Infrastructure resilience assessment through control design

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Abstract: Infrastructure resilience is a priority for homeland security in many nations around the globe. This paper describes a new approach for quantitatively assessing the resilience of critical infrastructure systems. The mathematics of optimal control design provides the theoretical foundation for this methodology. This foundation enables the inclusion of recovery costs within the resilience assessment approach, a unique capability for quantitative resilience assessment techniques. This paper describes the formulation of the optimal control problem for a set of representative infrastructure models. This example demonstrates the importance of recovery costs in quantitative resilience analysis, and the increased capability provided by this approach's ability to discern between varying levels of resilience.

Keywords: resilience; infrastructure assessment; optimal control.

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1 Introduction

Historically, US Federal Government policy toward critical infrastructure protection (CIP) has focused on physical protection and asset hardening (see Reagan, 1982; Clinton, 1998; Bush, 2002, 2003). In recent years, the inclusion of resilience enhancement strategies into CIP policies has become a priority in the USA and countries around the globe. Critical infrastructure resilience is a concept that describes the ability of infrastructure systems to absorb, adapt, and recover from the effects of a disruptive event while attempting to continue delivery of critical infrastructure services.

Though aspects of resilience have begun to become integrated into federal, state, and local CIP policies, no resilience definitions or evaluation methodologies have been uniformly accepted within the homeland security and CIP communities. Many definitions have been proposed in several academic disciplines (e.g., see Bruneau et al., 2003; Chang and Shinozuka, 2004; Rose and Liao, 2005), but few quantitative methods have been proposed for analysis of infrastructure and economics systems. Some of these quantitative methods (e.g., see Fisher et al., 2010) rely on subjective evaluations by subject matter experts (SMEs) of system features, such as redundancy, adaptivity, etc. These approaches are difficult to impose consistently across different infrastructure systems and types due to the subjective nature and variability of the SMEs' responses.

Some resilience measurement approaches have been developed that do not rely on these subjective evaluations. Bruneau et al. (2003) measure seismic resilience loss for communities by integrating the difference between optimal infrastructure quality and the degraded infrastructure quality following an earthquake. Chang and Shinozuka (2004) use a probabilistic formulation to compare time to recovery and the decrease in system performance, predicted through a set of Monte Carlo simulations, against pre-defined performance and duration standards. In this approach, the resilience of the system is the observed probability that both standards are met. Rose and Liao (2005) have developed resilience metrics for economic systems. Rose asserts that the static economic resilience of a system be measured as "the ratio of the avoided drop in [system] output and the maximum potential drop" in system output, and that dynamic economic resilience be measured as the cumulative difference between system outputs with and without hastened recovery efforts. Each of these approaches only consider the impact that a disturbance has on the state of the system or system outputs. They do not consider resources or costs expended during the recovery processes. Also, Bruneau et al.'s and Rose's do not directly consider that the recovery strategy employed will directly affect their estimates of resilience and resilience loss.

Resource allocation can be a critical concern during crisis events, and emergency responders need to decide how limited resources should be spent to minimise deleterious impacts and maximise response efficiencies. Vugrin et al. (2010) have proposed a resilience assessment framework that expands upon the aforementioned assessment approaches in two key areas. First, their mathematical formulation for measuring resilience costs is not reliant upon a specific modelling paradigm to represent the system, so it can be generally applied across various infrastructure and economics models. This flexibility is necessary for establishing resilience analysis standards across all critical infrastructure systems. Second, it explicitly considers the costs and resources expended during recovery efforts following infrastructure disruptions. Inclusion of recovery costs in resilience evaluations provides a more comprehensive accounting of disruption impacts. This approach also provides a means for assessing feedback loops consisting of recovery and system performance that can ultimately determine system resilience.

Vugrin et al. (2010) define system resilience as follows: Given the occurrence of a particular disruptive event (or set of events), the resilience of a system to that event (or events) is that system's ability to reduce efficiently both the magnitude and duration of the deviation from targeted system performance levels.

This definition provides the basis for the measurement of the two primary factors that determine the resilience costs. Systemic impact (*SI*) is the impact that a disruption has on system productivity and is measured by evaluating the difference between targeted and disrupted system performance. Total recovery effort (*TRE*) refers to the efficiency with which the system recovers from a disruption and is measured by analysing the amount of resources expended during the recovery process. The measurement of system resilience costs requires the quantification of both *SI* and *TRE*. The approach proposed by Vugrin et al. (2010) lends itself nicely to mathematical formulations used for the development of optimal feedback control laws. When applied to a system, feedback controller's use measured system outputs to regulate system behaviours to target conditions while simultaneously providing a measure of the cost in doing so. Incorporating feedback control in the quantitative description of resilience could enable automatic system recovery from disruption while providing a prediction of recovery cost.

This paper describes the mathematical formulation that Vugrin et al. (2010) developed for measurement of resilience costs and the application of optimal feedback control design methods to identify recovery strategies that optimise resilience costs. Specifically, this paper describes the formulation of the linear quadratic regulator (LQR) problem for a set of representative infrastructure models. Analysis of these models demonstrates how the resilience assessment approach can be used to assess resilience costs and identify optimal resilience strategies that minimise those costs.

