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Subject: Galerkin Reduced Order Model (ROM) for Compressible Flow: Enforcement of Boundary Conditions and Analysis of Coupled Fluid/Structure System

Abstract

This document presents a series of derivations performed by the author, Irina Kalashnikova, a Graduate Technical Intern in the Aerosciences Department at Sandia National Laboratories in Albuquerque, NM, working under the supervision of Matthew Barone. During the months of July - August 2007, the author:

- Formulated the implementation of a linearized no-penetration boundary condition (BC) on the plate boundary in terms of Reduced Order Model (ROM) coefficients.
- Formulated a weak implementation of a far-field approximately non-reflecting boundary condition for the fluid Reduced Order Model (ROM).
- Using the said boundary conditions and linearized equations, assembled the coupled linear system to be solved for the ROM coefficients.
- Proved that the said coupled system is stable under the uniform base flow assumption ($\nabla\bar{U} \equiv 0$) for $\bar{u} = 0$ using energy principles.
- Began to derive a 2nd order accurate explicit/implicit fluid/structure staggered time-integration procedure for the linear system.

The results enumerated above are summarized in this document.

1 Introduction

This document contains a number of derivations necessary for the implementation of a Reduced Order Model (ROM) of fluid flow over a solid plate representing, for example, the wing of an aircraft. Consider the flow of a compressible fluid past a thin, elastic plate lying in the $z = 0$ plane. Assume the flow is in the x-direction.

Let Ω be a finite region surrounding the plate and $\partial\Omega$ be the boundary of Ω. Note that Ω has two boundaries: the plate boundary, call it $\partial\Omega_P$ and the far-field boundary, call it $\partial\Omega_F$. Mathematically,

$$\partial\Omega = \partial\Omega_P \cup \partial\Omega_F \tag{1}$$

We begin by defining the relevant fluid variables.

Fluid Variable	Interpretation
$\mathbf{u} = \begin{pmatrix} u & v & w \end{pmatrix}^T$	fluid velocity vector
ρ	fluid density
$\zeta = 1/\rho$	specific volume
p	fluid pressure
c_p	specific heat at constant pressure
c_v	specific heat at constant volume
$\gamma = \frac{c_p}{c_v}$	specific heat ratio
$\bar{U} = \begin{pmatrix} \bar{u} & 0 & 0 & \bar{\zeta} & \bar{p} \end{pmatrix}^T$	base state fluid variables
$U' = \begin{pmatrix} u' & v' & w' & \zeta' & p' \end{pmatrix}^T$	perturbed fluid variables
U	$= \bar{U} + U'$

The governing fluid equations are the Euler equations, linearized about the base state $\bar{U}$. This linearization results in a system of the form

$$\frac{\partial U'}{\partial t} + A(\bar{U}) \cdot \nabla U' + C(\bar{U}, \nabla \bar{U}) U' = 0 \tag{2}$$

In (2), $A(\bar{U})$ is the following tensor

$$A(\bar{U}) \equiv \begin{pmatrix} A_x(\bar{U}) & A_y(\bar{U}) & A_z(\bar{U}) \end{pmatrix} \tag{3}$$

where

$$A_x(\bar{U}) = \begin{pmatrix} \bar{u} & 0 & 0 & 0 & \bar{\zeta} \\ 0 & \bar{u} & 0 & 0 & 0 \\ 0 & 0 & \bar{u} & 0 & 0 \\ -\bar{\zeta} & 0 & 0 & \bar{u} & 0 \\ \gamma\bar{p} & 0 & 0 & 0 & \bar{u} \end{pmatrix}, \quad A_y(\bar{U}) = \begin{pmatrix} \bar{v} & 0 & 0 & 0 & 0 \\ 0 & \bar{v} & 0 & 0 & \bar{\zeta} \\ 0 & 0 & \bar{v} & 0 & 0 \\ 0 & -\bar{\zeta} & 0 & \bar{v} & 0 \\ 0 & \gamma\bar{p} & 0 & 0 & \bar{v} \end{pmatrix}, \quad A_z(\bar{U}) = \begin{pmatrix} \bar{w} & 0 & 0 & 0 & 0 \\ 0 & \bar{w} & 0 & 0 & 0 \\ 0 & 0 & \bar{w} & 0 & \bar{\zeta} \\ 0 & 0 & -\bar{\zeta} & \bar{w} & 0 \\ 0 & 0 & \gamma\bar{p} & 0 & \bar{w} \end{pmatrix} \tag{4}$$

and

$$C(\bar{U}, \nabla \bar{U}) = \begin{pmatrix} \frac{\partial \bar{u}}{\partial x} & \frac{\partial \bar{u}}{\partial y} & \frac{\partial \bar{u}}{\partial z} & \frac{\partial \bar{p}}{\partial x} & 0 \\ \frac{\partial \bar{v}}{\partial x} & \frac{\partial \bar{v}}{\partial y} & \frac{\partial \bar{v}}{\partial z} & \frac{\partial \bar{p}}{\partial y} & 0 \\ \frac{\partial \bar{w}}{\partial x} & \frac{\partial \bar{w}}{\partial y} & \frac{\partial \bar{w}}{\partial z} & \frac{\partial \bar{p}}{\partial z} & 0 \\ \frac{\partial \bar{\zeta}}{\partial x} & \frac{\partial \bar{\zeta}}{\partial y} & \frac{\partial \bar{\zeta}}{\partial z} & -\left(\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z}\right) & 0 \\ \frac{\partial \bar{p}}{\partial x} & \frac{\partial \bar{p}}{\partial y} & \frac{\partial \bar{p}}{\partial z} & 0 & \gamma\left(\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z}\right) \end{pmatrix} \tag{5}$$

These matrices were derived in [1]. As expected,

$$\nabla U' \equiv \begin{pmatrix} \frac{\partial U'}{\partial x} & \frac{\partial U'}{\partial y} & \frac{\partial U'}{\partial z} \end{pmatrix} \tag{6}$$

(also a tensor), and similarly for $\nabla \bar{U}$.

We are interested in the flow field generated by small deformations of the elastic plate located in the $z = 0$ plane. The plate is square with $0 \leq x, y \leq L$. Assume that the deformations are small enough to cause only small perturbations in the fluid. Assume also that the deformations are restricted to the direction normal to the plate, leading to a z-direction displacement field, to be denoted $\eta = \eta(x,y,t)$. The following table summarizes the variables governing the motion of the solid plate.

Structure Variable	Interpretation
$\mathbf{q} = \begin{pmatrix} \beta & \gamma & \eta \end{pmatrix}^T$	displacement vector
ρ_s	density of solid plate material
ν	Poisson's ratio
$E = \frac{\text{tensile stress}}{\text{tensile strain}}$	Young's modulus
$D_{bend} = \frac{Eh^3}{12(1-\nu^2)}$	bending stiffness

A more detailed discussion of these variables, including their units, can be found in the "Structure Equations" section of this document.

The motion of the plate is governed by the von Karman equations, given in the "Structure Equations" section of this document as well as in [5]. The edges of the plate are assumed to be simply supported.

The boundary condition for the flow problem on the plate is the no-penetration condition $\mathbf{u} \cdot \mathbf{n} = \dot{\eta}$ where $\mathbf{n}$ is the surface unit normal vector. Since the plate is assumed to be in the $z = 0$ plane, in this case, $\mathbf{n} = e_z = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T$. The linearized no-penetration boundary condition on the plate is

$$w' + \bar{u}\frac{\partial \eta}{\partial x} = \dot{\eta} \quad \text{on} \quad \partial\Omega_P \tag{7}$$

The " $\dot{}$ " operator represents a time derivative, i.e., $\dot{\eta} \equiv \frac{\partial \eta}{\partial t}$. A non-reflecting boundary condition is assumed at the far-field boundary of the region surrounding the plate.

Both the fluid solution U' and the solid displacement (in the z-direction) are expanded in an orthonormal basis in the ROM:

Variable	Interpretation
$\mathbf{x} = \begin{pmatrix} x & y & z \end{pmatrix}^T$	direction vector
$\phi_j(\mathbf{x}) = \begin{pmatrix} \phi_j^1(\mathbf{x}) & \phi_j^2(\mathbf{x}) & \phi_j^3(\mathbf{x}) & \phi_j^4(\mathbf{x}) & \phi_j^5(\mathbf{x}) \end{pmatrix}^T$	fluid ROM orthonormal (vector) basis
$\xi_j(x,y)$	z-displacement ROM orthonormal (scalar) basis
M	number basis functions $\phi_j(\mathbf{x})$ kept in the fluid ROM
P	number basis functions $\xi_j(x,y)$ kept in the structure ROM

It follows that (expanding the fluid variable vector and the z-direction displacement in the appropriate orthonormal basis)

$$U' = \sum_{k=1}^{M} a_k(t)\phi_k(\mathbf{x}) \tag{8}$$

$$\eta = \sum_{j=1}^{P} b_j(t)\xi_j(x,y) \tag{9}$$

Here, $a_j(t)$ and $b_j(t)$ are the appropriate ROM coefficients, to be solved for in the Reduced Order Model.

This document is organized as follows. In §2, entitled "Fluid Equations", the no-penetration and far-field boundary conditions are derived in terms of the ROM bases $\phi_j(\mathbf{x})$ and $\xi_j(x,y)$. These expressions will be needed in the implementation of the said BCs in the ROM code. §3, entitled "Structure Equations", goes through the non-dimensionalization of the von Karman equations, governing the motion of the late. The "Coupled (Linearized) System" section (§4) contains a stability analysis of the coupled, linearized system of fluid and structure equations. Several stability results are derived using theorems proven in [3]. Current and future work is outlined in §5, in particular, the derivation of a 2nd order accurate explicit/implicit (fluid/structure) staggered time-integration scheme for the linearized coupled system assembled[1].

2 Fluid Equations

2.1 Integration by Parts

In the ROM model, equation (2) is projected onto the POD modes, $\phi_j(\mathbf{x})$, by forming the following inner product:

$$\left(\phi_j, \frac{\partial U'}{\partial t}\right)_H + \left(\phi_j, A(\bar{U})\cdot\nabla U'\right)_H + \left(\phi_j, C(\bar{U},\nabla\bar{U})U'\right)_H = 0 \tag{10}$$

In (10), $H(\bar{U})$ is the following symmetrizing operator for an arbitrary parameter $\alpha^2 \in \mathbb{R}$ to be specified in the implementation of the ROM[2]

$$H(\bar{U}) = \begin{pmatrix} \bar{\rho} & 0 & 0 & 0 & 0 \\ 0 & \bar{\rho} & 0 & 0 & 0 \\ 0 & 0 & \bar{\rho} & 0 & 0 \\ 0 & 0 & 0 & \alpha^2\gamma\bar{\rho}^2\bar{p} & \bar{\rho}\alpha^2 \\ 0 & 0 & 0 & \bar{\rho}\alpha^2 & \frac{(1+\alpha^2)}{\gamma\bar{p}} \end{pmatrix} \tag{11}$$

The H operator was derived in [1] to have the property that it is symmetric and that each of the HA_j for $j = x,y,z$ are also symmetric. The $(\cdot,\cdot)_H$ inner product is defined by

$$(u,v)_H \equiv \int_\Omega u^T H(\bar{U}) v d\Omega \tag{12}$$

Let

$$\tilde{A}(\bar{U}) \equiv \begin{pmatrix} \phi_j^T H(\bar{U})A_x(\bar{U}) & \phi_j^T H(\bar{U})A_y(\bar{U}) & \phi_j^T H(\bar{U})A_z(\bar{U}) \end{pmatrix} \in \mathbb{R}^{1\times5\times3} \tag{13}$$

As mentioned earlier, the $A_x(\bar{U}), A_y(\bar{U}), A_z(\bar{U})$ matrices are those derived in §3.4.1 of [1] and are given explicitly in (4) of this document. In implementing the no-penetration boundary condition (BC), one will need to express the second term of (10) as the sum of a boundary and a volume integral so as to apply the BC on the boundary. Integrating by parts:

[1] Note that the analysis of the proposed scheme is not complete at this time.
[2] The value of α^2 has not been chosen at this time. It may be that the H matrix will have better conditioning for certain values of α^2.

$$\boxed{\begin{array}{ll} I & \equiv \int_\Omega \phi_j^T H(\bar{U}) \left[A_x(\bar{U}) \frac{\partial U'}{\partial x} + A_y(\bar{U}) \frac{\partial U'}{\partial y} + A_z(\bar{U}) \frac{\partial U'}{\partial z} \right] d\Omega \\ & = \int_\Omega \tilde{A}(\bar{U}) \cdot \nabla U' d\Omega \\ & = \int_{\partial\Omega} \tilde{A}(\bar{U}) U' \cdot \mathbf{n} dS - \int_\Omega \left(\nabla \cdot \tilde{A}(\bar{U}) \right) U' d\Omega \\ & = \int_{\partial\Omega_P} \tilde{A}(\bar{U}) U' \cdot \mathbf{n} dS + \int_{\partial\Omega_F} \tilde{A}(\bar{U}) U' \cdot \mathbf{n} dS - \int_\Omega \left(\nabla \cdot \tilde{A}(\bar{U}) \right) U' d\Omega \\ & \equiv I_P + I_F + I_V \end{array}} \tag{14}$$

Using the fact that $\mathbf{n} = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T$ in the problem being considered here and expanding the divergence term in the last line of (14) gives

$$I = \int_{\partial\Omega_P} \phi_j^T H(\bar{U}) A_z(\bar{U}) U' dS + \int_{\partial\Omega_F} \phi_j^T H(\bar{U}) A(\bar{U}) \cdot \mathbf{n} U' dS - \int_\Omega \left(\frac{\partial \left\{ \phi_j^T H(\bar{U}) A_x(\bar{U}) \right\}}{\partial x} + \frac{\partial \left\{ \phi_j^T H(\bar{U}) A_y(\bar{U}) \right\}}{\partial y} + \frac{\partial \left\{ \phi_j^T H(\bar{U}) A_z(\bar{U}) \right\}}{\partial z} \right) U' d\Omega \tag{15}$$

One can check that all the matrix/vector multiplications (15) are defined (i.e., the dimensions make the matrix/vector multiplications possible) and each of the integrands is indeed a scalar, as it should be. The two surface integral terms in (15) will be written out explicitly in terms of the appropriate ROM coefficients and basis functions using the plate boundary and far-field boundary conditions. These integrals will be pre-computed prior to the assembly of the global linear system derived in §4 of this document.

2.2 Implementation of the No-Penetration Boundary Condition

In this section, the implementation of the weak no-penetration boundary condition on $\partial\Omega_P$ is formulated in terms of the $a_k(t)$ and $b_k(t)$ coefficients, basis functions $\phi_k(\mathbf{x})$ and $\xi_k(x,y)$ and pre-computable integrals.

To be consistent with the notation and derivations of [1], let

$$\bar{U} \equiv \begin{pmatrix} \bar{u} \\ 0 \\ 0 \\ \bar{\zeta} \\ \bar{p} \end{pmatrix}, \qquad U' \equiv \begin{pmatrix} u' \\ v' \\ w' \\ \zeta' \\ p' \end{pmatrix} \tag{16}$$

where $\zeta \equiv 1/\rho$ and $U' \equiv U - \bar{U}$. We are interested in applying the following no-penetration linearized[3] boundary condition on the elastic plate (lying in the plane $z = 0$ with flow in the x-direction)

$$w' + \bar{u} \frac{\partial \eta}{\partial x} = \dot{\eta} \quad \text{on} \quad \partial\Omega_P \tag{17}$$

to the plate boundary integral term of (15), namely

$$\boxed{I_P = \int_{\partial\Omega_P} \phi_j^T H(\bar{U}) A_z(\bar{U}) U' dS} \tag{18}$$

For easy reference, we repeat the matrices $H(\bar{U})$ and $A_z(\bar{U})$ here, which were derived in [1] (p. 9, §3.4.1) to be

[3] The general non-linear no-penetration BC is $\mathbf{u} \cdot \mathbf{n} = \dot{\eta} \equiv \frac{\partial \eta}{\partial t}$.

$$A_z(\bar{U}) = \begin{pmatrix} \bar{w} & 0 & 0 & 0 & 0 \\ 0 & \bar{w} & 0 & 0 & \bar{0} \\ 0 & 0 & \bar{w} & 0 & \zeta \\ 0 & 0 & -\bar{\zeta} & \bar{w} & 0 \\ 0 & 0 & \gamma\bar{p} & 0 & \bar{w} \end{pmatrix}, \quad H(\bar{U}) = \begin{pmatrix} \bar{\rho} & 0 & 0 & 0 & 0 \\ 0 & \bar{\rho} & 0 & 0 & 0 \\ 0 & 0 & \bar{\rho} & 0 & 0 \\ 0 & 0 & 0 & \alpha^2\gamma\bar{\rho}^2\bar{p} & \bar{\rho}\alpha^2 \\ 0 & 0 & 0 & \bar{\rho}\alpha^2 & \frac{(1+\alpha^2)}{\gamma\bar{p}} \end{pmatrix} \tag{19}$$

Multiplying these out and using the fact that $\rho\zeta = 1$ gives

$$H(\bar{U})A_z(\bar{U}) \quad = \begin{pmatrix} \bar{\rho}\bar{w} & 0 & 0 & 0 & 0 \\ 0 & \bar{\rho}\bar{w} & 0 & 0 & 0 \\ 0 & 0 & \bar{\rho}\bar{w} & 0 & 1 \\ 0 & 0 & 0 & \alpha^2\gamma\bar{\rho}^2\bar{p}\bar{w} & \alpha^2\bar{\rho}\bar{w} \\ 0 & 0 & 1 & \alpha^2\bar{\rho}\bar{w} & \frac{(1+\alpha^2)\bar{w}}{\gamma\bar{p}} \end{pmatrix} \tag{20}$$

Recall that α^2 is an arbitrary real parameter[4] and $\gamma = c_p/c_v$[5].

