

Rigorous component-based coupling of first-principles and data-driven models







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2 Motivation

The past decades have seen tremendous investment in **simulation frameworks** for **coupled multi-scale** and **multi-physics** problems.

- Frameworks rely on established mathematical theories to couple physics components.
- Most existing coupling frameworks are based on **traditional discretization methods**.



Complex System Model

- PDEs, ODEs
- Nonlocal integral
- Classical DFT
- Atomistic, ...

Traditional Methods

 N_{2}

- Mesh-based (FE, FV, FD)
- Meshless (SPH, MLS)
- Implicit, explicit

 N_3

• Eulerian, Lagrangian...

Coupled Numerical Model

Monolithic (Lagrange multipliers)

 N_{4}

 N_{5}

(EAM)

E³SI

Ocean (MPAS-

Land (ELM)

Land Ice (MALI) Sea Ice (MPAS-

- Partitioned (loose) coupling
- Iterative (Schwarz, optimization)

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N_1 N_2 N_4 N_3 N_5

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- PINNs
- Neural ODEs
- Projection-based ROMs, ...
- There is currently a big push to integrate data-driven methods into modeling & simulation toolchains.

Unfortunately, existing algorithmic and software infrastructures are **ill-equipped** to handle plug-and-play integration of **non-traditional**, **data-driven models**!

4 Current Projects on Coupling for Predictive Heterogeneous Models

fHNM: flexible Heterogeneous Numerical Methods

- Sandia Laboratory Directed Research & Development (LDRD) project (FY22-FY24)
 - Co-Pls: Pavel Bochev & Irina Tezaur; Team: 5 staff, 2 post docs, 3 students, 2 consultants
 - Academic Alliance: Prof. Arif Masud (UIUC)
- **Primary research objective:** discover the mathematical principles guiding the assembly of standard and **data-driven numerical models** in stable, accurate and physically consistent ways

M2dt: Multi-faceted Mathematics for Predictive Digital Twins

- Funded by DOE's Advanced Scientific Computing Research (ASCR) Mathematical Multifaceted Integrated Capability Centers (MMICC) Program (FY23-FY27)
- **Partnership** between UT Austin (Lead Institution), Sandia National Labs (SNL), Argonne National Lab (ANL), Brookhaven National Lab (BNL) and MIT
 - Directors: Karen Willcox & Omar Ghattas (UT Austin)
 - Sandia co-Pls: Irina Tezaur & Pavel Bochev; Sandia team: 6 staff, 1 post doc
- **Primary research objective:** establish a center for research and education on multifaceted mathematical foundations for predictive digital twins (DTs) for complex energy systems
 - > Central to DTs is: (1) tight two-way coupling of data and models, (2) structure preservation and (3) dynamic data assimilation











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5 Coupling Scenarios, Models and Methods



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Data-driven models: to be "mixed-and-matched" with each other and first-principles models

- Class A: projection-based reduced order models (ROMs)
- Class B: machine-learned models, i.e., Physics-Informed Neural Networks (PINNs)
- Class C: flow map approximation models, i.e., dynamic model decomposition (DMD) models

Coupling methods:

- Method 1: Alternating Schwarz-based coupling
- *Method* 2: Optimization-based coupling
- *Method 3*: Coupling via generalized mortar methods (GMMs)

6 Coupling Scenarios, Models and Methods



Data-driven models: to be "mixed-and-matched" with each other and first-principles models

• *Class A*: projection-based reduced order models (ROMs)

This talk

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- *Class B*: machine-learned models, i.e., Physics-Informed Neural Networks (PINNs)
- Class C: flow map approximation models, i.e., dynamic model decomposition (DMD) models

Coupling methods:

- Method 1: Alternating Schwarz-based coupling
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- This talk

This talk

- 1. The Alternating Schwarz Method for FOM*-ROM[#] and ROM-ROM Coupling
 - Method Formulation
 - ROM Construction and Implementation
 - Numerical Examples
- 2. A Lagrange Multiplier-based Partitioned Scheme for FOM-ROM and ROM-ROM Coupling
 - Method Formulation
 - ROM Construction and Implementation
 - Numerical Examples
- 3. Summary and Future Work





*Full-Order Model. #Reduced Order Model.

- 8 Outline
- 1. The Alternating Schwarz Method for FOM*-ROM[#] and ROM-ROM Coupling
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*Full-Order Model. #Reduced Order Model.

- Schwarz Alternating Method for Domain Decomposition
- Proposed in 1870 by H. Schwarz for solving Laplace PDE on irregular domains.

Crux of Method: if the solution is known in regularly shaped domains, use those as pieces to iteratively build a solution for the more complex domain.

Basic Schwarz Algorithm

Initialize:

- Solve PDE by any method on Ω_1 w/ initial guess for transmission BCs on Γ_1 . Iterate until convergence:
- Solve PDE by any method on Ω_2 w/ transmission BCs on Γ_2 based on values just obtained for Ω_1 .
- Solve PDE by any method on Ω_1 w/ transmission BCs on Γ_1 based on values just obtained for Ω_2 .





 Schwarz alternating method most commonly used as a *preconditioner* for Krylov iterative methods to solve linear algebraic equations.

Idea behind this work: using the Schwarz alternating method as a *discretization method* for solving multi-scale or multi-physics partial differential equations (PDEs).

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How We Use the Schwarz Alternating Method



¹¹ Spatial Coupling via (Multiplicative) Alternating Schwarz

Overlapping Domain Decomposition



Model PDE:
$$\begin{cases} N(\boldsymbol{u}) = \boldsymbol{f}, & \text{in } \Omega \\ \boldsymbol{u} = \boldsymbol{g}, & \text{on } \partial \Omega \end{cases}$$

• Dirichlet-Dirichlet transmission BCs [Schwarz 1870; Lions 1988; Mota *et al.* 2017; Mota *et al.* 2022]

Non-overlapping Domain Decomposition

$$\begin{cases} N\left(\boldsymbol{u}_{1}^{(n+1)}\right) = f, & \text{in } \Omega_{1} \\ \boldsymbol{u}_{1}^{(n+1)} = \boldsymbol{g}, & \text{on } \partial \Omega_{1} \setminus \Gamma \\ \boldsymbol{u}_{1}^{(n+1)} = \boldsymbol{\lambda}_{n+1}, & \text{on } \Gamma \end{cases}$$
$$\begin{cases} N\left(\boldsymbol{u}_{2}^{(n+1)}\right) = f, & \text{in } \Omega_{2} \\ \boldsymbol{u}_{2}^{(n+1)} = \boldsymbol{g}, & \text{on } \partial \Omega_{2} \setminus \Gamma \\ \boldsymbol{\nabla} \boldsymbol{u}_{2}^{(n+1)} \cdot \boldsymbol{n} = \boldsymbol{\nabla} \boldsymbol{u}_{1}^{(n+1)} \cdot \boldsymbol{n}, \text{ on } \Gamma \end{cases}$$
$$\boldsymbol{\lambda}_{n+1} = \theta \boldsymbol{u}_{2}^{(n)} + (1 - \theta) \boldsymbol{\lambda}_{n}, \text{ on } \Gamma, \text{ for } n \geq 1 \end{cases}$$



- Relevant for multi-material and multiphysics coupling
- Alternating Dirichlet-Neumann transmission BCs [Zanolli *et al.* 1987]
- Robin-Robin transmission BCs also lead to convergence [Lions 1990]
- $\theta \in [0,1]$: relaxation parameter (can help convergence)

Multiplicative Overlapping Schwarz

Additive Overlapping Schwarz



Model PDE: $\begin{cases}
N(u) = f, \text{ in } \Omega \\
u = g, \text{ on } \partial\Omega
\end{cases}$



- Multiplicative Schwarz: solves subdomain problems sequentially (in serial)
- Additive Schwarz: advance subdomains in parallel, communicate boundary condition data later
 - > Typically requires a few more **Schwarz iterations**, but does not degrade **accuracy**
 - Parallelism helps balance additional cost due to Schwarz iterations
 - > Applicable to both **overlapping** and **non-overlapping** Schwarz

¹³ Time-Advancement Within the Schwarz Framework



<u>Step 0</u>: Initialize i = 0 (controller time index).

Controller time stepper

Time integrator for Ω_1

Time integrator for Ω_2

Model PDE:	$\begin{cases} \dot{\boldsymbol{u}} + N(\boldsymbol{u}) = \boldsymbol{f}, \\ \boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{g}(t), \end{cases}$	in Ω on $\partial \Omega$
	$(\boldsymbol{u}(\boldsymbol{x},0)=\boldsymbol{u}_0,$	in Ω

14 Time-Advancement Within the Schwarz Framework



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<u>Step 1</u>: Advance Ω_1 solution from time T_i to time T_{i+1} using time-stepper in Ω_1 with time-step Δt_1 , using solution in Ω_2 interpolated to Γ_1 at times $T_i + n\Delta t_1$.

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15 **Time-Advancement Within the Schwarz Framework**



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DE: $\begin{cases} \dot{\boldsymbol{u}} + N(\boldsymbol{u}) = \boldsymbol{f}, & \text{in } \Omega\\ \boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{g}(t), & \text{on } \partial\Omega\\ \boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0, & \text{in } \Omega \end{cases}$
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17 Time-Advancement Within the Schwarz Framework



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different time steps within each domain!

¹⁹ Time-Advancement Within the Schwarz Framework



Time-stepping procedure is **equivalent** to doing Schwarz on **space-time domain** [Mota *et al.* 2022].

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Coupling is *concurrent* (two-way).

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- *Ease of implementation* into existing massively-parallel HPC codes.
- Scalable, fast, robust (we target real engineering • problems, e.g., analyses involving failure of bolted components!).
- Coupling does not introduce *nonphysical artifacts*.
- *Theoretical* convergence properties/guarantees¹. ۲
- "Plug-and-play" framework:
 - > Ability to couple regions with *different non-conformal meshes*, *different element types* and *different levels of refinement* to simplify task of *meshing complex geometries*.
 - > Ability to use *different solvers/time-integrators* in different regions.

Model Solid Mechanics PDEs:

Quasistatic:	Div $\boldsymbol{P} + \rho_0 \boldsymbol{B} = \boldsymbol{0}$ in Ω
Dynamic:	Div $\boldsymbol{P} + \rho_0 \boldsymbol{B} = \rho_0 \ddot{\boldsymbol{\varphi}}$ in $\Omega \times I$





