Growth and Defection in Hierarchical, Synergistic Networks

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Abstract

Certain classes of social and economic networks display a set of fundamentally similar characteristics: individuals benefit from being in groups, and the groups are internally organized in a hierarchical fashion where benefit is allocated unequally. To better understand how these networks evolve and maintain themselves, we developed a model that grows hierarchical networks through the local actions of agents who can both solicit others to join their group and defect to start their own independent organization. Group benefit is conceptualized as synergy, which increases more rapidly than the sum of the individual talent of group members. Synergy is allocated back to members within the hierarchy through a nonlinear payback function such that the higher an individual is within the hierarchy, the greater their payback. Parametric simulations where we consider the influence of the payback function and population size indicate that cluster size transitions from the size of the entire population where the maximum system synergy is obtained to smaller and smaller values as payback becomes increasingly inequitable and population size becomes large. For such sub-maximal situations, we find cluster size and total system synergy to fluctuate with well defined mean values that suggests the existence of a stochastically stable, asymptotic attractor.

1 Introduction

Groups of people, animals, and other entities form for many reasons, such as to be more productive (in the case of humans that collaborate) and to increase the

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likelihood of individual survival (in the case of animals that herd). One can think of similar benefits of creating tribes, trade unions, extended families, special interest groups, political action committees, and political parties. Over longer time scales, local communities form so as to expand, adapt and prosper; and federations of nations form to create international stability. In economics, the forces by which firms are formed are often discussed through the work of Coase[1], who postulated that economic firms composed of employees fundamentally cut the costs of people exchanging services and information — clustering individuals generates returns that are greater than the sum of their parts. Broadly, these varied grouping benefits are forms of *synergy*.[1]

An important factor influencing these groups is the returns that each group member receives. In natural and social systems, not every member receives the same portion of the overall "pie." Heads of corporations, for example, often receive orders of magnitude more than lower level employees. Leaders of political parties hold inordinate powers generated by the work of the party. Animals at the physical center of a herd are more protected than those at the edges. Since most groups are composed of overall group leaders, sub-group leaders, and base-level members, they resemble hierarchical networks, where leaders have sub-group- or group-wide powers, and receive above-average returns. If the returns for a member are not sufficient, the member leaves the group, either alone or with other group members that defer to the leaver in some way (e.g., an entire department of technology stock traders). Very large groups grow and adjust naturally through the addition and defection of members.

But how do these groups fundamentally form? How is the structure of a group, its speed of formation, and its stability affected by the structure of paybacks to its group members? Are some groups inherently unstable? Are there natural constraints that prevent the group from reaching the size that maximizes global benefit?

To better understand how groups or networks of individuals form and evolve, we construct a *recurrent-game* model of agents that form networks, where the returns created by the group are greater than the sum of the individuals' returns. Each group benefits from formation, and each returns the benefits to its members using a hierarchical payback scheme in which those higher in the organizational structure receive returns generally higher than those lower in the structure. Simulations based on different returns and payback schemes indicate that while in many situations members form stable single clusters, in some situations there is continual, dramatic fluctuations in these hierarchical structures and, most importantly, in total system synergy. Because our underling process is Markovian, the fluctuating states suggest *stochastic equilibria*, in which cluster structures oscillate stochastically within a reduced set of potential system states. Regardless, the resulting group structures

[1] E.g., *The American Heritage Dictionary of the English Language* (Fourth Edition Copyright 2000 by Houghton Mifflin Company) applies: "(1) The interaction of two or more agents or forces so that their combined effect is greater than the sum of their individual effects; (2) Cooperative interaction among groups, especially among the acquired subsidiaries or merged parts of a corporation, that creates an enhanced combined effect."

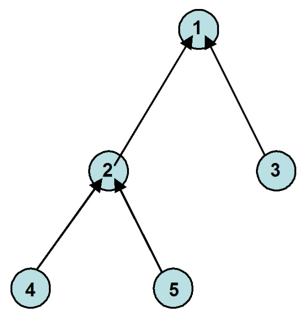

Figure 1: Hierarchical Cluster of Agents

in both single-cluster and multiple-cluster cases are a clear function of the payback structure.