Expanding the fluid solution in an orthonormal spectral basis gives.

$$U' \quad = \textstyle\sum_{k=1}^{M} a_k(t)\phi_k(\mathbf{x}). \tag{21}$$

Taking $\phi_k^i, i = 1, 2, \ldots, 5$ to be the components of the vector $\phi_k \in \mathbb{R}^5$, one can write (21) as

$$\begin{pmatrix} u' \\ v' \\ w' \\ \zeta' \\ p' \end{pmatrix} \equiv \begin{pmatrix} \sum_{k=1}^{M} a_k(t)\phi_k^1(\mathbf{x}) \\ \sum_{k=1}^{M} a_k(t)\phi_k^2(\mathbf{x}) \\ \sum_{k=1}^{M} a_k(t)\phi_k^3(\mathbf{x}) \\ \sum_{k=1}^{M} a_k(t)\phi_k^4(\mathbf{x}) \\ \sum_{k=1}^{M} a_k(t)\phi_k^5(\mathbf{x}) \end{pmatrix} \tag{22}$$

Applying the no-penetration boundary condition (17) to (22) on $\partial\Omega_P$ gives

$$U' = \begin{pmatrix} \sum_{k=1}^{M} a_k(t)\phi_k^1(\mathbf{x}) \\ \sum_{k=1}^{M} a_k(t)\phi_k^2(\mathbf{x}) \\ \dot{\eta} - \bar{u}\frac{\partial\eta}{\partial x} \\ \sum_{k=1}^{M} a_k(t)\phi_k^4(\mathbf{x}) \\ \sum_{k=1}^{M} a_k(t)\phi_k^5(\mathbf{x}) \end{pmatrix} \tag{23}$$

(recall that $\dot{\eta} \equiv \frac{\partial\eta}{\partial t}$). Substituting (23) and (20) into the boundary integral (18) gives

$$\begin{aligned} \phi_j^T\,[H(\bar{U})A_z(\bar{U})]\,U' = \quad & \bar{\rho}\bar{w}\left(\phi_j^1 \textstyle\sum_{k=1}^{M} a_k(t)\phi_k^1(\mathbf{x}) + \phi_j^2 \sum_{k=1}^{M} a_k(t)\phi_k^2(\mathbf{x})\right) + \left(\phi_j^3\bar{\rho}\bar{w} + \phi_j^5\right)\left(\dot{\eta} - \bar{u}\frac{\partial\eta}{\partial x}\right) \\ & + \alpha^2\bar{\rho}\bar{w}\left(\phi_j^4\gamma\bar{\rho}\bar{p} + \phi_j^5\right)\textstyle\sum_{k=1}^{M} a_k(t)\phi_k^4(\mathbf{x}) + \left(\phi_j^3 + \phi_j^4\alpha^2\bar{\rho}\bar{w} + \phi_j^5\frac{(1+\alpha^2)\bar{w}}{\gamma\bar{p}}\right)\sum_{k=1}^{M} a_k(t)\phi_k^5(\mathbf{x}) \end{aligned} \tag{24}$$

where again the boundary integral of interest is

$$I_P = \int_{\partial\Omega_P} \phi_j^T\,[H(\bar{U})A_z(\bar{U})]\,U' dS \tag{25}$$

[4] See §3.4.1 of [1]).

[5] $\gamma = 1.4$ for a diatonic gas at reasonable temperatures and pressures.

To complete the implementation of the no-penetration BC, we expand the plate deflection field η in the appropriate eigenmode, orthonormal (scalar) basis

$$\eta = \sum_{k=1}^{P} b_k(t)\xi_k(x,y) \tag{26}$$

In terms of this basis, the no-penetration BC on $\partial\Omega_P$ is

$$\dot{\eta} - \bar{u}\frac{\partial \eta}{\partial x} = \sum_{k=1}^{P} \dot{b}_k(t)\xi_k(x,y) - \bar{u}\sum_{k=1}^{P} b_k(t)\frac{\partial \xi_k(x,y)}{\partial x} \tag{27}$$

Plugging expression (27) into the second term in (24) yields

$$\left(\phi_j^3\bar{\rho}\bar{w}+\phi_j^5\right)\sum_{k=1}^{P}\dot{b}_k(t)\xi_k(x,y) - \bar{u}\left(\phi_j^3\bar{\rho}\bar{w}+\phi_j^5\right)\sum_{k=1}^{P} b_k(t)\frac{\partial \xi_k(x,y)}{\partial x} \tag{28}$$

For convenience, let us rewrite (24) grouping the coefficients of the summations over the $a_k(t)$, the $b_k(t)$ and the $\dot{b}_k(t)$ respectively:

$$\begin{aligned}\phi_j^T\left[H(\bar{U})A_z(\bar{U})\right]U' = & \sum_{k=1}^{M}\left[\bar{\rho}\bar{w}\left[\left(\phi_j^1\phi_k^1+\phi_j^2\phi_k^2\right)+\alpha^2\left(\phi_j^4\gamma\bar{\rho}\bar{p}+\phi_j^5\right)\phi_k^4\right]+\left(\phi_j^3+\phi_j^4\alpha^2\bar{w}+\phi_j^5\frac{(1+\alpha^2)\bar{w}}{\gamma\bar{p}}\right)\phi_k^5\right]a_k(t) \\ & +\sum_{k=1}^{P}\left[\left(\phi_j^3\bar{\rho}\bar{w}+\phi_j^5\right)\xi_k\right]\dot{b}_k(t) - \sum_{k=1}^{P}\left[\left(\phi_j^3\bar{\rho}\bar{w}+\phi_j^5\right)\bar{u}\frac{\partial \xi_k}{\partial x}\right]b_k(t)\end{aligned} \tag{29}$$

Recall that for this instance of the problem, $\bar{w}=0$. Thus (29) simplifies to

$$\phi_j^T\left[H(\bar{U})A_z(\bar{U})\right]U' = \sum_{k=1}^{M}\phi_j^3\phi_k^5 a_k(t) + \sum_{k=1}^{P}\phi_j^5\xi_k\dot{b}_k(t) - \sum_{k=1}^{P}\phi_j^5\bar{u}\frac{\partial \xi_k}{\partial x}b_k(t) \tag{30}$$

It might seem as though the orthonormality of each of the bases ϕ_k and ξ_k should be invoked at this point; however, remark that the expressions in (29) are in terms of the *components* of the basis vectors $\phi_k^i, i=1,2,\dots,5$. A trivial algebraic simplification of the first summation in (29) using orthonormality is therefore not possible. It follows that the boundary integral enforcing the linearized no-penetration BC is

$$\boxed{I_P = \int_{\partial\Omega_P}\phi_j^T\left[H(\bar{U})A_z(\bar{U})\right]U'dS = \sum_{k=1}^{M}a_k(t)\int_{\partial\Omega_P}\phi_j^3\phi_k^5 dS + \sum_{k=1}^{P}\dot{b}_k(t)\int_{\partial\Omega_P}\phi_j^5\xi_k dS - \sum_{k=1}^{P}b_k(t)\int_{\partial\Omega_P}\phi_j^5\bar{u}\frac{\partial \xi_k}{\partial x}dS} \tag{31}$$

$\square$

(31) is a linear system that is written in matrix/vector form later in this document.

2.3 Implementation of the Far-Field Boundary Condition

The boundary integral in (15) *needs to be evaluated over the entire domain boundary, not just the plate boundary. The formulation of the no-penetration BC on the plate boundary has enabled us to write I_P in terms of the basis functions and ROM coefficients. We now turn our attention to I_F, the far-field boundary portion of the boundary integral over $\partial\Omega$. An approximately non-reflecting boundary condition is desirable on the far-field boundary. A weak implementation of such a condition for the fluid ROM is formulated below.*

2.3.1 Properties of Hyperbolic Systems

Before proceeding to the implementation of the far-field boundary condition, let us briefly review some properties of hyperbolic systems of PDEs that justify the upcoming derivations, in particular the diagonalization of the matrix $A(\bar{U})\cdot\mathbf{n}$. Recall the 3D Euler equations for compressible flow, typically written as

$$U_t + F_1(U)_x + F_2(U)_y + F_3(U)_z = 0 \tag{32}$$

with

$$U = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ E \end{pmatrix}, \quad F_1(U) = \begin{pmatrix} \rho u \\ p+\rho u^2 \\ \rho uv \\ \rho uw \\ u(E+p) \end{pmatrix}, \quad F_2(U) = \begin{pmatrix} \rho v \\ \rho uv \\ p+\rho v^2 \\ \rho vw \\ v(E+p) \end{pmatrix}, \quad F_3(U) = \begin{pmatrix} \rho w \\ \rho uw \\ \rho vw \\ p+\rho w^2 \\ w(E+p) \end{pmatrix} \tag{33}$$

The first equation in (33) represents conservation of mass, the second three represent conservation of momentum (in each of the three directions, x, y and z), and the final equation represents conservation of energy. Recall also that the Euler equations (33) form a nonlinear hyperbolic system.

If one defines the tensor

$$F(U) = \begin{pmatrix} F_1(U) & F_2(U) & F_3(U) \end{pmatrix} \tag{34}$$

then (32) can be written as

$$\frac{\partial U}{\partial t} + \nabla_{x,y,z}\cdot F(U) = 0 \tag{35}$$

If $A_i(U)\in\mathbb{R}^{5\times 5}$ is defined to be the Jacobian of $F_i(U)$ for $i=1,2,3$, it is a well-known fact that since the system (34) is hyperbolic, the matrix $A(U)\equiv \alpha_1 A_1(U)+\alpha_2 A_2(U)+\alpha_3 A_3(U)$ has only real eigenvalues and is diagonalizable[6]. It is precisely this property that guarantees the diagonalization of the $A(\bar{U})\cdot\mathbf{n}$ matrix, performed in the next section..

2.3.2 Diagonalization of the Matrix $A(\bar{U})\cdot\mathbf{n}$

Let $A(\bar{U})$ denote the tensor

$$A(\bar{U}) \equiv \begin{pmatrix} A_x(\bar{U}) & A_y(\bar{U}) & A_z(\bar{U}) \end{pmatrix} \in \mathbb{R}^{5\times5\times3} \tag{36}$$

where $A_x(\bar{U}), A_y(\bar{U}), A_z(\bar{U})$ were derived in §3.4.1 of [1] to be

$$A_x(\bar{U}) = \begin{pmatrix} \bar{u} & 0 & 0 & 0 & \bar{\zeta} \\ 0 & \bar{u} & 0 & 0 & 0 \\ 0 & 0 & \bar{u} & 0 & 0 \\ -\bar{\zeta} & 0 & 0 & \bar{u} & 0 \\ \gamma\bar{p} & 0 & 0 & 0 & \bar{u} \end{pmatrix}, \quad A_y(\bar{U}) = \begin{pmatrix} \bar{v} & 0 & 0 & 0 & 0 \\ 0 & \bar{v} & 0 & 0 & \bar{\zeta} \\ 0 & 0 & \bar{v} & 0 & 0 \\ 0 & -\bar{\zeta} & 0 & \bar{v} & 0 \\ 0 & \gamma\bar{p} & 0 & 0 & \bar{v} \end{pmatrix}, \quad A_z(\bar{U}) = \begin{pmatrix} \bar{w} & 0 & 0 & 0 & 0 \\ 0 & \bar{w} & 0 & 0 & 0 \\ 0 & 0 & \bar{w} & 0 & \bar{\zeta} \\ 0 & 0 & -\bar{\zeta} & \bar{w} & 0 \\ 0 & 0 & \gamma\bar{p} & 0 & \bar{w} \end{pmatrix} \tag{37}$$

[6]See `http://en.wikipedia.org/wiki/Hyperbolic_equation` for more details or almost any textbook on numerical solutions to PDEs.

Let $\mathbf{n} \equiv \begin{pmatrix} n_1 & n_2 & n_3 \end{pmatrix}^T$ be the unit normal to the far-field boundary $\partial\Omega_F$[7]. In the general case of a boundary positioned at some angle θ relative to the plate (in $\mathbb{R}^3$), one will be interested in diagonalizing the matrix $A(\bar{U})\cdot\mathbf{n} \equiv S\Lambda S^{-1}$ so as to transition to the so-called "characteristic variables" $V' \equiv S^{-1}U'$. In the upcoming derivations, we use the following relations to simplify the expressions:

$$||\mathbf{n}||_2^2 \equiv n_1^2 + n_2^2 + n_3^2 = 1 \tag{38}$$

(i.e., $\mathbf{n}$ is chosen to be a *unit* normal vector) and define

$$c = \sqrt{\gamma\bar{p}\bar{\zeta}} = \sqrt{\frac{\gamma\bar{p}}{\bar{\rho}}} \tag{39}$$

to be the speed of sound. Note that (39) implies that

$$\sqrt{\frac{\gamma\bar{p}}{\bar{\zeta}}} = \frac{c}{\bar{\zeta}} \tag{40}$$

Letting

$$\bar{u}_n \equiv \bar{\mathbf{u}}\cdot\mathbf{n} \tag{41}$$

where $\bar{\mathbf{u}} \equiv \begin{pmatrix} \bar{u} & \bar{v} & \bar{w} \end{pmatrix}^T$, one has that

$$A(\bar{U})\cdot\mathbf{n} \quad = \begin{pmatrix} \bar{u}_n & 0 & 0 & 0 & \bar{\zeta}n_1 \\ 0 & \bar{u}_n & 0 & 0 & \bar{\zeta}n_2 \\ 0 & 0 & \bar{u}_n & 0 & \bar{\zeta}n_3 \\ -\bar{\zeta}n_1 & -\bar{\zeta}n_2 & -\bar{\zeta}n_3 & \bar{u}_n & 0 \\ \gamma\bar{p}n_1 & \gamma\bar{p}n_2 & \gamma\bar{p}n_3 & 0 & \bar{u}_n \end{pmatrix} \tag{42}$$

Diagonalizing the matrix $A(\bar{U})\cdot\mathbf{n}$ amounts to finding S and Λ such that

$$A(\bar{U})\cdot\mathbf{n} = S\Lambda S^{-1} \tag{43}$$

2.3.3 Jacobian Eigenvalues and Eigenvectors

We now go through the linear algebra of the computation of the eigenvalues and eigenvectors of the Jacobian matrix $A(\bar{U})\cdot\mathbf{n}$, defined in the previous section. The eigenvalue problem is equivalent to finding the constants λ such that $\det(\lambda I_5 - A(\bar{U})\cdot\mathbf{n}) = 0$, and the eigenvector problem is equivalent to finding the null vectors of $\lambda I_5 - A(\bar{U})\cdot\mathbf{n}$ where I_5 denotes the 5×5 identity matrix.

$$\lambda I_5 - A(\bar{U})\cdot\mathbf{n} = \begin{pmatrix} \lambda-\bar{u}_n & 0 & 0 & 0 & -\bar{\zeta}n_1 \\ 0 & \lambda-\bar{u}_n & 0 & 0 & -\bar{\zeta}n_2 \\ 0 & 0 & \lambda-\bar{u}_n & 0 & -\bar{\zeta}n_3 \\ \bar{\zeta}n_1 & \bar{\zeta}n_2 & \bar{\zeta}n_3 & \lambda-\bar{u}_n & 0 \\ -\gamma\bar{p}n_1 & -\gamma\bar{p}n_2 & -\gamma\bar{p}n_3 & 0 & \lambda-\bar{u}_n \end{pmatrix} \tag{44}$$

[7] For instance, the far-field boundary may be a square or rectangular box around the plate, in which case $\mathbf{n} = \mathbf{e}_x$, $\mathbf{n} = \mathbf{e}_y$ and $\mathbf{n} = \mathbf{e}_z$, depending on the side of the box one is considering.

Setting $\det(\lambda I_5 - A(\bar{U})\cdot\mathbf{n}) = 0$ results in the following Λ matrix (where the diagonal entries of Λ are the eigenvalues of $A(\bar{U})\cdot\mathbf{n}$):

$$\Lambda = \begin{pmatrix} \bar{u}_n & 0 & 0 & 0 & 0 \\ 0 & \bar{u}_n & 0 & 0 & 0 \\ 0 & 0 & \bar{u}_n & 0 & 0 \\ 0 & 0 & 0 & \bar{u}_n + c & 0 \\ 0 & 0 & 0 & 0 & \bar{u}_n - c \end{pmatrix} \tag{45}$$

The eigenvalues u_n, $u_n + c$ and $u_n - c$ are known as the "characteristic speeds" of the propagating waves.

We now turn our attention to computing the eigenvectors corresponding to each of these eigenvalues.

2.3.4 Eigenvectors corresponding to $\lambda = \bar{u}_n$

Let us begin by computing the eigenvectors corresponding to the triple eigenvalue, $\lambda = \bar{u}_n$[8].

It turns out that in order to avoid considering three separate cases, one should compute the *left* eigenvectors of $A(\bar{U})\cdot\mathbf{n}$ first. Let $\mathbf{l} \equiv \begin{pmatrix} l_1 & l_2 & l_3 & l_4 & l_5 \end{pmatrix} \in \mathbb{R}^{1\times 5}$ denote a left eigenvector of $A(\bar{U})\cdot\mathbf{n}$, i.e., a vector such that $\mathbf{l}\left[\bar{u}_n I_5 - A(\bar{U})\cdot\mathbf{n}\right] = 0$. Solving for the components of $\mathbf{l}$ gives rise to the following equations:

$$\begin{gathered} l_4 = \text{free} \\ l_1 n_1 + l_2 n_2 + l_3 n_3 = 0 \\ \gamma\bar{p} l_5 = \bar{\zeta} l_4 \end{gathered} \tag{46}$$

It is not hard to see that (46) offers three degrees of freedom, as expected, implying three linearly independent (unnormalized) eigenvectors, call them $\mathbf{l}^{(1)}, \mathbf{l}^{(2)}, \mathbf{l}^{(3)}$:

$$\mathbf{l}^{(1)} = \begin{pmatrix} 0 & n_3\mu_1^{(1)} & -n_2\mu_1^{(1)} & \mu_2^{(1)} & \frac{\bar{\zeta}\mu_2^{(1)}}{\gamma\bar{p}} \end{pmatrix} \tag{47}$$

$$\mathbf{l}^{(2)} = \begin{pmatrix} n_3\mu_1^{(2)} & 0 & -n_1\mu_1^{(2)} & \mu_2^{(2)} & \frac{\bar{\zeta}\mu_2^{(2)}}{\gamma\bar{p}} \end{pmatrix} \tag{48}$$

$$\mathbf{l}^{(3)} = \begin{pmatrix} n_2\mu_1^{(3)} & -n_1\mu_1^{(3)} & 0 & \mu_2^{(3)} & \frac{\bar{\zeta}\mu_2^{(3)}}{\gamma\bar{p}} \end{pmatrix} \tag{49}$$

where the $\mu_i^{(j)}, i \in \{1,2\}, j \in \{1,2,3\}$ are arbitrary nonzero constants.