Convergence Proof*

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ormulation of the Schwarz Alternating Method by defining the standard finite deformation variational formulation to establish notation before gin beformulation the coupling methods 'ariational Formulation on a Single Domain			
by defining the standard finite deformation variational formulation to establish notation before ng the formulation of the coupling method. /ariational Formulation on a Single Domain		1: $\mathbf{x}_{B}^{(1)} \leftarrow \mathbf{X}_{B}^{(1)}$ in $\Omega_{1}, \mathbf{x}_{h}^{(1)} \leftarrow \chi(\mathbf{X}_{h}^{(1)})$ on $\partial_{\mu}\Omega_{1}$, \triangleright initialize for Ω_{1}	Described of FOI
ng the formulation of the coupling method. /ariational Formulation on a Single Domain		2: $\mathbf{x}_B^{(2)} \leftarrow \mathbf{X}_B^{(2)}$ in $\Omega_2, \mathbf{x}_b^{(2)} \leftarrow \chi(\mathbf{X}_b^{(2)})$ on $\partial_{\phi}\Omega_2$, \diamond initialize for Ω_2 3: repeat \diamond Newton-Schwarz loop	Kernark that [50] $\tilde{S}_n = \tilde{\varphi}^{(n-1)} + \tilde{V}_i$ for $\tilde{\varphi}^{(n-1)} \in \tilde{S}_{n-1} \Rightarrow \tilde{\varphi}^{(n-1)} \in \tilde{S}_n$.
ariational Formulation on a Single Domain	(ISP F2 /) F1 IS2	4: $\left\{ \Delta \mathbf{x}_{\beta}^{(1)} \atop \Delta \mathbf{y}_{\beta}^{(2)} \right\} \leftarrow \left(\mathbf{K}_{AB}^{(1)} + \mathbf{K}_{AB}^{(1)} \mathbf{H}_{11} - \mathbf{K}_{AB}^{(1)} \mathbf{H}_{12} \atop \mathbf{K}^{(2)} + \mathbf{K}^{(2)} \mathbf{H}_{22} - \mathbf{K}^{(2)} \mathbf{H}_{22} \right) \setminus \left\{ -\mathbf{R}_{\beta}^{(1)} \right\} \rightarrow \text{linear system}$	Theorem 1. Assume that the energy functional $\Phi[\varphi]$ satisfies properties 1–5 above. Con-
		$(\overset{(m,p)}{\longrightarrow} y)$ $(\overset{(m,p)}{$	alternating method of Section 2 defined by $(9)-(13)$ and its equivalent form (39) . Then $(a) \Phi(\bar{a}^{(0)}) > \Phi(\bar{a}^{(1)}) > \Phi(\bar{a}^{(0)}) > \Phi(\bar{a}^{(0)})$
r a body as the open set $\Omega \subset \mathbb{R}^3$ undergoing a motion described by the mapping $x = \varphi(X) : \Omega \to \mathbb{R}^3$,		6: $\mathbf{x}_{B}^{(p)'} \leftarrow \mathbf{x}_{B}^{(p)'} + \Delta \mathbf{x}_{B}^{(p)'}$ 7: multi $\left[\left(\ \oplus \mathbf{x}^{(1)} \ \ \ _{\mathbf{w}}^{(1)} \ \right)^{2} + \left(\ \oplus \mathbf{x}^{(2)} \ \ \ \ \ ^{2} \right)^{2} \right]^{1/2} \leq \epsilon$, (5) there is a static relations	(a) Φ[φ ⁺ , i] ≥ Φ[φ ⁺ , i] ≥ ··· ≥ Φ[φ ⁺ , i] ≥ Φ[φ ⁺ , i] ≥ ··· ≥ Φ[φ], where φ is the m over S.
Assume that the boundary of the body is $\partial \Omega = \partial_{\varphi} \Omega \cup \partial_T \Omega$ with unit normal N, where $\partial_{\varphi} \Omega$ placement boundary, $\partial_T \Omega$ is a traction boundary, and $\partial_{\varphi} \Omega \cap \partial_T \Omega = \emptyset$. The prescribed boundary	Figure 1: Two subdomains $i rate and i rate and the corresponding boundaries \Gamma_1 and \Gamma_2 used by the Schwarz alternating method.$	Alexithe S Mendible Schwart Method	(b) the sequence $\{\tilde{\varphi}^{(n)}\}$ defined in (39) converges to the minimizer φ of $\Phi[\varphi]$ in S.
ments or Dirichlet boundary conditions are $\chi : \partial_{\varphi}\Omega \rightarrow \mathbb{R}^3$. The prescribed boundary tractions or in boundary conditions are $T : \partial_{\tau}\Omega \rightarrow \mathbb{R}^3$. Let $F := \text{Grad } \varphi$ be the deformation gradient. Let			(c) the Schwarz minimum values $\Phi[\hat{\varphi}^{(n)}]$ converge monotonically to the minimum value Φ
$\mathfrak{F}: \Omega \to \mathbb{R}^3$ be the body force, with R the mass density in the reference configuration. Furthermore, is the energy functional	that is i = 1 and j = 2 if n is odd, and i = 2 and j = 1 if n is even. Introduce the following definitions for each externation i:	[35, 34, 4]. Although we do not provide here formal convergence proofs for the remaining variants of the Schwarz method, we offer some numerical equilibrium their convergence in Section 4.	from any initial guess $\varphi \sim$. (d) if $W(\alpha)$ is Lincohitz continuous in a usiablearboad of α , then the sequence $L\bar{\alpha}^{(n)}$.
	Closure ⊠:= ∞/ @∞	Consider the energy functional $\Phi[\varphi]$ defined in (1). We will denote by (\cdot, \cdot) the usual L^2 inner product	(a) η * (φ) is expected, commons in a neighborhood of φ, then the nequence (φ ⁻¹) c rically to the minimizer φ. ³
$\Psi[\varphi] := \int_{\Omega} A(\mathbf{F}, \mathbf{Z}) dv - \int_{\Omega} R\mathbf{B} \cdot \boldsymbol{\varphi} dv - \int_{\partial_T \Omega} \mathbf{I} \cdot \boldsymbol{\varphi} ds,$ (1)	Dirichlet boundary: @ IZ) := @ IZN IZ).	over it, that is, $(\psi_1, \psi_2) := \int \psi_1 \cdot \psi_2 dV,$ (35)	Proof. See Appendix A.
A(F, Z) is the Helmholtz free-energy density and Z is a collection of internal variables. The weak the problem is obtained by minimizing the energy functional $\Phi[\alpha]$ over the Sobolev space $W_{\alpha}^{2}(\Omega)$	Neumann boundary: @ 🖾 := @ 🖾 1 🖾	for w_{i} , $w_{i} \in W^{1}(\Omega)$, with corresponding norm $\ \cdot \ $. The proof of the convergence of the Schwarr alternation	Finally, while most of works cited above present their analysis for the specific case of attention to multiple subdomains is in general straightforward. The case of multiple subdom
imprised of all functions that are square-integrable and have square-integrable first derivatives. Define	 Schwarz boundary: F₁ := @R>1 Ky. 	method requires that the functional $\Phi[\varphi]$ satisfy the following properties over the space S defined in (2):	specifically in Lions [35], Badea [4], and Li-Shan and Evans [34].
$S := \{\varphi \in W_2^1(\Omega) : \varphi = \chi \text{ on } \partial_{\varphi}\Omega\}$ (2)	Note that with these definitions we guarantee that $@ \boxtimes 1 @ \boxtimes = ;, @ \boxtimes 1 \Gamma_i = ;$ and $@ \boxtimes 1 \Gamma_i = ;$.	1. $\Phi[\varphi]$ is coercive.	4 Numerical Examples
$V := \{ \mathbf{F} \in W_{*}^{1}(\Omega) : \mathbf{F} = 0 \text{ on } \partial_{\mathbf{u}} \Omega \}$ (3)	Now define the spaces $S_{i} = \frac{1}{2} \frac{2}{3} \frac{W^{2}(m)}{(m)} \frac{1}{2} = W \exp(\frac{2m}{m}) \frac{1}{2} = B_{i} = \frac{1}{2} \frac{(m)^{2}}{(m)^{2}} = \frac{1}{2} \frac{m}{(m)}$	2. $\Phi[\varphi]$ is Fréchet differentiable, with $\Phi'[\varphi]$ denoting its Fréchet derivative.	4 Numerical Examples
$\in V$ is a test function. The potential energy is minimized if and only if $\Phi[\varphi] \le \Phi[\varphi + \epsilon\xi]$ for all	$S_1 := \{ : Z W_2(X_i) : : = \chi \text{ on } g_i(X_i) : : = P_{X_i} / r_i [(X_i^2)] \text{ on } 1_i , (7)$	 Φ[φ] is strictly convex. 	In this section, we present numerical examples of the behavior of the Schwarz alternating different implementations. First, we briefly describe the two implementations, one in MATI
nd $e \in \mathbb{R}$. It is straightforward to show that the minimum of $\Phi[\varphi]$ is the mapping $\varphi \in S$ that satisfies	$V_i := \{ \leftarrow 2 W_2^i(\mathbb{Z}) : \leftarrow = 0 \text{ on } \otimes \mathbb{Z} \setminus \{ \Gamma_i \},$ (8)	 Φ[φ] is lower semi-continuous. 	in the open source ALBANY finite element code [32]. Next, we discuss the error measures
method of Section 2 defined (a) $\Phi[\tilde{\varphi}^{(0)}] > \Phi[\tilde{\varphi}^{(1)}] > \cdot$	$\cdots > \Phi[\tilde{\varphi}^{(n-1)}] > \Phi[\tilde{\varphi}^{(n)}] > \cdot$	$\cdots > \Phi[\varphi]$, where φ is the minin	uizer of $\Phi[\varphi]$ over S.
method of Section 2 defined (a) $\Phi[\tilde{\varphi}^{(0)}] \ge \Phi[\tilde{\varphi}^{(1)}] \ge \cdot$ (b) The sequence $\{\tilde{\varphi}^{(n)}\} d$	$\cdots \ge \Phi[\tilde{\varphi}^{(n-1)}] \ge \Phi[\tilde{\varphi}^{(n)}] \ge \cdots$	form (39). Then $\cdots \geq \Phi[\varphi]$, where φ is the minin minimizer (φ of $\Phi[\varphi]$ in S	uzer of $\Phi[oldsymbol{arphi}]$ over \mathcal{S} .
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method of Section 2 defined (a) $\Phi[\tilde{\varphi}^{(0)}] \ge \Phi[\tilde{\varphi}^{(1)}] \ge \cdot$ (b) The sequence $\{\tilde{\varphi}^{(n)}\} d$ (c) The Schwarz minimum	$\cdots \ge \Phi[\tilde{\varphi}^{(n-1)}] \ge \Phi[\tilde{\varphi}^{(n)}] \ge \cdot$ efined in (39) converges to the <i>n</i> values $\Phi[\tilde{\varphi}^{(n)}]$ converge monot	form (39). Then $\cdots \geq \Phi[\varphi]$, where φ is the minin ninimizer φ of $\Phi[\varphi]$ in S. tonically to the minimum value Φ	nizer of $\Phi[\varphi]$ over S . [φ] in S starting from any
method of Section 2 defined (a) $\Phi[\tilde{\varphi}^{(0)}] \ge \Phi[\tilde{\varphi}^{(1)}] \ge \cdot$ (b) The sequence $\{\tilde{\varphi}^{(n)}\} d$ (c) The Schwarz minimum initial array $\tilde{\varphi}^{(0)}$	$\cdots \ge \Phi[\tilde{\varphi}^{(n-1)}] \ge \Phi[\tilde{\varphi}^{(n)}] \ge \cdot$ efined in (39) converges to the n values $\Phi[\tilde{\varphi}^{(n)}]$ converge monot	form (39). Then $\cdots \geq \Phi[\varphi]$, where φ is the minin ninimizer φ of $\Phi[\varphi]$ in S. tonically to the minimum value Φ	nizer of $\Phi[arphi]$ over S. [arphi] in S starting from any
 method of Section 2 defined (a) Φ[φ̃⁽⁰⁾] ≥ Φ[φ̃⁽¹⁾] ≥ . (b) The sequence {φ̃⁽ⁿ⁾} d (c) The Schwarz minimum initial guess φ̃⁽⁰⁾. 	$\psi = \Phi[\tilde{\varphi}^{(n-1)}] \ge \Phi[\tilde{\varphi}^{(n)}] \ge \psi$ efined in (39) converges to the n values $\Phi[\tilde{\varphi}^{(n)}]$ converge monot	form (39). Then $\cdots \geq \Phi[\varphi]$, where φ is the minin ninimizer φ of $\Phi[\varphi]$ in S. tonically to the minimum value Φ	nizer of $\Phi[arphi]$ over \mathcal{S} . [$arphi]$ in \mathcal{S} starting from any
method of Section 2 defined (a) $\Phi[\tilde{\varphi}^{(0)}] \ge \Phi[\tilde{\varphi}^{(1)}] \ge \cdot$ (b) The sequence $\{\tilde{\varphi}^{(n)}\} d$ (c) The Schwarz minimum initial guess $\tilde{\varphi}^{(0)}$.	$\psi = \Phi[\tilde{\varphi}^{(n-1)}] \ge \Phi[\tilde{\varphi}^{(n)}] \ge \psi$ efined in (39) converges to the n values $\Phi[\tilde{\varphi}^{(n)}]$ converge monot	form (39). Then $\cdots \geq \Phi[\varphi]$, where φ is the minin ninimizer φ of $\Phi[\varphi]$ in S . tonically to the minimum value Φ	nizer of $\Phi[arphi]$ over \mathcal{S} . [$arphi$] in \mathcal{S} starting from any
method of Section 2 defined (a) $\Phi[\tilde{\varphi}^{(0)}] \ge \Phi[\tilde{\varphi}^{(1)}] \ge \cdot$ (b) The sequence $\{\tilde{\varphi}^{(n)}\} d$ (c) The Schwarz minimum initial guess $\tilde{\varphi}^{(0)}$.	$\cdots \ge \Phi[\tilde{\varphi}^{(n-1)}] \ge \Phi[\tilde{\varphi}^{(n)}] \ge \cdots$ efined in (39) converges to the n values $\Phi[\tilde{\varphi}^{(n)}]$ converge monot	form (39). Then $\cdots \geq \Phi[\varphi]$, where φ is the minin ninimizer φ of $\Phi[\varphi]$ in S. tonically to the minimum value Φ	nizer of $\Phi[arphi]$ over \mathcal{S} . [$arphi]$ in \mathcal{S} starting from any
method of Section 2 defined (a) $\Phi[\tilde{\varphi}^{(0)}] \ge \Phi[\tilde{\varphi}^{(1)}] \ge \cdot$ (b) The sequence $\{\tilde{\varphi}^{(n)}\} d$ (c) The Schwarz minimum initial guess $\tilde{\varphi}^{(0)}$.	$ \psi = \Phi[\tilde{\varphi}^{(n-1)}] \geq \Phi[\tilde{\varphi}^{(n)}] \geq \cdot $ efined in (39) converges to the n values $\Phi[\tilde{\varphi}^{(n)}]$ converge monot $ (e^{[\varphi^{(n)}], c_0] - (e^{[\varphi^{(n)}], c_0]} = (e^{[\varphi^{(n)}], a_0] \geq e^{[\varphi^{(n)}], a_0]$	form (39). Then $\cdots \geq \Phi[\varphi]$, where φ is the minin minimizer φ of $\Phi[\varphi]$ in S . tonically to the minimum value Φ	nizer of $\Phi[\varphi]$ over S . [φ] in S starting from any
method of Section 2 defined (a) $\Phi[\tilde{\varphi}^{(0)}] \ge \Phi[\tilde{\varphi}^{(1)}] \ge \cdot$ (b) The sequence $\{\tilde{\varphi}^{(n)}\} d$ (c) The Schwarz minimum initial guess $\tilde{\varphi}^{(0)}$.	$ \begin{array}{l} \cdots \geq \Phi[\tilde{\varphi}^{(n-1)}] \geq \Phi[\tilde{\varphi}^{(n)}] \geq \cdot \\ efined \ in \ (39) \ converges \ to \ the \ n \\ values \ \Phi[\tilde{\varphi}^{(n)}] \ converge \ monot \\ \end{array} $	form (39). Then $\cdots \geq \Phi[\varphi]$, where φ is the minin minimizer φ of $\Phi[\varphi]$ in S . tonically to the minimum value Φ model of $(12.3.5)$, Since $\Theta[\Phi^{(0)}] = f(ax) \rightarrow \infty$. In the second s	nizer of $\Phi[\varphi]$ over S . [φ] in S starting from any
method of Section 2 defined (a) $\Phi[\tilde{\varphi}^{(0)}] \ge \Phi[\tilde{\varphi}^{(1)}] \ge \cdot$ (b) The sequence $\{\tilde{\varphi}^{(n)}\} d$ (c) The Schwarz minimum initial guess $\tilde{\varphi}^{(0)}$.	$ \begin{array}{l} \cdots \geq \Phi[\tilde{\varphi}^{(n-1)}] \geq \Phi[\tilde{\varphi}^{(n)}] \geq \cdot \\ efined \ in \ (39) \ converges \ to \ the \ n \\ values \ \Phi[\tilde{\varphi}^{(n)}] \ converge \ monot \\ \end{array} $	$form (59). Then \cdots \geq \Phi[\varphi], \text{ where } \varphi \text{ is the minin ninimizer } \varphi \text{ of } \Phi[\varphi] \text{ in } S. tonically to the minimum value \Phi \max_{\substack{[\varphi]^{(\alpha)} = \varphi^{(\alpha)} \varphi ^{(\alpha)} = \Phi(S)}} \sup_{\substack{[\varphi]^{(\alpha)} = \varphi^{(\alpha)} \varphi ^{(\alpha)} = \Phi(S)}} \sum_{\substack{[\varphi]^{(\alpha)} = \varphi^{($	nizer of $\Phi[\varphi]$ over S . $[\varphi]$ in S starting from any ${}^{(e^{i} \varphi^{(\alpha)} ,\varphi-\varphi^{(\alpha)}) \leq (e^{i} \varphi^{(\alpha)} ,\varphi-\varphi^{(\alpha)}) + \alpha_{\alpha} \varphi-\varphi^{(\alpha)} \leq \Phi \varphi = \Phi}$ ${}^{(e^{i} \varphi^{(\alpha)} ,\varphi-\varphi^{(\alpha)}) \leq (e^{i} \varphi^{(\alpha)} ,\varphi-\varphi^{(\alpha)}) + \alpha_{\alpha} \varphi-\varphi^{(\alpha)} \leq \Phi \varphi = \Phi}$
method of Section 2 defined (a) $\Phi[\tilde{\varphi}^{(0)}] \ge \Phi[\tilde{\varphi}^{(1)}] \ge \cdot$ (b) The sequence $\{\tilde{\varphi}^{(n)}\} d$ (c) The Schwarz minimum initial guess $\tilde{\varphi}^{(0)}$.	$ \psi(\varphi) = (15) \text{ and its equivalent} $ $ \psi \geq \Phi[\tilde{\varphi}^{(n-1)}] \geq \Phi[\tilde{\varphi}^{(n)}] \geq \cdot $ $ efined in (39) \text{ converges to the n} $ $ values \Phi[\tilde{\varphi}^{(n)}] \text{ converge monot} $ $ (\Psi(\varphi^{(n)}), \zeta) = (\Psi(\varphi^{(n)}), \zeta) \leq \Psi(\varphi^{(n)}) - \Psi(\varphi^{(n-1)}) \cdot \zeta . (G)) $ $ (\Psi(\varphi^{(n)}), \zeta) = (\Psi(\varphi^{(n)}), \zeta) \leq \Psi(\varphi^{(n)}) - \Psi(\varphi^{(n-1)}) \cdot \zeta . (G)) $ $ (\Psi(\varphi^{(n)}), \zeta) = (\Psi(\varphi^{(n)}), \zeta) \leq \Psi(\varphi^{(n)}) - \Psi(\varphi^{(n-1)}) \cdot \zeta . (G)) $ $ (\Psi(\varphi^{(n)}), \zeta) = (\Psi(\varphi^{(n)}), \zeta) \leq \Psi(\varphi^{(n)}) - \Psi(\varphi^{(n-1)}) \cdot \zeta . (G)) $ $ (\Psi(\varphi^{(n)}), \zeta) = (\Psi$	$ \cdots \geq \Phi[\varphi], \text{ where } \varphi \text{ is the mining ninimizer } \varphi \text{ of } \Phi[\varphi] \text{ in } S. $ $ \text{tonically to the minimum value } \Phi $ $ \text{tonically to the minimum value } \Phi $ $ \text{tonically to the minimum value } \Phi $	nizer of $\Phi[\varphi]$ over S. [φ] in S starting from any $\binom{(q' a^{(n)} ,q-a^{(n)})}{(q' a^{(n)} ,q-a^{(n)})+\alpha_n\ q-a^{(n)}\ } \leq \phi q -a^{(n)}$ the cost of the lattice of the
method of Section 2 defined (a) $\Phi[\tilde{\varphi}^{(0)}] \ge \Phi[\tilde{\varphi}^{(1)}] \ge \cdot$ (b) The sequence $\{\tilde{\varphi}^{(n)}\} d$ (c) The Schwarz minimum initial guess $\tilde{\varphi}^{(0)}$. 4.10 for 6 subscription of the full set Migran them the unique minimum initial guess $\tilde{\varphi}^{(0)}$. 4.2 By the Sumpachia theorem, the minimization of $\Phi[\varphi]$ is S in equivalent to finding $\varphi \in S$ and $(\Psi[\varphi], \xi - \varphi) \ge 0$ (1)	$ \begin{array}{l} \psi(\varphi) = (13) \text{ and its equivalent} \\ efined in (39) \text{ converges to the non-values} \\ \psi(\varphi) = (13) \text{ converge monot} \\ \psi(\varphi) = (13) converg$	$ \begin{array}{l} \cdots \geq \Phi[\varphi], \text{ then} \\ \cdots \geq \Phi[\varphi], \text{ where } \varphi \text{ is the minin } \\ \text{minimizer } \varphi \text{ of } \Phi[\varphi] \text{ in } S. \\ \text{tonically to the minimum value } \Phi \\ \end{array} $	nizer of $\Phi[\varphi]$ over S . [φ] in S starting from any $\left(e^{i[\phi^{(\alpha)}]} e^{-\phi^{(\alpha)}} \right) \leq \left(e^{i[\phi^{(\alpha)}]} e^{-\phi^{(\alpha)}} \right) + a_0 e^{-\phi^{(\alpha)}} \leq \Phi e - \Phi^{(\alpha)}$ inder $[\phi^{(\alpha)}] e^{-\phi^{(\alpha)}} \right) \leq \left(e^{i[\phi^{(\alpha)}]} e^{-\phi^{(\alpha)}} \right) + a_0 e^{-\phi^{(\alpha)}} \leq \Phi e - \Phi^{(\alpha)}$ inder $[\phi^{(\alpha)}] e^{-\phi^{(\alpha)}} \geq \left(e^{i[\phi^{(\alpha)}]} e^{-\phi^{(\alpha)}} \right) \leq \left\ e^{i[\phi^{(\alpha)}]} e^{-\phi^{(\alpha)}} \right\ \leq K e^{-\phi^{(\alpha)}} ^2.$ Hence, from (29).
method of Section 2 defined (a) $\Phi[\tilde{\varphi}^{(0)}] \ge \Phi[\tilde{\varphi}^{(1)}] \ge \cdot$ (b) The sequence $\{\tilde{\varphi}^{(n)}\} d$ (c) The Schwarz minimum initial guess $\tilde{\varphi}^{(0)}$. KAMME PROPERT I SCHWARZ MINIMUM (c) the Schwarz minimum initial guess $\tilde{\varphi}^{(0)}$. KAMME PROPERT I SCHWARZ MINIMUM (c) $\mu_{(2)}(z \to z) = (z \to z)$ (c) $(\varphi_{(2)}(z \to z) = (z \to z))$ (c) $(z \to z)$	$ \begin{array}{l} & \psi(\varphi) = (13) \text{ and its equivalent} \\ & \cdots \geq \Phi[\tilde{\varphi}^{(n-1)}] \geq \Phi[\tilde{\varphi}^{(n)}] \geq \cdot \\ efined in (39) \text{ converges to the n} \\ values \Phi[\tilde{\varphi}^{(n)}] \text{ converge monot} \\ \\ & \text{ ($ \theta^{(p)}(, \varsigma) - (\theta^{(p)(n-1)}, \varsigma) = (\theta^{(p)(n)} - \theta^{(p)(n-1)}(\varsigma) \leq \ \theta^{(p)(n)} - \theta^{(p)(n-1)}\ _{1} \ \zeta_{\varsigma}\ _{1} } (0) \\ & \text{ Again oning (57) and also (58) in (60) leads to} \\ & (\theta^{(p)(n)}(, \varsigma) - \theta^{(p)(n-1)}(, \varsigma) - (\theta^{(p)(n-1)}(\varsigma) \leq \ \theta^{(p)(n)} - \theta^{(p)(n-1)}\ _{1} \ \zeta_{\varsigma}\ _{1} } (0) \\ & \text{ and substituting (59) ino (61) we finally obtain that} \\ & (\theta^{(p)(n)}(, \varsigma) \leq \ \theta^{(p)(n-1)}\ _{1} \ \zeta\ _{1} (\infty) \\ & (\theta^{(p)(n)}(, \varsigma) \leq \ \theta^{(p)(n-1)}\ _{1} \ \zeta\ _{1}) = (0) \end{array} $	$ \begin{array}{l} \cdots \geq \Phi[\varphi], \text{ Interl} \\ \cdots \geq \Phi[\varphi], \text{ where } \varphi \text{ is the minim num rate } \\ \text{ninimizer } \varphi \text{ of } \Phi[\varphi] \text{ in } S. \\ \text{tonically to the minimum value } \Phi \\ \end{array} \\ \end{array}$	nizer of $\Phi[\varphi]$ over S. $[\varphi] \text{ in S starting from any}$ $\binom{\psi[\varphi^{[n]}] \varphi - \varphi^{[n]}}{\varphi^{[n]}} \leq \frac{\psi[\varphi^{[n]}] \varphi - \varphi^{[n]}}{\varphi^{[n]}} \leq \frac{\psi[\varphi^{[n]}]}{\varphi^{[n]}} \leq \frac{\psi[\varphi^{[n]}]}{\varphi^{[n]}} \leq \frac{\psi[\varphi^{[n]}] \varphi}{\varphi^{[n]}} \leq \frac{\psi[\varphi^{[n]}]}{\varphi^{[n]}} = \psi[\varphi^{[n]$
method of Section 2 defined (a) $\Phi[\tilde{\varphi}^{(0)}] \ge \Phi[\tilde{\varphi}^{(1)}] \ge \cdot$ (b) The sequence $\{\tilde{\varphi}^{(n)}\} d$ (c) The Schwarz minimum initial guess $\tilde{\varphi}^{(0)}$.	$ (\psi[\varphi^{(n)}], \zeta) = (\psi[\varphi^{(n-1)}]) \geq \Phi[[\tilde{\varphi}^{(n)}]] \geq \cdot $ efined in (39) converges to the n values $\Phi[[\tilde{\varphi}^{(n)}]]$ converge monot $ (\psi[\varphi^{(n)}], \zeta) = (\psi[\varphi^{(n)}], \zeta) = (\psi[\varphi^{(n)}], \zeta) \leq \psi[\varphi^{(n)}] - \psi[\varphi^{(n-1)}] + \zeta . (6) $ Again using (5) and also (5) in (60) leads to $ (\psi[\varphi^{(n)}] - \psi[\psi^{(n-1)}], \zeta) = (\psi[\varphi^{(n)}] - \psi[\varphi^{(n-1)}] + \zeta . (6) $ and substraining (5) in (6) to a that $ (\psi[\varphi^{(n)}], \zeta) = (\psi[\varphi^{(n)}], \zeta) = (\psi[\varphi^{(n)}] - \psi[\varphi^{(n-1)}] + \zeta . (6) $ and substraining (5) in (6) to a that $ (\psi[\varphi^{(n)}], \zeta) = \zeta_0[\psi[\varphi^{(n)}] - \psi[\varphi^{(n-1)}] + \zeta . (62) $ $ (\psi[\varphi^{(n)}], \zeta) = \zeta_0[\psi[\varphi^{(n)}] - \psi[\varphi^{(n-1)}] + \zeta . (62) $ $ (\psi[\varphi^{(n)}], \zeta) = \zeta_0[\psi[\varphi^{(n)}] - \psi[\varphi^{(n-1)}] + \zeta . (62) $ $ (\psi[\varphi^{(n)}], \zeta) = \zeta_0[\psi[\varphi^{(n)}] - \psi[\varphi^{(n-1)}] + \zeta . (62) $ $ (\psi[\varphi^{(n)}], \zeta) = \zeta_0[\psi[\varphi^{(n)}] - \psi[\varphi^{(n-1)}] + \zeta . (62) $	$ \cdots \geq \Phi[\varphi], \text{ where } \varphi \text{ is the mining ninimizer } \varphi \text{ of } \Phi[\varphi] \text{ in } S. $ $ \text{tonically to the minimum value } \Phi $ $ \text{for all } (1,2,4,\ldots) \text{ Since } \Theta[\varphi^{(n)}] \xrightarrow{\varphi} (1,2,4,\ldots) = 0 \text{ or } n + \infty. $ $ \text{tonically to the minimum value } \Phi $ $ \text{for all } (1,2,4,\ldots) = 0 \text{ or } n = 0. $ $ \text{for all } (1,2,4,\ldots) = 0 \text{ or } n = 0. $ $ \text{for all } (1,2,4,\ldots) = 0 \text{ or } n = 0. $ $ \text{for all } (1,2,4,\ldots) = 0 \text{ or } n = 0. $ $ \text{for all } (1,2,4,\ldots) = 0 \text{ or } n = 0. $ $ \text{for all } (1,2,4,\ldots) = 0 \text{ or } n = 0. $ $ \text{for all } (1,2,4,\ldots) = 0 \text{ or } n = 0. $ $ \text{for all } n = 0 \text{ or } n = 0. $ $ \text{for all } n = 0 \text{ or } n = 0. $ $ \text{for all } n = 0 \text{ or } n = 0. $ $ \text{for all } n = 0 \text{ or } n = 0. $ $ \text{for all } n = 0 \text{ or } n = 0. $ $ \text{for all } n = 0. $ $ for all$	nizer of $\Phi[\varphi]$ over S. $[\varphi] in S starting from any$ $(\psi[\varphi^{[n]}], \varphi - \varphi^{[n]}) \in \psi[\varphi^{[n]}], \varphi - \varphi^{[n]}] + \alpha_0 \ \varphi - \varphi^{[n]}\ \le \Phi[\varphi] - \Phi$ In the case of the starting from any of the lattice of the l
method of Section 2 defined (a) $\Phi[\tilde{\varphi}^{(0)}] \ge \Phi[\tilde{\varphi}^{(1)}] \ge \cdot$ (b) The sequence $\{\tilde{\varphi}^{(n)}\} d$ (c) The Schwarz minimum initial guess $\tilde{\varphi}^{(0)}$. 4.2 by the Superconduction between the 1 by the operative of $\varphi_{i}^{(1)}$ is follows from the Law Migram horses that a unique minimizer to initial guess $\tilde{\varphi}^{(0)}$. 4.3 by the descentive of $\varphi_{i}^{(1)}$ is follows from the Law Migram horses that a unique minimizer to (a) $\Phi(\varphi_{i}) = 0$ (c) 4.3 by the descention of $\varphi_{i}^{(1)}$ is follows from the Law Migram horses that a unique minimizer to (b) $\Phi(\varphi_{i}) = 0$ (c) 4.3 by the descention theorem, the minimization of $\varphi_{i}^{(1)}$ is a to equivalent to finding $\varphi \in S$ such (c) $\Phi(\varphi_{i}) = 0$ (c) (c) $\Phi(\varphi_{i}) = 0$ (c) (c)	$\begin{split} & \psi(\varphi) - (13) \text{ and its equivalent} \\ & \psi(\varphi) - (13) \text{ and its equivalent} \\ & \psi(\varphi) - (13) \text{ and its equivalent} \\ & \psi(\varphi) - (13) \text{ and its equivalent} \\ & \psi(\varphi) - (13) \text{ and its equivalent} \\ & efined in (39) converges to the non-set operator of the end $	$ \begin{array}{l} \cdots \geq \Phi[\varphi], \text{ then} \\ \cdots \geq \Phi[\varphi], \text{ where } \varphi \text{ is the minin } \\ \text{minimizer } \varphi \text{ of } \Phi[\varphi] \text{ in } S. \\ \text{tonically to the minimum value } \Phi \\ \end{array} \\ \end{array}$	nizer of $\Phi[\varphi]$ over S. [φ] in S starting from any $(e^{i[\phi^{(\alpha)}], \varphi - \phi^{(\alpha)}]} \in (e^{i[\phi^{(\alpha)}], \varphi - \phi^{(\alpha)}]} + \alpha_{\beta}[\varphi - \phi^{(\alpha)}] \le 4 \varphi - 0^{i}$ ind $e^{i[\varphi]}(0)$ we can write $e^{i[\varphi](0)}(0)$ we can write $e^{i[\varphi](0)}(0)$ we can write $e^{i[\varphi](0)}(0) = e^{i[\varphi](0)} \ge 4 \varphi ^{i} = 0^{i(\alpha)} \le 4 \varphi - \phi^{(\alpha)} ^2$. Hence, from (9), $e^{i[\varphi^{(\alpha)}], \varphi - \phi^{(\alpha)}]} \ge 4 \varphi ^{i} = 0^{i(\alpha)} \le 4 \varphi - \phi^{(\alpha)} ^2$. Hence, from (9), $e^{i[\varphi^{(\alpha)}], \varphi - \phi^{(\alpha)}]} = 4 \varphi ^{i} = 0^{i(\alpha)} \le 4 \varphi - \phi^{(\alpha)} ^2$.
method of Section 2 defined (a) $\Phi[\tilde{\varphi}^{(0)}] \ge \Phi[\tilde{\varphi}^{(1)}] \ge \cdot$ (b) The sequence $\{\tilde{\varphi}^{(n)}\} d$ (c) The Schwarz minimum initial guess $\tilde{\varphi}^{(0)}$. K4 Aumerproperties to connecting the features $\Phi^{(0)}$. K4 Aumerproperties to connect the teach float (c) The Schwarz minimum initial guess $\tilde{\varphi}^{(0)}$. K5 Aumerproperties to connect the teach float ($\varphi(\varphi) \in -\varphi) \ge 0$ (1) (c) φ standard the strict correctly property of $\varphi(\varphi)$ is the leaves the the $H^{-}(\varphi) = 0$ (2) (c) φ . (c) $\varphi(\varphi) = (\varphi(\varphi) = (\varphi(\varphi) - \varphi)) \ge 0$ (2) (c) φ . (c) $\varphi(\varphi) = (\varphi(\varphi) = (\varphi(\varphi) - \varphi)) \ge 0$ (2) (c) φ . (c) $\varphi(\varphi) = (\varphi(\varphi) = (\varphi(\varphi) - \varphi)) \ge 0$ (2) (c) φ . From (b), remark that if $\varphi(\varphi)$ is strictly converse over $S VR \in \mathbb{R}$ when that $R < \infty$, we can find (c) $\varphi(\varphi)$ the $Y = 0$ (c)	$ \begin{array}{l} \psi(\varphi) = (13) \text{ and its equivalent} \\ effined in (39) \text{ converges to the new values } \Phi[\widetilde{\varphi}^{(n)}] \text{ converge monot} \\ \psi(\varphi) = (12) - (12) \text{ and } \psi(\varphi) = (12) - (12) $	$ \begin{array}{l} \cdots \geq \Phi[\varphi], \text{ Then} \\ \cdots \geq \Phi[\varphi], \text{ where } \varphi \text{ is the minim num rate of } \\ \text{point } (S), \text{ where } \varphi \text{ of } \Phi[\varphi] \text{ in } S. \\ \text{tonically to the minimum value } \Phi \\ \end{array} \\ \end{array}$	nizer of $\Phi[\varphi]$ over S. $[\varphi] in S starting from any$ $(e^{i[\varphi^{(\alpha)]},\varphi-\varphi^{(\alpha)})} \le (e^{i[\varphi^{(\alpha)]},\varphi-\varphi^{(\alpha)})} + \alpha_{\alpha}[\varphi-\varphi^{(\alpha)}] \le \theta[\varphi] = \theta$ in eq. (a) to by the Calcely-Schwarz inequality followed by the application of the La index on a > 0. Now the Calcely-Schwarz inequality followed by the application of the La index (p) (e^{i(\varphi^{(\alpha)})}, \varphi-\varphi^{(\alpha)}) \le e^{i(\varphi^{(\alpha)})} \cdot e-\varphi^{(\alpha)} \le K \varphi-\varphi^{(\alpha)} ^2. Hence, from (29) $\theta[\varphi^{(\alpha)}] - \Phi[\varphi] \le K \varphi^{(\alpha)} - \varphi ^2.$ Moreover, by (13) wince $\Phi^{i}[\varphi] = \theta$ $\theta[\varphi^{(\alpha)}] - \Phi[\varphi] \ge \alpha_{\alpha}[\varphi^{(\alpha)} - \varphi ^2.$ Using (01) and (03) we obtain
method of Section 2 defined (a) $\Phi[\tilde{\varphi}^{(0)}] \ge \Phi[\tilde{\varphi}^{(1)}] \ge \cdot$ (b) The sequence $\{\tilde{\varphi}^{(n)}\} d$ (c) The Schwarz minimum initial guess $\tilde{\varphi}^{(0)}$.	$\cdots \ge \Phi[\tilde{\varphi}^{(n-1)}] \ge \Phi[\tilde{\varphi}^{(n)}] \ge \cdots$ efined in (39) converges to the n values $\Phi[\tilde{\varphi}^{(n)}]$ converge monot	form (39). Then $\cdots \geq \Phi[\varphi]$, where φ is the minin ninimizer φ of $\Phi[\varphi]$ in S. tonically to the minimum value Φ	nizer of $\Phi[arphi]$ over S. $[arphi]$ in S starting from any