Furthermore, the systems have different resilience enhancement features, such as system redundancy and emergency inventory (EI) stocks, and this paper confirms that the Vugrin et al. (2010) approach identifies the system that logic indicates is most resilient.

2 A quantitative framework for assessing infrastructure and economic resilience

Consider a dynamic system modelled as follows:

$$y(t) = f\left(x\left(u(t), d(t), t\right)\right), \tag{1}$$

where

- x is a time-varying state vector with dependence on the control term u and the disturbance d.
- u is a time-dependent control vector representing the means by which the system recovers, i.e., the recovery effort.
- d represents a time-dependent, piece-wise continuous, disturbance forcing term.

- y is the vector of system outputs under disturbance d, and is obtained by calculation of the function f.

Let z be an exogenous reference signal that represents the time-dependent, targeted system performance level.

The quantities $\overline{SI}$ and $\overline{TRE}$ are calculated according to

$$\overline{SI} = \int_{t_0}^{t_1} q^T(t)\left[z(t) - y(t)\right]dt, \tag{2}$$

$$\overline{TRE} = \int_{t_0}^{t_1} r^T(t)u(t)dt, \tag{3}$$

where $t_0 > 0$ is the time at which the disturbance initiates and t_1 is the time at which recovery is considered complete.

Since x and y are dependent upon u, two resilience cost measurements are defined. The first, recovery dependent resilience costs, are those costs resulting from a particular control strategy. They are calculated according to

$$\overline{RDR}\left(x\left(t_0\right), u, d\right) = \frac{\overline{SI} + \overline{TRE}}{\int_{t_0}^{t_1}\left\{q^T(t)z(t)\right\}dt}. \tag{4}$$

In (2) and (3) the weighting vector q converts differences between y and z into the units of $\overline{SI}$ and the weighting vector r is used to set the relative importance of $\overline{SI}$ to $\overline{TRE}$. The denominator in (4) is a normalising term that permits comparison of $\overline{RDR}$ values for systems of varying magnitudes.

The second resilience cost, namely optimal resilience costs, $\overline{OR}$, of system x to disturbance d are defined (when they exist) as

$$\overline{OR}\left(x\left(t_0\right), d\right) = \min_u \frac{\overline{SI} + \overline{TRE}}{\int_{t_0}^{t_1}\left\{q^T(t)z(t)\right\}dt}. \tag{5}$$

According to the above definitions, $\overline{SI}$ and $\overline{TRE}$ may be negative. For example, a particular disruption may actually cause y to exceed z, resulting in a negative integrand. However, frequently, it is not beneficial from an inventory or economic standpoint to greatly exceed target goals as this typically invokes even greater cost. Hence, it is reasonable to modify the definition of $\overline{SI}$ and $\overline{TRE}$ so that y tracks z, and all deviations from z are penalised in the objective function that is used to calculate $\overline{RDR}$ and $\overline{OR}$. Additionally, the definitions above can be simplified by assuming the weighting factors are constant with respect to time. In particular, we define the terms SI and TRE using fixed semi-definite and positive semi-definite weighting matrices Q and R according to

$$SI = \int_{t_0}^{t_1} [z - y]^T Q[z - y]dt, \tag{6}$$

$$TRE = \int_{t_0}^{t_1} u^T Ru\, dt. \tag{7}$$

These equations lead to a definition of *OR* as

$$OR\left(x\left(t_0\right), d\right) = \min_u \frac{SI + TRE}{\int_{t_0}^{t_1} \left\{[z]^T Qz\right\} dt}, \tag{8}$$

where the optimal resilience cost calculation is now specified in a manner amenable to an LQR control formulation.

3 Resilience from optimal feedback control

Equation (8) indicates how to measure resilience costs for an optimal recovery strategy, but it does not indicate how to identify that strategy. To do that, we appeal to the mathematics of optimal control.

In the work presented here, we assume the availability of a linear dynamical system model. There are several reasons for this. For applications that exhibit non-linear dynamics, a linearisation is often done around an equilibrium point, resulting in a linear system model from which linear feedback controllers can be developed. Often, the resulting linear controller is placed into the original non-linear plant with control performance being satisfactory for the non-linear system. System identification can also be used to develop linear system models from known input-output data. The direct application of non-linear control methods to highly non-linear system models is beyond the scope of this paper.

As previously mentioned, an overall quantitative description of resilience must include measurements of the cost of departure from target operating conditions as well as the cost associated with inputs necessary to maintain preferred operating conditions following a systemic disruption. These requirements lend themselves nicely to tracking feedback control formulations.