2.3.5 Eigenvector corresponding to $\lambda = \bar{u}_n + c$

This is similar to the previous case. Now one is interested in the null vector of $\mathbf{l}\left[(\bar{u}_n + c) I_5 - A(\bar{U})\cdot\mathbf{n}\right]$. The relevant system of equations describing the components of $\mathbf{l}^{(4)}$ is

$$\begin{gathered} l_5 = \text{free} \\ l_4 = 0 \\ c l_1 = \gamma\bar{p} n_1 l_5 \\ c l_2 = \gamma\bar{p} n_2 l_5 \\ c l_3 = \gamma\bar{p} n_3 l_5 \\ c l_5 = \bar{\zeta}(n_1 l_1 + n_2 l_2 + n_3 l_3) \end{gathered} \tag{50}$$

[8]One knows a priori that these will span an eigenspace of dimension three. This follows from the fact that the Euler equations are a hyperbolic system, as discussed earlier.

The (unnormalized) eigenvector corresponding to the eigenvalue $\lambda = \bar{u}_n + c$ is

$$\mathbf{l}^{(4)} = \begin{pmatrix} \frac{\gamma\bar{p}n_1\mu^{(4)}}{c} & \frac{\gamma\bar{p}n_2\mu^{(4)}}{c} & \frac{\gamma\bar{p}n_3\mu^{(4)}}{c} & 0 & \mu^{(4)} \end{pmatrix} \tag{51}$$

for some $0 \neq \mu^{(4)} \in \mathbb{R}$.

2.3.6 Eigenvector corresponding to $\lambda = \bar{u}_n - c$

One now solves for the null vector of the matrix $\mathbf{l}\left[(\bar{u}_n - c)I_5 - A(\bar{U}) \cdot \mathbf{n}\right]$. Doing so gives rise to the following system of equations for the components of $\mathbf{l}^{(5)}$:

$$\begin{aligned} l_5 &= \text{free} \\ l_4 &= 0 \\ cl_1 &= -\gamma\bar{p}n_1 l_5 \\ cl_2 &= -\gamma\bar{p}n_2 l_5 \\ cl_3 &= -\gamma\bar{p}n_3 l_5 \\ cl_5 &= -\bar{\zeta}(n_1 l_1 + n_2 l_2 + n_3 l_3) \end{aligned} \tag{52}$$

Similarly to the previous case, the (unnormalized) eigenvector corresponding to the eigenvalue $\lambda = \bar{u}_n - c$ is

$$\mathbf{l}^{(5)} = \begin{pmatrix} -\frac{\gamma\bar{p}n_1\mu^{(5)}}{c} & -\frac{\gamma\bar{p}n_2\mu^{(5)}}{c} & -\frac{\gamma\bar{p}n_3\mu^{(5)}}{c} & 0 & \mu^{(5)} \end{pmatrix} \tag{53}$$

for some $0 \neq \mu^{(5)} \in \mathbb{R}$.

2.3.7 Diagonalizing Matrices

In the previous subsection, we computed the eigenvectors of the matrix $A(\bar{U}) \cdot \mathbf{n}$ up to some degrees of freedom. From a theoretical perspective, the constants μ_i^j are completely arbitrary. However, it turns out that the invertibility of the diagonalizing matrix S depends critically on the choice of $\mu_i^{(j)}$; in fact, for most arbitrary $\mu_i^{(j)}$, S *will* be singular for some normal vector $\mathbf{n}$. This is a problem because then S^{-1} is undefined and hence the diagonalization $A(\bar{U}) \cdot \mathbf{n} = S\Lambda S^{-1}$ does not exist.

Careful inspection and analysis shows that a “good” choice of $\mu_i^{(j)}$, i.e., one for which S will always be nonsingular is

$$\begin{aligned} \mu_1^{(1)} = 1, \mu_2^{(1)} &= n_1 \\ \mu_1^{(2)} = 1, \mu_2^{(2)} &= -n_2 \\ \mu_1^{(3)} = 1, \mu_2^{(3)} &= n_3 \\ \mu^{(4)} = \mu^{(5)} &= \frac{c}{\gamma\bar{p}} \end{aligned} \tag{54}$$

Then

$$S^{-1} \quad \equiv L = \begin{pmatrix} \mathbf{l}^{(1)} \\ \mathbf{l}^{(2)} \\ \mathbf{l}^{(3)} \\ \mathbf{l}^{(4)} \\ \mathbf{l}^{(5)} \end{pmatrix} = \begin{pmatrix} 0 & n_3 & -n_2 & n_1 & \frac{\bar{\zeta}n_1}{\gamma\bar{p}} \\ n_3 & 0 & -n_1 & -n_2 & -\frac{\bar{\zeta}n_2}{\gamma\bar{p}} \\ n_2 & -n_1 & 0 & n_3 & \frac{\bar{\zeta}n_3}{\gamma\bar{p}} \\ n_1 & n_2 & n_3 & 0 & \frac{c}{\gamma\bar{p}} \\ -n_1 & -n_2 & -n_3 & 0 & \frac{c}{\gamma\bar{p}} \end{pmatrix} \tag{55}$$

Lemma 2.3.1. *The diagonalizing matrix L defined in* (55) *is non-singular and hence invertible* $\forall\, \mathbf{n} = \begin{pmatrix} n_1 & n_2 & n_3 \end{pmatrix}^T$.

Proof. Recall from linear algebra that a matrix M is non-singular if and only if $\det(M) \neq 0$. Computing the determinant of L defined above and using the relation $n_1^2 + n_2^2 + n_3^2 = 1$, one finds that

$$\det(L) = \frac{2c}{\gamma\bar{p}} \tag{56}$$

which is never zero provided $\gamma, \bar{p} \neq 0$ (which they would not be in physical applications).

□

Having verified the invertibility of L defined in (55) for all normals $\mathbf{n}$, we can find $R \equiv S$ by computing L's inverse:

$$S \quad \equiv R = L^{-1} = \begin{pmatrix} 0 & n_3 & n_2 & \frac{1}{2}n_1 & -\frac{1}{2}n_1 \\ n_3 & 0 & -n_1 & \frac{1}{2}n_2 & -\frac{1}{2}n_2 \\ -n_2 & -n_1 & 0 & \frac{1}{2}n_3 & -\frac{1}{2}n_3 \\ n_1 & -n_2 & n_3 & -\frac{\bar{\zeta}}{2c} & -\frac{\bar{\zeta}}{2c} \\ 0 & 0 & 0 & \frac{\gamma\bar{p}}{2c} & \frac{\gamma\bar{p}}{2c} \end{pmatrix} \tag{57}$$

We have finished deriving the desired diagonalization, $A(\bar{U}) \cdot \mathbf{n} = R\Lambda L = S\Lambda S^{-1}$ where Λ is as in (45).

The above result can be verified in MATLAB by multiplying out the computed matrices. In particular, one computes $S\Lambda S^{-1}$ to be:

$$\begin{aligned} S\Lambda S^{-1} \quad &= \begin{pmatrix} 0 & n_3 & n_2 & \frac{1}{2}n_1 & -\frac{1}{2}n_1 \\ n_3 & 0 & -n_1 & \frac{1}{2}n_2 & -\frac{1}{2}n_2 \\ -n_2 & -n_1 & 0 & \frac{1}{2}n_3 & -\frac{1}{2}n_3 \\ n_1 & -n_2 & n_3 & -\frac{\bar{\zeta}}{2c} & -\frac{\bar{\zeta}}{2c} \\ 0 & 0 & 0 & \frac{\gamma\bar{p}}{2c} & \frac{\gamma\bar{p}}{2c} \end{pmatrix} \begin{pmatrix} \bar{u}_n & 0 & 0 & 0 & 0 \\ 0 & \bar{u}_n & 0 & 0 & 0 \\ 0 & 0 & \bar{u}_n & 0 & 0 \\ 0 & 0 & 0 & \bar{u}_n + c & 0 \\ 0 & 0 & 0 & 0 & \bar{u}_n - c \end{pmatrix} \begin{pmatrix} 0 & n_3 & -n_2 & n_1 & \frac{\bar{\zeta} n_1}{\gamma\bar{p}} \\ n_3 & 0 & -n_1 & -n_2 & -\frac{\bar{\zeta} n_2}{\gamma\bar{p}} \\ n_2 & -n_1 & 0 & n_3 & \frac{\bar{\zeta} n_3}{\gamma\bar{p}} \\ n_1 & n_2 & n_3 & 0 & \frac{c}{\gamma\bar{p}} \\ -n_1 & -n_2 & -n_3 & 0 & \frac{c}{\gamma\bar{p}} \end{pmatrix} \\ &= \begin{pmatrix} \bar{u}_n & 0 & 0 & 0 & \frac{n_1 c^2}{\gamma\bar{p}} \\ 0 & \bar{u}_n & 0 & 0 & \frac{n_2 c^2}{\gamma\bar{p}} \\ 0 & 0 & \bar{u}_n & 0 & \frac{n_3 c^2}{\gamma\bar{p}} \\ -\bar{\zeta} n_1 & -\bar{\zeta} n_2 & -\bar{\zeta} n_3 & \bar{u}_n & 0 \\ \gamma\bar{p} n_1 & \gamma\bar{p} n_2 & \gamma\bar{p} n_3 & 0 & \bar{u}_n \end{pmatrix} \\ &= \begin{pmatrix} \bar{u}_n & 0 & 0 & 0 & \bar{\zeta} n_1 \\ 0 & \bar{u}_n & 0 & 0 & \bar{\zeta} n_2 \\ 0 & 0 & \bar{u}_n & 0 & \bar{\zeta} n_3 \\ -\bar{\zeta} n_1 & -\bar{\zeta} n_2 & -\bar{\zeta} n_3 & \bar{u}_n & 0 \\ \gamma\bar{p} n_1 & \gamma\bar{p} n_2 & \gamma\bar{p} n_3 & 0 & \bar{u}_n \end{pmatrix} \end{aligned} \tag{58}$$

using the relation (39) that says that $c^2 = \gamma\bar{p}\bar{\zeta}$. Comparing the last line of (58) with the matrix $A(\bar{U}) \cdot \mathbf{n}$ defined in (42), one sees that our diagonalization produces the desired result.

□

REMARK: The diagonalization in question implies that $c, \gamma, \bar{p} \neq 0$, all reasonable physical assumptions. One must be somewhat careful, however, since the diagonalizing matrices S and S^{-1} will become ill-conditioned for either very large or very small $\bar{\rho}$, γ and c.

2.3.8 Implementation of the Non-Reflecting Far-Field Boundary Condition

The far-field boundary condition will be written in terms of the "characteristic variables", i.e. the elements of the vector $S^{-1}U'$ decomposed in the components of the POD basis $\phi_j(\mathbf{x})$. Here we write out this vector and then formulate the explicit implementation of the far-field boundary integral for the ROM on $\partial\Omega_F$, like was done earlier for the solid wall BCs on the plate boundary $\partial\Omega_P$. The far-field BC implemented is an approximately non-reflecting condition, where the incoming characteristic variables (those corresponding to a negative eigenvalue) are set to zero and the outgoing characteristics (those corresponding to a positive eigenvalue) are left alone. It turns out that there are four cases to consider: supersonic inflow ($\bar{u}_n < -c$), subsonic inflow ($-c < \bar{u}_n < 0$), supersonic outflow ($\bar{u}_n > c$), and subsonic outflow ($0 < \bar{u}_n < c$). It is first assumed that the far-field boundary has only one of these cases; in general, multiple cases are possible, if the far-field boundary is, for example, spherical.

Recall from (16) the definition of the vector U', namely

$$U' \equiv \begin{pmatrix} u' \\ v' \\ w' \\ \zeta' \\ p' \end{pmatrix} \tag{59}$$

Since we will be working in the characteristic variables, i.e., $V' \equiv S^{-1}U'$, the first step in the derivation is to write out the V' vector using the S matrix which diagonalizes $A(\bar{U}) \cdot \mathbf{n}$ (computed in the previous section). Multiplying out gives

$$V' \equiv S^{-1}U' = \begin{pmatrix} (n_3 v' - n_2 w' + n_1 \zeta') + \frac{\bar{\zeta}}{\gamma \bar{p}} n_1 p' \\ (n_3 u' - n_1 w' - n_2 \zeta') - \frac{\bar{\zeta}}{\gamma \bar{p}} n_2 p' \\ (n_2 u' - n_1 v' + n_3 \zeta') + \frac{\bar{\zeta}}{\gamma \bar{p}} n_3 p' \\ u'_n + \frac{c}{\gamma \bar{p}} p' \\ -u'_n + \frac{c}{\gamma \bar{p}} p' \end{pmatrix} \equiv \begin{pmatrix} \mathbf{a}_{v'w'} \cdot \mathbf{n} + \left(\frac{\bar{\zeta}}{c}\right)^2 n_1 p' \\ \mathbf{a}_{u'w'} \cdot \mathbf{n} - \left(\frac{\bar{\zeta}}{c}\right)^2 n_2 p' \\ \mathbf{a}_{u'v'} \cdot \mathbf{n} + \left(\frac{\bar{\zeta}}{c}\right)^2 n_3 p' \\ u'_n + \frac{\bar{\zeta}}{c} p' \\ -u'_n + \frac{\bar{\zeta}}{c} p' \end{pmatrix} \tag{60}$$

where

$$\begin{array}{rcl} \mathbf{u}' & \equiv & \begin{pmatrix} u' & v' & w' \end{pmatrix}^T \\ \mathbf{n} & \equiv & \begin{pmatrix} n_1 & n_2 & n_3 \end{pmatrix}^T \\ u'_n & \equiv & \mathbf{u}' \cdot \mathbf{n} \\ \mathbf{a}_{v'w'} & \equiv & \begin{pmatrix} \zeta' & -w' & v' \end{pmatrix}^T \\ \mathbf{a}_{u'w'} & \equiv & \begin{pmatrix} -w' & -\zeta' & u' \end{pmatrix}^T \\ \mathbf{a}_{u'v'} & \equiv & \begin{pmatrix} -v' & u' & \zeta' \end{pmatrix}^T \end{array} \tag{61}$$

Recall that $\sqrt{\gamma \bar{p}/\bar{\zeta}} = c/\bar{\zeta}$.

In order to formulate the explicit implementation of the far-field non-reflecting BCs, one needs to write V' in terms of the orthonormal ROM (vector) basis $\phi_k(\mathbf{x}) \equiv \begin{pmatrix} \phi_k^1(\mathbf{x}) & \phi_k^2(\mathbf{x}) & \phi_k^3(\mathbf{x}) & \phi_k^4(\mathbf{x}) & \phi_k^5(\mathbf{x}) \end{pmatrix}^T \in \mathbb{R}^5$. Expanding the fluid solution in this orthonormal spectral basis as before:

$$
\begin{array}{ccc}
U' & = & \sum_{k=1}^{M} a_k(t)\phi_k(\mathbf{x}) \\
\begin{pmatrix} u' \\ v' \\ w' \\ \zeta' \\ p' \end{pmatrix} & = & \begin{pmatrix} \sum_{k=1}^{M} a_k(t)\phi_k^1(\mathbf{x}) \\ \sum_{k=1}^{M} a_k(t)\phi_k^2(\mathbf{x}) \\ \sum_{k=1}^{M} a_k(t)\phi_k^3(\mathbf{x}) \\ \sum_{k=1}^{M} a_k(t)\phi_k^4(\mathbf{x}) \\ \sum_{k=1}^{M} a_k(t)\phi_k^5(\mathbf{x}) \end{pmatrix}
\end{array} \tag{62}
$$

Then (60) becomes

$$
V' \equiv \begin{pmatrix} \mathbf{a}_{v'w'} \cdot \mathbf{n} + \left(\frac{\bar{\zeta}}{c}\right)^2 n_1 p' \\ \mathbf{a}_{u'w'} \cdot \mathbf{n} - \left(\frac{\bar{\zeta}}{c}\right)^2 n_2 p' \\ \mathbf{a}_{u'v'} \cdot \mathbf{n} + \left(\frac{\bar{\zeta}}{c}\right)^2 n_3 p' \\ u_n' + \frac{\bar{\zeta}}{c} p' \\ -u_n' + \frac{\bar{\zeta}}{c} p' \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^{M} \left[n_1 \left(\phi_k^4 + \left(\frac{\bar{\zeta}}{c}\right)^2 \phi_k^5 \right) - n_2 \phi_k^3 + n_3 \phi_k^2 \right] a_k(t) \\ \sum_{k=1}^{M} \left[-n_1 \phi_k^3 - n_2 \left(\phi_k^4 + \left(\frac{\bar{\zeta}}{c}\right)^2 \phi_k^5 \right) + n_3 \phi_k^1 \right] a_k(t) \\ \sum_{k=1}^{M} \left[-n_1 \phi_k^2 + n_2 \phi_k^1 + n_3 \left(\phi_k^4 + \left(\frac{\bar{\zeta}}{c}\right)^2 \phi_k^5 \right) \right] a_k(t) \\ \sum_{k=1}^{M} \left[n_1 \phi_k^1 + n_2 \phi_k^2 + n_3 \phi_k^3 + \frac{\bar{\zeta}}{c} \phi_k^5 \right] a_k(t) \\ \sum_{k=1}^{M} \left[-n_1 \phi_k^1 - n_2 \phi_k^2 - n_3 \phi_k^3 + \frac{\bar{\zeta}}{c} \phi_k^5 \right] a_k(t) \end{pmatrix} \tag{63}
$$

For completeness (and to see where this derivation is going), recall that the far-field integral of interest, denoted by I in (14) is

$$
\begin{aligned}
I &= \int_{\partial\Omega_P} \phi_j^T H(\bar{U}) \left[A(\bar{U}) \cdot \mathbf{n} \right] U' dS + \int_{\partial\Omega_F} \phi_j^T H(\bar{U}) \left[A(\bar{U}) \cdot \mathbf{n} \right] U' dS - \int_{\Omega} \left(\nabla \cdot \phi_j^T H(\bar{U}) A(\bar{U}) \right) U' d\Omega \\
&= I_P + I_F + I_V
\end{aligned} \tag{64}
$$

For ease of notation, let us denote the surface integral portion of (64) by I_S, so

$$
\boxed{I_S \equiv I_P + I_S = \int_{\partial\Omega_P} \phi_j^T H(\bar{U}) \left[A(\bar{U}) \cdot \mathbf{n} \right] U' dS + \int_{\partial\Omega_F} \phi_j^T H(\bar{U}) \left[A(\bar{U}) \cdot \mathbf{n} \right] U' dS} \tag{65}
$$

§2 was devoted to writing out I_P explicitly; we not turn our attention to the second surface integral term in (65), I_F. Using the diagonalization of $A(\bar{U}) \cdot \mathbf{n}$,

$$
\begin{aligned}
I_F &= \int_{\partial\Omega} \phi_j^T H(\bar{U}) \left[A(\bar{U}) \cdot \mathbf{n} \right] U' dS \\
&= \int_{\partial\Omega} \phi_j^T H(\bar{U}) S \Lambda S^{-1} U' dS \\
&= \int_{\partial\Omega} \phi_j^T \left[H(\bar{U}) S \Lambda \right] V' dS
\end{aligned} \tag{66}
$$

In (66), $(\phi_j^T H(\bar{U}) S)^T \in \mathbb{R}^5$, Λ is a diagonal matrix containing the eigenvalues of $A(\bar{U}) \cdot \mathbf{n}$ and $V' \in \mathbb{R}^5$ is the vector of characteristic variables, $S^{-1} U'$. Since the Λ matrix contains the eigenvalues of $A(\bar{U}) \cdot \mathbf{n}$, which represent the characteristic speeds, the decomposition in (66) makes it clear why it is convenient to transform to the characteristic variables V'.