The framework we have chosen is, by design, simple. While there are many extensions to this model that can explore more complex group returns (based specific physical network structures), learning routines (e.g., reinforcement, best reply), and the effects of local perturbations in group returns and the distribution of paybacks, our main finding is that, even with fixed relationships and parameters, group evolution can be quite complex.

Section 2 details our simple network model, including the calculation of group returns, the payback function that determines what fraction each member receives of the group's returns, and the rules for formation and defection; it concludes with a set of parameters that define the recurrent game. Section 3 describes the objectives and results of parametric simulations. Section 4 concludes.

2 The Network Model

2.1 Energy and Synergy

The network is composed of N agents. Each agent has a constant energy level e_n, and seeks to gain *synergy* from interacting with other agents. More energy is better than less: agent i can always achieve its own personal energy level, but it can also join a group or cluster of other agents and achieve additional synergy. Figure 1 illustrates an example of this hierarchical clustering of agents; agents 4 and 5 report to 2, who along with 3 reports to 1. Denoting the cluster of agents that is headed by agent i as C^i, the *energy* level of C^i is defined as

$$E^i = (\sum_{n \in C^i} e_n)^\rho, \tag{1}$$

where ρ is the measure of agent interactions, or synergy. The *synergy* of a cluster of agents, S^i, is defined as the difference between the energy level of the cluster when together and the sum of agents' individual energies:

$$S^i = (\sum_{n \in C^i} e_n)^\rho - (\sum_{n \in C^i} e_n^\rho). \tag{2}$$

When $\rho > 1$, cluster synergy is positive; the energy created together is greater than the sum of the individual energies. When $\rho < 1$, cluster synergy is negative; each agent is worse off in a cluster than being by itself.

The synergy that is returned to each agent is a function of its location in the cluster. If agent j belongs to cluster C^i, the synergy that agent j receives, e^i_j, is a fraction π^i_j of the total synergy level of cluster:

$$e^i_j = \pi^i_j S^i. \tag{3}$$

One can imagine a range of hierarchial pay structures that distribute the total synergy of a cluster back to its members: some structures could give an equal amount to each member while others gave more to those at the top of the "organization." To model this distribution we use the following functional form for π^i_j: defining the weighted contribution of agent j to cluster i to be the sum of its energy and all energies below it, raised to a power,

$$a^i_j = (\sum_{n \in C^i} e_n)^\gamma, \tag{4}$$

agent j's payback from the cluster's synergy is then the fraction

$$\pi^i_j = \frac{a^i_j}{\sum\limits_{n \in C^i} a^i_n}. \tag{5}$$

The exponent γ is our measure of the asymmetry of paybacks. When $\gamma = 0$, each member gets an equal share of the cluster's synergy; as γ increases above 0 higher fractions of total cluster synergy go to those in the upper parts of the hierarchy. Still, when $\rho > 1$, an agent will always prefer to be in a cluster than alone, since the synergy of a cluster of one agent is zero.[2]

2.2 The N-Person Recurrent Game

Each time period, an agent is selected at random and given two tasks. First, using equations (1) through (5), the agent computes how its payoff would change if it *and* those agents that report to it defected from their current network. If this predicted payoff is greater than what the selected agent is currently receiving, the agent and those underneath it defect to form their own new cluster, keeping the existing hierarchy.

Second, the agent attempts to add agents to its cluster, by randomly selecting another agent that is *not* in its cluster and offering it a payback equal to what the potential member will exactly receive this time period if it and its underlings join

[2]The returns for a one-agent cluster headed by agent i are

$$e^i_i = \frac{a^i_i}{a^i_i}\, e^\rho_i - e^\rho_i = e^\rho_i - e^\rho_i = 0. \tag{6}$$

Table 1. Payoffs for Defection Game

Defect	a
Do not defect	0

Table 2. Payoffs for Hiring Game

	Accept offer	Decline offer
Make offer	a, b	$0, 0$
Do not make offer	$0, 0$	$0, 0$

(the underlings have no say in this process). If the soliciting and solicited agents determine that they will each receive larger individual paybacks by joining, the solicited agent and its underlings defect from its current cluster and join the new cluster, reporting to the soliciting agent.