Given a linear system with state dynamics described by (9) to (10)

$$\dot{X}(t) = AX(t) + Bu(t), t > 0, \tag{9}$$

$$X(0) = X_0, \tag{10}$$

consider the cost function used in the tracking formulation of the LQR control problem

$$J\left(X_0, u\right) = \int_0^\infty \left\{(X - z)^T Q(X - z) + u^T Ru\right\} dt. \tag{11}$$

In (9), $X(t)$ is called the state vector with components $x_1(t)$, $x_2(t)$, …, $x_n(t)$ being state variables. Notation $\dot{X}$ denotes the differentiation of the state vector with respect to time. The control input to the system is denoted by $u(t)$. The coefficient matrices A and B are time invariant. In (11), the quantity $(X - z)$ is a measurement of the difference between the system measurement, X, and its target value z. Potential inputs to the system necessary to keep X close to z are captured in the control input term u. Quantity Q is a diagonal, symmetric, positive semi-definite matrix consisting of state weights. R is a diagonal, symmetric, positive definite matrix of control weights. The optimal control problem to consider is to minimise (11) over all square integrable control functions, i.e., $u \in L^2(0, \infty)$, subject to the constraints (9) to (10).

To solve the LQR tracking problem, one formulates the augmented dynamical system as follows. As the tracking signal is time-invariant, the dynamics of the system and the reference signal are given by

$$\begin{bmatrix} \dot{X} \\ z \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X \\ z \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u \tag{12}$$

$$= \bar{A}\bar{X} + \bar{B}u, \tag{13}$$

where we have defined the augmented state $\bar{X}$ as $\bar{X} = [X \quad z]^T$. The initial data of the augmented system is given by

$$\bar{X}_0 = \begin{bmatrix} X_0 \\ z(0) \end{bmatrix}. \tag{14}$$

For a controllable system, the tracking LQR problem has a unique solution (Kirk, 2004) of the form

$$u_{opt} = -K\bar{X} \tag{15}$$

$$= -\begin{bmatrix} K_1 & K_2 \end{bmatrix} \bar{X} \tag{16}$$

$$= -\begin{bmatrix} R^{-1}B^T\Pi_{11} & R^{-1}B^T\Pi_{12} \end{bmatrix} \bar{X}, \tag{17}$$

where Π_{11} is the unique symmetric, non-negative solution of the algebraic Riccati equation

$$A^T\Pi_{11} + \Pi_{11}A - \Pi_{11}BR^{-1}B^T\Pi_{11} + Q = 0. \tag{18}$$

The matrix Π_{12} in (17) satisfies the equation

$$\begin{bmatrix} A^T - \Pi_{11}BR^{-1}B^T \end{bmatrix} \Pi_{12} = Q. \tag{19}$$

There are many commercial software packages available that can solve the Riccati equation given by (18). With the Riccati matrix Π_{11} in hand, Π_{12} is found by simply solving matrix equation (19).

Once the gain matrix K is obtained, the feedback control law is placed into the augmented state-space equation. The resulting closed-loop system is of the form

$$\dot{\bar{X}} = (\bar{A} - \bar{B}K)\bar{X}, \tag{20}$$

$$\bar{X}(0) = \bar{X}_0. \tag{21}$$

LQR is only one of several optimal control methods suitable for linear systems. Other techniques include linear quadratic Gaussian (LQG) optimal control, a method utilised when measurements are affected by ‘white, Gaussian noise’ or if full state measurement is not possible. There are also robust control techniques, such as H-infinity control, that are used when it is necessary to stabilise a system in the presence of external disturbances. Zhou and Doyle (1996) provide an excellent discussion of robust and optimal control methods for linear systems.

4 Application to a stock management model

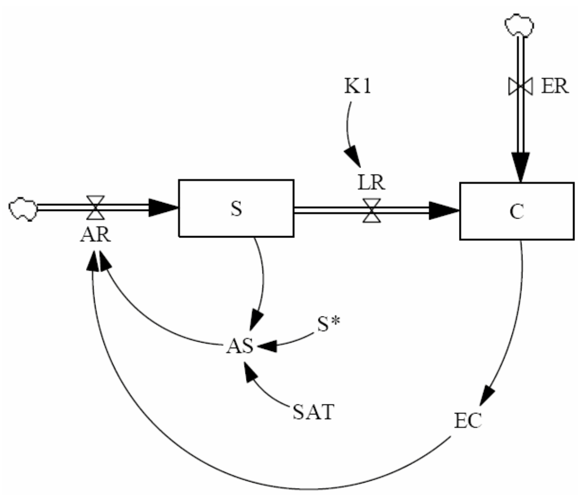
Having demonstrated the solution approach, the methodology is now applied to a set of relatively simple dynamical models. The goal of this application is two-fold. First, the application is intended to show the types of results one gets from using the approach. The second goal is to demonstrate that the approach can assess how different resilience enhancement features affect resilience and can recognise the system that logic indicates is most resilient. This application demonstrates how to measure resilience costs and how feedback control can be used to select optimal recovery strategies that minimise resilience costs.

4.1 Model description

Consider the stock management model illustrated with systems dynamics conventions in Figure 1. This model, a modification of Sterman's (2000) stock management structure model, includes a stock manager that attempts to control the stock acquisition rate (*SAR*) so that consumption rates (*C*) track an external demand function, *D*. The stock loss rate, *LR*, is proportional to the stock level, *S*. The acquisition rate is determined by three factors:

- The manager utilises a stock adjustment strategy, *AS*, that attempts to have stock levels track a desired stock profile, S^*. In this strategy, *AS*, is proportional to the difference between stock and desired stock levels.
- The SAR is the sum of *AS* and the average demand over the previous *td* time units.
- The system is also subject to external disruptions, *N*, that affect the SAR.