Let V_b' denote the value of V' at the far boundary. In the implementation of the far-field BC, therefore, one will overwrite $V \leftarrow V_b'$ on $\partial\Omega_F$. The entries of V_b' are determined by the sign of each of the eigenvalues in Λ. There will be four cases to consider. Before proceeding to these cases, let us write out the following matrix and vector, which will be needed shortly.

$$H(\bar{U})S\Lambda = \begin{pmatrix} 0 & \bar{\rho}n_3\bar{u}_n & \bar{\rho}n_2\bar{u}_n & \frac{1}{2}\bar{\rho}n_1(\bar{u}_n+c) & -\frac{1}{2}\bar{\rho}n_1(\bar{u}_n-c) \\ \bar{\rho}n_3\bar{u}_n & 0 & -\bar{\rho}n_1\bar{u}_n & \frac{1}{2}\bar{\rho}n_2(\bar{u}_n+c) & -\frac{1}{2}\bar{\rho}n_2(\bar{u}_n-c) \\ -\bar{\rho}n_2\bar{u}_n & -\bar{\rho}n_1\bar{u}_n & 0 & \frac{1}{2}\bar{\rho}n_3(\bar{u}_n+c) & -\frac{1}{2}\bar{\rho}n_3(\bar{u}_n-c) \\ \alpha^2\gamma\bar{\rho}^2\bar{p}n_1\bar{u}_n & -\alpha^2\gamma\bar{\rho}^2\bar{p}n_2\bar{u}_n & \alpha^2\gamma\bar{\rho}^2\bar{p}n_3\bar{u}_n & 0 & 0 \\ \alpha^2\bar{\rho}n_1\bar{u}_n & -\alpha^2\bar{\rho}n_2\bar{u}_n & \alpha^2\bar{\rho}n_3\bar{u}_n & \frac{1}{2c}(\bar{u}_n+c) & -\frac{1}{2c}(\bar{u}_n-c) \end{pmatrix} \tag{67}$$

Note that in order to simplify the last two rows of the matrix in (67), one uses the identity $\gamma\bar{p}\bar{\rho}^2 = c^2\bar{\rho}^2/\bar{\zeta} = c^2\bar{\rho}$.

$$\phi_j^T\,[H(\bar{U})S\Lambda] \quad = \begin{pmatrix} \left[\bar{\rho}n_3\phi_j^2 - \bar{\rho}n_2\phi_j^3 + \alpha^2\bar{\rho}n_1\left(\gamma\bar{\rho}\bar{p}\phi_j^4 + \phi_j^5\right)\right]\bar{u}_n \\ \left[\bar{\rho}n_3\phi_j^1 - \bar{\rho}n_1\phi_j^3 - \alpha^2\bar{\rho}n_2\left(\gamma\bar{\rho}\bar{p}\phi_j^4 + \phi_j^5\right)\right]\bar{u}_n \\ \left[\bar{\rho}n_2\phi_j^1 - \bar{\rho}n_1\phi_j^2 + \alpha^2\bar{\rho}n_3\left(\gamma\bar{\rho}\bar{p}\phi_j^4 + \phi_j^5\right)\right]\bar{u}_n \\ \frac{1}{2}\left[\bar{\rho}n_1\phi_j^1 + \bar{\rho}n_2\phi_j^2 + \bar{\rho}n_3\phi_j^3 + \frac{1}{c}\phi_j^5\right](\bar{u}_n+c) \\ \frac{1}{2}\left[-\bar{\rho}n_1\phi_j^1 - \bar{\rho}n_2\phi_j^2 - \bar{\rho}n_3\phi_j^3 - \frac{1}{c}\phi_j^5\right](\bar{u}_n-c) \end{pmatrix}^T \equiv \begin{pmatrix} d_1(\mathbf{x}) \\ d_2(\mathbf{x}) \\ d_3(\mathbf{x}) \\ d_4(\mathbf{x}) \\ d_5(\mathbf{x}) \end{pmatrix}^T \tag{68}$$

Remark that the entries of the vector in (68) are functions of ϕ_j for $j = 1,\dots,M$.

2.3.9 Case 1: Supersonic Inflow ($\bar{u}_n < -c$)

Note that $\bar{u}_n < -c < 0 \Rightarrow \bar{u}_n - c < 0$ and $\bar{u}_n + c < 0$, i.e., all the characteristics are coming into the box around our plate. The approximate non-reflecting BC mandates that all incoming characteristics be set to zero. The far-field BC is thus

$$V_b' \equiv \mathbf{0} \in \mathbb{R}^5 \tag{69}$$

It follows that I_F in (66) reduces to

$$I_F \equiv 0 \tag{70}$$

2.3.10 Case 2: Subsonic Inflow ($-c < \bar{u}_n < 0$)

Now, $\bar{u}_n < 0$, which implies $\bar{u}_n - c < 0$. However, $\bar{u}_n + c \in (0,c)$, in particular $\bar{u}_n + c > 0$. This means that the characteristics corresponding to the eigenvalues $\bar{u}_n$ and $\bar{u}_n - c$ are incoming whereas the characteristics corresponding to the eigenvalue $\bar{u}_n + c$ are outgoing. The non-reflecting BC says to leave the outgoing characteristics alone. Looking at the definition of Λ in (45), we see that the characteristics to be set to zero correspond to the first, second, third and fifth component of V'. Thus, the far-field BC is

$$V_b' \equiv \begin{pmatrix} 0 \\ 0 \\ 0 \\ u_n' + \frac{\bar{\zeta}}{c}p' \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \sum_{k=1}^M \left[n_1\phi_k^1 + n_2\phi_k^2 + n_3\phi_k^3 + \frac{\bar{\zeta}}{c}\phi_k^5\right]a_k(t) \\ 0 \end{pmatrix} \tag{71}$$

Using the vector derived in (68), one has that

$$\phi_j^T\,[H(\bar{U})S\Lambda]V_b' = d_4(\mathbf{x})\sum_{k=1}^M \left[n_1\phi_k^1 + n_2\phi_k^2 + n_3\phi_k^3 + \frac{\bar{\zeta}}{c}\phi_k^5\right]a_k(t) \tag{72}$$

It follows that the boundary integral I_F reduces to

$$
\begin{aligned}
I_F \quad &= \int_{\partial\Omega_F} \phi_j^T \, [H(\bar{U})S\Lambda] V_b' dS \\
&= \sum_{k=1}^M a_k(t) \int_{\partial\Omega_F} d_4(\mathbf{x}) \left[n_1\phi_k^1 + n_2\phi_k^2 + n_3\phi_k^3 + \frac{\bar{\zeta}}{c}\phi_k^5 \right] dS
\end{aligned}
\tag{73}
$$

2.3.11 Case 3: Subsonic Outflow ($0 < \bar{u}_n < c$)

In this case, $\bar{u}_n > 0$ implies that $\bar{u}_n + c > 0$ but $\bar{u}_n \in (0,c) \Rightarrow \bar{u}_n - c \in (-c,0)$, i.e., $\bar{u}_n - c < 0$. This means that the characteristics corresponding to $\bar{u}_n - c$ are incoming whereas the characteristics corresponding to the other two eigenvalues are outgoing. It follows that the far-field BC to be implemented is

$$
V_b' \equiv \begin{pmatrix} \mathbf{a}_{v'w'} \cdot \mathbf{n} + \left(\frac{\bar{\zeta}}{c}\right)^2 n_1 p' \\ \mathbf{a}_{u'w'} \cdot \mathbf{n} - \left(\frac{\bar{\zeta}}{c}\right)^2 n_2 p' \\ \mathbf{a}_{u'v'} \cdot \mathbf{n} + \left(\frac{\bar{\zeta}}{c}\right)^2 n_3 p' \\ u_n' + \frac{\bar{\zeta}}{c} p' \\ 0 \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^M \left[n_1 \left(\phi_k^4 + \left(\frac{\bar{\zeta}}{c}\right)^2 \phi_k^5 \right) - n_2\phi_k^3 + n_3\phi_k^2 \right] a_k(t) \\ \sum_{k=1}^M \left[-n_1\phi_k^3 - n_2 \left(\phi_k^4 + \left(\frac{\bar{\zeta}}{c}\right)^2 \phi_k^5 \right) + n_3\phi_k^1 \right] a_k(t) \\ \sum_{k=1}^M \left[-n_1\phi_k^2 + n_2\phi_k^1 + n_3 \left(\phi_k^4 + \left(\frac{\bar{\zeta}}{c}\right)^2 \phi_k^5 \right) \right] a_k(t) \\ \sum_{k=1}^M \left[n_1\phi_k^1 + n_2\phi_k^2 + n_3\phi_k^3 + \frac{\bar{\zeta}}{c}\phi_k^5 \right] a_k(t) \\ 0 \end{pmatrix}
\tag{74}
$$

Then

$$
\phi_j^T \, [H(\bar{U})S\Lambda] V_b' = \begin{aligned} & d_1(\mathbf{x}) \sum_{k=1}^M \left[n_1 \left(\phi_k^4 + \left(\frac{\bar{\zeta}}{c}\right)^2 \phi_k^5 \right) - n_2\phi_k^3 + n_3\phi_k^2 \right] a_k(t) \\ & + d_2(\mathbf{x}) \sum_{k=1}^M \left[-n_1\phi_k^3 - n_2 \left(\phi_k^4 + \left(\frac{\bar{\zeta}}{c}\right)^2 \phi_k^5 \right) + n_3\phi_k^1 \right] a_k(t) \\ & + d_3(\mathbf{x}) \sum_{k=1}^M \left[-n_1\phi_k^2 + n_2\phi_k^1 + n_3 \left(\phi_k^4 + \left(\frac{\bar{\zeta}}{c}\right)^2 \phi_k^5 \right) \right] a_k(t) \\ & + d_4(\mathbf{x}) \sum_{k=1}^M \left[n_1\phi_k^1 + n_2\phi_k^2 + n_3\phi_k^3 + \frac{\bar{\zeta}}{c}\phi_k^5 \right] a_k(t) \end{aligned}
\tag{75}
$$

and the desired boundary integral is

$$
I_F = \sum_{k=1}^M \begin{aligned} & a_k(t) \int_{\partial\Omega_F} \left\{ d_1(\mathbf{x}) \left[n_1 \left(\phi_k^4 + \left(\frac{\bar{\zeta}}{c}\right)^2 \phi_k^5 \right) - n_2\phi_k^3 + n_3\phi_k^2 \right] + d_2(\mathbf{x}) \left[-n_1\phi_k^3 - n_2 \left(\phi_k^4 + \left(\frac{\bar{\zeta}}{c}\right)^2 \phi_k^5 \right) + n_3\phi_k^1 \right] \right. \\ & \left. + d_3(\mathbf{x}) \left[-n_1\phi_k^2 + n_2\phi_k^1 + n_3 \left(\phi_k^4 + \left(\frac{\bar{\zeta}}{c}\right)^2 \phi_k^5 \right) \right] + d_4(\mathbf{x}) \left[n_1\phi_k^1 + n_2\phi_k^2 + n_3\phi_k^3 + \frac{\bar{\zeta}}{c}\phi_k^5 \right] \right\} dS \end{aligned}
\tag{76}
$$

in terms of the appropriate ROM coefficients and basis functions.

2.3.12 Case 4: Supersonic Outflow ($\bar{u}_n > c$)

Here, $\bar{u}_n > c \Rightarrow \bar{u}_n - c, \bar{u}_n + c > 0$, i.e., all the characteristics are outgoing. All the eigenvalues of $A(\bar{U}) \cdot \mathbf{n}$ are therefore to be left alone.

$$
V_b' = \begin{pmatrix} \mathbf{a}_{v'w'} \cdot \mathbf{n} + \left(\frac{\bar{\zeta}}{c}\right)^2 n_1 p' \\ \mathbf{a}_{u'w'} \cdot \mathbf{n} - \left(\frac{\bar{\zeta}}{c}\right)^2 n_2 p' \\ \mathbf{a}_{u'v'} \cdot \mathbf{n} + \left(\frac{\bar{\zeta}}{c}\right)^2 n_3 p' \\ u_n' + \frac{\bar{\zeta}}{c} p' \\ -u_n' + \frac{\bar{\zeta}}{c} p' \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^M \left[n_1 \left(\phi_k^4 + \left(\frac{\bar{\zeta}}{c}\right)^2 \phi_k^5 \right) - n_2 \phi_k^3 + n_3 \phi_k^2 \right] a_k(t) \\ \sum_{k=1}^M \left[-n_1 \phi_k^3 - n_2 \left(\phi_k^4 + \left(\frac{\bar{\zeta}}{c}\right)^2 \phi_k^5 \right) + n_3 \phi_k^1 \right] a_k(t) \\ \sum_{k=1}^M \left[-n_1 \phi_k^2 + n_2 \phi_k^1 + n_3 \left(\phi_k^4 + \left(\frac{\bar{\zeta}}{c}\right)^2 \phi_k^5 \right) \right] a_k(t) \\ \sum_{k=1}^M \left[n_1 \phi_k^1 + n_2 \phi_k^2 + n_3 \phi_k^3 + \frac{\bar{\zeta}}{c} \phi_k^5 \right] a_k(t) \\ \sum_{k=1}^M \left[-n_1 \phi_k^1 - n_2 \phi_k^2 - n_3 \phi_k^3 + \frac{\bar{\zeta}}{c} \phi_k^5 \right] a_k(t) \end{pmatrix} \tag{77}
$$

Then

$$
\begin{aligned}
\phi_j^T \left[H(\bar{U}) S \Lambda \right] V_b' = \; & d_1(\mathbf{x}) \sum_{k=1}^M \left[n_1 \left(\phi_k^4 + \left(\frac{\bar{\zeta}}{c}\right)^2 \phi_k^5 \right) - n_2 \phi_k^3 + n_3 \phi_k^2 \right] a_k(t) \\
& + d_2(\mathbf{x}) \sum_{k=1}^M \left[-n_1 \phi_k^3 - n_2 \left(\phi_k^4 + \left(\frac{\bar{\zeta}}{c}\right)^2 \phi_k^5 \right) + n_3 \phi_k^1 \right] a_k(t) \\
& + d_3(\mathbf{x}) \sum_{k=1}^M \left[-n_1 \phi_k^2 + n_2 \phi_k^1 + n_3 \left(\phi_k^4 + \left(\frac{\bar{\zeta}}{c}\right)^2 \phi_k^5 \right) \right] a_k(t) \\
& + d_4(\mathbf{x}) \sum_{k=1}^M \left[n_1 \phi_k^1 + n_2 \phi_k^2 + n_3 \phi_k^3 + \frac{\bar{\zeta}}{c} \phi_k^5 \right] a_k(t) \\
& + d_5(\mathbf{x}) \sum_{k=1}^M \left[-n_1 \phi_k^1 - n_2 \phi_k^2 - n_3 \phi_k^3 + \frac{\bar{\zeta}}{c} \phi_k^5 \right] a_k(t)
\end{aligned} \tag{78}
$$

The desired boundary integral is

$$
\begin{aligned}
I_F = \sum_{k=1}^M \; & a_k(t) \int_{\partial\Omega_F} \Big\{ d_1(\mathbf{x}) \left[n_1 \left(\phi_k^4 + \left(\frac{\bar{\zeta}}{c}\right)^2 \phi_k^5 \right) - n_2 \phi_k^3 + n_3 \phi_k^2 \right] \\
& + d_2(\mathbf{x}) \left[-n_1 \phi_k^3 - n_2 \left(\phi_k^4 + \left(\frac{\bar{\zeta}}{c}\right)^2 \phi_k^5 \right) + n_3 \phi_k^1 \right] \\
& + d_3(\mathbf{x}) \left[-n_1 \phi_k^2 + n_2 \phi_k^1 + n_3 \left(\phi_k^4 + \left(\frac{\bar{\zeta}}{c}\right)^2 \phi_k^5 \right) \right] \\
& + d_4(\mathbf{x}) \left[n_1 \phi_k^1 + n_2 \phi_k^2 + n_3 \phi_k^3 + \frac{\bar{\zeta}}{c} \phi_k^5 \right] \\
& + d_5(\mathbf{x}) \left[-n_1 \phi_k^1 - n_2 \phi_k^2 - n_3 \phi_k^3 + \frac{\bar{\zeta}}{c} \phi_k^5 \right] \Big\} \, dS
\end{aligned} \tag{79}
$$

2.3.13 Summary of Far-Field Boundary Condition Implementation

To ease the notation, note that for each of the four cases considered above, the boundary integral of interest over the far-field boundary $\partial\Omega_F$, denoted earlier by I_F (and defined in (66) has the form

$$
I_F = \sum_{k=1}^M a_k(t) \int_{\partial\Omega_F} h_k(\phi_j) dS \tag{80}
$$

where $h_k(\phi_j)$ is a function depending on the components of ϕ_j and ϕ_k, and which of the four cases considered above applies[9]. We given $h_k(\phi_j)$ explicitly below:

