown in a sequence of two images obtained through the use of a new moisture content visualization technique (see *Glass et al.* [1989*d*] for color prints). Infiltration into an unsaturated porous medium can be unstable when the flux through the system is less than its saturated conductivity. Water is supplied uniformly to this system at one-tenth the saturated conductivity through a thin top layer of low conductivity (dark rectangular region at top of image). Fingers form directly beneath this uniform supply surface. The dimensions of the medium are 45 cm wide, 76 cm high, and 1 cm thick (into the plane of visualization). If the thickness of the medium is less than the minimum finger width, a two-dimensional flow field is forced, as is the case here.

usually not occur in nature as disturbances or perturbations are always present. The general stability problem is formulated in the following way. A certain flow situation is governed by a set of hydrodynamic equations with solutions for the state variables, say, velocity and pressure. We consider an initial value problem with the values of the state variables different from their "base flow" or unperturbed solution. If the solution of the perturbed initial value problem approaches that of the base flow solution as time progresses, then the flow is stable with respect to the imposed perturbation. If the solution does not approach the base flow solution, but the perturbations do not grow in time, then the flow is considered to be neutrally stable. However, if the perturbations grow in time, then the solution is unstable and will not commonly be found in nature.

For general immiscible displacement we consider two immiscible fluids (denoted 1 and 2) of different viscosities and densities within a homogeneous isotropic porous medium. Fluid 1 displaces fluid 2 with a planar front or interface between the two fluids moving at a constant velocity, V. The interface is taken to be sharply defined such that its thickness is much thinner than the wavelengths of the perturbations. The z coordinate is taken parallel to the base flow velocity, U, and the position of the front is denoted by z_f. We take gravity to be acting in the direction of z.

Darcy's law governs incompressible flow through nondeformable porous media within the small Reynolds number regime where inertial forces are negligible relative to viscous forces,

$$u_i = -\frac{k}{\mu_i\theta_i}\nabla(p_i - \rho_i g z), \tag{1}$$

where u is the pore velocity $[L/t]$, k is the permeability $[L^2]$ considered to be a property of the medium and a function of θ, μ is the fluid viscosity $[M/Lt]$, θ is the fluid content $[L^3/L^3]$, p is the fluid pressure $[M/Lt^2]$, ρ is the fluid density $[M/L^3]$ considered here to be a constant, g is the gravitational acceleration $[L/t^2]$, and the subscript denotes the fluid (1 or 2). For k constant in space (a homogeneous, isotropic porous medium) and μ and θ constant within each region occupied by either fluid, we may define the velocity potential $\phi[L^2/t]$ as

$$\phi_i = -\frac{k}{\mu_i\theta_i}(p_i - \rho_i g z) \tag{2}$$

and (1) may be rewritten

$$u_i = \nabla\phi_i. \tag{3}$$

With continuity of mass

$$\nabla \cdot u_i = 0, \tag{4}$$

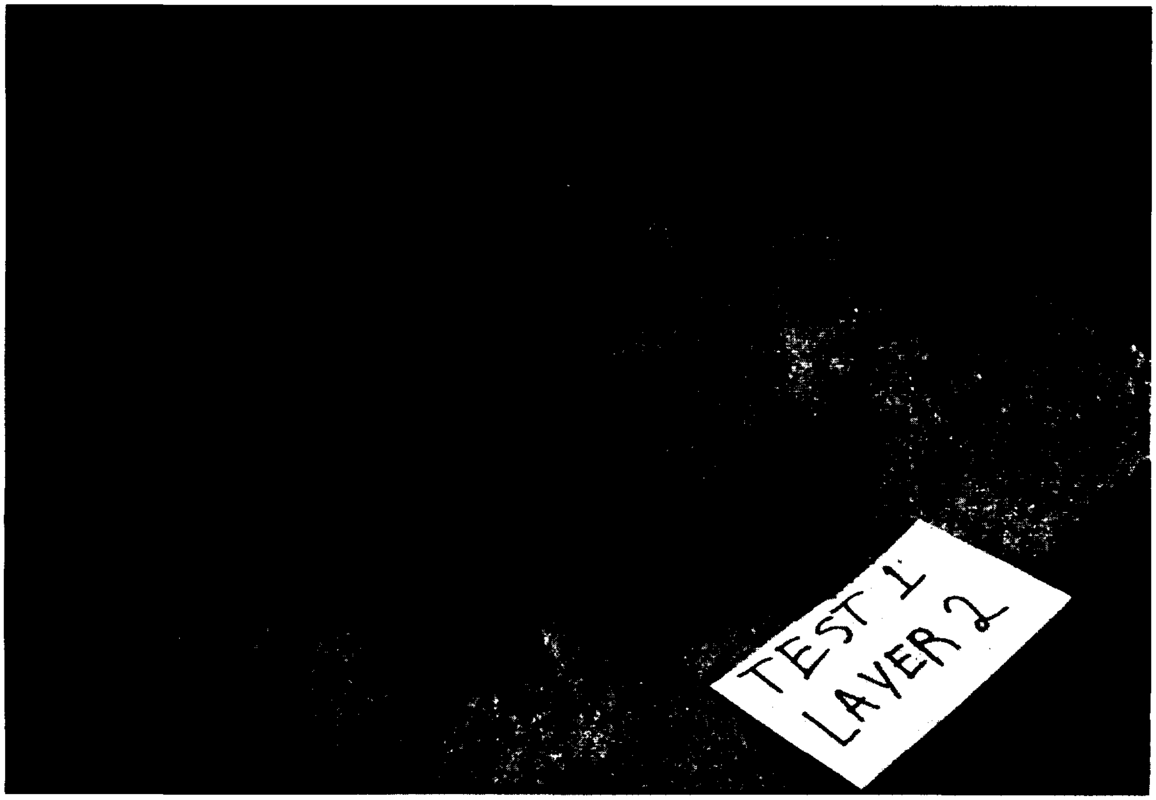

Fig. 2. Photograph of a 10-cm-high section removed from the 30-cm-diameter column used in the experiments of *Glass et al.* [1990]. Dry sand has been blown away from around the outside of the section leaving wet fingers standing.

we have for the velocity potential

$$\nabla^2 \phi_i = 0. \tag{5}$$

Thus in a homogeneous isotropic porous medium with Darcy's law obeyed, for constant fluid properties and constant θ_i within the region occupied by the fluid, Laplace's equation for the velocity potential is taken to govern the flow of each fluid within its region. Two boundary conditions must be imposed at the interface between the two fluids that reflect conservation of mass and momentum, respectively. To solve the hydrodynamic stability problem, the solution of (5), subject to these boundary conditions, is followed in time.