Figure 1 Stock management model



For a single stock system, this model is governed by the following set of delay-differential equations:

$$\frac{d}{dt}\begin{bmatrix} S(t) \\ C(t) \end{bmatrix} = \begin{bmatrix} SAT^{-1}\left(S^* - S\right) - k_1 S \\ k_1 S \end{bmatrix} + \begin{bmatrix} td^{-1}\left(C(t) - C(t - td)\right) \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}\hat{N}(t) \tag{22}$$

$$= \hat{A}_0 \begin{bmatrix} S(t) \\ C(t) \end{bmatrix} + \hat{A}_1 \begin{bmatrix} S(t - td) \\ C(t - td) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\hat{N}(t) + \begin{bmatrix} SAT^{-1} \\ 0 \end{bmatrix}\hat{S}^*(t), \tag{23}$$

where

- $\hat{A}_0 = \begin{bmatrix} -SAT^{-1} - k_1 & td^{-1} \\ k_1 & 0 \end{bmatrix},$
- $\hat{A}_1 = \begin{bmatrix} 0 & td^{-1} \\ 0 & 0 \end{bmatrix},$
- SAT is a constant and represents the stock adjustment time in the stock adjustment strategy, $AS = (S^* - S)\ SAT^{-1}$.
- k_1 represents the constant of proportionality between LR and S.

The measured output, y, for the system is denoted by C and calculated according to

$$y(t) = [0 \quad 1]\begin{bmatrix} S(t) \\ C(t) \end{bmatrix} = C(t). \tag{24}$$

This paper considers the base system and three variations of it:

- The EI system: This system is identical to the base system except that it includes an excess supply of stock. The emergency supply is activated when the consumption cannot meet the external demand signal. In this event, the emergency supply rate, *ER*, can supply that commodity directly to the consumer. The emergency supply enhances the restorative capacity of the system. Thus, it is expected that resilience costs for this system would decrease, relative to the base system, indicating greater resilience.
- The redundant stock (RS) system: This system is identical to the base system except that it has multiple locations for stock production instead of one location. In this paper, the system consists of four identical locations. The inherent redundancy enhances the absorptive capacity of the system. Thus, it is expected that resilience costs for this system are expected to decrease, relative to the base system, indicating greater resilience.
- The redundant stock, emergency inventory (RSEI) system: This system has multiple (four) stock production locations and an emergency supply rate. The inherent redundancy and emergency supply enhance the absorptive and restorative capacities

of the system. Thus, it is expected that resilience costs for this system are the least for all the systems, indicating this system is the most resilient.

Equation (22) is generalised to all the systems in the following manner. Let k_{1i} denote the proportionality constant between the i^{th} stock, S_i, and its respective loss rate LR_i. Parameter k_{2i} denotes the fractional contribution of the i^{th} stock to C, and SAT_i denotes the stock adjustment time for the i^{th} stock. Then,

$$\begin{aligned}\frac{dX(t)}{dt} &= A_0 X(t) + A_1 X(t-td) + B\big(ER(t)\big) \\ &+N(t) + td^{-1} S^*(t)\end{aligned} \tag{25}$$

where

- $X = [S_1\ S_2\ \dots\ S_M\ C]^T$ is the state vector.
- A_0 is the $(M+1)$ by $(M+1)$ matrix defined as

  $$A_0 = \begin{bmatrix} \alpha_1 & 0 & \dots & 0 & \beta_1 \\ 0 & \ddots & & & \vdots \\ \vdots & & \ddots & & \\ 0 & & & \alpha_M & \beta_M \\ k_{11}k_{21} & k_{12}k_{22} & \dots & k_{1M}k_{2M} & 0 \end{bmatrix}$$

  where $\alpha_i = -SAT_i^{-1} - k_i$, $\beta_i = k_{2i}(td)^{-1}$.
- A_1 is the $(M+1)$ by $(M+1)$ matrix defined as

  $$A_1 = \begin{bmatrix} 0 & \dots & 0 & -\beta_1 \\ \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & -\beta_M \\ 0 & \dots & 0 & 0 \end{bmatrix}$$
- B is the $(M+1)$ dimensional vector defined as $[0\ \dots\ 0b]^T$ where $b = 0$ for the base and RS systems, and $b = 1$ for the EI and RSEI systems.
- N is a time-dependent disturbance vector of length $(M+1)$. The last vector entry must be 0 since the consumption rate is not directly impacted by the disturbance.
- S^* is a time-dependent vector of length $(M+1)$. The i^{th} vector entry is the desired stock for the i^{th} stock, and the last vector entry is 0.
- The measured output, y, for the system is C and is calculated according to

  $$y(t) = \big[0 \dots 0\ 1\big] X(t) = C(t) \tag{26}$$

4.2 The optimal resilience problem

For these systems, the following questions are considered:

- Given the dynamics of the systems described in (25), what *ER* function optimises recovery by minimising the following resilience cost equation?

$$J\left(S(0), C(0), ER\right) = \frac{\int_0^{\infty} \left[C(t) - D(t)\right]^2 dt}{\int_0^{\infty} \left[D(t)\right]^2 dt} + \frac{\int_0^{\infty} R\left[ER(t)\right]^2 dt}{\int_0^{\infty} \left[D(t)\right]^2 dt}. \tag{27}$$

  In (27), $D(t)$ represents an exogenous demand signal and R is a positive, scalar constant that balances the cost of the SI with the recovery effort $ER(t)$. Note that (27) has the same form as the cost function in the LQR tracking cost function in (11).