While our sets of agent mechanics and decision rules are relatively simple, we construct our environment in the context of existing frameworks so that, in the future, we are prepared for natural extensions such as modeling learning and other adaptive processes. Following Young[2], each agent $i \in N$ has a pure strategy space X_i and utility function u_i that maps each N-tuple of strategies $\{x_1, x_2, ..., x_N\}$ to a payoff function $u_i(x)$. Each agent is involved in two sequential games, each with separate strategy spaces. The first strategy is to consider defection and the second to consider adding a new member (and any underlings) that will report to it. The first strategy is a single-person game, that is, the agent plays it against himself, where utility $u_i(x)$ is measured by current period payoff (Table 1). [3]

The value a is the return the agent will get *above* what it is currently receiving. This return are dynamic, i.e., it changes over time. If $a > 0$, the agent defects, otherwise it stays put. The second payoff function is for the two-person hiring game, where the hiring agent meets and plays against the sampled potential hiree (Table 2). Again, each agent's utility $u(x)$ is the current-period payoff of each strategy). In this second case, "Make offer" and "Accept offer" are the dominate strategy if $a > 0$ and $b > 0$, otherwise there will be no expansion of the offering agent's cluster.

Since each time period a new pair of agents from the population of N interacts, the game is an N-person *recurrent game.*[4] Defining the set O_i as the set of agents (i.e., opponents) eligible to play the game with agent i, the elements $\{X_i, u_i, O_i\}_{1 \le i \le N}$ completely define the N-person recurrent game. Our recurrent game is a *discrete-time Markov process*: first, at time t, past states of the game

[3] Analytically, we could alternatively combine these two strategy spaces, where the defecting/offering agent has four strategies: Defect & Make offer, Do not defect & Make offer, etc. For clarity of exposition, we keep them separate.

[4] Recurrent games are different from *repeated games* in that different pairs of agents interact each time step.

(e.g., $t-1$) have no influence on future states ($t+1$). Second, for any pair of states $\{z, z'\}$ in system state space Z, the transition probability $P_{z,z'} = P_{z,z'}(t)$ is independent of t. Third, the state space is finite.[5]

2.3 System States and System Stability

Since all of our agents are equal in energy e_n, the more important set of states to observe is the *number*, and to a lesser degree the structure, of clusters in the overall system. Some systems of agents will follow any number of evolutionary paths to a single cluster, i.e., the process to a single cluster is *ergodic*. Other systems will eventually oscillate between 1, 2, and 3 clusters of varying sizes, but may be *asymptotically stable*, that is, the system of clusters oscillates around and tends toward a particular number of clusters. Stability is generally a function of the relationship between N, ρ, and γ.

3 Parametric Simulations

To formally investigate the relationship between N, ρ, and γ, an initial series of simulations where conducted, using a RePast simulation environment,[6] where $N = \{6, 8, 10, 12, ..., 50\}$, $\rho = \{1.1, 1.3, 1.5\}$, and $\gamma = \{0.0, 0.25, 0.50, ..., 2.0\}$. Based on the results from these, additional simulations were conducted to investigate specific sub-areas within this space. Each simulation was run for a maximum of $(50 \times N)$ steps, but was stopped if the system had reached a stable configuration as defined by no change within the past $(10 \times N)$ steps. For those simulations where a stable, unchanging configuration was not reached, summary statistics such as mean and standard deviation were calculated for S and C after an initial transient period of $(5 \times N)$ steps.