[9] Note that multiple cases may apply along a smooth far-field boundary, e.g., a sphere. For example, if the flow changed from supersonic inflow to supersonic outflow along a spherical boundary, one would need to determine the precise coordinates at which this transition occurs and compute I_F as a sum of two integrals, one for the supersonic inflow (up to the transition point) and one for the supersonic outflow (after the transition point)

$$
h_k(\phi_j) = \begin{cases} 0, & \bar{u}_n < -c \\ d_4(\mathbf{x}) \left[n_1\phi_k^1 + n_2\phi_k^2 + n_3\phi_k^3 + \frac{\bar{\zeta}}{c}\phi_k^5 \right], & -c < \bar{u}_n < 0 \\ \begin{array}{l} d_1(\mathbf{x}) \left[n_1 \left(\phi_k^4 + \left(\frac{\bar{\zeta}}{c}\right)^2 \phi_k^5 \right) - n_2\phi_k^3 + n_3\phi_k^2 \right] + d_2(\mathbf{x}) \left[-n_1\phi_k^3 - n_2 \left(\phi_k^4 + \left(\frac{\bar{\zeta}}{c}\right)^2 \phi_k^5 \right) + n_3\phi_k^1 \right] \\ +d_3(\mathbf{x}) \left[-n_1\phi_k^2 + n_2\phi_k^1 + n_3 \left(\phi_k^4 + \left(\frac{\bar{\zeta}}{c}\right)^2 \phi_k^5 \right) \right] + d_4(\mathbf{x}) \left[n_1\phi_k^1 + n_2\phi_k^2 + n_3\phi_k^3 + \frac{\bar{\zeta}}{c}\phi_k^5 \right] \end{array}, & 0 < \bar{u}_n < c \\ \begin{array}{l} d_1(\mathbf{x}) \left[n_1 \left(\phi_k^4 + \left(\frac{\bar{\zeta}}{c}\right)^2 \phi_k^5 \right) - n_2\phi_k^3 + n_3\phi_k^2 \right] + d_2(\mathbf{x}) \left[-n_1\phi_k^3 - n_2 \left(\phi_k^4 + \left(\frac{\bar{\zeta}}{c}\right)^2 \phi_k^5 \right) + n_3\phi_k^1 \right] \\ +d_3(\mathbf{x}) \left[-n_1\phi_k^2 + n_2\phi_k^1 + n_3 \left(\phi_k^4 + \left(\frac{\bar{\zeta}}{c}\right)^2 \phi_k^5 \right) \right] + d_4(\mathbf{x}) \left[n_1\phi_k^1 + n_2\phi_k^2 + n_3\phi_k^3 + \frac{\bar{\zeta}}{c}\phi_k^5 \right] \\ +d_5(\mathbf{x}) \left[-n_1\phi_k^1 - n_2\phi_k^2 - n_3\phi_k^3 + \frac{\bar{\zeta}}{c}\phi_k^5 \right] \end{array}, & \bar{u}_n > c \end{cases} \tag{81}
$$

The $d_i(\mathbf{x})$, $i = 1, \ldots, 5$ are the entries of the $\phi_j^T \; [H(\bar{U})S\Lambda]$ vector given in (68). These depend on the components of $\phi_j(\mathbf{x})$. Note that the cases when $\bar{u}_n = \pm c$ can be encompassed into the four cases summarized in (81).

It follows that the surface integral (over the entire boundary $\partial\Omega = \partial\Omega_P \cup \partial\Omega_F$) is:

$$
\begin{aligned} I_S &= I_P + I_F \\ &= \sum_{k=1}^M a_k(t) \int_{\partial\Omega_P} \phi_j^3 \phi_k^5 dS + \sum_{k=1}^P \dot{b}_k(t) \int_{\partial\Omega_P} \phi_j^5 \xi_k dS - \sum_{k=1}^P b_k(t) \int_{\partial\Omega_P} \phi_j^5 \bar{u} \frac{\partial \xi_k}{\partial x} dS + \sum_{k=1}^M a_k(t) \int_{\partial\Omega_F} h_k(\phi_j) dS \\ &= \sum_{k=1}^M a_k(t) \left(\int_{\partial\Omega_P} \phi_j^3 \phi_k^5 dS + \int_{\partial\Omega_F} h_k(\phi_j) dS \right) + \sum_{k=1}^P \dot{b}_k(t) \int_{\partial\Omega_P} \phi_j^5 \xi_k dS - \sum_{k=1}^P b_k(t) \int_{\partial\Omega_P} \phi_j^5 \bar{u} \frac{\partial \xi_k}{\partial x} dS \end{aligned} \tag{82}
$$

2.3.14 An Alternate Expression for the Diagonalizing Matrix S^{-1}

In the previous section, the diagonalizing matrix S^{-1} was used to define the transformation to the characteristic variables $V' \equiv S^{-1}U'$. Recall that S^{-1} can be chosen up to some degrees of freedom, denoted $\mu_i^{(j)}$ earlier in this document. A point was made that one must be very careful in selecting these constants because otherwise the S matrix may fail to be invertible for some unit normal vectors $\mathbf{n}$.

In some implementations, it may be useful to have an S^{-1} such that one of the components of V' can be split into entropic, vortical and acoustic waves. It would be nice, for example, if one of the components of V' contains only entropy perturbations (meaning it has only p' and ζ' disturbances), two components of V' contain only velocity disturbances (u', v', and/or w') and the remaining two components of V' are associated with acoustic pressure waves (having the form $u_n' \pm \text{const} \cdot p'$). One can see from (60) that we get the latter two components (the acoustic ones) for free in the previous derivation. However, looking at (60), it is clear that the first three components of the V' vector are a linear combination of the entropic and vortical waves.

It turns out that one *can* define matrices S^{-1} such that $V' = S^{-1}U'$ has the desired splitting by specifying alternate values of the constants $\mu_i^{(j)}$ in (54). Although these new matrices, call them $\bar{S}^{-1}$, are "nicer" in that they make V' have the desired form, there is a price to pay: one now needs to consider three cases, depending on whether any of the components of $\mathbf{n}$ are zero. In the implementation code, this amounts to writing a set of "for" loops, which would decrease efficiency and can make the program run much slower.

Recall that S^{-1} was derived to be

$$S^{-1} = \begin{pmatrix} 0 & n_3\mu_1^{(1)} & -n_2\mu_1^{(1)} & \mu_2^{(1)} & \frac{\bar{\zeta}}{\gamma\bar{p}}\mu_2^{(1)} \\ n_3\mu_1^{(2)} & 0 & -n_1\mu_1^{(2)} & \mu_2^{(2)} & \frac{\bar{\zeta}}{\gamma\bar{p}}\mu_2^{(2)} \\ n_2\mu_1^{(3)} & -n_1\mu_1^{(3)} & 0 & \mu_2^{(3)} & \frac{\bar{\zeta}}{\gamma\bar{p}}\mu_2^{(3)} \\ \frac{\gamma\bar{p}n_1\mu^{(4)}}{c} & \frac{\gamma\bar{p}n_2\mu^{(4)}}{c} & \frac{\gamma\bar{p}n_3\mu^{(4)}}{c} & 0 & \mu^{(4)} \\ -\frac{\gamma\bar{p}n_1\mu^{(5)}}{c} & -\frac{\gamma\bar{p}n_2\mu^{(5)}}{c} & -\frac{\gamma\bar{p}n_3\mu^{(5)}}{c} & 0 & \mu^{(5)} \end{pmatrix} \tag{83}$$

up to the degrees of freedom, the $\mu_i^{(j)}$. The following table gives one choice of constants $\mu_i^{(j)}$, depending on whether one wants to split the entropic and vortical waves and which, if any, of the components of **n** are zero.

	S^{-1}	$\bar{S}_1^{-1}$	$\bar{S}_2^{-1}$	$\bar{S}_3^{-1}$
Condition:	no enrtropic/vortical split	split, $n_1 \neq 0$	split, $n_2 \neq 0$	split, $n_3 \neq 0$
$\mu_1^{(1)}$	1	0	1	1
$\mu_2^{(1)}$	n_1	1	0	0
$\mu_1^{(2)}$	1	1	0	1
$\mu_2^{(2)}$	$-n_2$	0	1	0
$\mu_1^{(3)}$	1	1	1	0
$\mu_2^{(3)}$	n_3	0	0	1
$\mu^{(4)}$	$\frac{c}{\gamma\bar{p}}$	$\frac{c}{\gamma\bar{p}}$	$\frac{c}{\gamma\bar{p}}$	$\frac{c}{\gamma\bar{p}}$
$\mu^{(5)}$	$\frac{c}{\gamma\bar{p}}$	$\frac{c}{\gamma\bar{p}}$	$\frac{c}{\gamma\bar{p}}$	$\frac{c}{\gamma\bar{p}}$

When the values in the above table are plugged into (83), one obtains the following matrices and transformed vectors V'.

2.3.15 Case 1: $n_1 \neq 0$

$$A(\bar{U}) \cdot \mathbf{n} = \bar{S}_1 \Lambda \bar{S}_1^{-1} \tag{84}$$

$$\bar{S}_1^{-1} = \begin{pmatrix} 0 & 0 & 0 & 1 & \frac{\bar{\zeta}}{\gamma\bar{p}} \\ n_3 & 0 & -n_1 & 0 & 0 \\ n_2 & -n_1 & 0 & 0 & 0 \\ n_1 & n_2 & n_3 & 0 & \frac{c}{\gamma\bar{p}} \\ -n_1 & -n_2 & -n_3 & 0 & \frac{c}{\gamma\bar{p}} \end{pmatrix} \tag{85}$$

$$\bar{S}_1 = \begin{pmatrix} 0 & n_3 & n_2 & \frac{1}{2}n_1 & -\frac{1}{2}n_1 \\ 0 & \frac{n_2 n_3}{n_1} & -\frac{n_1^2+n_3^2}{n_1} & \frac{1}{2}n_2 & -\frac{1}{2}n_2 \\ 0 & -\frac{n_1^2+n_2^2}{n_1} & \frac{n_2 n_3}{n_1} & \frac{1}{2}n_3 & -\frac{1}{2}n_3 \\ 1 & 0 & 0 & -\frac{\bar{\zeta}}{2c} & -\frac{\bar{\zeta}}{2c} \\ 0 & 0 & 0 & \frac{\gamma\bar{p}}{2c} & \frac{\gamma\bar{p}}{2c} \end{pmatrix} \tag{86}$$

$$V_1' \equiv \bar{S}_1^{-1} U' = \begin{pmatrix} \zeta' + \left(\frac{\bar{\zeta}}{c}\right)^2 p' \\ n_3 u' - n_1 w' \\ n_2 u' - n_1 v' \\ u_n' + \frac{\bar{\zeta}}{c} p' \\ -u_n' + \frac{\bar{\zeta}}{c} p' \end{pmatrix} = \begin{pmatrix} \text{entropic} \\ \text{vortical} \\ \text{vortical} \\ \text{acoustic} \\ \text{acoustic} \end{pmatrix} \tag{87}$$

2.3.16 Case 2: $n_2 \neq 0$

$$A(\bar{U}) \cdot \mathbf{n} = \bar{S}_2 \Lambda \bar{S}_2^{-1} \tag{88}$$

$$\bar{S}_2^{-1} = \begin{pmatrix} 0 & n_3 & -n_2 & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{\bar{\zeta}}{\gamma \bar{p}} \\ n_2 & -n_1 & 0 & 0 & 0 \\ n_1 & n_2 & n_3 & 0 & \frac{\bar{c}}{\gamma \bar{p}} \\ -n_1 & -n_2 & -n_3 & 0 & \frac{\bar{c}}{\gamma \bar{p}} \end{pmatrix} \tag{89}$$

$$\bar{S}_2 = \begin{pmatrix} \frac{n_1 n_3}{n_2} & 0 & \frac{n_2^2+n_3^2}{n_2} & \frac{1}{2}n_1 & -\frac{1}{2}n_1 \\ n_3 & 0 & -n_1 & \frac{1}{2}n_2 & -\frac{1}{2}n_2 \\ -\frac{n_1^2+n_2^2}{n_2} & 0 & -\frac{n_1 n_3}{n_2} & \frac{1}{2}n_3 & -\frac{1}{2}n_3 \\ 0 & 1 & 0 & -\frac{\bar{\zeta}}{2\bar{c}} & -\frac{\bar{\zeta}}{2\bar{c}} \\ 0 & 0 & 0 & \frac{\gamma \bar{p}}{2\bar{c}} & \frac{\gamma \bar{p}}{2\bar{c}} \end{pmatrix} \tag{90}$$

$$V_2' \equiv \bar{S}_2^{-1} U' = \begin{pmatrix} n_3 v' - n_2 w' \\ \zeta' + \left(\frac{\bar{\zeta}}{\bar{c}}\right)^2 p' \\ n_2 u' - n_1 v' \\ u_n' + \frac{\bar{\zeta}}{\bar{c}} p' \\ -u_n' + \frac{\bar{\zeta}}{\bar{c}} p' \end{pmatrix} = \begin{pmatrix} \text{vortical} \\ \text{entropic} \\ \text{vortical} \\ \text{acoustic} \\ \text{acoustic} \end{pmatrix} \tag{91}$$

2.3.17 Case 3: $n_3 \neq 0$

$$A(\bar{U}) \cdot \mathbf{n} = \bar{S}_3 \Lambda \bar{S}_3^{-1} \tag{92}$$

$$\bar{S}_3^{-1} = \begin{pmatrix} 0 & n_3 & -n_2 & 0 & 0 \\ n_3 & -n_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{\bar{\zeta}}{\gamma \bar{p}} \\ n_1 & n_2 & n_3 & 0 & \frac{\bar{c}}{\gamma \bar{p}} \\ -n_1 & -n_2 & -n_3 & 0 & \frac{\bar{c}}{\gamma \bar{p}} \end{pmatrix} \tag{93}$$

$$\bar{S}_3 = \begin{pmatrix} -\frac{n_1 n_2}{n_3} & \frac{n_2^2+n_3^2}{n_3} & 0 & \frac{1}{2}n_1 & -\frac{1}{2}n_1 \\ \frac{n_1^2+n_3^2}{n_3} & -\frac{n_1 n_2}{n_3} & 0 & \frac{1}{2}n_2 & -\frac{1}{2}n_2 \\ -n_2 & -n_1 & 0 & \frac{1}{2}n_3 & -\frac{1}{2}n_3 \\ 0 & 0 & 1 & -\frac{\bar{\zeta}}{2\bar{c}} & -\frac{\bar{\zeta}}{2\bar{c}} \\ 0 & 0 & 0 & \frac{\gamma \bar{p}}{2\bar{c}} & \frac{\gamma \bar{p}}{2\bar{c}} \end{pmatrix} \tag{94}$$

$$V_3' \equiv \bar{S}_3^{-1} U' = \begin{pmatrix} n_3 v' - n_2 w' \\ n_3 u' - n_1 w' \\ \zeta' + \left(\frac{\bar{\zeta}}{\bar{c}}\right)^2 p' \\ u_n' + \frac{\bar{\zeta}}{\bar{c}} p' \\ -u_n' + \frac{\bar{\zeta}}{\bar{c}} p' \end{pmatrix} = \begin{pmatrix} \text{vortical} \\ \text{vortical} \\ \text{entropic} \\ \text{acoustic} \\ \text{acoustic} \end{pmatrix} \tag{95}$$

3 Structure Equations

3.1 Notation and Units

Parameter	Meaning	Formula	Units
h	thickness of plate		m
ρ_s	density of plate material	$\frac{\text{mass}}{\text{volume}}$	kg/m^3
ν	Poisson's ratio	$-\frac{\text{transverse strain}}{\text{axial strain}}$	$-$
E	Young's modulus	$\frac{\text{tensile stress}}{\text{tensile strain}} = \frac{F/A_0}{\Delta L/L_0}$	N/m^2
D_{bend}	bending stiffness	$\frac{Eh^3}{12(1-\nu^2)}$	$N \cdot m$
$\mathbf{q} = \begin{pmatrix} \beta & \gamma & \eta \end{pmatrix}^T$	displacement vector	stiffness $\times$ force	m
$\frac{\partial^2 \eta}{\partial t^2}$	acceleration	$\frac{\text{force}}{\text{mass}}$	N/kg

3.2 Assumptions and Boundary Conditions

Assume the plate is square and positioned in the $z = 0$ plane and is square $L \times L$ for some length $L \in \mathbb{R}$: $0 \leq x, y \leq L$. Assume the edges of the plate are simply supported. Mathematically, this is written as the following boundary conditions:

$$\eta(0,y,t) = 0 = \eta(L,y,t) \tag{96}$$

$$\eta(x,0,t) = 0 = \eta(x,L,t) \tag{97}$$

$$\frac{\partial^2 \eta}{\partial x^2}(0,y,t) = 0 = \frac{\partial^2 \eta}{\partial x^2}(L,y,t) \tag{98}$$

$$\frac{\partial^2 \eta}{\partial y^2}(x,0,t) = 0 = \frac{\partial^2 \eta}{\partial y^2}(x,L,t) \tag{99}$$

It turns out that this assumption results in some nice properties of the eigenmode basis $\xi_j(x,y)$ in which the η displacement field was decomposed (see (9)).

3.3 Linearized Dimensional von Karman Plate Equations for $\beta = \gamma = 0$

The *dimensional* plate equation (disregarding the x and y components of the displacement vector $\mathbf{q}$) with a source is[10]

$$D_{bend}\left(\nabla^4 \eta\right) + \rho_s h \frac{\partial^2 \eta}{\partial t^2} = g(x,y,t) \tag{100}$$

where

$$\nabla^4 \eta \equiv \frac{\partial^4 \eta}{\partial x^4} + 2\frac{\partial^4 \eta}{\partial x^2 \partial y^2} + \frac{\partial^4 \eta}{\partial y^4} \tag{101}$$

The net units in (100) are N/m^2, as the units of $\rho_s h \frac{\partial^2 \eta}{\partial t^2}$ are $\frac{kg}{m^3} \cdot m \cdot \frac{N}{kg} = \frac{N}{m^2}$.