Remark 4 By property 5, the uniform continuity of $\Phi'[\varphi]$, there exists a modulus of continuity $\omega > 0$, with $||\Phi'(\psi_1) - \Phi'(\psi_2)|| \le \omega(||\psi_1 - \psi_2||),$ (54)the $\forall \psi_1, \psi_2 \in \mathcal{K}_R$. By definition, $\omega(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$. **Remark 5** It was shown in [35] that in the case $\Omega_1 \cap \Omega_2 \neq \emptyset$, $\forall \varphi \in S$, there exist $\zeta_1 \in S_1$ and $\zeta_2 \in S_2$. such that Co $\varphi = \zeta_1 + \zeta_2$, (55) and $\max(||\zeta_1||, ||\zeta_2||) \le C_0 ||\varphi||,$ (56) for some $C_0 > 0$ independent of φ . Remark 6 Note that (39) can be written as $(\Phi'[\hat{\varphi}^{(n)}], \xi^{(i)}) = 0$, for $\hat{\varphi}^{(n)} \in \hat{S}_n, \forall \xi^{(i)} \in S_i$, (57) for $i \in \{1, 2\}$ and $n \in \{0, 1, 2, ...\}$ (recall from (6) the relation between i and n). This is due to the uniqueness of the solution to each minimization problem over S_n and the definition of $\phi^{(n)}$ as the minimizer of $\Phi[\varphi]$ over S_n . Remark 7 Let $\tilde{\varphi}^{(n)} \in \tilde{S}_n$, and let $\xi \in S$. By Remark 5, there exist $\zeta_1 \in S_1$ and $\zeta_2 \in S_2$ such that $(\Phi'[\tilde{\varphi}^{(n)}], \xi) = (\Phi'[\tilde{\varphi}^{(n)}], \zeta_1 + \zeta_2).$ (58)

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$(\Phi'[\tilde{\varphi}^{(n)}], \zeta_2) - (\Phi'[\tilde{\varphi}^{(n-1)}], \zeta_2) = (\Phi'[\tilde{\varphi}^{(n)}] - \Phi'[\tilde{\varphi}^{(n-1)}], \zeta_2) \le \Phi'[\tilde{\varphi}^{(n)}] - \Phi'[\tilde{\varphi}^{(n-1)}] \cdot \zeta_2 .$	(60)
gain using (57) and also (58) in (60) leads to	
$(\Phi'[\tilde{\varphi}^{(n)}] - \Phi'[\tilde{\varphi}^{(n-1)}], \xi_2) = (\Phi'[\tilde{\varphi}^{(n)}], \xi) \le \Phi'[\tilde{\varphi}^{(n)}] - \Phi'[\tilde{\varphi}^{(n-1)}] \cdot \xi_2 ,$	(61)
nd substituting (56) into (61) we finally obtain that	
$(\Phi'[\bar{\varphi}^{(n)}], \xi) \le C_0 \Phi'[\bar{\varphi}^{(n)}] - \Phi'[\bar{\varphi}^{(n-1)}] \cdot \xi ,$	(62)
$\xi \in S$.	
Remark 8 For part (d) of Theorem 1, recall the definition of geometric convergence:	
$E_{n+1} \leq CE_n$,	(63)
$n \in \{0, 1, 2, \dots\}$ for some $C > 0$, where	
$E_n := \bar{\varphi}^{(n+1)} - \bar{\varphi}^{(n)} .$	(64)
Remark 9 Recall from the definition of continuity that if $\Phi'[\varphi]$ is Lipshitz continuous at $\phi^{(\alpha)}$ near φ here exists a constant $K \ge 0$ such that	, then
$\frac{ \Phi'[\bar{\varphi}^{(n)}] - \Phi'[\varphi] }{ \bar{\varphi}^{(n)} - \varphi } \le K.$	(65)
considering that $\Phi'[\varphi] = 0$ since φ is the minimizer of $\Phi[\varphi]$, (65) is equivalent to	
$ \Phi'[\hat{\varphi}^{(n)}] \le K \hat{\varphi}^{(n)} - \varphi .$	(66)
Proof of Theorem 1	
$\begin{array}{l} Proof of(a). \mbox{Let} \tilde{\varphi}^{(1)} = \arg\min_{\psi \in \mathcal{S}_1} \Phi[\varphi], \mbox{By (40)}, \tilde{\varphi}^{(1)} \in \mathcal{S}_2. \mbox{Let} \tilde{\varphi}^* \mbox{be the minimizer of } \Phi[\varphi] \mbox{ or and suppose } \Phi[\tilde{\varphi}^*] > \Phi[\tilde{\varphi}^{(1)}], \mbox{But this is a contradiction, since we can like \tilde{\varphi}^* = \tilde{\varphi}^{(1)}. \mbox{Hence, II can that } \Phi[\tilde{\varphi}^{(1)}] = \Phi[\tilde{\varphi}^{(1)}] \mbox{mode } \Phi[\tilde{\varphi}^{(2)}] \mbox{mode } \Phi[\tilde{\varphi}^{(2)}] \mbox{By induction that } \Phi[\tilde{\varphi}^{(1)}] \mbox{By induction that } \Phi[\tilde{\varphi}^{(1)}] \mbox{By induction that } \Phi[\tilde{\varphi}^{(2)}] By induct$	er \tilde{S}_2 10t be
$\Phi[\hat{arphi}^{(n)}] \leq \Phi[\hat{arphi}^{(n-1)}]$	(67)
for $n \in \{1, 2, 3,\}$. Now let φ be the minimizer of $\Phi[\varphi]$ over S . Since the problem is well-posed unique. Hence $\Phi[\varphi] \leq \Phi[\varphi^{(n)}]$ for all $n \in \{1, 2, 3,\}$.	lφis □
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From (68), we have that $y = y(z(n) - z(n+1)y^2 - \alpha$	
$\lim_{n\to\infty} \varphi^{(n)} - \varphi^{(n-1)} ^2 = 0,$	(09)
from which we can conclude that $\tilde{\varphi}^{(n)} = \tilde{\varphi}^{(n+1)} \rightarrow 0$ as $n \rightarrow \infty$. We must now show that $\tilde{\varphi}^{(n)}$ converges to φ , the minimizer of $\Phi[\varphi]$ on S . By (53) with $\psi_{\parallel} = \psi_{\parallel} = \tilde{\varphi}^{(n)}$, we have	φ and
$\ \varphi - \bar{\varphi}^{(n)}\ ^2 \le \frac{1}{\alpha_R} \left\{ \Phi[\varphi] - \Phi[\bar{\varphi}^{(n)}] - \left(\Phi'[\bar{\varphi}^{(n)}], \varphi - \bar{\varphi}^{(n)} \right) \right\}.$	(70)
Since φ is the minimum of $\Phi[\varphi]$, by (a) we have that $\Phi[\varphi] \le \Phi[\tilde{\varphi}^{(n)}]$. It follows that	
$-\Phi[\varphi]-\Phi[\bar{\varphi}^{(n)}]-\left(\Phi'[\bar{\varphi}^{(n)}],\varphi-\bar{\varphi}^{(n)}\right)\leq -\left(\Phi'[\bar{\varphi}^{(n)}],\varphi-\bar{\varphi}^{(n)}\right)=\left(\Phi'[\bar{\varphi}^{(n)}],\bar{\varphi}^{(n)}-\varphi\right).$	(71)
Subsituting (71) into (70) we have	
$ \varphi - \bar{\varphi}^{(n)} ^2 \le \frac{1}{\alpha_R} \left(\Phi'[\bar{\varphi}^{(n)}], \bar{\varphi}^{(n)} - \varphi \right).$	(72)
Now by (62) (Remark 7).	
$\left(\Phi'[\tilde{\varphi}^{(n)}], \tilde{\varphi}^{(n)} - \varphi\right) \le C_0 \Phi'[\tilde{\varphi}^{(n)}] - \Phi'[\tilde{\varphi}^{(n-1)}] \cdot \tilde{\varphi}^{(n)} - \varphi .$	(73)
Substituting (73) into (72) leads to	
$ \bar{\varphi}^{(n)} - \varphi \le \frac{C_0}{\alpha_R} \Phi'[\bar{\varphi}^{(n)}] - \Phi'[\bar{\varphi}^{(n-1)}] .$	(74)
Applying the uniform continuity assumption (54), we obtain	
$ \tilde{\varphi}^{(n)} - \varphi \le \frac{C_0}{\alpha_R} \omega \left(\tilde{\varphi}^{(n)} - \tilde{\varphi}^{(n-1)} \right).$	(75)
By (69), $ \tilde{\varphi}^{(n)} - \tilde{\varphi}^{(n-1)} \rightarrow 0$ as $n \rightarrow \infty$. From this we obtain the result, namely that $\tilde{\varphi}^{(n)} \rightarrow n \rightarrow \infty$.	φas
Proof of (c). This follows immediately from (a) and (b).	
Proof of (d). By (b), for large enough n, there exists some $C_1 > 0$ independent of n such that	
$ \tilde{\varphi}^{(n)} - \varphi ^2 \le C_1 \tilde{\varphi}^{(n+1)} - \tilde{\varphi}^{(n)} ^2$.	(76)
Let us choose C_1 such that $C_1 > \alpha_R/K$, where K is the Lipshitz continuity constant in (66). Com (68) with (76) leads to	bining
$\frac{1}{\alpha_R} \left(\Phi[\tilde{\varphi}^{(n)}] - \Phi[\tilde{\varphi}^{(n+1)}] \right) \ge \tilde{\varphi}^{(n+1)} - \tilde{\varphi}^{(n)} ^2 \ge \frac{1}{C_1} \tilde{\varphi}^{(n)} - \varphi ^2.$	(77)