3.1 Cluster Structure and Evolution

The payback exponent γ affects the returns that members of a cluster receive and therefore the structures of clusters. When $\gamma = 0$, each member receives an equal share of the pie, and there is never sufficient incentive to defect (the new cluster's per-member returns will be smaller, since the cluster is smaller). Figure 2 illustrates a single cluster that resulted from $N = 100, \rho = 1.1, \gamma = 0.0$.

When γ is larger than 0, agents within clusters begin to have sufficient incentives to defect, resulting in different hierarchical structures. For example, Figure 3 shows a single cluster that resulted from $N = 100, \rho = 1.1, \gamma = 1.1$. Comparing the two

[5]The actual number of states, while finite, is difficult to calculate. We know that each agent has N possible states, since it can either report to one of the other $(N-1)$ agents or not belong, i.e., it report to itself; the maximum possible states is then N^N. However, we impose on the model that there not be any circular reporting — for example, we prohibit states where, either directly or indirectly, agent i reports to j, who reports to i. This reduces the number of states from N^N.

[6]RePast 2.01 (repast.sourceforge.net), using the Java JRE 1.4.1 (www.java.sun.com) and Eclipse 2.1.0 (www.eclipse.org).

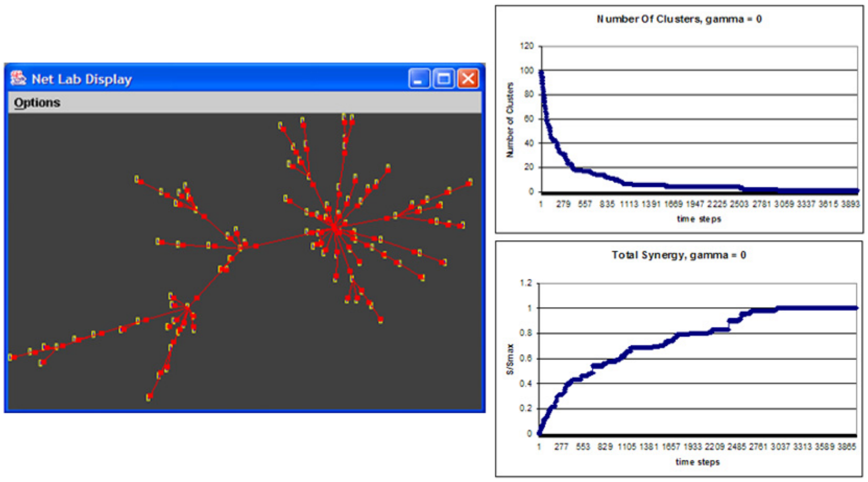

Figure 2: $N = 100, \rho = 1.1, \gamma = 0.0$

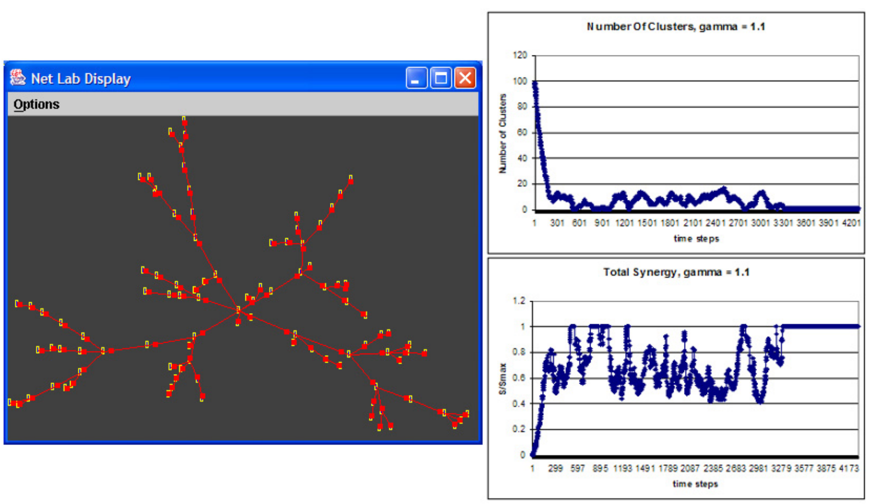

Figure 3: $N = 100, \rho = 1.1, \gamma = 1.1$

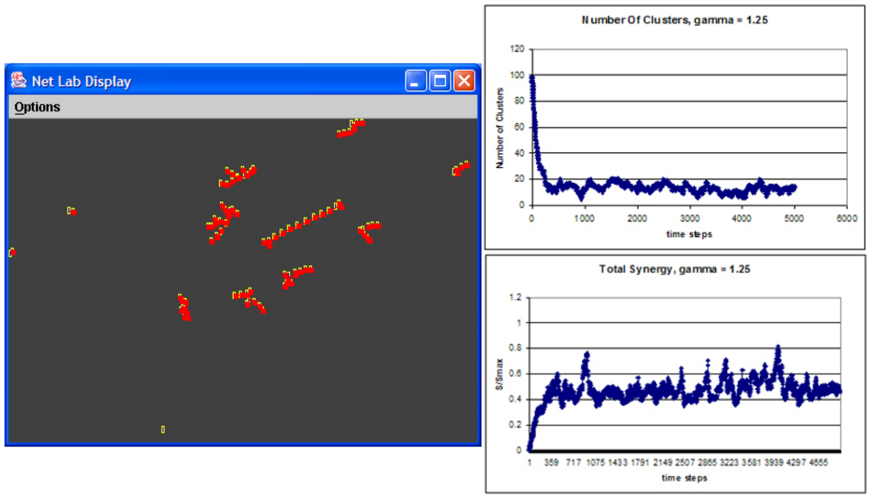

Figure 4: $N = 100, \rho = 1.1, \gamma = 1.25$

figures, increasing γ extends the branching of nodes or members out to the outer edges of clusters. Still larger values of γ create sufficient incentives to prevent the formation of one stable cluster. Figure 4 shows a set of multiple fluctuating clusters that resulted from $N = 100, \rho = 1.1, \gamma = 1.25$.