[10] The general von Karman equations are considered in §3.4. A more detailed discussion of these equations can be found in [5].

3.3.1 Non-dimensionalization of Linearized von Karman Plate Equations ($\beta = \gamma = 0$)

Usually, the dimensional and non-dimensional plate equations are pulled from textbooks and it is not clear precisely how to transform from the former set of variables to the latter. Since the fluid equations are *dimensional* but the structure code used in the implementation solves the *dimensionless* plate equations, one would like to derive the relation between the dimensional and the dimensionless von Karman equations. The goal here is to figure out how to non-dimensionalize the equation (100).

In (100), $\eta(x,y,t)$ is the dependent variable and x,y,t are the independent variables. To perform the non-dimensionalization, define the following non-dimensional variables:

$$x^* = \frac{x}{x_0} \tag{102}$$

$$y^* = \frac{y}{y_0} \tag{103}$$

$$t^* = \frac{t}{t_0} \tag{104}$$

$$\eta^* = \frac{\eta}{\eta_0} \tag{105}$$

By the chain rule,

$$\frac{\partial \eta^*}{\partial x^*} = \frac{\partial \eta^*/\partial x}{\partial x^*/\partial x} = \frac{\frac{\partial \eta^*}{\partial \eta} \cdot \frac{\partial \eta}{\partial x}}{\frac{\partial x^*}{\partial x}} = \frac{\frac{1}{\eta_0} \cdot \frac{\partial \eta}{\partial x}}{\frac{1}{x_0}} = \frac{x_0}{\eta_0}\frac{\partial \eta}{\partial x} \Rightarrow \frac{\partial \eta}{\partial x} = \frac{\eta_0}{x_0}\frac{\partial \eta^*}{\partial x^*} \tag{106}$$

$$\frac{\partial^2 \eta^*}{\partial x^{*2}} = \frac{\partial}{\partial x^*}\left(\frac{\partial \eta^*}{\partial x^*}\right) = \frac{\partial\left(\frac{\partial \eta^*}{\partial x^*}\right)/\partial x}{\partial x^*/\partial x} = \frac{\frac{x_0}{\eta_0}\frac{\partial^2 \eta}{\partial x^2}}{\frac{1}{x_0}} = \frac{x_0^2}{\eta_0}\frac{\partial^2 \eta}{\partial x^2} \Rightarrow \frac{\partial^2 \eta}{\partial x^2} = \frac{\eta_0}{x_0^2}\frac{\partial^2 \eta^*}{\partial x^{*2}} \tag{107}$$

and similarly for the other derivatives. It follows that

$$\nabla^4 \eta = \left(\frac{\eta_0}{x_0^4}\right)\frac{\partial^4 \eta^*}{\partial x^{*4}} + 2\left(\frac{\eta_0}{x_0^2 y_0^2}\right)\frac{\partial^4 \eta^*}{\partial x^{*2} y^{*2}} + \left(\frac{\eta_0}{y_0^4}\right)\frac{\partial^4 \eta^*}{\partial y^{*4}} \tag{108}$$

Substituting (108) into (100) gives

$$\left(\frac{\rho_s h \eta_0}{t_0^2}\right)\frac{\partial^2 \eta^*}{\partial t^{*2}} + \left(\frac{D_{bend}\eta_0}{x_0^4}\right)\frac{\partial^4 \eta^*}{\partial x^{*4}} + 2\left(\frac{D_{bend}\eta_0}{x_0^2 y_0^2}\right)\frac{\partial^4 \eta^*}{\partial x^{*2} y^{*2}} + \left(\frac{D_{bend}\eta_0}{y_0^4}\right)\frac{\partial^4 \eta^*}{\partial y^{*4}} = g \tag{109}$$

Dividing through by the constant in front of the $\partial^2 \eta^*/\partial t^{*2}$ term,

$$\frac{\partial^2 \eta^*}{\partial t^{*2}} + \left(\frac{D_{bend}t_0^2}{\rho_s h x_0^4}\right)\frac{\partial^4 \eta^*}{\partial x^{*4}} + 2\left(\frac{D_{bend}t_0^2}{\rho_s h x_0^2 y_0^2}\right)\frac{\partial^4 \eta^*}{\partial x^{*2} y^{*2}} + \left(\frac{D_{bend}t_0^2}{\rho_s h y_0^4}\right)\frac{\partial^4 \eta^*}{\partial y^{*4}} = \frac{g t_0^2}{\rho_s h \eta_0} \tag{110}$$

To non-dimensionalize, set

$$\boxed{x_0 = y_0 = \eta_0 \equiv h, \qquad t_0^2 \equiv \frac{\rho_s h^5}{D_{bend}}} \tag{111}$$

where again h is the thickness of the plate. Then (110) becomes

$$\boxed{\frac{\partial^2 \eta^*}{\partial t^{*2}} + \nabla^4 \eta^* = \tilde{g}} \tag{112}$$

where $\tilde{g} \equiv \frac{gh^3}{D_{bend}}$. Equation (112) has the same form as the non-dimensional equation given in [5]. The dimensionlessness of the equation is easily verified by checking the units of $\tilde{g}$:

$$\text{units } \tilde{g}: \qquad \frac{gh^3}{D_{bend}} = \frac{\frac{N}{m^2} \cdot m^3}{N \cdot m} = \frac{N \cdot m}{N \cdot m} = 1 \tag{113}$$

□

3.4 General Nonlinear Dimensional von Karman Equations for $\beta, \gamma \neq 0$

Recall our notation for the components of the displacement vector: $\mathbf{q} = \begin{pmatrix} \beta & \gamma & \eta \end{pmatrix}$. Before, we dealt only with the z-component, η, assuming the other two components are zero. This reduced the von Karman equations substantially. We now consider the more general case when $\beta, \gamma \neq 0$. Then the von Karman equations, taken from [5], are

$$g(x,y,t) = \begin{array}{l} D_{bend}\left(\nabla^4 \eta\right) + \rho_s h \frac{\partial^2 \eta}{\partial t^2} - \frac{12 D_{bend}}{h^2}\Big[(1-\nu)\left(\frac{\partial \eta}{\partial x}\right)\left(\frac{\partial \eta}{\partial y}\right)\left(\frac{\partial^2 \eta}{\partial x \partial y}\right) \\ + \left(\frac{\partial \beta}{\partial x} + \frac{1}{2}\left(\frac{\partial \eta}{\partial x}\right)^2\right)\left(\frac{\partial^2 \eta}{\partial x^2} + \nu \frac{\partial^2 \eta}{\partial y^2}\right) + (1-\nu)\left(\frac{\partial \beta}{\partial y}\right)\left(\frac{\partial^2 \eta}{\partial x \partial y}\right) \\ + \left(\frac{\partial \gamma}{\partial y} + \frac{1}{2}\left(\frac{\partial \eta}{\partial y}\right)^2\right)\left(\frac{\partial^2 \eta}{\partial y^2} + \nu \frac{\partial^2 \eta}{\partial x^2}\right) + (1-\nu)\left(\frac{\partial \gamma}{\partial x}\right)\left(\frac{\partial^2 \eta}{\partial x \partial y}\right)\Big] \end{array} \tag{114}$$

$$\frac{\partial^2 \beta}{\partial x^2} + \left(\frac{1-\nu}{2}\right)\frac{\partial^2 \beta}{\partial y^2} + \left(\frac{1+\nu}{2}\right)\frac{\partial^2 \gamma}{\partial x \partial y} = -\Omega_1 \tag{115}$$

$$\frac{\partial^2 \gamma}{\partial y^2} + \left(\frac{1-\nu}{2}\right)\frac{\partial^2 \gamma}{\partial x^2} + \left(\frac{1+\nu}{2}\right)\frac{\partial^2 \beta}{\partial x \partial y} = -\Omega_2 \tag{116}$$

where

$$\Omega_1 \equiv \left(\frac{\partial \eta}{\partial x}\right)\left(\frac{\partial^2 \eta}{\partial x^2}\right) + \left(\frac{1+\nu}{2}\right)\left(\frac{\partial \eta}{\partial y}\right)\left(\frac{\partial^2 \eta}{\partial x \partial y}\right) + \left(\frac{1-\nu}{2}\right)\left(\frac{\partial \eta}{\partial x}\right)\left(\frac{\partial^2 \eta}{\partial y^2}\right) \tag{117}$$

$$\Omega_2 \equiv \left(\frac{\partial \eta}{\partial y}\right)\left(\frac{\partial^2 \eta}{\partial y^2}\right) + \left(\frac{1+\nu}{2}\right)\left(\frac{\partial \eta}{\partial x}\right)\left(\frac{\partial^2 \eta}{\partial x \partial y}\right) + \left(\frac{1-\nu}{2}\right)\left(\frac{\partial \eta}{\partial y}\right)\left(\frac{\partial^2 \eta}{\partial x^2}\right) \tag{118}$$

Defining x^*, y^*, t^* and η^* as before and

$$\beta^* = \frac{\beta}{\beta_0} \tag{119}$$

$$\gamma^* = \frac{\gamma}{\gamma_0} \tag{120}$$

one can apply the chain rule and substitute the derivatives in terms of the $*$ variables into (114), (115) and (116). Doing so gives

$$
g = \begin{array}{l}
\left(\frac{\rho_s h \eta_0}{t_0^2}\right)\frac{\partial^2 \eta^*}{\partial t^{*2}} + \left(\frac{D_{bend}\eta_0}{x_0^4}\right)\frac{\partial^4 \eta^*}{\partial x^{*4}} + 2\left(\frac{D_{bend}\eta_0}{x_0^2 y_0^2}\right)\frac{\partial^4 \eta^*}{\partial x^{*2} y^{*2}} + \left(\frac{D_{bend}\eta_0}{y_0^4}\right)\frac{\partial^4 \eta^*}{\partial y^{*4}} - 12\Bigg[(1-\nu)\left(\frac{D_{bend}\eta_0^3}{h^2 x_0^2 y_0^2}\right)\left(\frac{\partial \eta^*}{\partial x^*}\cdot\frac{\partial \eta^*}{\partial y^*}\cdot\frac{\partial^2 \eta^*}{\partial x^* \partial y^*}\right) \\
+\left(\left(\frac{D_{bend}\beta_0}{h^2 x_0}\right)\frac{\partial \beta^*}{\partial x^*} + \frac{1}{2}\left(\frac{D_{bend}\eta_0^2}{h^2 x_0^2}\right)\left(\frac{\partial \eta^*}{\partial x^*}\right)^2\right)\left(\left(\frac{\eta_0}{x_0^2}\right)\frac{\partial^2 \eta^*}{\partial x^{2*}} + \nu\left(\frac{\eta_0}{y_0^2}\right)\frac{\partial^2 \eta^*}{\partial y^{*2}}\right) + (1-\nu)\left(\frac{D_{bend}\beta_0\eta_0}{h^2 x_0 y_0^2}\right)\left(\frac{\partial \beta^*}{\partial y^*}\cdot\frac{\partial^2 \eta^*}{\partial x^* \partial y^*}\right) \\
+\left(\left(\frac{D_{bend}\gamma_0}{h^2 y_0}\right)\frac{\partial \gamma^*}{\partial y^*} + \frac{1}{2}\left(\frac{D_{bend}\eta_0^2}{h^2 y_0^2}\right)\left(\frac{\partial \eta^*}{\partial y^*}\right)^2\right)\left(\left(\frac{\eta_0}{y_0^2}\right)\frac{\partial^2 \eta^*}{\partial y^{*2}} + \nu\left(\frac{\eta_0}{x_0^2}\right)\frac{\partial^2 \eta^*}{\partial x^{*2}}\right) + (1-\nu)\left(\frac{D_{bend}\gamma_0\eta_0}{h^2 x_0^2 y_0}\right)\left(\frac{\partial \gamma^*}{\partial x^*}\cdot\frac{\partial^2 \eta^*}{\partial x^* \partial y^*}\right)\Bigg]
\end{array} \tag{121}
$$

$$
\left(\frac{\beta_0}{x_0^2}\right)\frac{\partial^2 \beta^*}{\partial x^{*2}} + \left(\frac{1-\nu}{2}\right)\left(\frac{\beta_0}{y_0^2}\right)\frac{\partial^2 \beta^*}{\partial y^{*2}} + \left(\frac{1+\nu}{2}\right)\left(\frac{\gamma_0}{x_0 y_0}\right)\frac{\partial^2 \gamma^*}{\partial x^* \partial y^*} = -\Omega_1 \tag{122}
$$

$$
\left(\frac{\gamma_0}{y_0^2}\right)\frac{\partial^2 \gamma^*}{\partial y^{*2}} + \left(\frac{1-\nu}{2}\right)\left(\frac{\gamma_0}{x_0^2}\right)\frac{\partial^2 \gamma^*}{\partial x^{*2}} + \left(\frac{1+\nu}{2}\right)\left(\frac{\beta_0}{x_0 y_0}\right)\frac{\partial^2 \beta^*}{\partial x^* \partial y^*} = -\Omega_2 \tag{123}
$$

with

$$
\Omega_1 = \left(\frac{\eta_0^2}{x_0^3}\right)\left(\frac{\partial \eta^*}{\partial x^*}\cdot\frac{\partial^2 \eta^*}{\partial x^{*2}}\right) + \left(\frac{1+\nu}{2}\right)\left(\frac{\eta_0^2}{x_0 y_0^2}\right)\left(\frac{\partial \eta^*}{\partial y^*}\cdot\frac{\partial^2 \eta^*}{\partial x^* \partial y^*}\right) + \left(\frac{1-\nu}{2}\right)\left(\frac{\eta_0^2}{x_0 y_0^2}\right)\left(\frac{\partial \eta^*}{\partial x^*}\cdot\frac{\partial^2 \eta^*}{\partial y^{*2}}\right) \tag{124}
$$

$$
\Omega_2 = \left(\frac{\eta_0^2}{y_0^3}\right)\left(\frac{\partial \eta^*}{\partial y^*}\cdot\frac{\partial^2 \eta^*}{\partial y^{*2}}\right) + \left(\frac{1+\nu}{2}\right)\left(\frac{\eta_0^2}{x_0^2 y_0}\right)\left(\frac{\partial \eta^*}{\partial x^*}\cdot\frac{\partial^2 \eta^*}{\partial x^* \partial y^*}\right) + \left(\frac{1-\nu}{2}\right)\left(\frac{\eta_0^2}{x_0^2 y_0}\right)\left(\frac{\partial \eta^*}{\partial y^*}\cdot\frac{\partial^2 \eta^*}{\partial x^{*2}}\right) \tag{125}
$$

To non-dimensionalize, set

$$
\boxed{x_0 = y_0 = \beta_0 = \gamma_0 = \eta_0 \equiv h, \qquad t_0^2 \equiv \frac{\rho_s h^5}{D_{bend}}} \tag{126}
$$

where h is the thickness of the plate (in m, for example). Then the above equations become

$$
g = \begin{array}{l}
\left(\frac{D_{bend}}{h^3}\right)\frac{\partial^2 \eta^*}{\partial t^{*2}} + \left(\frac{D_{bend}}{h^3}\right)\frac{\partial^4 \eta^*}{\partial x^{*4}} + 2\left(\frac{D_{bend}}{h^3}\right)\frac{\partial^4 \eta^*}{\partial x^{*2} y^{*2}} + \left(\frac{D_{bend}}{h^3}\right)\frac{\partial^4 \eta^*}{\partial y^{*4}} - 12\Big[(1-\nu)\left(\frac{D_{bend}}{h^3}\right)\left(\frac{\partial \eta^*}{\partial x^*}\cdot\frac{\partial \eta^*}{\partial y^*}\cdot\frac{\partial^2 \eta^*}{\partial x^* \partial y^*}\right) \\
+\left(\left(\frac{D_{bend}}{h^2}\right)\frac{\partial \beta^*}{\partial x^*} + \frac{1}{2}\left(\frac{D_{bend}}{h^2}\right)\left(\frac{\partial \eta^*}{\partial x^*}\right)^2\right)\left(\left(\frac{1}{h}\right)\frac{\partial^2 \eta^*}{\partial x^{2*}} + \nu\left(\frac{1}{h}\right)\frac{\partial^2 \eta^*}{\partial y^{*2}}\right) + (1-\nu)\left(\frac{D_{bend}}{h^3}\right)\left(\frac{\partial \beta^*}{\partial y^*}\cdot\frac{\partial^2 \eta^*}{\partial x^* \partial y^*}\right) \\
+\left(\left(\frac{D_{bend}}{h^2}\right)\frac{\partial \gamma^*}{\partial y^*} + \frac{1}{2}\left(\frac{D_{bend}}{h^2}\right)\left(\frac{\partial \eta^*}{\partial y^*}\right)^2\right)\left(\left(\frac{1}{h}\right)\frac{\partial^2 \eta^*}{\partial y^{*2}} + \nu\left(\frac{1}{h}\right)\frac{\partial^2 \eta^*}{\partial x^{*2}}\right) + (1-\nu)\left(\frac{D_{bend}}{h^3}\right)\left(\frac{\partial \gamma^*}{\partial x^*}\cdot\frac{\partial^2 \eta^*}{\partial x^* \partial y^*}\right)\Big]
\end{array} \tag{127}
$$

$$
\left(\frac{1}{h}\right)\frac{\partial^2 \beta^*}{\partial x^{*2}} + \left(\frac{1-\nu}{2}\right)\left(\frac{1}{h}\right)\frac{\partial^2 \beta^*}{\partial y^{*2}} + \left(\frac{1+\nu}{2}\right)\left(\frac{1}{h}\right)\frac{\partial^2 \gamma^*}{\partial x^* \partial y^*} = -\Omega_1 \tag{128}
$$

$$
\left(\frac{1}{h}\right)\frac{\partial^2 \gamma^*}{\partial y^{*2}} + \left(\frac{1-\nu}{2}\right)\left(\frac{1}{h}\right)\frac{\partial^2 \gamma^*}{\partial x^{*2}} + \left(\frac{1+\nu}{2}\right)\left(\frac{1}{h}\right)\frac{\partial^2 \beta^*}{\partial x^* \partial y^*} = -\Omega_2 \tag{129}
$$

with

$$
\Omega_1 = \left(\frac{1}{h}\right)\left(\frac{\partial \eta^*}{\partial x^*}\cdot\frac{\partial^2 \eta^*}{\partial x^{*2}}\right) + \left(\frac{1+\nu}{2}\right)\left(\frac{1}{h}\right)\left(\frac{\partial \eta^*}{\partial y^*}\cdot\frac{\partial^2 \eta^*}{\partial x^* \partial y^*}\right) + \left(\frac{1-\nu}{2}\right)\left(\frac{1}{h}\right)\left(\frac{\partial \eta^*}{\partial x^*}\cdot\frac{\partial^2 \eta^*}{\partial y^{*2}}\right) \tag{130}
$$

$$\Omega_2 = \left(\frac{1}{h}\right)\left(\frac{\partial\eta^*}{\partial y^*}\cdot\frac{\partial^2\eta^*}{\partial y^{*2}}\right) + \left(\frac{1+\nu}{2}\right)\left(\frac{1}{h}\right)\left(\frac{\partial\eta^*}{\partial x^*}\cdot\frac{\partial^2\eta^*}{\partial x^*\partial y^*}\right) + \left(\frac{1-\nu}{2}\right)\left(\frac{1}{h}\right)\left(\frac{\partial\eta^*}{\partial y^*}\cdot\frac{\partial^2\eta^*}{\partial x^{*2}}\right) \tag{131}$$