Even though (5) is linear, the boundary conditions at the interface are not, and thus to follow the evolution of the interface in time generally requires numerical solution. A simplification can be made, however, if we restrict ourselves to small perturbations (amplitude/wavelength small) which allow the equations to be linearized, that is, terms of quadratic or higher degree in the perturbations and their derivatives may be neglected. The solution to the set of linear equations will then admit solutions containing an exponential time factor, exp (ωt). If the real part of the growth rate, ω, is positive, then the flow is unstable; if it is negative, the flow is stable; and if it is zero, the flow is considered neutrally stable.

For two-dimensional disturbances a linear stability analysis of (5) was first accomplished by *Saffman and Taylor* [1958] and *Chouke et al.* [1959] and later by *Parlange and Hill* [1976]. The results of each of these analyses are different as the assumptions in the formulation of the boundary conditions at the interface are different.

Saffman and Taylor assumed continuity of normal velocity and pressure to first order at the interface to obtain ω,

$$\omega = \frac{m}{(\mu_1 + \mu_2)\theta_F}\,[g(\rho_1 - \rho_2)k_F + \theta_F V(\mu_2 - \mu_1)], \tag{6}$$

where m is the wave number of the disturbance; for simplicity here, and in the rest of the paper, we take $\theta_1 = \theta_2 = \theta_F$ and $k_F = k_1(\theta_F) = k_2(\theta_F)$, that is, the pore volume conducting either fluid is the same. Equation (6) predicts instability to occur when the quantity in the bracket is positive. We may specialize (6) to the case of vertical water infiltration where $\rho_1 \gg \rho_2$ and $\mu_1 \gg \mu_2$ to yield the condition for instability

$$\theta_F V < \frac{k_F g \rho_1}{\mu_1} \qquad \text{or} \qquad q_s < K_F, \tag{7}$$

where $q_s(L/t)$ is the flux through the system $(\theta_F V)$, and $K_F(L/t)$ is the value of the hydraulic conductivity at $\theta_F (K_F = k_F g \rho_1/\mu_1)$.

It can be seen from (6) and (7) that the wave number of the perturbation does not influence the stability criterion. The growth rate, however, increases without bound as m increases. Thus infinitesimal perturbations are predicted to grow the fastest. If either the conservation of mass or momentum conditions at the interface include a term dependent on the curvature of the front, then an additional constraint for stability, dependent on the wave number of the perturbation, is found. With this dependence on wave number the most rapidly growing wavelength, which should yield the expected finger width, will be finite and may be found by extremization. However, since linear stability is only valid for small perturbations, there is no guarantee that the wavelength that initially grows the most rapidly will yield the dominant finger width, especially in a highly nonlinear system.

Chouke et al. [1959] assumed a relationship analogous to the Laplace-Young relation exists between macroscopic frontal curvature and a pressure jump across the front:

$$p_1 - p_2 = \sigma_* \left[\frac{1}{r_1} + \frac{1}{r_2}\right], \tag{8}$$

where σ_* is the "effective macroscopic" surface tension, and r_1 and r_2 are the two principal radii of curvature for the macroscopic front. With (8) and the assumption of continuity of normal velocity at the interface they obtained to first order for ω,

$$\omega = \frac{m}{(\mu_1 + \mu_2)\theta_F} [g(\rho_1 - \rho_2)k_F + \theta_F V(\mu_2 - \mu_1) - \sigma_* k_F m^2]. \tag{9}$$

Thus for a given density difference, viscosity difference, and flow rate where (6) would stipulate instability, (9) states that only wave numbers less than a critical value or wavelengths above a critical wavelength, λ_c, given by

$$\lambda_c = 2\pi \left[\frac{\sigma_* k_F}{V\theta_F(\mu_2 - \mu_1) + k_F g(\rho_1 - \rho_2)}\right]^{1/2}, \tag{10}$$

will be unstable. The expected two-dimensional finger width, w, should be given by one-half the most rapidly growing wavelength; ω is extremized with respect to λ and specialized to the air/water system to yield

$$w = \pi \left[\frac{3\sigma_*}{\rho_1 g} \frac{1}{1 - q_s/K_F}\right]^{1/2}. \tag{11}$$

While the Laplace-Young relation does apply to Hele-Shaw cells and within the pores of a porous medium, its validity at a macroscopic level in porous media, as proposed in (8), is questionable.

The analysis of *Parlange and Hill* [1976] uses a method applied previously to laminar flame front stability by *Markstein* [1951] to derive a relationship between frontal velocity and curvature as a function of traditionally measured porous media properties. Parlange and Hill argue that in porous media where the fluid/fluid interface is not sharp in a macroscopic sense, continuity of mass at the interface must incorporate a term to account for capillary-induced "diffusion" at the front. They derive a relationship between the curvature of the front and the base flow velocity, U, to first order as

$$U = V - \Gamma\left[\frac{1}{r_1} + \frac{1}{r_2}\right], \tag{12}$$

where V is the velocity of an unperturbed, flat front, and r_1 and r_2 are the two principal radii of curvature. $\Gamma[L^2/t]$ was found for either two- or three-dimensional fronts to be

$$\Gamma = \int_{\psi_0}^{\psi_F} \frac{K\, d\psi}{\theta_F - \theta_0}, \tag{13}$$

where $\psi(L)$ is the pressure head of the invading fluid ($\psi = p/\rho g$), and the subscripts 0 and F denote values at the front and in back of the diffuse zone, respectively. Γ is defined for the fluid that would spontaneously displace the other by capillary action (that is, the fluid that wets the medium), and for the remainder of the paper we take this to be fluid 1. For the air/water system in water wettable soil, Γ is defined for the water. Using the approximate formula of *Parlange* [1975] for the sorptivity, $S[L/t^{1/2}]$,

$$S_F^2 = \int_{\psi_0}^{\psi_F} (\theta + \theta_F - 2\theta_0)K\, d\psi, \tag{14}$$

where θ and K are both functions of ψ, (13) may be written in terms of the sorptivity of the medium as

$$\Gamma = \frac{S_F^2}{2(\theta_F - \theta_0)^2}, \tag{15}$$

where the subscript F denotes evaluation of S between ψ_0 and ψ_F. The base flow velocity is thus found to be a function of the interfacial curvature and Γ, which in turn is a function of porous media properties and initial conditions.