Generally, for $\gamma = 0$ (egalitarian allocation) no defection occurs and clusters simply grow to eventually form a single giant cluster. Total system synergy S increases monotonically and the number of clusters decreases monotonically (Figure 2). As γ increases, defection begins to occur. When a cluster splits through defection, S decreases and the number of clusters increases (Figure 3). For $\gamma \leq 1.0$ and $N \leq 50$, the system always forms a single cluster. However, for $\gamma \geq 1.25$ and arbitrarily large N, the system does not evolve to a single cluster (at least for the time periods considered in our simulations) but rather reduces its state space to what may be a set of stochastically oscillating states (Figure 4). Interestingly, for some combinations of $\{\gamma, N\}$, the system does occasionally reach the maximized, single cluster configuration, only to fragment at a later time.

3.2 Effects of N on Long-Term Asymptotic Behavior

The number of nodes N affects the average cluster size $\overline{C}$ and thus the maximum synergy the system can attain. Figure 5 shows the relation between N and average cluster size, while Figure 6 shows the relation between N and maximum attained synergy, expressed as S/S_{max}. Looking at the two figures, for $\gamma \leq 1.0$, the system converges to a stable giant cluster. For $\gamma = 2.0$ and $N = 6$, the system stabilized to 3 clusters of 2 with $S/S_{max} = 0.6$. For $N \geq 6$, the system fluctuates stochastically between system states. For $1.0 < \gamma \leq 2.0$ there is a behavioral transition to multiple fluctuating clusters as N increases.