- What are the resilience costs under this optimal recovery function?

4.3 Methodology

To perform the resilience analysis, we use the LQR optimal feedback control approach described in Section 3. This approach is well-suited to this particular system since the model is linear. The presence of the delay term, $C(t - td)$, results in an infinite dimensional state. Hence, in order to perform simulation, the system must be approximated and discretised. The averaging scheme described in Banks and Burns (1978), Gibson (1983), and Kappel (1991) is used to do so. This approach is selected since feedback control approximations developed with the averaging scheme converge (in norm) to the optimal control for the infinite dimensional system (Kappel, 1991; Rosen, 1991). Kappel (1991) describes in detail how to develop the matrices required for the LQR methodology. Empirical testing indicated that relatively low order ($N \approx 8$) approximations sufficiently represent the solution.

4.4 Analysis

Table 1 lists the system parameter values that were implemented for the resilience analysis.

Table 1 System parameters

Parameter/system	*Base*	*EI*	*RS*	*RSEI*
k_{1i}	.5	.5	.5	.5
k_{2i}	.25	.25	.25	.25
SAT_i	8	8	8	8
td	1	1	1	1
R	.01	.01	.01	.01

Equations (28) to (30) define the history functions (i.e., functions $S(t)$ and $C(t)$ evaluated over the interval $[-td, 0))$ and S^*. Selection of these parameters results in identical dynamics for the stocks of all systems for undisrupted conditions.

$$S_i(t) = [3 + .5\sin(4t)], \quad t \in [-1, 0), \quad \forall i \tag{28}$$

$$C(t) = 0, \quad t \in [-1, 0), \tag{29}$$

$$S_i^*(t) = [3 + .5\sin(4t)], \quad \forall i. \tag{30}$$

Figures 2 and 3 show the dynamics of the supply-consumption system for undisrupted conditions. The nominal consumption function represents the demand function, *D*, that is tracked by *C*.

Figure 2 Nominal stock dynamics (see online version for colours)

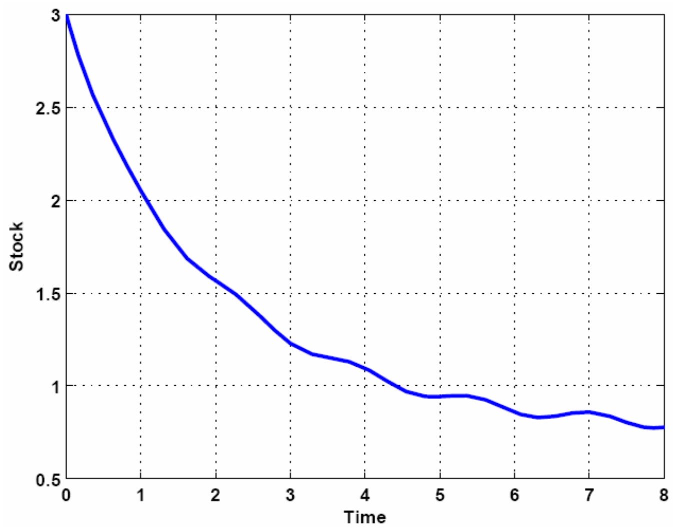
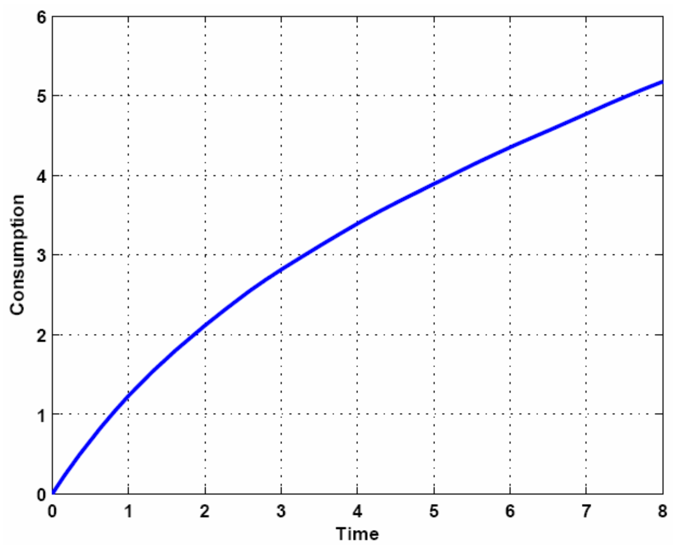


Figure 3 Nominal consumption dynamics (see online version for colours)



Figures 4 and 5 display the dynamics for the base system when imposing the disruption $N(t)$ defined by

$$N(t)=\begin{cases}-1, & 1\le t\le 1.2,\\ 0, & \text{otherwise}\end{cases} \tag{31}$$

to the acquisition rate. In the RS and RSEI systems, this disruption affects a single stock.