$(\Phi[\varphi^{(\alpha)}], \varphi - \varphi^{(\alpha)}) \le (\Phi[\varphi^{(\alpha)}], \varphi - \varphi^{(\alpha)}) + \alpha_R \varphi - \varphi^{(\alpha)} \le \Phi[\varphi] - \Phi[\varphi^{(\alpha)}]$	(79)
since $\alpha_R \ge 0$. Now, by the Cauchy-Schwarz inequality followed by the application of the Lipshitz conti of $\Phi'[\phi]$ (66) we can write	nuity
$\left(\Phi'[\bar{\varphi}^{(n)}], \varphi - \bar{\varphi}^{(n)}\right) \leq \Phi'[\bar{\varphi}^{(n)}] \cdot \varphi - \bar{\varphi}^{(n)} \leq K \varphi - \bar{\varphi}^{(n)} ^2.$	(80)
Hence, from (79), $\Phi[\tilde{\varphi}^{(n)}] - \Phi[\varphi] \le K \tilde{\varphi}^{(n)} - \varphi ^2.$	(81)
Moreover, by (53) since $\Phi'[\varphi] = 0$,	
$\Phi[\bar{\varphi}^{(n)}] - \Phi[\varphi] \ge \alpha_R \bar{\varphi}^{(n)} - \varphi ^2.$	(82)
Using (81) and (82) we obtain	
$\left(\Phi[\tilde{\varphi}^{(n)}] - \Phi[\varphi]\right) - \left(\Phi[\tilde{\varphi}^{(n+1)}] - \Phi[\varphi]\right) \leq K \tilde{\varphi}^{(n)} - \varphi ^2 - \alpha_R \tilde{\varphi}^{(n+1)} - \varphi ^2.$	(83)
Combining (83) and (78) leads to	
$\frac{\alpha_R}{C_1} \bar{\varphi}^{(n)} - \varphi ^2 \leq \left(\Phi[\bar{\varphi}^{(n)}] - \Phi[\varphi] \right) - \left(\Phi[\bar{\varphi}^{(n+1)}] - \Phi[\varphi] \right) \leq K \bar{\varphi}^{(n)} - \varphi ^2 - \alpha_R \bar{\varphi}^{(n+1)} - \Phi[\varphi] \bar{\varphi}^{(n+1)} ^2 + C_1 \bar{\varphi}^{(n)} - \varphi ^2 + C_1 \bar{\varphi}^{(n)} - C_1 \bar{\varphi}^{(n)} - \varphi ^2 + C_1 \bar{\varphi}^{(n)} - C_$	$\varphi \parallel^2$,
or	(84)
$ \hat{\varphi}^{(n+1)} - \varphi \le B \hat{\varphi}^{(n)} - \varphi $	(85)
with $B := \sqrt{\frac{K}{\alpha_R} - \frac{1}{C_1}},$	(86)
and $B \in \mathbb{R}$ as we chose $C_1 > \alpha_B/K$. Furthermore, since the sequence $\{\overline{\varphi}^{(n)}\}$ converges monotonica the minimizer φ of $\Phi[\varphi]$ by (b) and (c), it follows that $B \in (0, 1)$. Define $C := 1 - B \in (0, 1)$, then	lly to 1 (85)
can be recast as $\ \tilde{\varphi}^{(n+1)} - \tilde{\varphi}^{(n)}\ \le C \ \tilde{\varphi}^{(n)} - \tilde{\varphi}^{(n-1)}\ $	(87)
whereupon the claim is proven.	
B Analytic Solution for Linear-Elastic Singular Bar	
As reference, herein we provide the solution of the singular bar of Section 4.3 for linear elasticity, equilibrium equation is	The
$P = \sigma(X)A(X) = \text{const.}, \sigma(X) = E\epsilon(X), \epsilon(X) := u'(X), A(X) = A_0\left(\frac{X}{T}\right)^{\frac{1}{2}},$	(88)

*A. Mota, I. Tezaur, C. Alleman. "The Schwarz Alternating Method in Solid Mechanics", CMAME 319 (2017), 19-51.

Schwarz for Multiscale FOM-FOM Coupling in Solid Mechanics*



Figure above: tension specimen simulation coupling composite TET10 elements with HEX elements in Sierra/SM.

Figures right: bolted joint simulation coupling composite TET10 elements with HEX elements in Sierra/SM.



*Mota et al. 2017; Mota et al. 2022.

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*Full-Order Model. #Reduced Order Model.

Projection-Based Model Order Reduction via the POD/LSPG* Method

Full Order Model (FOM): $\frac{\partial q}{\partial t} = f(q, t; \mu)$

*Least-Squares Petrov-Galerkin



ROM = projection-based Reduced Order Model



HROM = Hyper-reduced ROM

Schwarz Extensions to FOM-ROM and ROM-ROM Couplings

Choice of domain decomposition

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- **Overlapping** vs. **non-overlapping** domain decomposition?
 - > Non-overlapping more flexible but typically requires more Schwarz iterations
- FOM vs. ROM subdomain assignment?
 - > Do not assign ROM to subdomains where they have no hope of approximating solution

Snapshot collection and reduced basis construction

• Are subdomains **simulated independently** in each subdomains (Scenario I) or together?

Enforcement of boundary conditions (BCs) in ROM at Schwarz boundaries

- Strong vs. weak BC enforcement?
 - > Strong BC enforcement difficult for some models (e.g., cell-centered finite volume, PINNs)
- **Optimizing parameters** in Schwarz BCs for non-overlapping Schwarz?

Choice of hyper-reduction

- What hyper-reduction method to use?
 - > Application may require particular method (e.g., ECSW for solid mechanics problems)
- How to **sample Schwarz boundaries** in applying hyper-reduction?
 - > Need to have enough sample mesh points at Schwarz boundaries to apply Schwarz

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*Full-Order Model. #Reduced Order Model.

²⁷ Model Problem 1:2D Inviscid Burgers Equation

Popular analog for fluid problems where **shocks** are possible, and particularly **difficult** for conventional projection-based ROMs

X

 Ω_1

0

3

100

$$\frac{\partial u}{\partial t} + \frac{1}{2} \left(\frac{\partial (u^2)}{\partial x} + \frac{\partial (uv)}{\partial y} \right) = 0.02 \exp(\mu_2 x)$$
$$\frac{\partial v}{\partial t} + \frac{1}{2} \left(\frac{\partial (vu)}{\partial x} + \frac{\partial (v^2)}{\partial y} \right) = 0$$
$$u(0, y, t; \boldsymbol{\mu}) = \mu_1$$
$$u(x, y, 0) = v(x, y, 0) = 1$$

Problem setup:

- $\Omega = (0, 100)^2, t \in [0, 25]$
- Two **parameters** $\mu = (\mu_1, \mu_2)$ defining source and BC terms, respectively

FOM discretization:

- Spatial discretization given by a **Godunov-type** scheme with N = 250 elements in each dimension
- Implicit temporal discretization: trapezoidal method with fixed $\Delta t = 0.05$



Single Domain Predictive ROM

- Uniform sampling of $\mathcal{D} = [4.25, 5.50] \times [0.015, 0.03]$ by a 3 × 3 grid \Rightarrow 9 training parameters characterized by $\Delta \mu_1 = 0.625$, $\Delta \mu_2 = 0.0075$
 - > 200 POD modes required to capture 99% snapshot energy
- Queried but **unsampled parameter** point $\mu = [4.75, 0.02]$
- Reduced mesh resulting from solving non-negative least squares problem defining ECSW gives $n_e = 5,689$ elements (9.1% of $N_e = 62,500$ elements).



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Figure above: Reduced mesh of single domain HROM



results at various time steps

	100							
olem defining ECSW								
% SV Energy	М	MSE* (%)	CPU time* (s)					
95	69	1.1	138					
00	177	0 17	117					

* Numbers in table are w/o hyper-reduction





Schwarz Coupling Details

Choice of domain decomposition

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- Overlapping DD of Ω into 4 subdomains coupled via multiplicative Schwarz
- Solution in Ω_1 is **most difficult** to capture by ROM

Snapshot collection and reduced basis construction

• Single-domain FOM on Ω used to generate snapshots/POD modes

Enforcement of boundary conditions (BCs) in ROM at Schwarz boundaries

• BCs imposed strongly using Method 1 of [Gunzburger et al., 2007] at indices i_{Dir}

 $\boldsymbol{q}(t) \approx \overline{\boldsymbol{q}} + \boldsymbol{\Phi} \widehat{\boldsymbol{q}}(t)$

> POD modes made to satisfy homogeneous DBCs: $\Phi(i_{\text{Dir}},:) = 0$

> BCs imposed by modifying \overline{q} : $\overline{q}(i_{\text{Dir}}) \leftarrow \chi_q$

Choice of hyper-reduction

- Energy Conserving Sampling & Weighting (ECSW) method for hyper-reduction
- All points on Schwarz boundaries are included in the sample mesh



All-ROM Coupling

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- Method converges in only 3
 Schwarz iterations per controller time-step
- Errors O(1%) or less
- 1.47× speedup over all-FOM coupling for 95% SV energy retention case

	95% SV Energy			99% SV Energy		
Subdomains	М	MSE (%)	CPU time (s)	М	MSE (%)	CPU time (s)
Ω_1	57	1.1	85	146	0.18	295
Ω_2	44	1.2	56	120	0.18	216
Ω_3	24	1.4	43	60	0.16	89
Ω_4	32	1.9	61	66	0.25	100
Total			245			700



FOM-HROM-HROM-HROM Coupling





 Ω_4

100

x cell index

 Ω_3

 Ω_1

 Ω_2

 Ω_{2}

 Ω_4

1 SD

- FOM in Ω_1 as this is "hardest" subdomain for ROM
- HROMs in Ω_2 , Ω_3 , Ω_4 capture **99% snapshot energy**
- Method converges in **3 Schwarz iterations** per controller time-step
- Errors O(0.1%) with 0 error in Ω_1
- 2.26× speedup achieved over all-FOM coupling

Further **speedups** possible via **code optimizations**, **additive Schwarz** and **reduction** of # **sample mesh points**.

³² Model Problem 2: 2D Shallow Water Equations (SWE)

Hyperbolic PDEs modeling **wave propagation** below a pressure surface in a fluid (e.g., atmosphere, ocean).

$$\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} + \frac{\partial (hv)}{\partial y} = 0$$
$$\frac{\partial (hu)}{\partial t} + \frac{\partial}{\partial x} \left(hu^2 + \frac{1}{2}gh^2 \right) + \frac{\partial}{\partial y} (huv) = -\mu v$$
$$\frac{\partial (hv)}{\partial t} + \frac{\partial}{\partial x} (huv) + \frac{\partial}{\partial y} \left(hv^2 + \frac{1}{2}gh^2 \right) = \mu u$$

Problem setup:

- $\Omega = (-5,5)^2$, $t \in [0, 10]$, Gaussian initial condition
- Coriolis parameter $\mu \in \{-4, -3, -2, -1, 0\}$ for training, and $\mu \in \{-3.5, -2.5, -1.5, -0.5\}$ for testing

FOM discretization:

- Spatial discretization given by a first-order **cell-centered finite volume** discretization with N = 300 elements in each dimension
- Implicit first order temporal discretization: **backward Euler** with fixed $\Delta t = 0.01$
- Implemented in Pressio-demoapps (<u>https://github.com/Pressio/pressio-demoapps</u>)



Figure above: FOM solutions to SWE for $\mu = -0.5$ (left) and $\mu = -3.5$ (right).





*https://github.com/Pressio/pressio-demoapps

Green: different from Model Problem 1

- Non-overlapping DD of Ω into 4 subdomains coupled via additive Schwarz
 - OpenMP parallelism with 1 thread/subdomain
- All-ROM or All-HROM coupling via Pressio*

Schwarz Coupling Details

Choice of domain decomposition

Snapshot collection and reduced basis construction

• Single-domain FOM on Ω used to generate snapshots/POD modes

Enforcement of boundary conditions (BCs) in ROM at Schwarz boundaries

- BCs are imposed approximately by fictitious ghost cell states
 - > Implementing Neumann and Robin BCs is challenging
- Ghost cells introduce some overlap even with non-overlapping DD
 - Dirichlet-Dirichlet non-overlapping Schwarz is stable/convergent!
- Choice of hyper-reduction

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- Collocation for hyper-reduction: min residual at small subset DOFs
- Assume fixed budget of sample mesh points at Schwarz boundaries





Figure above: sample mesh (yellow) and stencil (white) cells



Ghost

cells

Schwarz All-ROM Domain Overlap Study



Study of Schwarz convergence for all-ROM coupling as a function of N_o := cell width of overlap region (not including ghost cells).



Movie above: FOM (left), 4 subdomain ROM coupled via non-overlapping Schwarz (middle), and 4 subdomain ROM coupled via overlapping Schwarz (right) for predictive SWE problem with $\mu = -0.5$. All ROMs have K = 80 POD modes.

- Schwarz iterations decrease (very roughly) with $N_o^{0.25}$ (figure, right) whereas evaluating r(q) scales with N_o^2
 - ➤ ⇒ there is no reason not to do nonoverlapping coupling for this problem

 Dirichlet-Dirichlet coupling with no-overlap (N_o= 0) performs well with no convergence issues (movie, left) and errors comparable to Dirichlet-Dirichlet coupling with overlap (figure below, left)



Figures above: relative error and average # Schwarz iterations as a function of μ and N_o . Black μ : training, red μ : testing.

Schwarz Boundary Sampling for All-HROM Coupling

<u>Key question</u>: how many Schwarz boundary points need to be included in sample mesh when performing HROM coupling?

Schwarz Boundary Sampling for All-HROM Coupling

<u>Key question</u>: how many Schwarz boundary points need to be included in sample mesh when performing HROM coupling?

• Naïve/sparsely-sampled Schwarz boundary results in failure to transmit coupling information during Schwarz





Movie above: FOM (left) and all HROM with $N_b = 5\%$ (right). ROMs have K = 100 modes and $N_s = 0.5\%N$ sample mesh points.

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Figure above: example sample mesh with sampling rate $N_b = 5\%$.
<u>Key question</u>: how many Schwarz boundary points need to be included in sample mesh when performing HROM coupling?

• Naive/sparsely-sampled Schwarz boundary results in failure to transmit coupling information during Schwarz





Movie above: FOM (left) and all HROM with $N_b = 5\%$ (right). ROMs have K = 100 modes and $N_s = 0.5\%N$ sample mesh points.

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Figure above: example sample mesh with sampling rate $N_b = 0$.

<u>Key question</u>: how many Schwarz boundary points need to be included in sample mesh when performing HROM coupling?

• Naive/sparsely-sampled Schwarz boundary results in failure to transmit coupling information during Schwarz





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Figure above: example sample mesh with sampling rate $N_b = 5\%$.

<u>Key question</u>: how many Schwarz boundary points need to be included in sample mesh when performing HROM coupling?

• Naive/sparsely-sampled Schwarz boundary results in failure to transmit coupling information during Schwarz





Movie above: FOM (left) and all HROM with $N_b = 5\%$ (right). ROMs have K = 100 modes and $N_s = 0.5\% N$ sample mesh points.

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Figure above: example sample mesh with sampling rate $N_b = 10\%$.

<u>Key question</u>: how many Schwarz boundary points need to be included in sample mesh when performing HROM coupling?

• Naive/sparsely-sampled Schwarz boundary results in failure to transmit coupling information during Schwarz



Figure above: example sample mesh with sampling rate $N_b = 15\%$.

Movie above: FOM (left) and all HROM with $N_b = 5\%$ (right). ROMs have K = 100 modes and $N_s = 0.5\%N$ sample mesh points.

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<u>Key question</u>: how many Schwarz boundary points need to be included in sample mesh when performing HROM coupling?

• Naive/sparsely-sampled Schwarz boundary results in failure to transmit coupling information during Schwarz





Movie above: FOM (left), all HROM with $N_b = 5\%$ (middle) and all HROM with $N_b = 10\%$ (left). ROMs have K = 100 modes and $N_s = 0.5\%N$ sample mesh points.

Figure above: example sample mesh with sampling rate $N_b = 10\%$

- Including too many Schwarz boundary points (Nb) in sample mesh given fixed budget of Ns sample mesh
 points may lead to too few sample mesh points in interior
- For SWE problem, we can get away with ~10% boundary sampling (movie above, right-most frame)

Coupled HROM Performance

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- For a fixed ROM dimension, Schwarz delivers lower error and comparable cost!
- There are noticeable **cost savings** relative to **monolithic FOM**!
- Accuracy similar for **predictive** μ (red) and **non-predictive** μ (black) cases.