The transition between single-cluster states and fluctuating states appears to

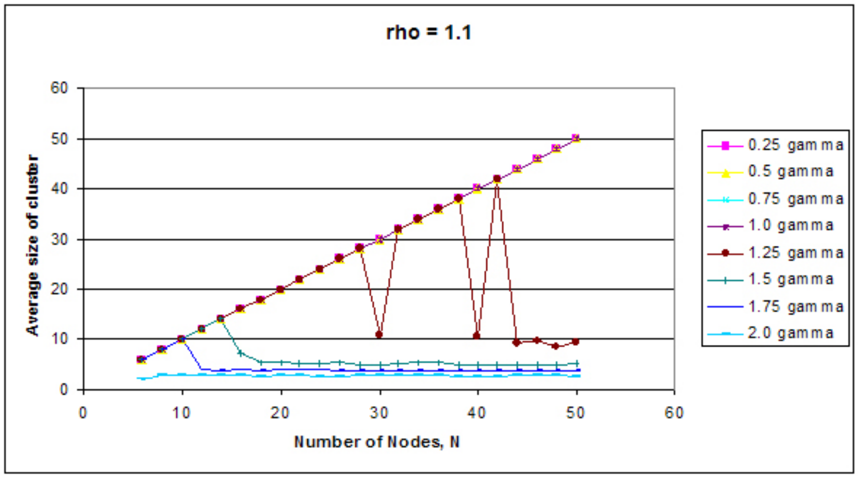

Figure 5: Average Size of Cluster as a Function of N and γ ($\rho = 1.1$)

be sharp for $\gamma > 1.25$, and decreases with increasing gamma. For $\gamma = 1.25$, the transition zone is elongated and takes place for $30 \leq N \leq 44$. This blurring could be due to the limited simulation period (i.e., for simulations where $N < 44$, stable giant clusters may have been found for longer simulation periods; we are currently running longer simulations within this region in an attempt to resolve this issue).

The increase in the transition value for N for decreasing γ suggests that with large enough N, simulations with $\gamma < 1.25$ will also undergo a transition to the stochastically fluctuating regime. This hypothesis was considered for $\gamma = 1.0$ with a second set of simulations that extended N up to 600 in increments of 100. Results clearly showed the fluctuating regime for N of 400 and above. Additional longer simulations are needed to better define this transition zone.

To consider behavior for large N, we plot average cluster size $\overline{C}$ vs γ in Figure 7 for $N = \{50, 400\}$ and see that to a first approximation, $\overline{C}$ is independent of N within the fluctuating regime. Let us now assume that systems for $\gamma > 0$ will eventually transition into the fluctuating regime for N large enough. For $\gamma = 0.0$ and $N \to \infty$, an infinite cluster will form. With this constraint, we fit a power law to the data as shown in the figure. If our assumptions are correct, this relation predicts the average cluster for asymptotically large populations. Populations that are smaller than this cluster size will be constrained to yield a single cluster while those sufficiently above will fall in the fluctuating regime.

Finally, we note that this prediction of $\overline{C}$ also allows prediction of S/S_{max} through a simple product of S calculated for the $\overline{C}$ and the number of such clusters for the given N. With this simple calculation, the slowly downward trend in S/S_{max} for a given gamma is explained (Figure 6).

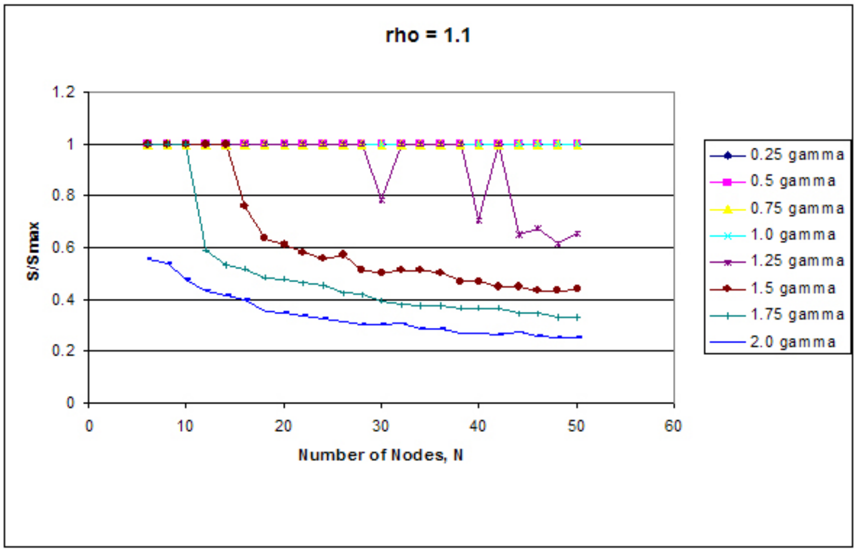

Figure 6: Ratio S/S_{max} as a Function of N and γ ($\rho = 1.1$)