Assuming the pressure to be continuous across the front, the application of (12) yields for ω

$$\omega = \frac{m}{(\mu_1 + \mu_2)\theta_F} [g(\rho_1 - \rho_2)k_F + \theta_F V(\mu_2 - \mu_1) - \Gamma(\mu_1 + \mu_2)\theta_F m] \tag{16}$$

and for λ_c

$$\lambda_c = \frac{2\pi\Gamma(\mu_1 + \mu_2)\theta_F}{V\theta_F(\mu_2 - \mu_1) + k_F g(\rho_1 - \rho_2)}. \tag{17}$$

Parlange and Hill [1976] gave a more restricted form of (16) assuming $\theta_F = \theta_s$, the saturated moisture content, specialized for the air/water system. Without this assumption, one can formulate for the finger width in the air/water system

$$w = \pi \frac{S_w^2}{K_s(\theta_s - \theta_0)} \left[\frac{1}{1 - q_s/K_F}\right], \tag{18}$$

where S_w is evaluated between ψ_0 and ψ_w [*Glass et al.*, 1989*c*]; ψ_w is the value of the pressure head that is first reached at θ_s as the porous media is wetted (below ψ_w, θ no longer varies with ψ). In deriving (18), the ratio $S_F^2/[(\theta_F - \theta_0)K_F]$ has been replaced by $S_w^2/[(\theta_s - \theta_0)K_s]$, a valid approximation for θ_F near θ_s.

Comparing (11) and (18) to experimental results by *Glass et al.* [1989*b*, *c*] has shown that for two-dimensional fingers, finger width obeys the results of Parlange and Hill. The fact that agreement between the linear theory of Parlange and Hill and experimental results presented by *Glass et al.* [1989*b*, *c*] is so close is likely due to the approximations used in their analysis. While the analysis linearizes with respect to the perturbation so that only small disturbances are treated, the fundamental nonlinearity in porous media property is preserved. Therefore we now derive stability

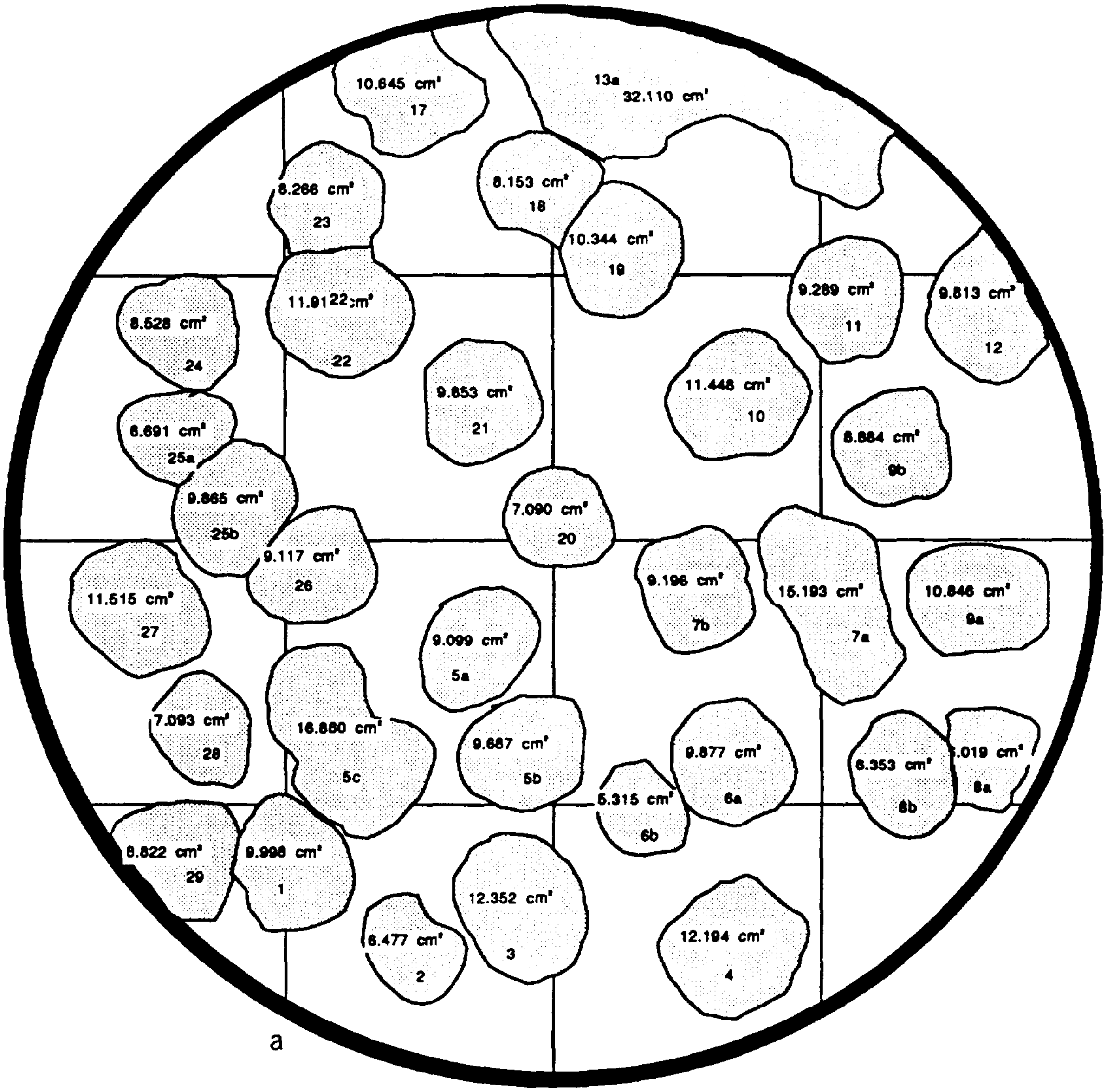

Fig. 3. Drawings of chamber cross sections showing fingers (shaded near circles) for experiments reported by *Glass et al.* [1990]: (*a*) experiment 1, (*b*) experiment 2, (*c*) experiment 4. Numbers denote cross-sectional areas of individual fingers.

criteria for three-dimensional systems and a formulation for finger diameter using the approach of *Parlange and Hill* [1976]. For purposes of comparison we also extend the formulation of *Chouke et al.* [1959] to three-dimensional axisymmetric fingers.