Figure 4 Disrupted case: directly affected stocks (see online version for colours)

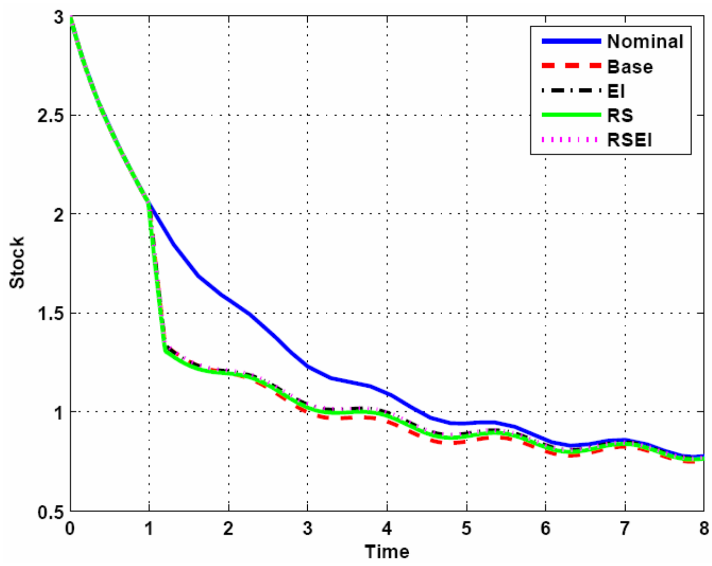


Figure 5 Disrupted case: indirectly affected stocks (see online version for colours)

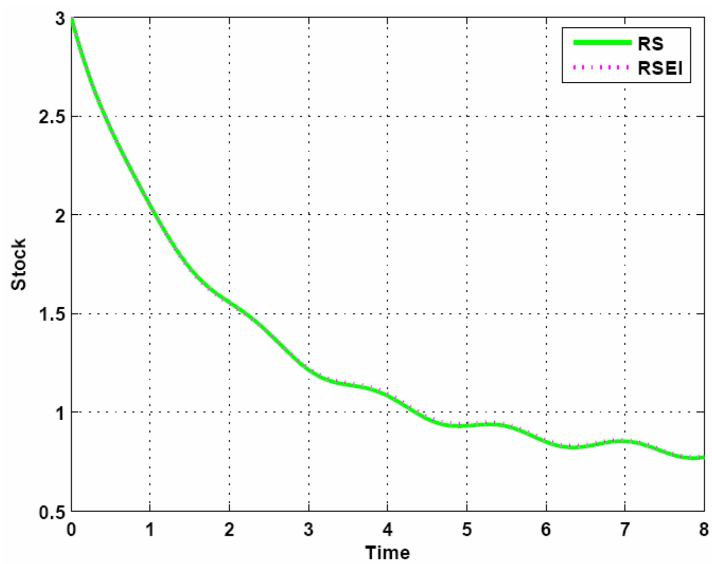


Figure 6 Close-up of disrupted case: directly affected stocks (see online version for colours)

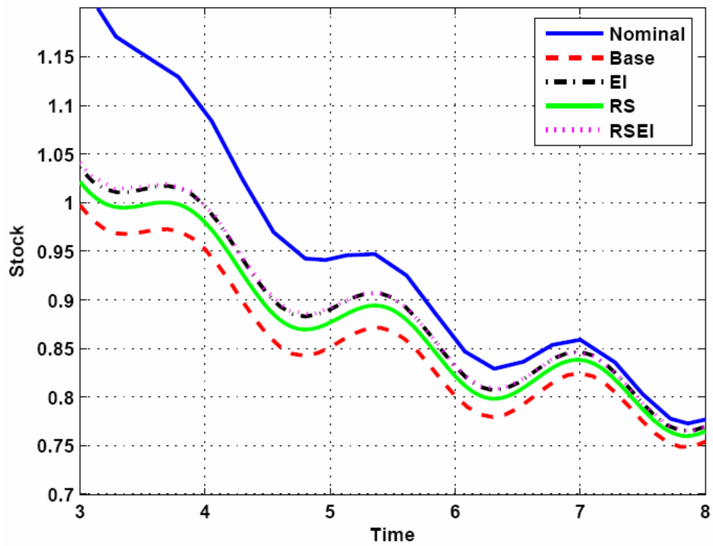
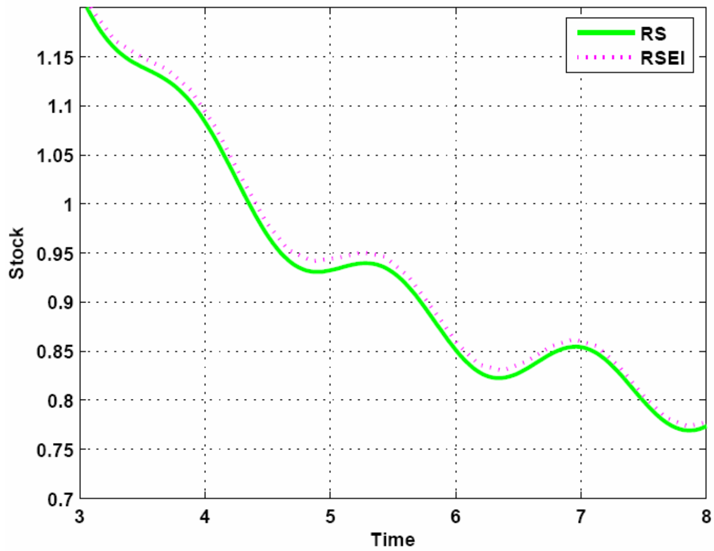