Extension to PINN-PINN Coupling

1.0

0.8 -

0.6

0.4

0.2

1.0

0.8

(x) n 0.6

0.2

0.0

with Will Snyder (Virginia Tech),

and Sigi Ma, Jinny Chung, Peter

Krenek (Stanford U)

Goal: investigate the use of the Schwarz alternating method as a means to couple Physics-Informed Neural Networks (PINNs)



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Learnings (using Schwarz to facilitate PINN training for 1D advection-diffusion PDE): n

- PINNs are very difficult to train even for 1D linear advection-diffusion PDE, if Pe > 100!
- Schwarz convergence is **sensitive** to how **BCs** are enforced in the PINN.
- Training can be facilitated greatly through **PINN-FOM** coupling.
- **Tuning libraries** like RayTune and HyperOpt can autotune PINN/Schwarz parameters to improve performance.
- Could not get **non-overlapping Schwarz** to work.



Figures above: untuned (left) & tuned (right) results. Gray = no convergence.

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*Full-Order Model. #Reduced Order Model.

⁴⁵ Lagrange Multiplier-Based Partitioned Coupling Formulation

Model problem: time-dependent **advection-diffusion** problem on $\Omega = \Omega_1 \cup \Omega_2$ with $\Omega_1 \cap \Omega_2 = \emptyset$

 $\begin{aligned} \dot{c}_i - \nabla \cdot F_i(c_i) &= f_i, & \text{in} \quad \Omega_i \times [0, T] \\ c_i &= g_i, & \text{on} \quad \Gamma_i \times [0, T] \\ c_i(\boldsymbol{x}, 0) &= c_{i,0}(\boldsymbol{x}), & \text{in} \quad \Omega_i \end{aligned}$ (1)

- $i \in \{1,2\}$
- c_i: unknown scalar solution field
- f_i : body force, g_i : boundary data on Γ_i
- $F_i(c_i) \coloneqq \kappa_i \nabla c_i uc_i$: total flux function
- κ_i : non-negative diffusion coefficient
- *u*: given advection velocity field

Compatibility conditions: on interface $\Gamma \times [0, T]$

- **Continuity of states:** $c_1(x, t) c_2(x, t) = 0$
- Continuity of total flux: $F_1(x,t) \cdot n_{\Gamma} = F_1(x,t) \cdot n_{\Gamma}$
- \Rightarrow Imposed weakly using Lagrange multiplier (LM) λ



Figure above: example nonoverlapping domain decomposition (DD) of $\Omega = \Omega_1 \cup \Omega_2$

Lagrange Multiplier-Based Partitioned FOM-FOM Coupling



FEM-FEM coupling for high Peclet transport problem



Coupling of nonconforming meshes



Patch test (ALEGRA-Sierra/SM coupling)



"Plug-and-play" framework:

- Ability to couple regions with different non-conformal meshes, different element types and different levels of refinement to simplify task of meshing complex geometries
- Ability to use *different solvers/time-integrators* in different regions^{1,2}
- Coupling is *non-iterative* (single pass)

Method is theoretically rigorous³:

- Coupling does not introduce *nonphysical artifacts*
- **Theoretical convergence** properties/guarantees including wellposedness of coupling force system
- **Preserves** the **exact solution** for conformal meshes

Method has been applied to several application spaces:

- Transport (unsteady advection-diffusion)
- Ocean-atmosphere coupling
- *Elasticity* (e.g., ALEGRA-Sierra/SM coupling)

¹Connors et al. 2022. ²Sockwell et al. 2023. ³Peterson et al. 2019.

A Lagrange Multiplier-Based Partitioned Scheme

Hybrid semi-discrete coupled formulation: obtained by differentiating interface conditions in time and discretizing hybrid problem using FEM in space

$$\begin{pmatrix} \boldsymbol{M}_1 & \boldsymbol{0} & \boldsymbol{G}_1^T \\ \boldsymbol{0} & \boldsymbol{M}_2 & -\boldsymbol{G}_2^T \\ \boldsymbol{G}_1 & -\boldsymbol{G}_2 & \boldsymbol{0} \end{pmatrix} \begin{pmatrix} \dot{\boldsymbol{c}}_1 \\ \dot{\boldsymbol{c}}_2 \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \boldsymbol{f}_1 - \boldsymbol{K}_1 \boldsymbol{c}_1 \\ \boldsymbol{f}_2 - \boldsymbol{K}_2 \boldsymbol{c}_2 \\ \boldsymbol{0} \end{pmatrix}$$
(2)

- *M_i*: mass matrices
- $K_i := D_i + A_i$: stiffness matrices, where D_i and A_i are matrices for diffusive and advective terms, respectively
- G_i: constraints matrices enforcing constraints in weak sense

Decoupling via Schur complement: equation (2) is equivalent to

Equations decouple if using explicit or IMEX time-integration!

$$\begin{pmatrix} \boldsymbol{M}_1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{M}_2 \end{pmatrix} \begin{pmatrix} \dot{\boldsymbol{c}}_1 \\ \dot{\boldsymbol{c}}_2 \end{pmatrix} = \begin{pmatrix} \boldsymbol{f}_1 - \boldsymbol{K}_1 \boldsymbol{c}_1 - \boldsymbol{G}_1^T \boldsymbol{\lambda} \\ \boldsymbol{f}_2 - \boldsymbol{K}_2 \boldsymbol{c}_2 + \boldsymbol{G}_2^T \boldsymbol{\lambda} \end{pmatrix}$$
(3)

where $(\boldsymbol{G}_1 \boldsymbol{M}_1^{-1} \boldsymbol{G}_1^T + \boldsymbol{G}_2 \boldsymbol{M}_2^{-1} \boldsymbol{G}_2^T) \boldsymbol{\lambda} = \boldsymbol{G}_1 \boldsymbol{M}_1^{-1} (\boldsymbol{f}_1 - \boldsymbol{K}_1 \boldsymbol{c}_1) - \boldsymbol{G}_2 \boldsymbol{M}_2^{-1} (\boldsymbol{f}_2 - \boldsymbol{K}_2 \boldsymbol{c}_2)$ (4)

Time integration schemes and *time-steps* in Ω_1 and Ω_2 can be *different*!

Implicit Value Recovery (IVR) Algorithm [Peterson *et al*. 2019]

- Pick explicit or IMEX timeintegration scheme for Ω_1 and Ω_2
- Approximate LM space as trace of FE space on Ω_1 or Ω_2^*
- Compute matrices M_i , K_i , G_i and vectors f_i
- For each timestep t^n :
 - > Solve Schur complement system (4) for the LM λ^n
 - Update the state variables c_iⁿ
 by advancing (3) in time

* Ensures that dual Schur complement of (2) is s.p.d.

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Projection-Based Model Order Reduction via the POD/Galerkin 49 Method Full Order Model (FOM): $M \frac{du}{dt} + Ku = f$ 1. Acquisition 3. Projection-Based Reduction Number of time steps $\boldsymbol{u}(t) \approx \widetilde{\boldsymbol{u}}(t) = \boldsymbol{\Phi}\widehat{\boldsymbol{u}}(t)$ Reduce the lumber of State Variables number of unknowns Solve ODE at different Perform Save solution data $\boldsymbol{\Phi}^{T}\boldsymbol{M}\boldsymbol{\Phi}\frac{d\hat{\boldsymbol{u}}}{dt} + \boldsymbol{\Phi}^{T}\boldsymbol{K}\boldsymbol{\Phi}\hat{\boldsymbol{u}} = \boldsymbol{\Phi}^{T}\boldsymbol{f}$ design points Galerkin 2. Learning projection Proper Orthogonal Decomposition (POD): Hyper-reduce $f(\Phi \hat{u}) \approx$ Α $f(\Phi \hat{u})$ nonlinear terms **X** = \mathbf{V}^{T} ΦU =

ROM = projection-based Reduced Order Model

HROM = Hyper-reduced ROM

⁵⁰ ROM-ROM Coupling: Full Subdomain Bases & Full LM Spaces

- Collect snapshots using suitable monolithic FOM solve for equation (1) and subtract DBC data on $\Gamma_1 \cup \Gamma_2$
- Partition modified snapshots into subdomain snapshot matrices X_1 and X_2 on Ω_1 and Ω_2 , respectively
- Calculate "full" subdomain POD bases $\boldsymbol{\Phi}_1$ and $\boldsymbol{\Phi}_2$ of dimensions M_1 and M_2 from SVD of \boldsymbol{X}_1 and \boldsymbol{X}_2
- Approximate the solution as a linear combination of the POD modes in each subdomain:

$$\boldsymbol{c}_1(t) \approx \tilde{\boldsymbol{c}}_1(t) \coloneqq \bar{\boldsymbol{c}}_1 + \boldsymbol{\Phi}_1 \hat{\boldsymbol{c}}_1(t), \qquad \boldsymbol{c}_2(t) \approx \tilde{\boldsymbol{c}}_2(t) \coloneqq \bar{\boldsymbol{c}}_2 + \boldsymbol{\Phi}_2 \hat{\boldsymbol{c}}_2(t)$$
(5)

• Substitute (5) into (2) and **project** (3) onto POD modes to obtain system of the form:

$$\begin{pmatrix} \widetilde{\boldsymbol{M}}_{1} & \boldsymbol{0} & \widetilde{\boldsymbol{G}}_{1}^{T} \\ \boldsymbol{0} & \widetilde{\boldsymbol{M}}_{2} & -\widetilde{\boldsymbol{G}}_{2}^{T} \\ \widetilde{\boldsymbol{G}}_{1} & -\widetilde{\boldsymbol{G}}_{2} & \boldsymbol{0} \end{pmatrix} \begin{pmatrix} \dot{\widehat{\boldsymbol{c}}}_{1} \\ \dot{\widehat{\boldsymbol{c}}}_{2} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \boldsymbol{s}_{1} \\ \boldsymbol{s}_{2} \\ \boldsymbol{0} \end{pmatrix} \qquad \text{where } \widetilde{\boldsymbol{M}}_{i} \coloneqq \boldsymbol{\Phi}_{i}^{T} \boldsymbol{M}_{i} \boldsymbol{\Phi}_{i}, \ \widetilde{\boldsymbol{G}}_{i} \coloneqq \boldsymbol{G}_{i} \boldsymbol{\Phi}_{i}, \\ \boldsymbol{s}_{i} \coloneqq \boldsymbol{\Phi}_{i}^{T} \boldsymbol{f}_{i} - \boldsymbol{\Phi}_{i}^{T} \boldsymbol{K}_{i} \boldsymbol{\Phi}_{i} \hat{\boldsymbol{c}}_{i} - \boldsymbol{\Phi}_{i}^{T} \boldsymbol{K}_{i} \overline{\boldsymbol{c}}_{i} - \boldsymbol{\Phi}_{i}^{T} \boldsymbol{M}_{i} \dot{\overline{\boldsymbol{c}}}_{i} \end{pmatrix}$$
(6)

Online ROM-ROM IVR Solution Algorithm with Full Subdomain Bases & LM Spaces: at each time step t^n

- \succ Use \hat{c}_1^n and \hat{c}_2^n to compute updated RHS s_1^n and s_2^n
- > Solve the Schur complement system for λ^n :

$$(\widetilde{\boldsymbol{G}}_{1}\widetilde{\boldsymbol{M}}_{1}^{-1}\widetilde{\boldsymbol{G}}_{1}^{T}+\widetilde{\boldsymbol{G}}_{2}\widetilde{\boldsymbol{M}}_{2}^{-1}\widetilde{\boldsymbol{G}}_{2}^{T})\boldsymbol{\lambda}^{n}=\widetilde{\boldsymbol{G}}_{1}\widetilde{\boldsymbol{M}}_{1}^{-1}\boldsymbol{s}_{1}^{n}-\widetilde{\boldsymbol{G}}_{2}\widetilde{\boldsymbol{M}}_{2}^{-1}\boldsymbol{s}_{2}^{n}$$

> Advance the following systems forward in time: $\widetilde{M}_1 \dot{\widetilde{c}}_1^n = s_1^n - \widetilde{G}_1 \lambda^n$ and $\widetilde{M}_2 \dot{\widetilde{c}}_2^n = s_2^n + \widetilde{G}_2 \lambda^n$

51 ROM-ROM Coupling: What Could Go Wrong?

A provably non-singular dual Schur complement requires:

1. Symmetric positive-definite projected mass matrices \widetilde{M}_i



Explicit synchronous partitioned scheme for coupled reduced order models based on composite reduced bases 🛪

<u>Amy de Castro a b</u> ⊠, <u>Pavel Bochev</u> b <u>A</u> ⊠, <u>Paul Kuberry</u> b <u>B</u>, <u>Irina Tezaur</u> c <u>B</u>

2. Projected constraint matrix $(\tilde{G}_1, \tilde{G}_2)^T$ must have full column rank

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- 2. Projected constraint matrix $(\tilde{G}_1, \tilde{G}_2)^T$ must have full column rank
 - ⊗ Not guaranteed for "full" LM space, taken as trace of underlying FEM discretization space

⁵³ ROM-ROM Coupling: What Could Go Wrong?

A provably non-singular dual Schur complement requires:

- 1. Symmetric positive-definite projected mass matrices \widetilde{M}_i
 - ${oxed {\Theta}}$ Not guaranteed *a priori* with full subdomain bases ${oldsymbol {\Phi}}_1$ and ${oldsymbol {\Phi}}_2$
 - ^(C) Remedied by creating separate "split" reduced bases $\Phi_{i,\Gamma}$ and $\Phi_{i,0}$, for interface and interior DOFs
 - > Columns of each basis matrix will have full column rank



Explicit synchronous partitioned scheme for coupled reduced order models based on composite reduced bases 🛪

<u>Amy de Castro a b</u> ⋈, <u>Pavel Bochev</u> Q ⋈, <u>Paul Kuberry</u> K ⋈, <u>Irina Tezaur</u> K

- 2. Projected constraint matrix $(\tilde{G}_1, \tilde{G}_2)^T$ must have full column rank
 - ⊗ Not guaranteed for "full" LM space, taken as trace of underlying FEM discretization space
 - © Remedied by reducing LM space to ensure satisfaction of discrete *inf-sup* condition for (6)
 - ▶ Reduce size of LM space to size $N_{R,\Gamma} < N_{R,1\Gamma} + N_{R,2\Gamma}$, where $N_{R,i\Gamma} = \#$ POD modes in $\Phi_{i,\Gamma}$
 - > For now, approximate $\lambda \approx \Phi_{LM} \hat{\lambda}$ where $\Phi_{LM} = \Phi_{i,\Gamma}$ for i = 1,2, so that $N_{R,\Gamma} = N_{R,i\Gamma}$

⁵⁴ ROM-ROM Coupling: Split Bases & Reduced LM Spaces

• Consider two separate expansions for interface and interior DOFs for i = 1,2:

 $\boldsymbol{c}_{i,0}(t) \approx \tilde{\boldsymbol{c}}_{i,0}(t) \coloneqq \bar{\boldsymbol{c}}_{i,0} + \boldsymbol{\Phi}_{i,0} \hat{\boldsymbol{c}}_{i,0}(t), \quad \boldsymbol{c}_{i,\Gamma}(t) \approx \tilde{\boldsymbol{c}}_{i,\Gamma}(t) \coloneqq \bar{\boldsymbol{c}}_{i,\Gamma} + \boldsymbol{\Phi}_{i,\Gamma} \hat{\boldsymbol{c}}_{i,\Gamma}(t)$

• Substituting above expansions into (2) and projecting equations onto reduced bases gives system of the form:

Reduced LM space also helps prevent overconstraining for full subdomain basis implementation.

$$\begin{pmatrix} \tilde{M}_{1,\Gamma} \tilde{M}_{1,\Gamma 0} & \mathbf{0} & \mathbf{0} & \tilde{G}_{1}^{T} \\ \tilde{M}_{1,0\Gamma} \tilde{M}_{1,0} & \tilde{M}_{2,\Gamma} \tilde{M}_{2,\Gamma 0} - \tilde{G}_{2}^{T} \\ \mathbf{0} & \mathbf{0} & \tilde{M}_{2,0\Gamma} \tilde{M}_{2,0} - \tilde{G}_{2}^{T} \\ \mathbf{0} & \mathbf{0} & \tilde{M}_{2,0\Gamma} \tilde{M}_{2,0} & \mathbf{0} \\ \tilde{G}_{1} & \mathbf{0} & -\tilde{G}_{2} & \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \dot{\hat{c}}_{1,\Gamma} \\ \dot{\hat{c}}_{1,0} \\ \dot{\hat{c}}_{2,\Gamma} \\ \dot{\hat{c}}_{1,0} \\ \hat{\lambda} \end{pmatrix} = \begin{pmatrix} s_{1,\Gamma} \\ s_{1,0} \\ s_{2,\Gamma} \\ s_{2,0} \\ \mathbf{0} \end{pmatrix}$$

Split basis + reduced LM space guarantees ROM-ROM coupling has **non-singular dual Schur complement***.

Online ROM-ROM IVR Solution Algorithm with Split Bases & Reduced LM Spaces: at each time step t^n

- > Use $\hat{c}_{i,0}^n$ and $\hat{c}_{i,\Gamma}^n$ to compute updated RHS $s_{i,0}^n$ and $s_{i,\Gamma}^n$ for i = 1,2.
- Define $\widetilde{M}_{i,jk} \coloneqq \Phi_{i,jk}^T M_{i,jk} \Phi_{i,k}$, $\widetilde{G}_i \coloneqq \Phi_{LM}^T G_i \Phi_{i,\Gamma}$, $\widetilde{P}_i \coloneqq \widetilde{M}_{i,\Gamma} \widetilde{M}_{i,\Gamma 0} M_{i,0}^{-1} \widetilde{M}_{i,\Gamma 0}$ for $\{j,k\} \in \{0,\Gamma\}$ and solve:

 $(\widetilde{\boldsymbol{G}}_{1}\widetilde{\boldsymbol{P}}_{1}^{-1}\widetilde{\boldsymbol{G}}_{1}^{T}+\widetilde{\boldsymbol{G}}_{2}\widetilde{\boldsymbol{P}}_{2}^{-1}\widetilde{\boldsymbol{G}}_{2}^{T})\widehat{\boldsymbol{\lambda}}^{n}=\widetilde{\boldsymbol{G}}_{1}\widetilde{\boldsymbol{P}}_{1}^{-1}(\boldsymbol{s}_{1,\Gamma}^{n}-\widetilde{\boldsymbol{M}}_{1,\Gamma 0}\boldsymbol{M}_{1,0}^{-1}\boldsymbol{s}_{1,0}^{n})-\widetilde{\boldsymbol{G}}_{2}\widetilde{\boldsymbol{P}}_{2}^{-1}(\boldsymbol{s}_{2,\Gamma}^{n}-\widetilde{\boldsymbol{M}}_{2,\Gamma 0}\boldsymbol{M}_{2,0}^{-1}\boldsymbol{s}_{2,0}^{n})$

> Advance the following systems forward in time:

$$\begin{pmatrix} \widetilde{\boldsymbol{M}}_{i,\Gamma} & \widetilde{\boldsymbol{M}}_{i,\Gamma0} \\ \widetilde{\boldsymbol{M}}_{i,\Gamma0} & \widetilde{\boldsymbol{M}}_{i,\Gamma} \end{pmatrix} \begin{pmatrix} \dot{\widehat{\boldsymbol{c}}}_{i,\Gamma}^n \\ \dot{\widehat{\boldsymbol{c}}}_{i,0}^n \end{pmatrix} = \begin{pmatrix} \boldsymbol{s}_{i,\Gamma}^n + (-1)^i \widetilde{\boldsymbol{G}}_i^T \widehat{\boldsymbol{\lambda}}^n \\ \boldsymbol{s}_{i,0}^n \end{pmatrix}$$
 * See

* See [de Castro *et al.,* 2023]

55 Extension to ROM-FOM Coupling

• Assume WLOG ROM is in Ω_1 and FOM is in Ω_2 , so that Schur complement takes the form:

 $\widehat{\boldsymbol{S}} \coloneqq \widehat{\boldsymbol{G}}_1 \widetilde{\boldsymbol{M}}_1^{-1} \widehat{\boldsymbol{G}}_1^T + \widehat{\boldsymbol{G}}_2 \boldsymbol{M}_2^{-1} \widehat{\boldsymbol{G}}_2^T$

- There are multiple choices for LM space that guarantee an s.p.d. Schur complement and *inf-sup* stability:
 - > May use full LM (fLM) space, defined as trace of FE space on Ω_1
 - > May use **reduced LM (rLM) space**, as in ROM-ROM coupling



• Proven using a variational (rather than discrete) approach in [de Castro et al., 2023]:



Formulations yield provably nonsingular Schur complements, independent of mesh size or reduced basis dimension.