Dividing through by the appropriate coefficients results in the following non-dimensional set of von Karman plate equations:

$$\boxed{\tilde{g} \equiv \frac{gh^3}{D_{bend}} = \begin{array}{l} \frac{\partial^2\eta^*}{\partial t^{*2}} + \nabla^4\eta^* - 12\Big[(1-\nu)\left\{\frac{\partial\eta^*}{\partial x^*}\cdot\frac{\partial\eta^*}{\partial y^*} + \frac{\partial\beta^*}{\partial y^*} + \frac{\partial\gamma^*}{\partial x^*}\right\}\frac{\partial^2\eta^*}{\partial x^*\partial y^*} \\ + \left(\frac{\partial\beta^*}{\partial x^*} + \frac{1}{2}\left(\frac{\partial\eta^*}{\partial x^*}\right)^2\right)\left(\frac{\partial^2\eta^*}{\partial x^{2*}} + \nu\frac{\partial^2\eta^*}{\partial y^{*2}}\right) + \left(\frac{\partial\gamma^*}{\partial y^*} + \frac{1}{2}\left(\frac{\partial\eta^*}{\partial y^*}\right)^2\right)\left(\frac{\partial^2\eta^*}{\partial y^{*2}} + \nu\frac{\partial^2\eta^*}{\partial x^{*2}}\right)\Big] \end{array}} \tag{132}$$

$$\boxed{\frac{\partial^2\beta^*}{\partial x^{*2}} + \left(\frac{1-\nu}{2}\right)\frac{\partial^2\beta^*}{\partial y^{*2}} + \left(\frac{1+\nu}{2}\right)\frac{\partial^2\gamma^*}{\partial x^*\partial y^*} = -\Omega_1} \tag{133}$$

$$\boxed{\frac{\partial^2\gamma^*}{\partial y^{*2}} + \left(\frac{1-\nu}{2}\right)\frac{\partial^2\gamma^*}{\partial x^{*2}} + \left(\frac{1+\nu}{2}\right)\frac{\partial^2\beta^*}{\partial x^*\partial y^*} = -\Omega_2} \tag{134}$$

where

$$\boxed{\Omega_1 = \left(\frac{\partial\eta^*}{\partial x^*}\cdot\frac{\partial^2\eta^*}{\partial x^{*2}}\right) + \left(\frac{1+\nu}{2}\right)\left(\frac{\partial\eta^*}{\partial y^*}\cdot\frac{\partial^2\eta^*}{\partial x^*\partial y^*}\right) + \left(\frac{1-\nu}{2}\right)\left(\frac{\partial\eta^*}{\partial x^*}\cdot\frac{\partial^2\eta^*}{\partial y^{*2}}\right)} \tag{135}$$

$$\boxed{\Omega_2 = \left(\frac{\partial\eta^*}{\partial y^*}\cdot\frac{\partial^2\eta^*}{\partial y^{*2}}\right) + \left(\frac{1+\nu}{2}\right)\left(\frac{\partial\eta^*}{\partial x^*}\cdot\frac{\partial^2\eta^*}{\partial x^*\partial y^*}\right) + \left(\frac{1-\nu}{2}\right)\left(\frac{\partial\eta^*}{\partial y^*}\cdot\frac{\partial^2\eta^*}{\partial x^{*2}}\right)} \tag{136}$$

One can check that these equations are really dimensionless by checking the units of $\tilde{g}$, as before:

$$\text{units } \tilde{g} : \frac{gh^3}{D_{bend}} = \frac{\frac{N}{m^2}\cdot m^3}{N\cdot m} = \frac{N\cdot m}{N\cdot m} = 1 \tag{137}$$

The boxed equations above are consistent with what is given on the first page of [5].

□

4 The Coupled (Linearized) System

Our aim here is to write the coupled system of equations describing the fluid motion around a solid plate in the $z = 0$ plane having the form:

$$\boxed{\begin{array}{rcl} \begin{pmatrix} \dot{F} \\ \dot{S} \end{pmatrix} & = & \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} F \\ S \end{pmatrix} \\ \begin{pmatrix} F \\ S \end{pmatrix}_{t=0} & = & \begin{pmatrix} F_0 \\ S_0 \end{pmatrix} \end{array}} \tag{138}$$

The coupling matrices B and C come from the implementation of the appropriate BCs.

Recall that $U' \equiv \begin{pmatrix} u' & v' & w' & \zeta' & p' \end{pmatrix}^T \in \mathbb{R}^5$ are the perturbed fluid variables (so $U = \bar{U} + U'$ where $\bar{U}$ is the base state) and $\mathbf{q} = \begin{pmatrix} \beta & \gamma & \eta \end{pmatrix}^T \in \mathbb{R}^3$ is the displacement in each of the x, y, z coordinates respectively (so β and γ are displacements in the plane and η is the displacement out of the plane[11]. In this Reduced Order Model, we expanded U' and the z-component[12] of the displacement vector in the respective orthonormal basis:

$$U' = \sum_{k=1}^{M} a_k(t)\phi(\mathbf{x}) \tag{139}$$

$$\eta(x,y,t) = \sum_{k=1}^{P} b_k(t)\xi_k(x,y) \tag{140}$$

Since the original coupled fluid/structure system of PDEs will become a coupled system of ODEs in the context of the ROM, the F and S vectors in (138) will contain the $a_k(t)$ coefficients and $b_k(t)$ and $\dot{b}_k(t)$ respectively:

$$F \equiv \begin{pmatrix} \mathbf{a}(t) \end{pmatrix} = \begin{pmatrix} a_1(t) \\ \vdots \\ a_M(t) \end{pmatrix} \in \mathbb{R}^M \tag{141}$$

$$S \equiv \begin{pmatrix} \mathbf{b}(t) \\ \dot{\mathbf{b}}(t) \end{pmatrix} = \begin{pmatrix} b_1(t) \\ \vdots \\ b_P(t) \\ \dot{b}_1(t) \\ \vdots \\ \dot{b}_P(t) \end{pmatrix} \in \mathbb{R}^{2P} \tag{142}$$

The matrices A, B, C and D describing the system (138) will be determined by the linearized Euler equations for the fluid flow, the linear von Karman plate equations for the plate motion (with all non-linear terms in the full set of equations omitted) and the appropriate no-penetration boundary condition, whose implementation was discussed in §2.2.

4.1 Fluid Side

For the fluid side, we consider the linearized Euler equations:

$$\dot{U}' + A_x(\bar{U})\frac{\partial U'}{\partial x} + A_y(\bar{U})\frac{\partial U'}{\partial y} + A_z(\bar{U})\frac{\partial U'}{\partial z} + C(\bar{U}, \nabla\bar{U})U' = 0 \tag{143}$$

where the $A_x(\bar{U}), A_y(\bar{U}), A_y(\bar{U})$ and $C(\bar{U}, \nabla\bar{U})$ matrices are derived in [1] and are given in (4) and (5) respectively. Let

$$A(\bar{U}) \equiv \begin{pmatrix} A_x(\bar{U}) & A_y(\bar{U}) & A_z(\bar{U}) \end{pmatrix} \tag{144}$$

and define the linear operator

$$\mathscr{L}U' \equiv -A(\bar{U}) \cdot \nabla U' - C(\bar{U}, \nabla\bar{U})U' \tag{145}$$

[11] In Dan's notation, $\beta \equiv u$, $\gamma \equiv v$ and $\eta \equiv w$. The notation has been changed in this document to avoid confusion with the fluid velocities, denoted by u, v and w. Note also that in the notation of [5], $b_j(t) \equiv \alpha_j(t)$, $\xi_j(x,y) \equiv W_j(x,y)$, $\theta_{mn}(x,y) \equiv U_{mn}(x,y)$, $\phi_{mn}(x,y) \equiv V_{mn}(x,y)$.

[12] The x and y components will be omitted in this analysis to ensure the resulting system is linear. See the "Structure Equations" section for more on this.

Taking the Galerkin projection of equation (143) onto each POD mode ϕ_j gives

$$\left(\dot{U}',\phi_j\right)_H = \left(\mathscr{L}U',\phi_j\right)_H \tag{146}$$

for

$$\boxed{(u,v)_H \equiv \int_\Omega u^T H(\bar{U}) v d\Omega} \tag{147}$$

The $H(\bar{U})$ operator is defined in [1] and is stated explicitly in (11). Using the orthonormality of the basis functions ϕ_j, (146) reduces to the following linear system of ODEs:

$$\dot{a}_k(t) = \sum_{l=1}^{M} a_l(t)\,(\phi_k,\mathscr{L}\phi_l)_H \tag{148}$$

where $k = 1,\ldots,M$. Applying the definition of the inner product (147) to (148) and integrating by parts (where the integral in (149) is over the *entire* boundary $\partial\Omega = \partial\Omega_P \cup \partial\Omega_F$) gives

$$\begin{aligned}\dot{a}_k(t) &= \textstyle\sum_{l=1}^{M} a_l(t) \int_\Omega \phi_k^T H(\bar{U}) \mathscr{L}\phi_l d\Omega \\ &= \textstyle\sum_{l=1}^{M} a_l(t) \left\{ \int_\Omega \phi_k^T H(\bar{U}) \left[-A(\bar{U}) \cdot \nabla\phi_l - C(\bar{U},\nabla\bar{U})\phi_l \right] d\Omega \right\} \\ &= \textstyle\sum_{l=1}^{M} a_l(t) \left\{ \int_{\partial\Omega} -\phi_k^T H(\bar{U})[A(\bar{U})\cdot\mathbf{n}]\phi_l dS + \int_\Omega \left(\nabla\cdot\phi_k^T H(\bar{U})A(\bar{U}) - \phi_k^T H(\bar{U})C(\bar{U},\nabla\bar{U})\right)\phi_l d\Omega \right\} \\ &= \textstyle\int_{\partial\Omega} -\phi_k^T H(\bar{U})[A(\bar{U})\cdot\mathbf{n}]U' dS + \sum_{l=1}^{M} a_l(t) \int_\Omega \left(\nabla\cdot\phi_k^T H(\bar{U})A(\bar{U}) - \phi_k^T H(\bar{U})C(\bar{U},\nabla\bar{U})\right)\phi_l d\Omega \\ &= \textstyle -I_S + \sum_{l=1}^{M} a_l(t) \int_\Omega \left(\nabla\cdot\phi_k^T H(\bar{U})A(\bar{U}) - \phi_k^T H(\bar{U})C(\bar{U},\nabla\bar{U})\right)\phi_l d\Omega\end{aligned} \tag{149}$$

Recall that the linearized no-penetration boundary condition in the context of the problem in question (with the fluid flow in the x-direction and a plate lying in the $z = 0$ plane)

$$\dot{\eta} = w' + \bar{u}\frac{\partial\eta}{\partial x} \quad \text{on} \quad \partial\Omega_P \tag{150}$$

was implemented in §2 and an approximate non-reflecting far-field boundary condition was implemented in §3. In particular, one had in (82) that

$$I_S = \sum_{k=1}^{M} a_k(t) \left(\int_{\partial\Omega_P} \phi_j^3\phi_k^5 dS + \int_{\partial\Omega_F} h_k(\phi_j) dS \right) + \sum_{k=1}^{P} \dot{b}_k(t) \int_{\partial\Omega_P} \phi_j^5 \xi_k dS - \sum_{k=1}^{P} b_k(t) \int_{\partial\Omega_P} \phi_j^5 \bar{u}\frac{\partial\xi_k}{\partial x} dS \tag{151}$$

Substituting (151) into (149) yields, for $k = 1,\ldots,M$,

$$\begin{aligned}\dot{a}_k(t) = &\textstyle\sum_{l=1}^{M} a_l(t) \left\{ -\left(\int_{\partial\Omega_P} \phi_k^3\phi_l^5 dS + \int_{\partial\Omega_F} h_l(\phi_k) dS \right) + \int_\Omega \left(\nabla\cdot\phi_k^T H(\bar{U})A(\bar{U}) - \phi_k^T H(\bar{U})C(\bar{U},\nabla\bar{U})\right)\phi_l d\Omega \right\} \\ &\textstyle + \sum_{l=1}^{P} b_l(t) \int_{\partial\Omega_P} \left(\phi_k^5 \bar{u}\frac{\partial\xi_l}{\partial x} \right) dS + \sum_{l=1}^{P} \dot{b}_l(t) \int_{\partial\Omega_P} \left(-\phi_k^5\xi_l\right) dS\end{aligned} \tag{152}$$

where $h_l(\phi_k)$ depends on the four cases considered §2.3 and is given explicitly in (81). (152) defines a system of equations that can also be written in matrix form:

$$\boxed{\dot{F} = \begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} F \\ S \end{pmatrix}} \tag{153}$$

where the entries of the $A \in \mathbb{R}^{M\times M}$ and $B \in \mathbb{R}^{M\times 2P}$ are

$$\boxed{A(i,j) = -\left(\int_{\partial\Omega_P} \phi_i^3 \phi_j^5 dS + \int_{\partial\Omega_F} h_j(\phi_i) dS\right) + \int_\Omega \left(\nabla\cdot \phi_i^T H(\bar{U})A(\bar{U}) - \phi_i^T H(\bar{U})C(\bar{U},\nabla\bar{U})\right)\phi_j d\Omega, \qquad 1 \le i,j \le M} \tag{154}$$

$$\boxed{B(i,j) = \begin{cases} \int_{\partial\Omega_P} \left(\phi_i^5 \bar{u}\frac{\partial \xi_j}{\partial x}\right) dS, & 1 \le i \le M, \ \ 1 \le j \le P \\ \int_{\partial\Omega} \left(-\phi_i^5 \xi_{j-P}\right) dS, & 1 \le i \le M, \ \ (P+1) \le j \le 2P \end{cases}} \tag{155}$$

These equations will be written in matrix form shortly. Note that the integrals in (155) and the surface integral in (154) are L^2 inner products.

4.2 Structure Side

At this stage in the analysis, we require the stability of only the coupled *linear* system (138). As discussed in the "Structure Equations" section, the motion of the plate is assumed to follow the von Karman plate equations, which are in general non-linear. In order to end up with a linear set of structure equations, we disregard the non-linear terms, which entails ignoring the in-plane displacements and thereby keeping only the η component of the displacement vector[13] $\mathbf{q} \equiv \begin{pmatrix} \beta & \gamma & \eta \end{pmatrix}^T$. Let us consider the *dimensional* equation (100). Doing so will make it easier to interpret the energy matrices derived in a physical context.

Recall that the *dimensional* linear von Karman equation for the z-component of the displacement is

$$(\rho_s h)\ddot{\eta} = -D_{bend}\left(\nabla^4\eta\right) + g \tag{156}$$

where $\nabla^4 = (\nabla^2)^2 = (\Delta)^2$ is the biharmonic operator with the derivatives with respect to z omitted, given by

$$\nabla^4 = \frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} \tag{157}$$

and D_{bend} is a constant, namely the bending stiffness. The function $g(x,y,t)$ will be defined later in this document to be the pressure (and indeed has the units of pressure: N/m^2). We will project equation (156) onto the orthogonal POD modes $\xi_k(x,y)$ using the standard L^2 inner product

$$(u,v)_{L^2} \equiv \int_{\partial\Omega} uv dS. \tag{158}$$

Let $(u,v)_{L^2}^P$ denote the L^2 inner product over the plate boundary $\partial\Omega_P$ and $(u,v)_{L^2}^F$ denote the L^2 inner product over the far-field boundary $\partial\Omega_F$. Earlier,the z-component of the displacement was expanded in the appropriate orthonormal eigenmode basis, i.e., the $\xi_j(x,y)$:

$$\eta = \sum_{j=1}^{P} b_j(t)\xi_j(x,y) \tag{159}$$

Substituting (159) into (156) one has, for $k = 1,\ldots,P$,

[13] See §3.3.

$$
\begin{array}{rcl}
\rho_s h \left(\sum_{l=1}^{P} \ddot{b}_l(t)\xi_l, \xi_k\right)_{L^2}^{P} & = & \left(D_{bend}\nabla^4\left(-\sum_{l=1}^{P} b_l(t)\xi_l\right) + g, \xi_k\right)_{L^2}^{P} \\
\rho_s h \sum_{l=1}^{P} \ddot{b}_l(t) \left(\xi_l, \xi_k\right)_{L^2}^{P} & = & -D_{bend} \sum_{l=1}^{P} \left(b_l(t)\nabla^4\xi_l, \xi_k\right)_{L^2}^{P} + \left(g, \xi_k\right)_{L^2}^{P} \\
\rho_s h \ddot{b}_k(t) & = & -D_{bend} \sum_{l=1}^{P} b_l(t) \left(\nabla^4\xi_l, \xi_k\right)_{L^2}^{P} + \left(g, \xi_k\right)_{L^2}^{P}
\end{array}
\tag{160}
$$

REMARK: Note that the units of the POD basis functions $\xi_j(x,y)$ are taken to be units of length, e.g., m. This means the L^2 inner product $(\xi_j, \xi_j)_{L^2}^P = 1$ has units of area, i.e., m^2. It follows that $\rho_s h = \rho_s h(\xi_j, \xi_j)_{L^2}^P$ actually has units of mass, not mass per unit area. We call attention to this fact here because it will come into play when the energy matrices for the coupled system are derived later in this document.