ANALYSIS OF THREE-DIMENSIONAL AXISYMMETRIC PERTURBATIONS

To analyze the linear stability of the interface between the two fluids moving at the base flow velocity, U, we suppose that the position of the interface is perturbed, which we denote by the function $A(x, y, t)$, about its location at z_f. In the three-dimensional problem we use cylindrical coordinates and explore axisymmetric perturbations represented by a series in Bessel's functions. As for the two-dimensional case, we need only to explore one component of this series as any disturbance with axial symmetry may be represented as a linear combination of orthogonal Bessel's functions. Thus

$$A(r, t) = ae^{\omega t}Jo(\kappa r), \tag{19}$$

where $r[L]$ is the radial component, and κ is the Bessel's function equivalent of a wave number. Laplace's equation may be written in cylindrical coordinates (r, z) as

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial \phi_i}{\partial r}\right] + \frac{\partial^2 \phi_i}{\partial z^2} = 0. \tag{20}$$

We may solve (20) by separation of variables. The solutions for which the perturbation diminishes as $z \to -\infty$ and $+\infty$, respectively, and remains finite at $r = 0$, yield for ϕ_1 and ϕ_2, respectively,

$$\phi_1 = B_1 Jo(\kappa r)e^{\kappa z} + f_1 z + g_1 r + h_1 zr + C_1 \tag{21}$$

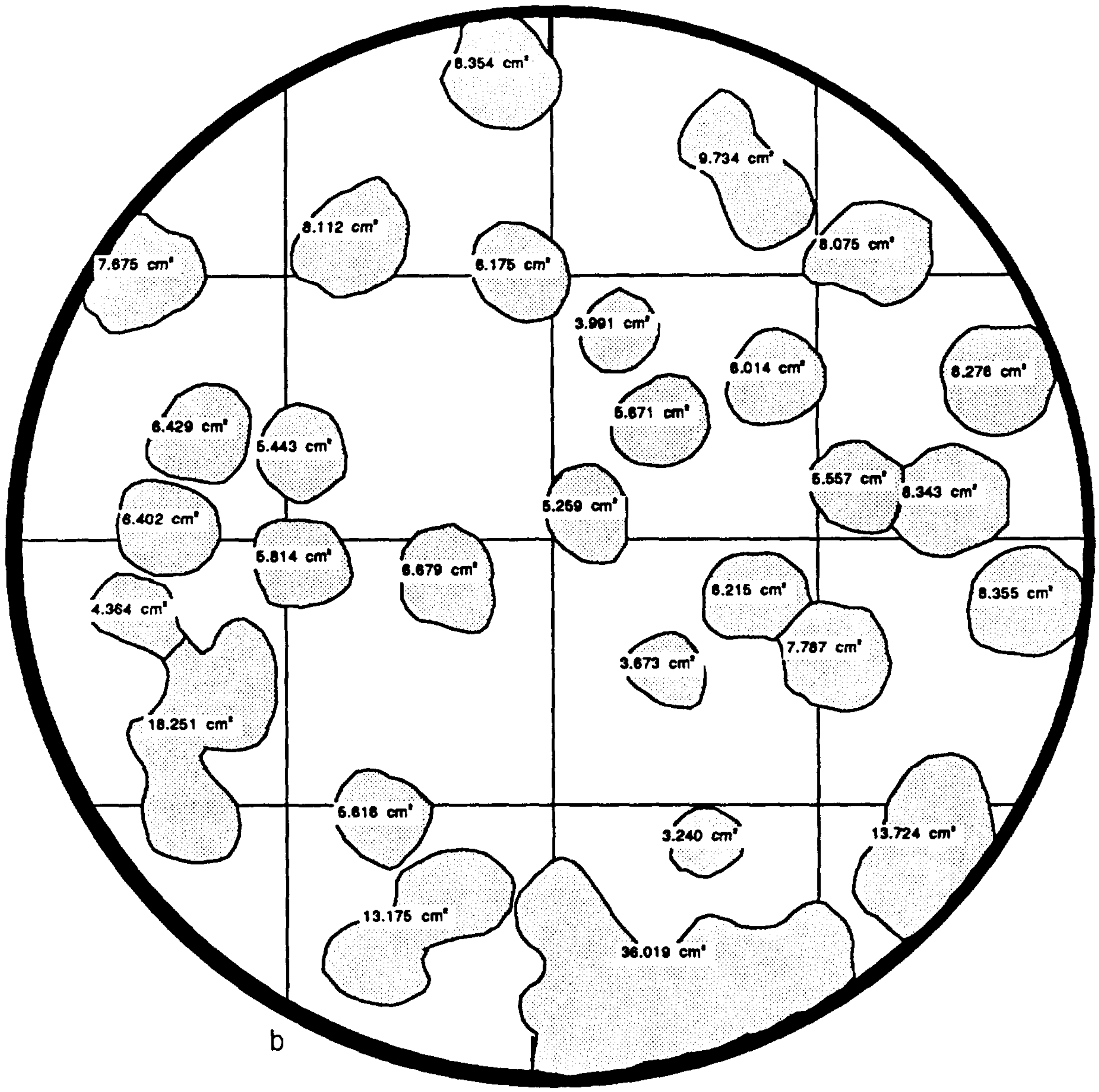

Fig. 3. (continued)

$$\phi_2 = B_2 Jo(\kappa r)e^{-\kappa z} + f_2 z + g_2 r + h_2 zr + C_2 \qquad (22)$$

where B, C, f, g, and h are constants to be determined.