- 56 Outline
- 1. The Alternating Schwarz Method for FOM*-ROM[#] and ROM-ROM Coupling
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 - ROM Construction and Implementation
 - Numerical Examples
- 2. A Lagrange Multiplier-based Partitioned Scheme for FOM-ROM and ROM-ROM Coupling
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 - Numerical Examples
- 3. Summary and Future Work





*Full-Order Model. #Reduced Order Model.

Model Problem: 2D High Peclet Transmission Problem

Figure left: initial

condition.

Figure right: mesh

and DD.



Initial conditions at t = 0

Problem setup:

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- $\Omega = (0,1)^2$, DD into 2 subdomains (top right)
- Homogeneous source and homogeneous Dirichlet boundary conditions
- Cone, cylinder and smooth hump initial condition
- Rotating advection field (0.5 y, x 0.5) for one full rotation
- Viscosity κ_i can vary across subdomains: $\kappa_1 \neq \kappa_2$

FOM discretization:

- Spatial discretization given by finite element method with N = 64 elements in each dimension
- IMEX Crank-Nicholson discretization in time (treating LM explicitly) with fixed $\Delta t = 6.734 \times 10^{-3}$ for $\kappa_i < 10^{-2}$ and $\Delta t = 9.156 \times 10^{-4}$ for $\kappa_i = 10^{-2}$

58 POD/Galerkin ROM Setup

- Prediction across κ_i : training parameters $\kappa_1 = \kappa_2 = 10^{-2}$ and $\kappa_1 = \kappa_2 = 10^{-8}$, testing parameters $\kappa_1 = 10^{-5}$, $\kappa_2 = 10^{-4}$
- Snapshots collected by **restricting** singledomain solution to Ω_i
- $M_1 = 23$, $M_2 = 19$ interior modes and 5 interface modes capture **99% of snapshot energy**
- Full LM (fLM) space has dimension of 63 (# nodes on Γ) .
- Reduced LM (rLM) space has dimension: $N_{R,i\Gamma} = \min\left\{\frac{1}{4}N_{R,i0}, 63\right\}$



Figure above: snapshot energies as a function of the basis sizes.



Figure above left: relative errors at final time 2π w.r.t. single-domain FOM solution. Figure above right: Schur complement condition numbers for RR, FR and FF couplings.

- All stable couplings **converge** with basis refinement
- FR-rLM formulation has much larger errors for small basis sizes (left figure)
- Using rLM space improves condition number (right figures)
- Condition number of stable couplings with rLM is O(1) independent of the reduced basis size

Comparison of Solutions at Final Time



Naïve ROM-ROM coupling with fLM Lagrange multiplier space Provably-stable ROM-ROM coupling with rLM Lagrange multiplier space FOM-FOM coupling

G

Provably-stable ROM-ROM (and FOM-ROM) formulations deliver **artifactfree** solutions unlike naïve (unstable) coupling formulations!

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Comparison of Interface States at Final Time



The label "m/n modes" corresponds to minterior and n interface modes. • All formulations **converge** to monolithic solution with basis **refinement**

 Oscillations in FR-rLM formulations with "small" basis sizes are due to accumulation of interface errors during time-integration caused by the approximate enforcement of the coupling condition

Model	CPU time (s)
Monolithic FOM	90.79
FOM-FOM	105.89
ROM-ROM, rLM, 90/60 modes	45.57
ROM-ROM, rLM, 60/40 modes	25.36
ROM-ROM, rLM, 15/10 modes	10.19

Accurate ROM-ROM couplings offer 1.99-3.58× speedup w.r.t. monolithic FOM!

Ongoing Work: Approximation of Interface Flux with Data-Driven Surrogates

• Bottleneck in GMM-based coupling approach is solving the Schur system given by $S \coloneqq G_1 M_1^{-1} G_1^T + G_2 M_2^{-1} G_2^T$, especially when coupling involves FOMs

<u>Key idea:</u> use data-driven techniques to create **efficient surrogates** that approximate the dynamics of the **interface flux**, to avoid **expensive Schur complement** solves in GMM.

We consider two different surrogates $\lambda = \mathcal{F}(y)$ for the interface flux dynamics (to replace (4)) using similar states y:

• DMD surrogate: $y_{k+1} = Ay_k$

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- **nODE surrogate:** $\frac{dy}{dt} = f(t, y, u; \theta) = \text{feed-forward NN}$
- **Training data** consists of both the flux (λ_{k-1}) and patches of the states near the interface
- DMD or nODE trained to learn the mapping from $y_{k-1} \coloneqq (\lambda_{k-1}, u_{1,k}(\delta_1), u_{2,k}(\delta_2))^T$ to y_k
- Preliminary results indicate that the new DMD approach is more accurate than lumped mass GMM approach and around 20× times cheaper than a consistent mass GMM approach (Figure 11)





Figure above: CPU times (left) and relative errors (right) for GMM method with consistent mass (IVR(C)), lumped mass (IVR(L)) and a DMD surrogate (DMD-FS)

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 Ω_1

 Γ_2

 Γ_1

 Ω_2

 Ω_2



64 Summary

- Two domain decomposition-based methods for coupling projection-based ROMs with each other and with conventional full order models have been proposed
 - An iterative coupling formulation based on the Schwarz alternating method and an overlapping or non-overlapping DD
 - A Lagrange multiplier-based single-pass (non-iterative) partitioned scheme based on non-overlapping DD
- Numerical results show promise in using the proposed methods to create heterogeneous coupled models comprised of arbitrary combinations of ROMs and/or FOMs
 - > Coupled models can be **computationally efficient** w.r.t analogous FOM-FOM couplings
 - > Coupling introduces **no numerical artifacts** into the solution

Opinion: *hybrid FOM-ROM models are the future!*

 FOM-ROM and ROM-ROM have potential to improve the predictive viability of projectionbased ROMs, by enabling the spatial localization of ROMs (via DD) and the online integration of high-fidelity information into these models (via FOM coupling)

Ongoing & Future Work

Alternating Schwarz-based Coupling

- Complete study involving Euler Riemann problem with moving shocks
- Journal article in preparation



- Extension to coupling of **non-intrusive ROMs** (dynamic mode decomposition or DMD, operator inference or OpInf, neural networks or NNs)
 - > With Ian Moore (summer intern starting May 2024, from Virginia Tech)

Lagrange Multiplier-Based Partitioned Coupling

- Extension to **nonlinear** problems with hyper-reduction
- Alternate constructions for reduced Lagrange multiplier space (e.g., from snapshots of fluxes)
- DMD or nODE **flux surrogates** to reduce computational cost of Schur complement interface problem

General

٠

- Numerical comparison of alternating Schwarz and LM-based partitioned coupling methods
- Development of smart domain decomposition approaches, to determine optimal placement of ROM and FOM in a computational domain (including on-the-fly ROM-FOM switching)
- Development of "bottom-up" subdomain ROMs that are trained separately
- Application to other problems, including multi-physics problems, and Sandia production applications



66 Team & Acknowledgments



Irina Tezaur



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Alejandro Mota



Chris Wentland



Francesco Rizzi







Pavel Bochev



Amy De Castro



Paul Kuberry



 $\int \mathcal{M}^2 dt$

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<u>Students</u>: please consider Sandia and other national labs as a potential employer for summer internships and when you graduate!



- Sandia is a multidisciplinary national lab and Federally Funded Research & Development Center (FFRDC).
- Contractor for U.S. DOE's National Nuclear Security Administration (NNSA).
- Two main sites: Albuquerque, NM and Livermore, CA



Careers at Sandia National Labs

<u>Students</u>: please consider Sandia and other national labs as a potential employer for summer internships and when you graduate!

- Sandia is a *great* place to work!
 - > Very *collaborative* environment
 - Lots of *interesting* problems that require *fundamental research* in applied math/computational science and impact *mission-critical applications*.
 - Great work/life balance.
- **Opportunities** at/with Sandia:
 - Interns (summer, year-round)
 - Post docs
 - Several prestigious post doctoral fellowships (von Neumann, Truman, Hruby, Collis)
 - > Staff

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Please see: <u>www.sandia.gov/careers</u> for info about current opportunities.



Start of Backup Slides

Model Problem 3: 2D Euler Equations Riemann Problem 71



$$\frac{\partial}{\partial t} \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho E \end{pmatrix} + \frac{\partial}{\partial x} \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ (E+p)u \end{pmatrix} + \frac{\partial}{\partial y} \begin{pmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ (E+p)v \end{pmatrix} = \mathbf{0}$$
$$p = (\gamma - 1) \left(\rho E - \frac{1}{2} \rho (u^2 + v^2) \right)$$

Problem setup:

- $\Omega = (0,1)^2, t \in [0,0.8]$, homogeneous Neumann BCs
- Fix $\rho_1 = 1.5$, $u_1 = v_1 = 0$, $p_3 = 0.029$
- Vary p_1 ; IC from compatibility conditions*
 - ▶ Training: $p_1 \in [1.0, 1.25, 1.5, 1.75, 2.0]$
 - ▶ Testing: $p_1 \in [1.125, 1.375, 1.625, 1.875]$

$p_1 = 1.5$ - 1.50 - 1.50 - 1.35 1.35 0.8 0.8 - 1.20 0.6 0.6 - 0.75 **B** > 0.4 0.4 · 0.60 0.60 0.45 0.2 -0.2 0.30 0.30 - 0.15 0.15 - 0.00 0.00 0.2 0.4 0.6 0.8 0.2 0.4 0.8 0.6

 $p_1 = 0.875$

х

Figure above: FOM solutions to Euler Riemann problem for $p_1 = 0.875$ (left) and $p_1 = 1.5$ (right).

Preliminary results (WIP)

FOM discretization:

- Spatial discretization given by a first-order cell-centered finite volume discretization with N = 300 or N =N = 100 elements in each dimension
- Implicit first order temporal discretization: **backward Euler** with fixed $\Delta t = 0.005$ ٠
- Implemented in Pressio-demoapps (<u>https://github.com/Pressio/pressio-demoapps</u>) ٠

*Schulz-Rinne, 1993.

х

Schwarz Coupling Details

Choice of domain decomposition

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- Overlapping and non-overlapping DD of Ω into 4 subdomains coupled via additive/multiplicative Schwarz
- All-ROM or All-HROM coupling via Pressio*

Snapshot collection and reduced basis construction

• Single-domain FOM on Ω used to generate snapshots/POD modes

Enforcement of boundary conditions (BCs) in ROM at Schwarz boundaries

- BCs are imposed **approximately** by fictitious ghost cell states
- Dirichlet-Dirichlet BCs for both overlapping and non-overlapping

Choice of hyper-reduction

- Collocation and gappy POD for hyper-reduction
- Assume fixed budget of sample mesh points at Schwarz boundaries



Figure above: DD of Ω into 4 subdomains



Figure above: Slow decay of POD energy for Euler problem

Pressio
Model Problem 3: All-ROM Coupling + Overlapping Schwarz

- For smaller basis sizes and larger p_1 , monolithic ROM is **unstable** whereas **Schwarz ROM** gives **accurate solution**!
- Increased overlap degrades accuracy (top right)

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- Shock transmission error significantly increases with overlap
- ~4.4 average # Schwarz iterations with additive Schwarz vs.
 ~3.6 for multiplicative Schwarz
- With **additive Schwarz**, can achieve **lower error** than monolithic ROM for **same CPU time** (bottom right)



Movie above: FOM (left), K = 50 monolithic ROM (middle), and K = 50 overlapping Schwarz ROM with $N_o = 4$ (left) for $p_1 = 1.875$.



Model Problem 3: All-HROM Coupling + Non-Overlapping Schwarz



- Hyper-reduction via collocation works better than gappy POD
- Schwarz can give **improved accuracy** relative to monolithic ROM
- Achieving **cost-savings** w.r.t. monolithic FOM is WIP

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Movie above: FOM (left), HROM (middle) and Schwarz All-HROM (right) solution. HROMs have 5% sampling rate and 200 POD modes.

Preliminary results (WIP)



Figure above: collocation and gappy POD relative errors for K=200, 1% sampling rate.



Figure above: monolithic vs. decomposed HROM errors with 5% sampling rate no overlap.

Other Ongoing Work: Optimization-Based Coupling (OBC)

<u>Key Idea:</u> introduce control as the shared Neumann BC on the interface Γ satisfying the continuity of flux, and form a loss function that, when minimized, will enforce the continuity of states.

• In each time-step, find $(u_1^n, u_2^n, g^n) \in X_1^n \times X_2^n \times L^2(\Gamma)$ that **minimizes**

 $J_{\delta}(u_1^n, u_2^n, g^n) := \frac{1}{2} ||u_1^n - u_2^n||_{\Gamma}^2 + \frac{1}{2} \delta ||g^n||_{\Gamma}^2$

subject to

$$\frac{1}{\Delta t} \left(u_i^n - u_i^{n-1}, v \right) + \left(\sigma_i(u_i^n), \nabla v \right) = (f_i^n, v) + (-1)^i (g^n, v)_{\Gamma}, \ \forall v \in V_i, \ i = 1, 2$$

- We relax the constrained optimization problem with a Lagrange multiplier μ_i
- Past related work extended [Gunzburger, 1999; Gunzburger, 2000; Kuberry, 2013] to ROM-ROM and ROM-FOM coupling
- Accurate results for ROM-ROM coupling when using FEM adjoints
- Linear patch tests pass to the tolerance of the penalty parameter δ
- Ongoing work investigating alternative snapshot matrices onto which the adjoint equations are projected to enable using fewer modes



The **control** g^n is common to both subdomains, **implicitly** enforcing **continuity of flux**

Figure above: ROM-ROM coupling at final timestep.



Comparison of Methods

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Alternating Schwarz-based Coupling Method

- Can do FOM-FOM, FOM-ROM, ROM-ROM coupling
- Overlapping or non-overlapping DD
- **Iterative** formulation (less intrusive but likely requires more CPU time)
- Can couple different mesh resolutions and element types
- Can use **different time-integrators** with **different time-steps** in different subdomains
- No interface bases required
- Sequential subdomain solves in multiplicative Schwarz variant
 - Parallel subdomain solves possible with additive Schwarz variant
- Extensible in straightforward way to PINN/DMD data-driven model

Lagrange Multiplier-Based Partitioned Coupling Method

- Can do FOM-FOM, FOM-ROM, ROM-ROM coupling
- Non-overlapping DD
- **Monolithic** formulation requiring hybrid formulation (more intrusive but more efficient)
- Can couple different mesh resolutions and element types
- Can use **different explicit time-integrators** with **different time-steps** in different subdomains
- Provably convergent variant requires interface bases
- **Parallel subdomain solves** if explicit or IMEX time-integrator is employed

• Extensions to PINN/DMD data-driven models are not obvious

77 Numerical Examples: ID Dynamic Wave Propagation Problem

- **Basis sizes** M_1 and M_2 vary from 60 to 300
 - > Larger ROM used in Ω_1 , since solution has steeper gradient here
- For couplings involving FOM and ROM/HROM, FOM is placed in Ω_1 , since solution has steeper gradient here
- Non-negative least-squares optimization problem for ECSW weights solved using MATLAB's Isqnonneg function with early termination criterion (solution step-size tolerance = 10^{-4})
 - > Ensures all HROMs have **consistent termination criterion** w.r.t. MATLAB implementation
 - > However, relative error tolerance of selected reduced elements will differ
 - Switching to termination criterion based on relative error is work in progress and expected to improve HROM results
 - > Convergence tolerance determines size of sample mesh $N_{e,i}$
 - > Boundary points must be in sample mesh for application of Schwarz BC

									<i>Figure left</i> : sample sample mesh for
0	50	100	150	200	250	300	350	400	1D wave propagation problem
				nz = 130					

J. Barnett, I. Tezaur, A. Mota. "The Schwarz alternating method for the seamless coupling of nonlinear reduced order models and full order models", in <u>Computer Science Research Institute</u> <u>Summer Proceedings 2022</u>, S.K. Seritan and J.D. Smith, eds., Technical Report SAND2022-10280R, Sandia National Laboratories, 2022, pp. 31-55. (<u>https://arxiv.org/abs/2210.12551</u>)

Theoretical Foundation

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Using the Schwarz alternating as a *discretization method* for PDEs is natural idea with a sound *theoretical foundation*.