Figure 7 Close-up of disrupted case: indirectly affected stocks (see online version for colours)



As seen in Figure 4, stock levels for all disrupted stocks drop sharply as soon as the disruption occurs, and eventually return to near undisrupted performance levels by $t = 8$. Stock levels are similar for all disrupted stocks. However, systems with emergency supply rates, EI and RSEI, have the highest stock levels after the disruption, followed by stocks for the RS and then the base system, as seen in Figures 6 and 7. Undisrupted stock levels for the RSEI system exceed their counterparts for the EI system. This relative increase is caused by the use of the emergency supply inventory. When the ESR function initiates at $t = 1$, consumption levels for RSEI system exceed those of the EI system.

Since the SARs are a function of the average consumption rate, RSEI SARs increase relative to the EI rates, leading to increased RSEI stock levels.

Consumption dynamics for all systems are shown in Figure 8. Again, the systems with emergency supply inventories, EI and RSEI, track the target demand most closely, followed by the RS and base systems. Since the base and RS systems do not have emergency inventories, they have no means for overcoming the initial disruption. The RS system is not as severely affected as the base system since only one of the four stocks (25%) are affected, instead of 100% in the base system.

The optimal recovery functions, *ER*, for the EI and RSEI systems are shown in Figure 9. *ER* for the EI system exceeds its RSEI counterpart at all times after the disruption initiation; that is, the EI system requires a ‘greater’ recovery effort than does the RSEI system. This occurs since only 25% of the stocks are affected in the RSEI system, instead of 100% in the EI system.

The SIs, TREs, and optimal resilience costs, as calculated with equation (27), are shown in Table 2. As expected, the base system has the largest *SI* value, an order of magnitude larger than the RS system, and two orders of magnitude larger than the EI and RSEI systems. Because the base and RS systems do not have a recovery mechanism (the EI), the TREs for those systems are 0. The *SI* values for the EI and RSEI systems are equal, but the *TRE* value for the EI system exceeds the RSEI value by an order of magnitude. This difference indicates that the EI system requires a greater recovery effort to maintain the same performance levels as the RSEI system and provides a quantitative demonstration that the redundancy within the RSEI system enhances system resilience. Consequently, optimal resilience costs, *OR*, are least for the RSEI system, followed by the EI system, then the RS system, and the base system. Hence, the RSEI is considered the most resilient system, followed by the EI system, then the RS system, and the base system.

Figure 8 Disrupted consumption dynamics (see online version for colours)

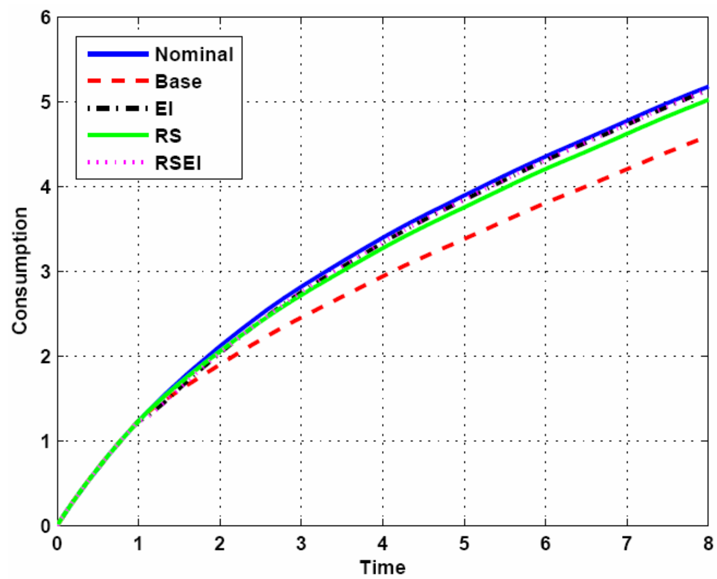


Figure 9 Optimal emergency supply rates (see online version for colours)