The spatial disturbance in interfacial position also causes a perturbation in Darcy velocity u_i which we denote with a prime,

$$u_i = U + u_i'. \qquad (23)$$

Conservation of mass at the interface requires to first order that the z component of u be equivalent in both fluids or

$$\frac{\partial \phi_1}{\partial z} = \frac{\partial \phi_2}{\partial z} \qquad (24)$$

on $z = z_f + A$. Substitution of (12) into (23) yields

$$\frac{\partial \phi_i}{\partial z} = V - \Gamma\left[\frac{1}{r_1} + \frac{1}{r_2}\right] + \frac{\partial \phi_i'}{\partial z} \qquad (25)$$

on $z = z_f + A$. The last term on the right-hand side is the velocity of the perturbation in the z direction which may be approximated to first order by the partial derivative of A with respect to t. For small perturbations we may also replace the curvature in (25) to first order by $\nabla^2 A$. Thus

$$\frac{\partial \phi_i}{\partial z} - V - \frac{\partial A}{\partial t} = -\Gamma\left[\frac{1}{r}\frac{\partial A}{\partial r} + \frac{\partial^2 A}{\partial r^2}\right] \qquad (26)$$

evaluated at $z = z_f + A$. Substituting (19) into (26) and making use of the properties of Bessel's function with respect to derivatives, we have

$$\frac{\partial \phi_i}{\partial z} = V + (\omega + \Gamma\kappa^2)Jo(\kappa r)ae^{\omega t}. \qquad (27)$$

Applying (27) to (21) and (22) on $z = z_f + A$, we have $g = h = 0$, $f = V$ and B given by

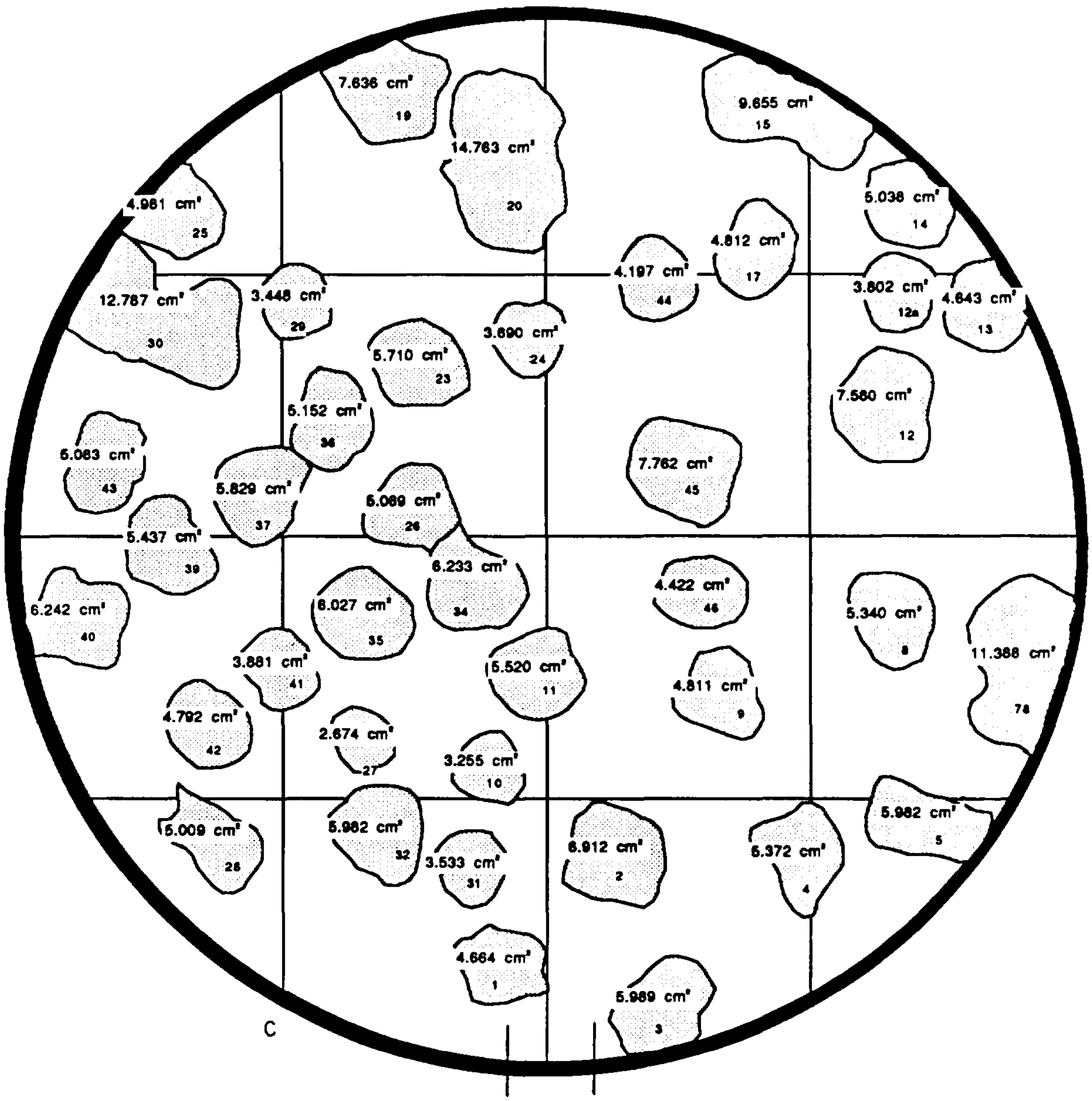

Fig. 3. (continued)

$$B_1 = \left(\Gamma\kappa + \frac{\omega}{\kappa}\right) a e^{-\kappa z_f} \tag{28}$$

$$B_2 = -\left(\Gamma\kappa + \frac{\omega}{\kappa}\right) a e^{\kappa z_f}. \tag{29}$$

Thus

$$\phi_1 = \left(\Gamma\kappa + \frac{\omega}{\kappa}\right) Jo(\kappa r) a e^{\omega t + \kappa(z - z_f)} + Vz + C_1 \tag{30}$$

$$\phi_2 = -\left(\Gamma\kappa + \frac{\omega}{\kappa}\right) Jo(\kappa r) a e^{\omega t - \kappa(z - z_f)} + Vz + C_2. \tag{31}$$

To obtain C and ω, we apply conservation of momentum at the interface which we take as simply

$$p_1 = p_2 + p_F \tag{32}$$

on $z = z_f + A$, where p_F is the pressure at the wet side of the wetting front and constitutes a constant jump in pressure across the macroscopic air/water interface. Thus we have by (2) with $\theta_1 = \theta_2 = \theta_F$ and $k_F = k_1(\theta_F) = k_2(\theta_F)$

$$\rho_1 g z - \frac{\theta_F \mu_1 \phi_1}{k_F} = \rho_2 g z - \frac{\theta_F \mu_2 \phi_2}{k_F} + p_F \tag{33}$$

on $z = z_f + A$. Application of (33) yields to first order in A

$$C = \frac{k_F}{(\mu_2 - \mu_1)\theta_F}[g z_f(\rho_2 - \rho_1) + p_F] - V z_f \tag{34}$$

where $C_1 = C_2 = C$, and

$$\omega = \frac{\kappa}{(\mu_1 + \mu_2)\theta_F}[g(\rho_1 - \rho_2)k_F + \theta_F V(\mu_2 - \mu_1) - \Gamma(\mu_1 + \mu_2)\theta_F \kappa]. \tag{35}$$

Thus the system is unstable when the quantity in the bracket of (35) is positive. We find stability governed by the viscosity difference, the density difference, the imposed flow velocity, and the "wavelength" of the disturbance. We also note that the linear analysis does not predict traveling waves at the interface because ω cannot be imaginary.