- <u>S.L. Sobolev (1936)</u>: posed Schwarz method for *linear elasticity* in variational form and *proved method's convergence* by proposing a convergent sequence of energy functionals.
- <u>S.G. Mikhlin (1951)</u>: proved convergence of Schwarz method for general linear elliptic PDEs.
- <u>P.-L. Lions (1988)</u>: studied convergence of Schwarz for *nonlinear monotone elliptic problems* using max principle.
- <u>A. Mota, I. Tezaur, C. Alleman (2017)</u>: proved *convergence* of the alternating Schwarz method for *finite deformation quasi-static nonlinear PDEs* (with energy functional Φ[φ]) with a *geometric convergence rate.*

$$\boldsymbol{\Phi}[\boldsymbol{\varphi}] = \int_{B} A(\boldsymbol{F}, \boldsymbol{Z}) \, dV - \int_{B} \boldsymbol{B} \cdot \boldsymbol{\varphi} \, dV$$
$$\nabla \cdot \boldsymbol{P} + \boldsymbol{B} = \boldsymbol{0}$$



S.L. Sobolev (1908 - 1989)



S.G. Mikhlin (1908 – 1990)



P.- L. Lions (1956-)



A. Mota, I. Tezaur, C. Alleman

Convergence Proof*

A. Mona, I. Tegane, C. Alleman Schwarz Alternating Method in Solid Mechanics	A. Mota, I. Tezaur, C. Alleman Schwarz Alternating Method in Solid Mechanics	A. Mota, I. Tezaur, C. Alleman Schwarz Alternating Method in Solid Mechanics	A. Mota, I. Tezaur, C. Alleman Schwarz Alternating Method in Solid Mechanics
2 Formulation of the Schwarz Alternating Method We ture by defining the stundard failed deformation variational formulation to establish notation before pre-writer the formulation after committee method.		$\begin{array}{l} \ x_{k}^{(1)} - X_{k}^{(1)} \otimes \Omega_{1}, x_{k}^{(1)} - \chi(X_{k}^{(1)} \otimes \partial_{0}\Omega_{1}), & \text{somation for } \Omega_{1} \\ \ x_{k}^{(2)} - X_{k}^{(1)} \otimes \partial_{0}, & \text{somation for } \Omega_{1} \\ \ x_{k}^{(2)} - X_{k}^{(2)} \otimes \partial_{0}, & \text{somation for } \Omega_{1} \\ \ x_{k}^{(2)} - X_{k}^{(1)} + \chi(X_{k}^{(1)} + X_{k}^{(2)} H_{k}) - \chi(X_{k}^{(1)} + X_{k}) \\ \ x_{k}^{(2)} - X_{k}^{(2)} - \chi(X_{k}^{(2)} + X_{k}^{(2)} H_{k}) - \chi(X_{k}^{(2)} + X_{k}) \\ \ x_{k}^{(2)} - x_{k}^{(2)} - \chi(X_{k}^{(2)} + X_{k}^{(2)} H_{k}) - \chi(X_{k}^{(2)} + X_{k}) \\ \ x_{k}^{(2)} - x_{k}^{(2)} - \chi(X_{k}^{(2)} + X_{k}) \\ \ x_{k}^{(2)} - \chi(X_{k}^{(2)} - \chi(X_{k}) \\ \ x_{k}^{(2)} - \chi(X_{k}) \\ \ x_$	Remark that [50] $\tilde{S}_{\alpha} = \phi^{(\alpha-1)} + \tilde{Y}_{\alpha}$ for $\phi^{(\alpha-1)} \in \tilde{S}_{\alpha-1} \Rightarrow \phi^{(\alpha-1)} \in \tilde{S}_{\alpha}$. (40) There is the dense for all order of $\delta(\alpha)$ and the second set $[1, 2]$ is the constant of $\delta(\alpha)$.
2.1 Variational Formulation on a Single Domain		$ \begin{array}{c} \kappa \\ \left(\Delta x_{B}^{(2)} \right)^{-\kappa} \left(- K_{AB}^{(2)} H_{H_{1}} - K_{AB}^{(2)} + K_{AB}^{(2)} H_{20} \right)^{-1} \left(- R_{A}^{(2)} \right)^{-p} \operatorname{Horizypen} \\ S = a_{B}^{(1)} \leftarrow a_{B}^{(1)} + \Delta a_{B}^{(1)} \end{array} $	alternating method of Section 2 defined by (1) and its equivalent form (3)). Then (1) $R(M) = R(M) = R(M)$
Consider a body as the open set $\Omega \subset \mathbb{R}^3$ undergoing a motion described by the mapping $x = \varphi(X) : \Omega \to \mathbb{R}^3$, $X \in \Omega$. Assume that the boundary of the body is $\partial \Omega = \partial_0 \Omega \cup \partial_1 \Omega$ with muit normal N , where $\partial_0 \Omega$ is a displacement boundary, $\partial_1 \Omega \cap \Omega$ is a metricito boundary, and $2, \Omega \cap \Omega \cap \Omega \cap \Omega$. The prescribed boundary	Figure 1: Two subdomains Trip and Trip and the corresponding boundaries (", and (", used by the Schwarz alternating method.	$\frac{n}{n} \propto \frac{\omega_{0}^{2}}{\omega_{0}^{2}} \approx \frac{\omega_{0}^{2}}{\omega_{0}^{2}} \left[\left(\left \left \left(\omega_{0}^{2} \right) \right \right) \right ^{2} + \left(\left \left \left(\left \left(\omega_{0}^{2} \right) \right \right) \right ^{2} \right)^{2} \right)^{1/2} \le \epsilon_{malow} \right] > \epsilon_{malow} > \epsilon_{m$	 (ii) ∀(µ[*]₁) ≥ ∀(µ[*]₁) ≥ ··· ≥ ∀(µ[*]₁) ≥ ··· ≥ ∀(µ[*]₁) ≥ ··· ≥ ∀(µ[*]₁), where µ is the number (i) ∀(µ[*]₁) over (ii) (b) the sequence (µ^(h)) defined in (39) converges to the minimizer φ of Φ(µ) is S.
displacements or Dirichlet boundary conditions are $\chi : a_d J \rightarrow \mathbb{R}^n$. The presented boundary tractions or Neumann boundary conditions are $T : \partial_t \Omega \rightarrow \mathbb{R}^n$. Let $F :=$ Gond $\phi =$ but deforming rankint. Let also $RB : \Omega \rightarrow \mathbb{R}^n$ be the body force, with R the mass density in the reference configuration. Furthermore, introduce the curver functional	that is $j = 1$ and $j = 2$ if n is odd, and $i = 2$ and $j = 1$ if n is even. Introduce the following definitions for each subdomain i :		(c) the Schwarz minimum rather Φ(µ ⁽ⁿ⁾) converge monotonically to the minimum value Φ(µ) in S marting from any initial guess φ ⁽⁰⁾ . (d) if Ψ(n) (c) functor continuous in a metroblowhood of as them the securics ⁽¹ (a) ⁽ⁿ⁾) converse ensure-
$\Psi(\varphi) := \int A(F, Z) dV - \int RB \cdot \varphi dV - \int -T \cdot \varphi dS.$ (1)	 Clease To := 10 (BC) 	Consider the energy functional $\Phi[\varphi]$ defined in (1). We will denote by (\cdot, \cdot) the usual L^2 inner product over Ω , that is,	rically to the minimizer φ . ⁵
Ja Ja Ja Ja	Dirichlet boundary: @ ⊠) := @ ⊠1 型).	$(\psi_1, \psi_2) := \int \psi_1 \cdot \psi_2 dV,$ (35)	Proof. See Appendix A.
in which $A(F, Z)$ is the Holmhötz free-energy density and Z is a collection of internal variables. The weak form of the problem is obtained by minimizing the energy functional $\Phi(\varphi)$ over the Sobelev space $W_2^1(\Omega)$ that is comprised of all functions that are space-integrable and variable weak space. Define	 Neumann boundary: @ ap := @ ap 1 Tb. Schwarz boundary: F ;:= @p 1 Tp. 	J_{01} for $\psi_1, \psi_2 \in W_2^1(\Omega)$, with corresponding neem $ \cdot $. The proof of the convergence of the Schwarz alternating method requires that the functional $\Psi_2^i \varphi $ satisfy the following properties over the space S defined in (1):	Finally, while most of works cited above present their analysis for the specific case of two subdomains, extension to multiple subdomains is in general straightforward. The case of multiple subdomains is considered specifically in Lions (512) Budei (41, and 11-Shra and Evans (14).
$S := \{\varphi \in W_2^1(\Omega) : \varphi = \chi \text{ on } \partial_\varphi \Omega\}$ and (2)	Note that with these definitions we guarantee that $@ \Box \downarrow @ \Box \downarrow = ; @ \Box \downarrow \Gamma_i = ;$ and $@ \Box \downarrow \Gamma_i = ;$. Now define the spaces	 Φ(φ) is corrective. Φ(φ) is Fréchet differentiable, with Φ'(φ) denoting its Fréchet derivative. 	4 Numerical Examples
$V := \{\xi \in W_2^1(\Omega) : \xi = 0 \text{ on } \partial_{\varphi}\Omega\}$ (3)	$S_i := \{ 2 W_2^2(33) : ' = \chi \text{ on } (0.33), ' = P_{33/2} _{\Gamma_1} \{ (33) \} \text{ on } \Gamma_1 ,$ (7)	 θ(φ) is strictly convex. 	In this section, we present numerical examples of the behavior of the Schwarz alternating method for two
where $\xi \in V$ is a test function. The potential energy is minimized if and only if $\Phi[\varphi] \le \Phi[\varphi + \epsilon \xi]$ for all $\xi \in V$ and $\epsilon \in \mathbb{R}$. It is straightforward to show that the minimum of $\Phi[\varphi]$ is the mapping $\varphi \in S$ that satisfies	and $V_i := \{ e \ge W_2^1(\infty) : e = 0 \text{ on } (0, \infty) \in \Gamma_i \}$, (8)	4. $\Phi(\varphi)$ is lower semi-continuous.	different implementations. First, we briefly describe the two implementations, one in MATLAN and the other in the open-source ALBANY finite element code [32]. Next, we discuss the error measures used throughout
$D \Phi(\varphi)(\xi) = \int_{\Omega} \mathbf{P} : \text{Grad} \xi dV - \int_{\Omega} R \mathbf{B} \cdot \xi dV - \int_{\partial_T \Omega} T \cdot \xi dS = 0,$ (4)	where the symbol $P_{0,1}$, $r, [:]$ denotes the projection from the subdomain 130 onto the Schwarz boundary Γ_1 . This projection operator plays a central role in the Schwarz alternating method. Its form and implementation	5. $\Psi[\varphi]$ is uniformly continuous on K_R , where $K_R := \{\varphi \in S : \Phi[\varphi] < R, R \in \mathbb{R}, R < \infty\}$, (36)	use manuscove examples - trans, we common with 10th examples that definition attention that the source of the Schwarz alternating method and our implementations. The first example, a one-dimensional singular bar, is used to demonstrate the behavior of the four Schwarz variants of Section 2.4. The second example, a cuboid
where $P = \partial A/\partial F$ denotes the first Piola Kirchhoff stress. The Euler-Lagrange equation corresponding to the variational statement (4) is	are discussed in subsequent sections. For the moment it is sufficient to assume that the operator is able to project a field ' from one subdomain to the Schwarz boundary of the other subdomain. The Schwarz alternation method solves a sequence of problems on 502 and 50. The solution ' ^(A) for the	It can be shown that the energy functional $ \Phi _{\varphi}$ defined in (1) is strictly convex in S (property 3) provided	body of square base, aims to study the effect of the size of the overlap region on the convergence of the method. The objective of the third example, a notched cylinder, is to analyze the numerical error in the results and the demonstrate the ability of the method to consule different alongent theories. The lot range a laser

Theorem 1. Assume that the energy functional $\Phi[\varphi]$ satisfies properties 1–5 above. Consider the Schwarz alternating method of Section 2 defined by (9)–(13) and its equivalent form (39). Then

- (a) Φ[φ̃⁽⁰⁾] ≥ Φ[φ̃⁽¹⁾] ≥ ··· ≥ Φ[φ̃⁽ⁿ⁻¹⁾] ≥ Φ[φ̃⁽ⁿ⁾] ≥ ··· ≥ Φ[φ], where φ is the minimizer of Φ[φ] over S.
 (b) The sequence {φ̃⁽ⁿ⁾} defined in (39) converges to the minimizer φ of Φ[φ] in S.
- (c) The Schwarz minimum values $\Phi[\tilde{\varphi}^{(n)}]$ converge monotonically to the minimum value $\Phi[\varphi]$ in S starting from any initial guess $\tilde{\varphi}^{(0)}$.

Remark 1 By the coercivity of $\Phi[\varphi]$, it follows from the Lax-Milgram theorem that a unique minimizer to	Anticuting (\$7) and also (\$9) in (69) lands to	$\lim \phi^{(n)} - \phi^{(n+1)} ^2 = 0, (69)$	$\left(\Phi'[\hat{\varphi}^{(m)}], \varphi - \hat{\varphi}^{(n)}\right) \le \left(\Phi'[\hat{\varphi}^{(m)}], \varphi - \hat{\varphi}^{(n)}\right) + \alpha_R \varphi - \hat{\varphi}^{(m)} \le \Phi[\varphi] - \Phi[\hat{\varphi}^{(n)}]$ (79)
this functional over S exists, i.e., the minimization of $\Phi[\phi]$ is well-posed. Remark 2. By the Stamparchia theorem, the minimization of $\Phi[\phi]$ in S is equivalent to finding $\phi \in S$ such	$(\Phi^{i}[\hat{\varphi}^{(n)}] - \Phi^{i}[\hat{\varphi}^{(n-1)}], \zeta_{2}) = (\Phi^{i}[\hat{\varphi}^{(n)}], \xi) \le \Phi^{i}[\hat{\varphi}^{(n)}] - \Phi^{i}[\hat{\varphi}^{(n-1)}] \cdot \zeta_{2} , (61)$	from which we can conclude that $\bar{\phi}^{(n)} - \bar{\phi}^{(n+1)} \rightarrow 0$ as $n \rightarrow \infty$. We must now show that $\bar{\phi}^{(n)}$ converges to so the minimizer of $\theta(n)$ on S. By (53) with $a_{2n}^{(n)} = 0$ and	since $\alpha_R \ge 0$. Now, by the Cauchy-Schwarz inequality followed by the application of the Lipshitz continuity of $\Phi^*[\varphi]$ (66) we can write
that $(\Phi' \varphi , \xi - \varphi) \ge 0$ (51)	and substituting (56) into (61) we finally obtain that	$\psi_2 = \hat{\varphi}^{(n)}$, we have	$\left(\Phi^{\prime}[\hat{\phi}^{(n)}], \varphi - \hat{\phi}^{(n)}\right) \le \Phi^{\prime}[\hat{\phi}^{(n)}] \cdot \varphi - \hat{\phi}^{(n)} \le K \varphi - \hat{\phi}^{(n)} ^{2}.$ (80)
for all $\xi \in S$.	$\langle \Phi'[\dot{\varphi}^{(n)}], \xi \rangle \le C_0 \Phi'[\dot{\varphi}^{(n)}] - \Phi'[\dot{\varphi}^{(n-1)}] \cdot \xi ,$ (62)	$ \varphi - \overline{\varphi}^{(n)} ^2 \le \frac{1}{\alpha_R} \left\{ \Phi(\varphi) - \Phi[\overline{\varphi}^{(n)}] - \left(\Phi'[\overline{\varphi}^{(n)}], \varphi - \overline{\varphi}^{(n)} \right) \right\}.$ (70)	Hence, from (79),
Remark 3 Recall that the strict convexity property of $\Phi[\phi]$ can be written as	$\forall \xi \in S$.	Since φ is the minimum of $\Phi[\varphi]$, by (a) we have that $\Phi[\varphi] \leq \Phi[\hat{\varphi}^{(n)}]$. It follows that	$\Phi[\hat{\varphi}^{(n)}] - \Phi[\varphi] \le K \hat{\varphi}^{(n)} - \varphi ^2.$ (81)
$\Phi[\psi_2] - \Phi[\psi_1] - (\Phi'[\psi_1], \psi_2 - \psi_1) \ge 0,$ (52)	Remark 8 For part (d) of Theorem 1, recall the definition of geometric convergence:	$\Phi[\phi] - \Phi[\hat{\mu}^{(n)}] - (\Phi'[\hat{\mu}^{(n)}], \phi - \hat{\mu}^{(n)}) \le - (\Phi'[\hat{\mu}^{(n)}], \phi - \hat{\mu}^{(n)}) = (\Phi'[\hat{\mu}^{(n)}], \hat{\mu}^{(n)} - \phi).$ (71)	Moreover, by (53) since $\Phi' \rho = 0$, $\Phi(\bar{\alpha}^{(n)}) - \Phi(\alpha) \ge c_n \bar{\alpha}^{(n)} - \alpha ^2$. (87)
$\forall \psi_1, \psi_2 \in S$. From (36), remark that if $\Phi[\varphi]$ is strictly convex over $S \forall R \in \mathbb{R}$ such that $R < \infty$, we can find an $\alpha_R > 0$ such that $\forall \psi_1, \psi_2 \in \mathcal{K}_R$ we have	$E_{w+1} \le CE_w$, (63)	Substituting (71) into (70) we have	Using (81) and (82) we obtain
$\Phi[\psi_2] - \Phi[\psi_1] - (\Psi'[\psi_1], \psi_2 - \psi_1) \ge \alpha_R \psi_2 - \psi_1 ^2.$ (53)	$\forall n \in \{0, 1, 2,\}$ for some $C > 0$, where $E_{} = a ^{(n+1)} - a^{(n)} $ (64)	$ \varphi - \dot{\varphi}^{(\alpha)} ^2 \le \frac{1}{\alpha_R} \left(\Phi'(\dot{\varphi}^{(\alpha)}), \dot{\varphi}^{(\alpha)} - \varphi \right).$ (72)	$\left(\Phi[\tilde{\varphi}^{(n)}] - \Phi[\varphi]\right) - \left(\Phi[\tilde{\varphi}^{(n+1)}] - \Phi[\varphi]\right) \le K \tilde{\varphi}^{(n)} - \varphi ^2 - \alpha_R \tilde{\varphi}^{(n+1)} - \varphi ^2.$ (83)
Remark 4. By property 5, the uniform continuity of $\Phi'[\varphi]$, there exists a modulus of continuity $\omega > 0$, with $\omega = 1 - \omega = 1$ for each that	$\omega_{ij} := \varphi^{ij} - \varphi^{ij} _{1}^{1}$ Remark 9 Recall from the definition of continuity the H($\theta^{ij} _{12}$) is 1 indice continuous at $\lambda^{(j)}$ near to the	Now by (62) (Remark 7),	Combining (83) and (78) leads to
$\ \Phi'(\psi_1) - \Phi'(\psi_2)\ \le \omega(\psi_1 - \psi_2),$ (54)	there exists a constant $K \ge 0$ such that	$\left(\Psi'[\hat{\varphi}^{(n)}], \hat{\varphi}^{(n)} - \varphi\right) \le C_0 \Psi'[\hat{\varphi}^{(n)}] - \Psi'[\hat{\varphi}^{(n-1)}] \cdot \hat{\varphi}^{(n)} - \varphi .$ (73)	$\frac{\alpha_B}{C_1} \hat{\varphi}^{(n)} - \varphi ^2 \le \left(\Phi \hat{\varphi}^{(n)} - \Phi \varphi \right) - \left(\Phi \hat{\varphi}^{(n+1)} - \Phi \varphi \right) \le K \hat{\varphi}^{(n)} - \varphi ^2 - \alpha_B \hat{\varphi}^{(n+1)} - \varphi ^2,$
$\forall \psi_1, \psi_2 \in K_R$. By definition, $\omega(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$.	$ \Phi^{i}(\hat{\varphi}^{(\eta)}) - \Phi^{i}(\varphi) _{x \in \mathbb{R}^{n}}$	Substituting (73) into (72) leads to	(84)
Remark 5 It was shown in [35] that in the case $\Omega_1 \cap \Omega_2 \neq \emptyset$, $\forall \varphi \in S$, there exist $\zeta_1 \in S_1$ and $\zeta_2 \in S_2$	$ \tilde{\varphi}^{(u)} - \varphi \le K.$ (65)	$ \hat{\varphi}^{(n)} - \varphi \le \frac{C_0}{2} \Psi' \hat{\varphi}^{(n)} - \Psi' \hat{\varphi}^{(n-1)} .$ (74)	$ \tilde{\varphi}^{(n+1)} - \varphi \le B \tilde{\varphi}^{(n)} - \varphi $ (85)
such that $\varphi = \zeta_1 + \zeta_2.$ (55)	Considering that $\Phi'[\varphi] = 0$ since φ is the minimizer of $\Phi[\varphi]$, (65) is equivalent to	Applying the uniform continuity assumption (54), we obtain	with $B := \sqrt{\frac{K}{L} - \frac{1}{L}}$. (86)
and $\max (\zeta_1 , \zeta_2) \le C_0 \varphi $, (56)	$ \Psi \varphi^{(n)} \le K \varphi^{(n)} - \varphi .$ (66)	$ \hat{\varphi}^{(n)} - \varphi \le \frac{C_0}{\pi} \omega \left(\hat{\varphi}^{(n)} - \hat{\varphi}^{(n-1)} \right).$ (75)	$V a_R = C_1$ and $R \in \mathbb{R}$ as we show $C \gg - K$. Forthermore, time the community $(G(0))$ community manufacture C
for some $C_0 > 0$ independent of φ .	Proof of Theorem 1	The state link (a) a discribing the second state with the state of the	the minimizer φ of $\Phi[\varphi]$ by (b) and (c), it follows that $B \in \{0, 1\}$. Define $C := 1 - B \in \{0, 1\}$, then (85)
Remark 6 Note that (39) can be written as	Proof of (a). Let $\tilde{\varphi}^{(1)} = \arg \min_{\varphi \in S_1} \Phi[\varphi]$. By (40), $\tilde{\varphi}^{(1)} \in \tilde{S}_2$. Let $\tilde{\varphi}^*$ be the minimizer of $\Phi[\varphi]$ over \tilde{S}_2 and suppose $\Phi[\tilde{\varphi}^*] > \Phi[\tilde{\varphi}^{(1)}]$. But this is a contradiction, since we can take $\tilde{\omega}^* = \tilde{\omega}^{(1)}$. Hence, it cannot be	By (eq. $ \varphi^{n+1} = \varphi^{n-1} = 0$ as $n \to \infty$. From this we common the result, namely that $\varphi^{n+1} \Rightarrow \varphi$ is $n \to \infty$.	can be recast as $ \tilde{\phi}^{(n+1)} - \tilde{\phi}^{(n)} \le C \tilde{\phi}^{(n)} - \tilde{\phi}^{(n-1)} $ (87)
$(\Phi^{i}[\tilde{\varphi}^{(n)}], \xi^{(i)}) = 0, \text{ for } \tilde{\varphi}^{(n)} \in \tilde{S}_{n}, \forall \xi^{(i)} \in S_{i},$ (57)	that $\Phi[\phi^{(1)}] < \Phi[\phi^{(2)}]$ where $\phi^{(2)} = \arg \min_{\varphi \in S_2} \Phi[\varphi]$. It follows by induction that	Proof of (c). This follows immediately from (a) and (b).	whereupon the claim is proven.
for $i \in \{1, 2\}$ and $n \in \{0, 1, 2,\}$ (recall from (6) the relation between i and n). This is due to the uniqueness of the solution to each minimization problem over S_n and the definition of $\phi^{(n)}$ as the minimizer of $\Phi(\phi)$ over	$\Phi[\hat{\varphi}^{(v)}] \le \Phi[\hat{\varphi}^{(v-1)}]$ (67)	Proof of (d). By (b), for large enough n, there exists some $C_1 > 0$ independent of n such that $ \tilde{\varphi}^{(n)} - \varphi ^2 \le C_1 \tilde{\varphi}^{(n+1)} - \tilde{\varphi}^{(n)} ^2$. (76)	B Analytic Solution for Linear-Elastic Singular Bar
S_n . Remark 7 Let $\phi^{(n)} \in \tilde{S}_n$, and let $\xi \in S$. By Remark 5, there exist $\zeta_1 \in S_1$ and $\zeta_2 \in S_2$ such that	for $n \in \{1, 2, 3,\}$. Now let φ be the minimizer of $\Phi[\varphi]$ over S . Since the problem is well-posed φ is unique. Hence $\Phi[\varphi] \le \Phi[\tilde{\varphi}^{(n)}]$ for all $n \in \{1, 2, 3,\}$.	Let us choose C_1 such that $C_1 > \alpha_R/K$, where K is the Lipshitz continuity constant in (66). Combining (68) with (76) leads to	As reference, herein we provide the solution of the singular bar of Section 4.3 for linear elasticity. The equilibrium equation is
$(\Phi'[\hat{\varphi}^{(n)}], \xi) = (\Phi'[\hat{\varphi}^{(n)}], \zeta_1 + \zeta_2).$ (58)		$\frac{1}{\alpha_R} \left(\Phi(\varphi^{(\alpha)}) - \Phi(\varphi^{(\alpha+1)}) \right) \ge \varphi^{(\alpha+1)} - \varphi^{(\alpha)} ^2 \ge \frac{1}{C_1} \varphi^{(\alpha)} - \varphi ^2. (77)$	$P = \sigma(X)A(X) = \text{const.}, \sigma(X) = Ee(X), c(X) := a'(X), A(X) = A_0\left(\frac{X}{L}\right)^{\frac{1}{2}},$ (88)
	35	16	