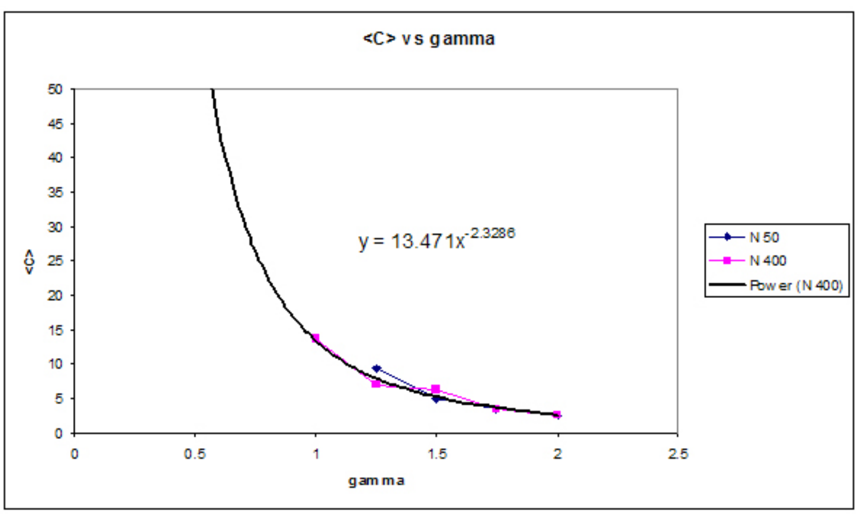

Figure 7: Average Size of Cluster as a Function of γ ($N = 100, \rho = 1.1$)

4 Conclusions

The system displays a number of useful characteristics associated with Markov-process networks. At the start of simulations, the set of agents, with vary large number of possible set of state configurations, converges rapidly to a smaller set of irreducible states. This smaller set may be one state (i.e., a single stable cluster) or a subset of all possible states, each with a fixed probability of transition to the other states in the irreducible set. In this sense, while not being a single stable system, it is likely a *stochastically stable equilibria*, where the system as evolved to a smaller, irreducible set of states that oscillate stochastically. Within the small range of potential parameters for $\{N, \rho, \gamma\}$, we were able to create a range of single-cluster and what appear to be asymptotically stable clusters.

Our hierarchical synergistic network exhibits some of the fundamental behaviors of *self-organized critical* systems. From a set of unorganized individual nodes or agents, simple rules of interaction create structured, often stable single-cluster configurations that maximize overall system performance. Other times — through the random interaction of agents within these clusters — dramatic changes in cluster and system structure occur. Still other systems display oscillating cluster structure as agents act on the benefits of defecting from current clusters and joining others.

Compared with other social interaction models, our model has a number of limitations that deserve further exploration. First, when an agent considers defection, adding new agents, and joining another cluster, it has perfect information about expected current-period information, but no information about future earnings. Additionally, agents do not adapt their strategies based on past actions, using for example an agent-level reinforcement process. Second, a stable cluster or system of stochastically stable clusters may change dramatically if there are minor stochastic disturbances in ρ, γ, or even N. Since these system perturbations may cause dramatic changes in system structure and synergy, additional simulations should be conducted. Finally, while analysis of the data indicates that systems that create more than one cluster exhibit the properties of asymptotic stability, more work needs to be done on the characteristics of this stability. However, our simulations indicate that even without such enhancements, networks can grow and evolve in interesting new ways.

References

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[2] H.P. Young (1998) *Individual Strategy and Social Structure: An Evolutionary Theory of Institutions*, Princeton University Press: Princeton, New Jersey.