The most rapidly growing or most unstable mode denoted κ_* is found by extremizing (35) with respect to κ:

$$\kappa_* = \frac{\theta_F V(\mu_2 - \mu_1) + k_F g(\rho_1 - \rho_2)}{2\Gamma(\mu_1 - \mu_2)\theta_F}. \tag{36}$$

Taking the finger diameter, d, to be

$$J_o\left[\frac{\kappa_* d}{2}\right] = 0, \tag{37}$$

we find

$$d = \frac{4.8}{\kappa_*}. \tag{38}$$

Specializing (36) for the case of downward water infiltration into an air-filled porous medium where $\rho_1 \gg \rho_2$ and $\mu_1 \gg \mu_2$, we have

$$d = \frac{9.6\ \Gamma\theta_F}{(K_F - q_s)}. \tag{39}$$

For a constant nonzero initial water content, θ_0, within the region occupied by air, θ_F may be replaced by $(\theta_F - \theta_0)$. Because for soils the ratio $S_F^2/[(\theta_F - \theta_0)K_F]$ is a weak function of θ near θ_s and we know from experiment that fingers are at or near θ_s, we may replace the values of S_F, K_F, and θ_F in this ratio with their values at saturation. Note that S is now evaluated between ψ_0 and ψ_w, denoted S_w. Substitution for Γ from (15) yields

$$d = 4.8\ \frac{S_w^2}{K_s(\theta_s - \theta_0)}\left[\frac{1}{1 - q_s/K_F}\right]. \tag{40}$$

Following a similar stability analysis of three-dimensional axisymmetric perturbations using the Chouke conditions at the interface, that is, (8) and $U = V$, we arrive at the relation for ω,

$$\omega = \frac{\kappa}{(\mu_1 + \mu_2)\theta_F}[g(\rho_1 - \rho_2)k_F + \theta_F V(\mu_2 - \mu_1) - \sigma_* k_F \kappa^2] \tag{41}$$

and the predicted finger diameter specialized for the water/air system

$$d = 4.8\left[\frac{3\sigma_*}{\rho_1 g}\,\frac{1}{1 - q_s/K_F}\right]^{1/2}. \tag{42}$$

A form of (41) was first derived by *Peters and Flock* [1981].

DISCUSSION

The results for three-dimensional axisymmetric perturbations using either the *Parlange and Hill* [1976] (equations (35) and (40)) or the *Chouke et al.* [1959] (equations (41) and (42)) approach are identical to their counterparts for two-dimensional perturbations (equations (16) and (18), and (9) and (11), respectively) except for a constant factor which in the three-dimensional case is 4.8 and the two-dimensional case is π.

Equations (40) and (42) predict a minimum finger width as $q_s \to 0$ to be given by $4.8S_s^2/[(\theta_s - \theta_0)K_s]$ and $4.8(3\sigma_*/\rho g)^{1/2}$, respectively. While (40) contains traditionally measured porous media properties, σ_* must be found by fitting (42) to experimental data from fingering flow fields. Thus the value of $d(0)$ cannot be used to distinguish the models. The functional behavior with q_s, however, could allow such a distinction because the exponent in (40) and (42) differs by a power of $\frac{1}{2}$; a possibility we explore further.

From both two- and three-dimensional experiments we have found that the flux through the finger q_F is essentially equal to K_F, and unit gradient drives the flow within the finger [*Glass et al.*, 1989c]. We have also found that as q_S increases, q_F and thus K_F also increase as does the average finger width or diameter, d. We may also write

$$q_F = \beta q_S, \tag{43}$$

where β is the reciprocal fractional cross-sectional area of the system in fingers and is itself a function of q_F, the form of which must be found through experimentation. We may now rewrite (40) and (42) as

$$d = d(0)\left[\frac{1}{(1 - 1/\beta)}\right]^n, \tag{44}$$

where n is 1 for (40) and $\frac{1}{2}$ for (42).

We can test (44) on a preliminary basis using the three two-layer experiments reported by *Glass et al.* [1990] where an entire unstable flow field was generated and both β and d were measured (experiments 1, 2, and 4). Drawings of chamber cross sections taken near the bottom of the chamber for these three experiments are shown in Figure 3 (fingers are shaded, near circles). Average β and d for these experiments give the three points in Figure 4; $d(0)$ can be estimated from the smallest finger diameter observed in the experiments, found to be 1.9 cm (experiment 4, depth-averaged d for finger 27). This value represents an upper bound on $d(0)$ because it was not measured at near-zero system flux. However, we will use it conservatively to curve fit (44) yielding the two curves in Figure 4. While the data are indeed limited, $n = 1$ appears to give a slightly better fit than $n = \frac{1}{2}$. We may consider the relationship predicted among $d(0)$, S^2, K_s, and θ_s as an additional check on (40). As reported by *Glass et al.* [1990], the value for S^2 of 5.5 cm^2/min for the coarse sand used in the experiment had a high degree of uncertainty associated with it (a standard deviation larger than the value itself). Taking the upper bound on $d(0)$ (1.9 cm), K_s (24.7 ± 3 cm/min), and θ_s (0.41), we calculate from (40) a value of S^2 (6.1 cm^2/min) which is well within the error of its measurement. Development of a measurement technique for S in coarse material is required to conclusively test the applicability of (40).