*A. Mota, I. Tezaur, C. Alleman. "The Schwarz Alternating Method in Solid Mechanics", CMAME 319 (2017), 19-51.

Schwarz Alternating Method for Dynamic Multiscale Coupling: Theory



- Like for quasistatics, dynamic alternating Schwarz method converges provided each single-domain problem is *well-posed* and *overlap region* is *non-empty*, under some *conditions* on Δt .
- Well-posedness for the dynamic problem requires that action functional $S[\boldsymbol{\varphi}]\coloneqq$

 $\int_{I} \int_{\Omega} L(\boldsymbol{\varphi}, \dot{\boldsymbol{\varphi}}) dV dt \text{ be strictly convex or strictly concave, where } L(\boldsymbol{\varphi}, \dot{\boldsymbol{\varphi}}) \coloneqq T(\dot{\boldsymbol{\varphi}}) + V(\boldsymbol{\varphi}) \text{ is the Lagrangian.}$

> This is studied by looking at its second variation $\delta^2 S[\boldsymbol{\varphi}_h]$

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• We can show assuming a *Newmark* time-integration scheme that for the *fully-discrete* problem:

$$\delta^2 S[\boldsymbol{\varphi}_h] = \boldsymbol{x}^T \left[\frac{\gamma^2}{(\beta \Delta t)^2} \boldsymbol{M} - \boldsymbol{K} \right] \boldsymbol{x}$$

- $\succ \delta^2 S[\boldsymbol{\varphi}_h]$ can always be made positive by choosing a *sufficiently small* Δt
- > Numerical experiments reveal that Δt requirements for **stability/accuracy** typically lead to automatic satisfaction of this bound.

⁸¹ Numerical Examples: Linear Elastic Wave Propagation Problem

- Linear elastic *clamped beam* with Gaussian initial condition.
- Simple problem with analytical exact solution but very stringent test for discretization/coupling methods.
- *Couplings tested:* FOM-FOM, FOM-ROM, ROM-ROM, implicit-explicit, implicit-implicit, explicit-explicit.
- ROMs are *reproductive* and based on the *POD/Galerkin* method.
 - 50 POD modes capture ~100% snapshot energy





Linear Elastic Wave Propagation Problem: FOM-ROM and ROM-82 ROM Couplings

Coupling delivers accurate solution if each subdomain model is reasonably accurate, can couple different discretizations with different Δx , Δt and basis sizes.







¹Implicit 40 mode POD ROM, Δt =1e-6, Δx =1.25e-3 ²Implicit FOM, Δt =1e-6, Δx =8.33e-4 ³Explicit 50 mode POD ROM, Δt =1e-7, Δx =1e-3



3 overlapping subdomain ROM¹-FOM²-ROM³





2 non-overlapping subdomain FOM⁴-ROM⁵ ($\theta = 1$)



⁵Implicit FOM, Δt =2.25e-7, Δx =1e-6 ⁴Explicit 50 mode POD ROM, Δt =2.25e-7, Δx =1e-6

Linear Elastic Wave Propagation Problem: FOM-ROM and ROM-ROM Couplings

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Coupled models are reasonably accurate w.r.t. FOM-FOM coupled analogs and convergence with respect to basis refinement for FOM-ROM and ROM-ROM coupling is observed.

	disp MSE ⁶	velo MSE	acce MSE
Overlapping ROM ¹ -FOM ² -ROM ³	1.05e-4	1.40e-3	2.32e-2
Non-overlapping FOM ⁴ -ROM ⁵	2.78e-5	2.20e-4	3.30e-3

¹Implicit 40 mode POD ROM, Δt =1e-6, Δx =1.25e-3 ²Implicit FOM, Δt =1e-6, Δx =8.33e-4 ³Explicit 50 mode POD ROM, Δt =1e-7, Δx =1e-3 ⁴Implicit FOM, Δt =2.25e-7, Δx =1e-6 ⁵Explicit 50 mode POD ROM, Δt =2.25e-7, Δx =1e-6

⁶MSE= mean squared error =
$$\sqrt{\sum_{n=1}^{N_t} \left\| \widetilde{\mathbf{u}}^n(\boldsymbol{\mu}) - \mathbf{u}^n(\boldsymbol{\mu}) \right\|_2^2} / \sqrt{\sum_{n=1}^{N_t} \left\| \mathbf{u}^n(\boldsymbol{\mu}) \right\|_2^2}.$$

⁸⁴ Linear Elastic Wave Propagation Problem: ROM-ROM Couplings

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ROM-ROM coupling gives errors < O(1e-6) & speedups over FOM-FOM coupling for basis sizes > 40.



- Smaller ROMs are not the fastest: less accurate & require more Schwarz iterations to converge.
- All couplings converge in ≤ 4 Schwarz iterations on average (FOM-FOM coupling requires average of 2.4 Schwarz iterations).

Overlapping implicit-implicit coupling with $\Omega_1 = [0, 0.75], \Omega_2 = [0.25, 1]$

⁸⁵ Linear Elastic Wave Propagation Problem: FOM-ROM Couplings

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FOM-ROM coupling shows convergence with basis refinement. FOM-ROM couplings are 10-15% slower than comparable FOM-FOM coupling due to increased # Schwarz iterations.



Linear Elastic Wave Propagation Problem: FOM-ROM and ROM-ROM Couplings

Inaccurate model + accurate model ≠ accurate model.



Accuracy can be improved by "gluing" several smaller, spatially-local models



30 mode POD - 15 mode POD



- ⁸⁷ Energy-Conserving Sampling and Weighting (ECSW)
 - **Project-then-approximate** paradigm (as opposed to approximate-then-project)

$$r_{k}(q_{k}, t) = W^{T}r(\tilde{u}, t)$$
$$= \sum_{e \in \mathcal{E}} W^{T}L_{e}^{T}r_{e}(L_{e}+\tilde{u}, t)$$

- $L_e \in \{0,1\}^{d_e \times N}$ where d_e is the **number of degrees of freedom** associated with each mesh element (this is in the context of meshes used in first-order hyperbolic problems where there are N_e mesh elements)
- $L_{e^+} \in \{0,1\}^{d_e \times N}$ selects degrees of freedom necessary for flux reconstruction
- Equality can be **relaxed**



Augmented reduced mesh: \odot represents a selected node attached to a selected element; and \otimes represents an added node to enable the full representation of the computational stencil at the selected node/element

⁸⁸ ECSW: Generating the Reduced Mesh and Weights

- Using a subset of the same snapshots $u_i, i \in 1, ..., n_h$ used to generate the state basis V, we can train the reduced mesh
- Snapshots are first **projected** onto their associated basis and then **reconstructed**

$$\begin{split} c_{se} &= W^T L_e^T r_e \left(L_{e^+} \left(u_{ref} + V V^T \left(u_s - u_{ref} \right) \right), t \right) \in \mathbb{R}^n \\ d_s &= r_k (\tilde{u}, t) \in \mathbb{R}^n, \quad s = 1, \dots, n_h \end{split}$$

• We can then form the **system**

$$\boldsymbol{C} = \begin{pmatrix} c_{11} & \dots & c_{1N_e} \\ \vdots & \ddots & \vdots \\ c_{n_h 1} & \dots & c_{n_h N_e} \end{pmatrix}, \quad \boldsymbol{d} = \begin{pmatrix} d_1 \\ \vdots \\ d_{n_h} \end{pmatrix}$$

- Where $C\xi = d, \xi \in \mathbb{R}^{N_e}, \xi = 1$ must be the solution
- Further relax the equality to yield **non-negative least-squares problem**:

 $\boldsymbol{\xi} = \arg \min_{\boldsymbol{x} \in \mathbb{R}^n} ||\boldsymbol{C}\boldsymbol{x} - \boldsymbol{d}||_2$ subject to $\boldsymbol{x} \ge \boldsymbol{0}$

 Solve the above optimization problem using a non-negative least squares solver with an early termination condition to promote sparsity of the vector ξ

- ⁸⁹ Numerical Examples: ID Dynamic Wave Propagation Problem
- Alternating **Dirichlet-Neumann** Schwarz BCs with **no relaxation** ($\theta = 1$) on Schwarz boundary Γ



θ	Min # Schwarz Iters	Max # Schwarz Iters	Total # Schwarz Iters
1.10	3	9	59,258
1.00	1	4	24,630
0.99	1	5	35,384
0.95	3	6	45,302
0.90	3	8	56,114

> A parameter sweep study revealed $\theta = 0$ gave best performance (min # Schwarz iterations)

• All couplings were **implicit-implicit** with $\Delta t_1 = \Delta t_2 = \Delta T = 10^{-7}$ and $\Delta x_1 = \Delta x_2 = 10^{-3}$

> Time-step and spatial resolution chosen to be small enough to resolve the propagating wave

- All reproductive cases run on the same RHEL8 machine and all predictive cases run on the same RHEL7 machine, in MATLAB
- Model accuracy evaluated w.r.t. analogous FOM-FOM coupling using mean square error (MSE):

$$\varepsilon_{MSE}(\widetilde{\boldsymbol{u}}_i) \coloneqq \frac{\sqrt{\sum_{n=1}^{S} ||\widetilde{\boldsymbol{u}}_i^n - \boldsymbol{u}_i^n||_2^2}}{\sqrt{\sum_{n=1}^{S} ||\boldsymbol{u}_i^n||_2^2}}$$

Overlapping Coupling, Nonlinear Henky MM, 2 Subdomains

• $\Omega = [0, 0.7] \cup [0.3, 1]$, implicit-implicit FOM-FOM coupling, dt = 1e-7, dx=1e-3.



Multiplicative Schwarz

Additive Schwarz

Overlapping Coupling, Nonlinear Henky MM, 2 Subdomains





- Ω = [0, 0.7]U[0.3,1], implicit-implicit FOM-FOM coupling, dt = 1e-7, dx=1e-3.
- Additive Schwarz requires slightly more Schwarz iterations but is actually faster.
- Solutions agree effectively to machine precision in mean square (MS) sense.

	Additive	Multiplicative
Total # Schwarz iters	24495	24211
CPU time	2.03e3s	2.16e3
MS difference in disp	6.34e-13/6.12e-13	
MS difference in velo	1.35e-11/1.86e-11	
MS difference in acce	5.92e-10/1.07e-9	

Overlapping Coupling, Nonlinear Henky MM, 3 Subdomains





- Ω = [0, 0.3]∪[0.25, 0.75]∪[0.7, 1], implicit-implicit-explicit
 FOM-FOM-FOM coupling, dt = 1e-7, dx = 0.001.
- Solutions agree effectively to machine precision in mean square (MS) sense.
- Additive Schwarz has slightly more Schwarz iterations but is slightly faster than multiplicative.

	Additive	Multiplicative
Total # Schwarz iters	26231	25459
CPU time	1.89e3s	2.05e3s
MS difference in disp	5.3052e-13/9.	3724e-13/6.1911e-13
MS difference in velo	7.2166e-12/2.	2937e-11/2.4975e-11
MS difference in acce	2.8962e-10/1.	1042e-09/1.6994e-09

Non-overlapping Coupling, Nonlinear Henky MM, 2 Subdomains

• $\Omega = [0, 0.3] \cup [0.3, 1]$, implicit-implicit FOM-FOM coupling, dt = 1e-7, dx = 1e-3.



Multiplicative Schwarz

Additive Schwarz





- Ω = [0, 0.3]∪[0.3,1], implicit-implicit FOM-FOM coupling, dt = 1e-7, dx = 1e-3.
- Additive Schwarz requires 1.81x Schwarz iterations (and 1.9x CPU time) to converge. CPU time could be reduced through added parallelism of additive Schwarz.
 - > Note blue square for additive Schwarz...
- Additive and multiplicative solutions differ in mean square (MS) sense by O(1e-5).

	Additive	Multiplicativ e
Total # Schwarz iters	44895	24744
CPU time	1.87e3s	982.5s
MS difference in disp	4.26e-5	/2.74e-5
MS difference in velo	1.02e-5/5.91e-6	
MS difference in acce	5.84e-5/1.21e-5	

Non-overlapping Coupling, Nonlinear Henky MM, 3 Subdomains





- Ω = [0, 0.3]U[0.3,0.7]U[0.7,1], implicit-implicitexplicit FOM-FOM-FOM coupling, dt = 1e-7, dx = 0.001.
- Additive Schwarz has about 1.94x number Schwarz iterations and is about 2.06x slower - similar to 2 subdomain variant of this problem. No "blue square".
 - Results suggest you could win with additive Schwarz if you parallelize and use enough domains.
- Additive/multiplicative solutions differ by O(1e-5), like for 2 subdomain variant of this problem.

	Additive	Multiplicative	
Total # Schwarz iters	53413	27509	
CPU time	5.91e3s	2.87e3s	
MS difference in disp	2.8036e-05/3.	1142e-05/ 8.8395e-06	
MS difference in velo	1.4077e-05/1.2104e-05/6.5771e-06		
MS difference in acce	8.7885e-05/3.	2707e-05/1.3778e-05	