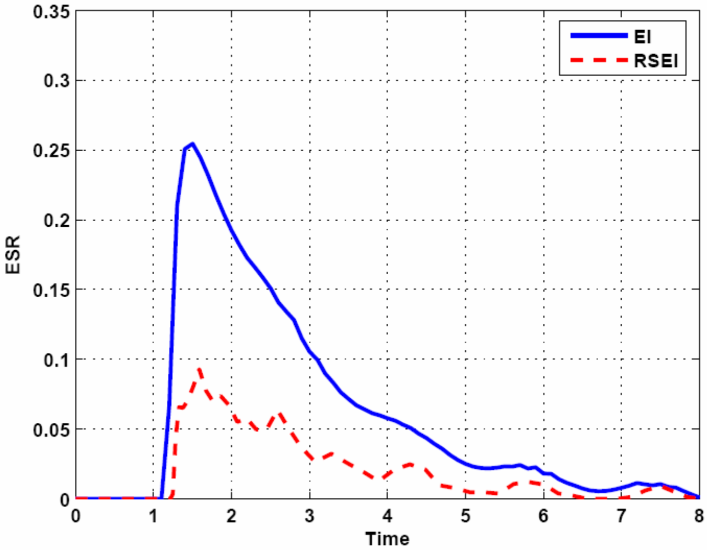


Table 2 Resilience costs

System	*Base*	*EI*	*RS*	*RSEI*
SI	1.4e-0	2.0e-2	1.0e-1	2.0e-2
TRE	0.0e-0	7.1e-2	0.0e-0	7.6e-3
$\int D^2(t)dt$	24.5	24.5	24.5	24.5
OR	5.7e-2	8.4e-4	4.0e-3	8.1e-4
Rank	4	2	3	1

If one wanted to enhance the base system to improve its resilience to this type of disruption, one should first consider the addition of emergency supply inventory. This addition made the single largest improvement for decreasing resilience costs and SIs. Furthermore, maintaining additional inventory is likely a cheaper overall investment than building three new facilities for stock production. Additional resilience enhancement could be added by increasing system redundancy, but one would likely need to perform a cost-benefit analysis to determine if the small improvement to overall resilience costs is worth the investment necessary to create the redundancy.

It should be noted that resilience measurement approaches such as those developed by Rose and Bruneau et al., that measure only SI and do not consider the TRE, would consider the RSEI and EI systems to be equally resilient. Table 3 shows the resilience loss and resilience calculations using the methods proposed by Bruneau et al. (2003), and Rose and Liao (2005), respectively. In these approaches, the additional redundancy in the RSEI system adds no additional resilience benefit relative to the EI system that only has EI available to it. This conclusion is counterintuitive since only 25% of the stocks are affected in the RSEI system, instead of 100% in the EI system. This example demonstrates the importance of the TRE term in quantitative resilience analysis, and the increased ability of Vugrin et al.'s (2010) approach to discern between varying levels of resilience.

Table 3 Alternative resilience calculations

System	*Base*	*EI*	*RS*	*RSEI*
Resilience loss (Bruneau et al., 2003)	8.0e-1	1.3e-1	2.1e-1	1.3e-1
Rank	4	1	3	1
Static resilience (Rose and Liao, 2005)	0.0e-0	7.9e-1	7.3e-1	7.9e-1
Rank	4	1	3	1
Dynamic resilience (Rose and Liao, 2005)	0.0e-0	2.6e-0	2.2e-0	2.6e-0
Rank	4	1	3	1

5 Summary and future work

This paper presents a quantitative framework for critical infrastructure resilience assessment. This framework has the unique capability to explicitly include recovery processes and the consumption of resources during those processes. The importance of this capability is demonstrated in the numerical example presented in this paper. Using an approach that includes the TRE term, one concludes that the RSEI system is the most resilient of the four systems studied. The combination of RS systems with the EI enhances the absorptive and restorative capacities of the system and results in the lowest costs. This quantitative analysis confirms what one intuitively knows and demonstrates this approach's increased capability, relative to alternative resilience assessment techniques, to discern between varying levels of resilience.

The development of quantitative resilience methods still requires much work to be done, including:

- *Investigate the use of more generally applicable optimisation methods to use for resilience analysis*: Any optimal control problem can be posed as a more general optimisation problem. The use of optimal feedback control algorithms can make solution of these problems more efficient, but the application of these methods requires linearity of the system. When it is not possible or preferable to apply or develop a non-linear control method, non-linear optimisation methods can be utilised. It would be worthwhile to investigate the use of non-linear optimisation techniques for resilience analysis methods.
- *Explicitly include recovery resource constraints in numerical optimisation methods*: Emergency responders and infrastructure managers frequently face a common dilemma: how does one utilise limited resources when repairing and restoring infrastructure systems following a disruptive event? Though the LQR control problem includes a penalty in its objective function for increased use of control input, i.e., resources expended in recovery processes, it does not explicitly include resource constraints. If constraints exist, one can tune the LQR problem by varying the R term in the objective function so that the optimal recovery strategy meets the constraint. However, the tuning process is inexact and can be time consuming. Many optimisation approaches, such as linear and non-linear programming, consider these

constraints. Therefore, the use of those numerical solution techniques should be investigated in the context of quantitative resilience assessment.

- *Integrate order reduction into resilience assessment simulations*: Real infrastructure systems can have a large number of components and can be fairly complex, so simulation of these systems for resilience assessment can be computationally intensive. Reduced-order modelling is often used to reduce the dimension of dynamical systems in order to lessen the burden of computation (Antoulas, 2005; Holmes et al., 1996). Inclusion of reduced order modelling can facilitate predictive resilience assessment and should be investigated for this purpose.

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