Additional experimentation in full three-dimensional unstable flow fields for $1/\beta$ above 0.6 would be of use to distinguish the models. However, experimentation at such high $1/\beta$ is problematical. For the same fractional area in fingers for two- and three-dimensional geometries, fingers in three dimensions will be closer than in two dimensions. In

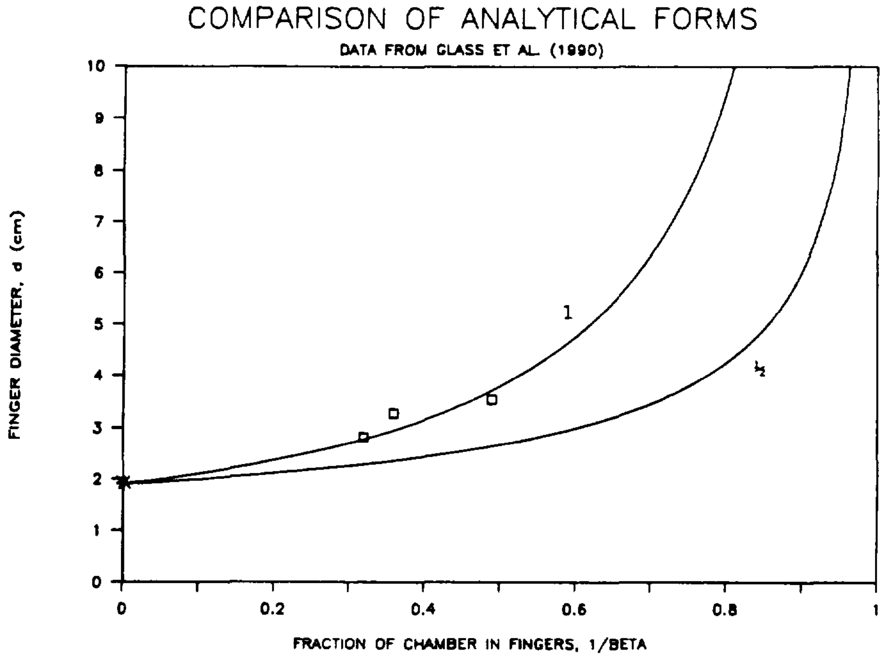

Fig. 4. Finger diameter versus the fraction of chamber area in fingers ($1/\beta$). Squares represent experiments 1, 2, and 4. The smallest observed finger diameter is shown as the star and used to curve fit equation (44) for $n = 1$ and $\frac{1}{2}$ (solid curves).

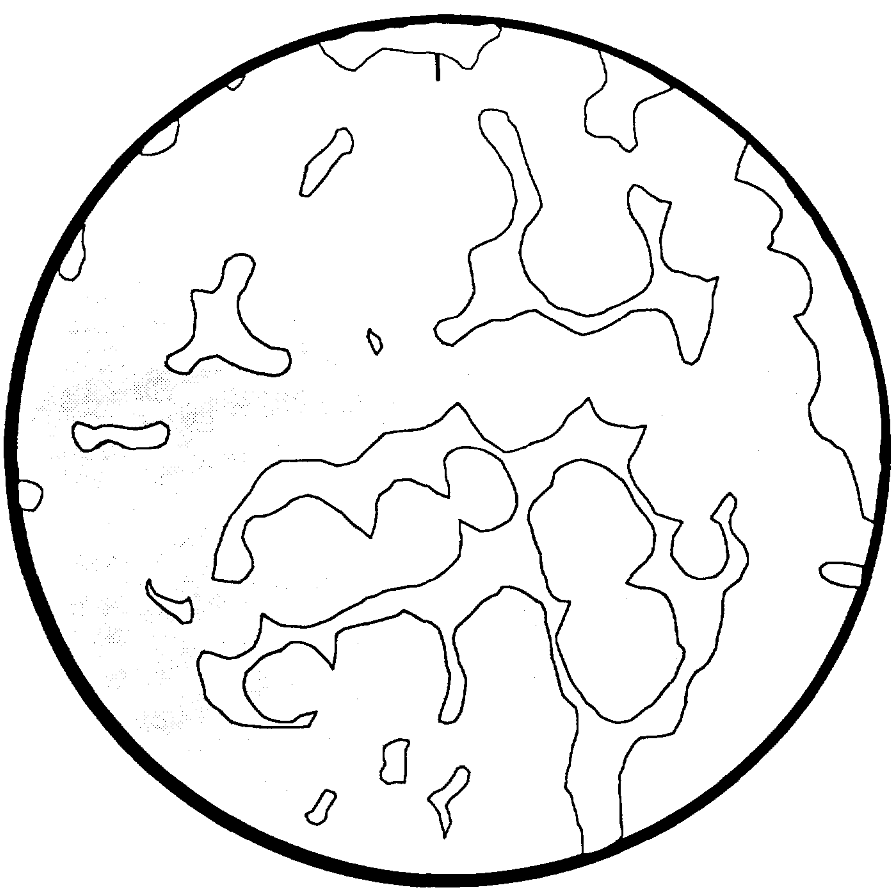

Fig. 5. Drawing of chamber cross section showing wetted region (shaded) for experiment 3 from *Glass et al.* [1990].

fact, fingers will touch if they are arranged in a square lattice for $1/\beta = \pi/4$. As an example, Figure 5 shows the cross section of experiment 3 in *Glass et al.* [1990] where $1/\beta$ was 0.82. As one can see, almost all fingers are touching each other and the determination of clear finger diameters is impossible.

CONCLUSION

Two different approaches to the formulation of conservation of mass and momentum at the interface between two immiscible fluids are seen to yield functional forms for the behavior of expected finger diameter as a function of the flux through the system that differ by the exponent n (either 1 or $\frac{1}{2}$). While the approach based on that of *Parlange and Hill* [1976] ($n = 1$) appears to fit experimental data slightly better, in practice, it may not be very important to know the exact value of n. For most practical situations where fingering will play a major role, $1/\beta$ will be much less than 1. It can be seen in Figure 4 that d varies only slightly over the range $0 < 1/\beta < 0.5$, and within this range either form will yield reasonable results. The form based on the analysis of Parlange and Hill however, also contains only traditionally measured hydraulic properties of the porous medium and yields directly the minimum finger diameter, while the form based on the analysis of *Chouke et al.* [1959] contains the ill-defined effective macroscopic surface tension.

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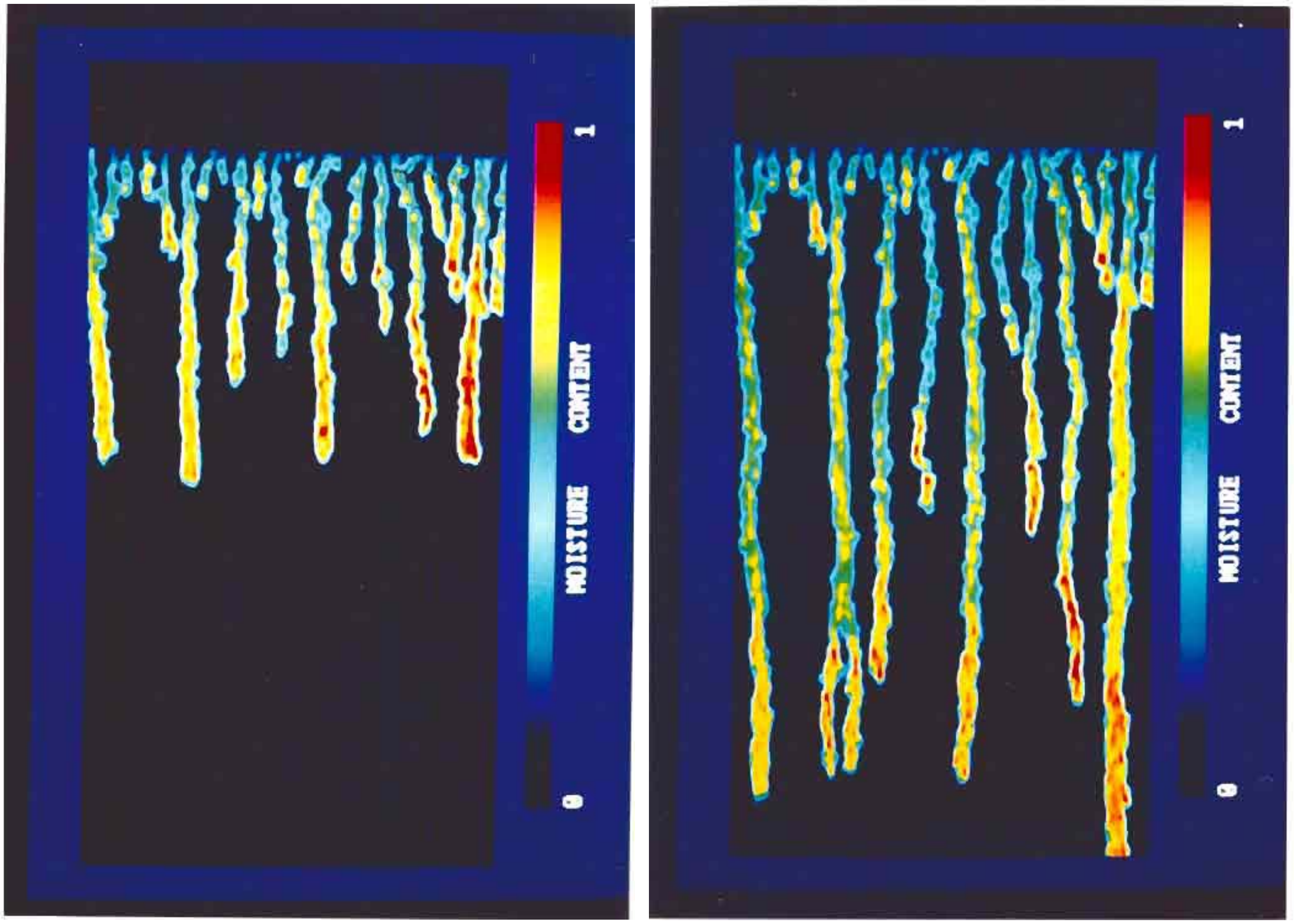
(Received December 29, 1989;
revised March 5, 1991;
accepted March 14, 1991.)



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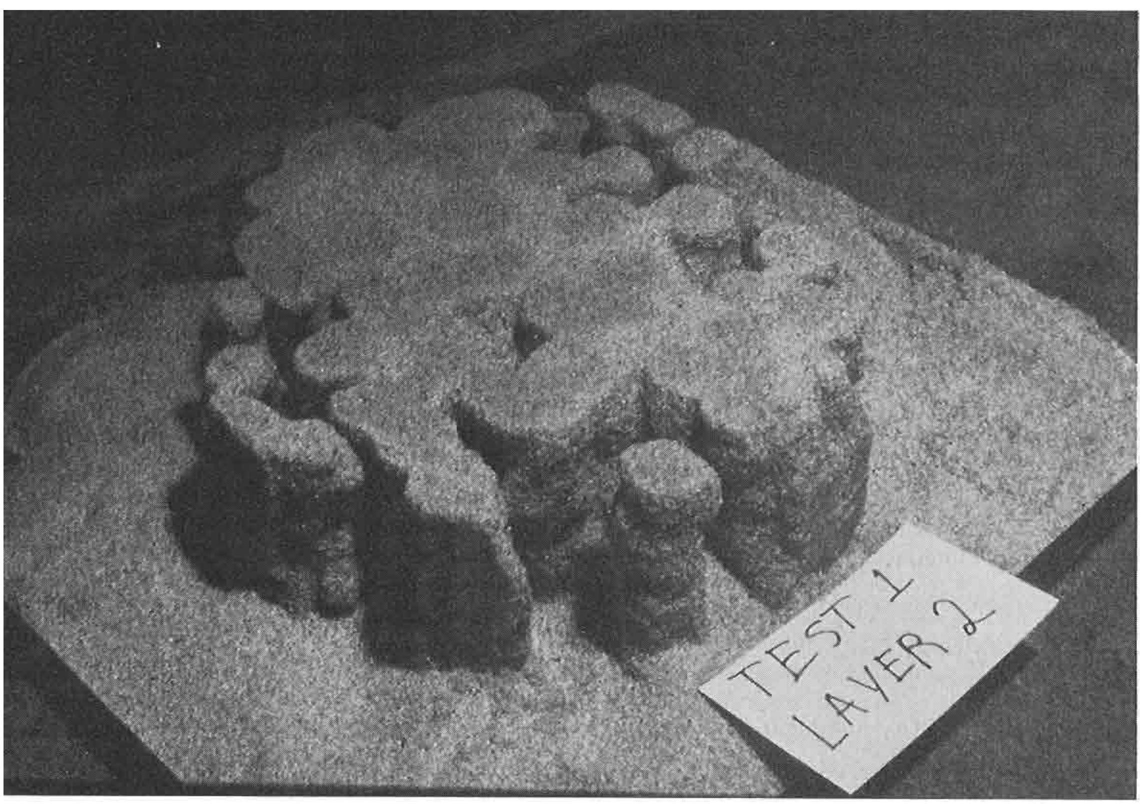

Fig. 2. Photograph of a 10-cm-high section removed from the 30-cm-diameter column used in the experiments of *Glass et al.* [1990]. Dry sand has been blown away from around the outside of the section leaving wet fingers standing.