If we consider a square plate in the $z = 0$ plane with $0 \leq x, y \leq L$ and impose the BC of simply supported edges, it turns out that the functions $\nabla^4\xi_k$ and ξ_j are orthogonal, i.e., $\left(\nabla^4\xi_k, \xi_j\right)_{L^2}^P = 0$ for all $j \neq k$. Then the last line of (160) reduces to

$$
(\rho_s h)\, \ddot{b}_k(t) = -D_{bend} \left(\nabla^4\xi_k, \xi_k\right)_{L^2}^{P} b_k(t) + (g, \xi_k)_{L^2}^{P}
\tag{161}
$$

The ultimate purpose of this derivation will be to derive an energy matrix, call it E_D. It will be desired to decompose this matrix into two submatrices, one representing the mass and one the stiffness[14]. Defining

$$
\omega_k^2 \equiv D_{bend} \left(\nabla^4\xi_k, \xi_k\right)_{L^2}^{P} = D_{bend} \int_{\partial\Omega_P} \nabla^4\xi_k \xi_k dS
\tag{162}
$$

$$
G_k(t) \equiv (g, \xi_k)_{L^2}^{P} = \int_{\partial\Omega_P} g\xi_k dS
\tag{163}
$$

(161) becomes

$$
\boxed{(\rho_s h)\, \ddot{b}_k + \omega_k^2 b_k = G_k(t)}
\tag{164}
$$

(164) is actually the dimensional version of (14) on p. 4 of [5] (with the non-linear terms omitted).

Let us now specify the right hand side term $g(x,y,t)$ in (160) to be the unsteady fluid pressure loading, so $G_k(t)$ is the pressure loading term. The pressure is applied downward and the coordinate system is chosen in the direction of increasing z. It follows that

$$
g(x,y,t) = p(x,y,0,t)
\tag{165}
$$

and

$$
G_k(t) = (\xi_k, p)_{L^2}^{P} = \int_{\partial\Omega_P} \xi_k p(x,y,0,t) dS
\tag{166}
$$

Expanding the pressure in the appropriate component of the ϕ_j basis vector,

$$
p(x,y,0,t) = \sum_{i=1}^{M} a_i(t)\phi_i^5(\mathbf{x})
\tag{167}
$$

Substituting (167) into (166) gives

[14] See §5.2.3.

$$G_k(t) = \sum_{i=1}^{M} a_i(t) \int_{\partial\Omega_P} \phi_i^5 \xi_k dS = \sum_{i=1}^{M} \left(\phi_i^5, \xi_k\right)_{L^2}^{P} a_i(t) \tag{168}$$

Let

$$G \equiv \left(\begin{array}{c} 0 \\ \vdots \\ 0 \\ \hline G_1(t) \\ \vdots \\ G_P(t) \end{array}\right) \tag{169}$$

Then (160) can be written as a linear system in terms of the S vector, defined in (142):

$$\begin{array}{ccccccc} (\rho_s h)\,\frac{d}{dt}\begin{pmatrix} b_1(t) \\ \vdots \\ b_P(t) \\ \dot{b}_1(t) \\ \vdots \\ \dot{b}_P(t) \end{pmatrix} & = & \left(\begin{array}{c|c} 0_{P\times P} & (\rho_s h)\, I_{P\times P} \\ \hline -\tilde{L}_{P\times P} & 0_{P\times P} \end{array}\right) & \begin{pmatrix} b_1(t) \\ \vdots \\ b_P(t) \\ \dot{b}_1(t) \\ \vdots \\ \dot{b}_P(t) \end{pmatrix} & + & \left(\begin{array}{c} 0 \\ \vdots \\ 0 \\ \hline G_1(t) \\ \vdots \\ G_P(t) \end{array}\right) \\ (\rho_s h)\,\dot{S} & \equiv & L & S & + & G \end{array} \tag{170}$$

where $I_{P\times P}$ denotes the $P \times P$ identity matrix and $0_{P\times P}$ is the zero $P \times P$ matrix and

$$\tilde{L}_{P\times P} \equiv \begin{pmatrix} D_{bend}\left(\nabla^4 \xi_1, \xi_1\right)_{L^2}^{P} & 0 & \cdots \\ \vdots & \ddots & \vdots \\ 0 & \cdots & D_{bend}\left(\nabla^4 \xi_P, \xi_P\right)_{L^2}^{P} \end{pmatrix} \tag{171}$$

Here, $L \in \mathbb{R}^{2P\times 2P}$. Using (168), the vector G in (169) can be written as

$$G = JF \tag{172}$$

where

$$J \equiv \left(\begin{array}{ccc} & 0_{P\times M} & \\ \hline \left(\phi_1^5, \xi_1\right)_{L^2}^{P} & \cdots & \left(\phi_M^5, \xi_1\right)_{L^2}^{P} \\ \vdots & \ddots & \vdots \\ \left(\phi_1^5, \xi_P\right)_{L^2}^{P} & \cdots & \left(\phi_M^5, \xi_P\right)_{L^2}^{P} \end{array}\right) \in \mathbb{R}^{2P\times M} \tag{173}$$

Alternatively, one can write (170) as

$$\dot{S} = CF + DS \tag{174}$$

where $C \equiv \frac{1}{\rho_s h} J$ and $D \equiv \frac{1}{\rho_s h} L$.

4.3 Summary: Coupled Fluid/Structure System

We are now ready to write down the equations for the coupled fluid/structure system in the form (138). Recalling the definitions of the vectors F and S, the first containing the fluid ROM coefficients and the second containing the solid ROM coefficients and their time derivatives, the system has the form

$$\boxed{\begin{array}{ccccc} \begin{pmatrix} \dot{F} \\ \dot{S} \end{pmatrix} & = & \begin{pmatrix} A & B \\ C & D \end{pmatrix} & \begin{pmatrix} F \\ S \end{pmatrix} \\ \\ \dot{X} & = & K & X \end{array}} \tag{175}$$

Define

$$(u,v)_{dHA} \equiv \int_\Omega \nabla \cdot \left(u^T H(\bar{U}) A(\bar{U})\right) v d\Omega \tag{176}$$

Lemma 4.3.1. $(\cdot,\cdot)_{dHA} : \mathbb{R}^5 \times \mathbb{R}^5 \to \mathbb{R}$ *defined in* (176) *satisfies the first three axioms of an inner product.*

Proof. We check the first three inner product axioms:

1. The first axiom is that $(u+v,w)_{dHA} = (u,w)_{dHA} + (v,w)_{dHA}$. This follows immediately from the linearity of the divergence $\nabla\cdot$.
2. The second axiom to check is that $(\alpha u,v)_{dHA} = \alpha(u,v)_{dHA}$ for any scalar α. This property is also trivial, following directly from the linearity of the divergence operator.
3. We must show that the operator is symmetric, namely $(u,v)_{dHA} = (v,u)_{dHA}$. Applying the divergence theorem,

$$\begin{aligned} (u,v)_{dHA} &= \int_\Omega \nabla \cdot \left(u^T H(\bar{U}) A(\bar{U})\right) v d\Omega \\ &= \int_{\partial\Omega} u^T H(\bar{U}) \left[A(\bar{U}) v \cdot \mathbf{n}\right] dS - \int_\Omega u^T H(\bar{U}) \left[A(\bar{U}) \cdot \nabla v\right] d\Omega \\ &= \int_{\partial\Omega} \left[H(\bar{U}) u\right]^T \left[A(\bar{U}) v \cdot \mathbf{n}\right] dS - (u, A(\bar{U}) \cdot \nabla v)_H \end{aligned} \tag{177}$$

The first term in the last line of (177) is a surface integral of a vector dot product (because H is symmetric), which is an inner product and hence symmetric. The matrix H in the second term of the last line of (177) was actually derived such that $(\cdot,\cdot)_H$ defines an inner product and thus satisfies symmetry. It follows that, since $(u,v)_{dHA}$ is the sum of two symmetric operators, it too satisfies the symmetry property.

$\square$

Now, $A \in \mathbb{R}^{M\times M}$, $B \in \mathbb{R}^{M\times 2P}$, $C \in \mathbb{R}^{2P\times M}$ and $D \in \mathbb{R}^{2P\times 2P}$ matrices are given by

$$\boxed{A \equiv \begin{pmatrix} -\left(\phi_1^3,\phi_1^5\right)_{L^2}^P - (h_1(\phi_1),1)_{L^2}^F - (\phi_1, C(\bar{U},\nabla\bar{U})\phi_1)_H + (\phi_1,\phi_1)_{dHA} & \cdots & \cdots \\ -\left(\phi_2^3,\phi_1^5\right)_{L^2}^P - (h_1(\phi_2),1)_{L^2}^F - (\phi_2, C(\bar{U},\nabla\bar{U})\phi_1)_H + (\phi_2,\phi_1)_{dHA} & \cdots & \cdots \\ \vdots & & \vdots \\ \vdots & \ddots & \vdots \\ \vdots & & \vdots \\ -\left(\phi_M^3,\phi_1^5\right)_{L^2}^P - (h_1(\phi_M),1)_{L^2}^F - (\phi_M, C(\bar{U},\nabla\bar{U})\phi_1)_H + (\phi_M,\phi_1)_{dHA} & \cdots & \cdots \end{pmatrix} \in \mathbb{R}^{M\times M}} \tag{178}$$

$$
B \equiv \left(\begin{array}{cccc|cccc}
\left(\phi_1^5, \overline{u}\frac{\partial \xi_1}{\partial x}\right)_{L^2}^P & \cdots & \left(\phi_1^5, \overline{u}\frac{\partial \xi_P}{\partial x}\right)_{L^2}^P & & -\left(\phi_1^5, \xi_1\right)_{L^2}^P & \cdots & -\left(\phi_1^5, \xi_P\right)_{L^2}^P \\
\vdots & & \vdots & & \vdots & & \vdots \\
\vdots & \ddots & \vdots & & \vdots & \ddots & \vdots \\
\vdots & & \vdots & & \vdots & & \vdots \\
\left(\phi_M^5, \overline{u}\frac{\partial \xi_M}{\partial x}\right)_{L^2}^P & \cdots & \left(\phi_M^5, \overline{u}\frac{\partial \xi_P}{\partial x}\right)_{L^2}^P & & -\left(\phi_M^5, \xi_1\right)_{L^2}^P & \cdots & -\left(\phi_M^5, \xi_P\right)_{L^2}^P
\end{array}\right) \equiv \left(\begin{array}{c|c} \tilde{B}_{M\times P} & \tilde{C}^T_{P\times M} \end{array}\right) \tag{179}
$$

$$
C \equiv \left(\begin{array}{ccc}
 & 0_{P\times M} & \\
\hline
\frac{1}{\rho_s h}\left(\phi_1^5, \xi_1\right)_{L^2}^P & \cdots & \frac{1}{\rho_s h}\left(\phi_M^5, \xi_1\right)_{L^2}^P \\
\vdots & \ddots & \vdots \\
\frac{1}{\rho_s h}\left(\phi_1^5, \xi_P\right)_{L^2}^P & \cdots & \frac{1}{\rho_s h}\left(\phi_M^5, \xi_P\right)_{L^2}^P
\end{array}\right) \equiv \left(\begin{array}{c} 0_{P\times M} \\ \hline \frac{1}{\rho_s h}\tilde{C}_{P\times M} \end{array}\right) \tag{180}
$$

$$
D \equiv \left(\begin{array}{ccc|c}
 & 0_{P\times P} & & I_{P\times P} \\
\hline
-\frac{D_{bend}}{\rho_s h}\left(\nabla^4 \xi_1, \xi_1\right)_{L^2}^P & 0 & \cdots & \\
\vdots & \ddots & \vdots & 0_{P\times P} \\
0 & \cdots & -\frac{D_{bend}}{\rho_s h}\left(\nabla^4 \xi_P, \xi_P\right)_{L^2}^P &
\end{array}\right) = \left(\begin{array}{c|c} 0_{P\times P} & I_{P\times P} \\ \hline -\frac{1}{\rho_s h}\tilde{L}_{P\times P} & 0_{P\times P} \end{array}\right) \tag{181}
$$

5 Stability Analysis

The goal of this system is to prove the stability of the coupled system (170). To do so, we must make one additional assumption, namely that the flow is uniform. Mathematically: $\nabla\overline{U} \equiv \mathbf{0}$. A consequence of this assumption is that the $C(\overline{U}, \nabla\overline{U})$ matrix is identically zero. This property is nice because it makes the A matrix symmetric which makes possible the upcoming stability analysis.

5.1 Useful Prior Results

In analyzing the stability of the coupled system (170), we will make use of the following theorems, proven in [3]. First, a definition, quoted from [3]:

Definition 3.1 in [3]. *We say that K is 'stable' if and only if:*

1. *K is diagonalizable in $\mathbb{C}$.*
2. *$\forall\lambda \in Sp(K), \mathscr{R}(\lambda) \leq 0$.*

Theorem 3.1 in [3]. *A real, symmetric positive definite (RSPD) matrix E_K is an energy matrix for K if and only if for all X that solve $\dot{X} = KX$, $\frac{1}{2}\frac{d}{dt}\left(X^T E_K X\right) \leq 0$.*

Theorem 3.4 in [3]. *If A and D are two real, stable matrices with energy matrices E_A and E_D, then*

$$\left\{E_A B + (E_D C)^T = 0\right\} \Rightarrow \left\{K = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \text{ is a stable matrix.}\right\} \tag{182}$$

5.2 Energy Matrices (Structure Equations in Dimensional Coordinates)

We now derive the energy matrices E_A and E_D for the problem in question.

5.2.1 The E_A Matrix

If the flow is assumed to be uniform, the A matrix reduces to

$$A = \begin{pmatrix} -\left(\phi_1^3,\phi_1^5\right)_{L^2}^P - (h_1(\phi_1),1)_{L^2}^F + (\phi_1,\phi_1)_{dHA} & \cdots & -\left(\phi_1^3,\phi_M^5\right)_{L^2}^P - (h_M(\phi_1),1)_{L^2}^F + (\phi_1,\phi_M)_{dHA} \\ -\left(\phi_2^3,\phi_1^5\right)_{L^2}^P - (h_1(\phi_2),1)_{L^2}^F + (\phi_2,\phi_1)_{dHA} & \cdots & -\left(\phi_2^3,\phi_M^5\right)_{L^2}^P - (h_M(\phi_2),1)_{L^2}^F + (\phi_2,\phi_M)_{dHA} \\ \vdots & & \vdots \\ \vdots & \ddots & \vdots \\ \vdots & & \vdots \\ -\left(\phi_M^3,\phi_1^5\right)_{L^2}^P - (h_1(\phi_M),1)_{L^2}^F + (\phi_M,\phi_1)_{dHA} & \cdots & -\left(\phi_M^3,\phi_M^5\right)_{L^2}^P - (h_M(\phi_M),1)_{L^2}^F + (\phi_M,\phi_M)_{dHA} \end{pmatrix} \tag{183}$$

If $F \equiv (a_1(t) \;\; a_2(t) \;\; \cdots \;\; a_M(t))^T \in \mathbb{R}^M$ is the vector of fluid ROM coefficients, F satisfies the system

$$\dot{F} = AF \tag{184}$$

Recall the definition of the L^2 inner product:

$$(u,v)_{L^2} \equiv \int_{\partial\Omega} uv dS \tag{185}$$

Under the uniform flow assumption, the $(\cdot,\cdot)_{dHA}$ integral reduces to

$$\begin{aligned} (u,v)_{dHA} &= \int_\Omega \nabla \cdot \left(u^T H(\bar{U})A(\bar{U})\right) v d\Omega \\ &= \int_\Omega v^T \nabla \cdot (H(\bar{U})A(\bar{U})u)\, d\Omega \\ &= \int_\Omega v^T \left[\nabla \cdot (H(\bar{U})A(\bar{U})\, u + H(\bar{U})A(\bar{U}) \cdot (\nabla u)\right] d\Omega \\ &= \int_\Omega v^T \left[H(\bar{U})A(\bar{U})\right] \cdot (\nabla u)\, d\Omega \end{aligned} \tag{186}$$

By Theorem 3.1, in order for E_A to be an energy matrix with respect to A, it must be that

$$\frac{1}{2}\frac{d}{dt}\left(F^T E_A F\right) = F^T E_A \dot{F} = F^T [E_A A] F \leq 0 \tag{187}$$

Lemma 5.2.1. *$E_A = I$ is an energy matrix for* (184)*.*

Proof. Let $E_A = I$ in (187). Then

$$\frac{1}{2}\left(F^T E_A F\right) = \frac{1}{2}\left(F^T F\right) = \frac{1}{2}\sum_{k=1} a_k^2(t) = \frac{1}{2}(U',U')_H = E \tag{188}$$

using the orthonormal POD representation $U' = \sum_{k=1}^{M} a_k(t)\phi_k(\mathbf{x})$, the fact that $\int_\Omega \phi_j H(\bar{U})\phi_k = \delta_{jk}$ and the definition of $E = \frac{1}{2}(U',U')_H$, an energy-like quantity derived from the flow disturbances. It was shown in [1] that $\frac{d}{dt}(U',U')_H = 0$ for $C(\overline{U},\nabla\overline{U}) \equiv 0$. It follows from (188) that $\frac{1}{2}\frac{d}{dt}\left(F^T E_A F\right) = 0$ for $E_A = I$, meaning (187) holds. By Theorem 3.1 in [3], $E_A = I$ is an energy matrix.

□

5.2.2 The E_D Matrix

We now derive an energy matrix E_D. Recall that

$$\begin{array}{ccccccc}
(\rho_s h)\,\frac{d}{dt}\begin{pmatrix} b_1(t) \\ \vdots \\ b_P(t) \\ \dot{b}_1(t) \\ \vdots \\ \dot{b}_P(t) \end{pmatrix} & = & \left(\begin{array}{c|c} 0_{P\times P} & (\rho_s h)\, I_{P\times P} \\ \hline -\tilde{L}_{P\times P} & 0_{P\times P} \end{array}\right) & \begin{pmatrix} b_1(t) \\ \vdots \\ b_P(t) \\ \dot{b}_1(t) \\ \vdots \\ \dot{b}_P(t) \end{pmatrix} & + & \left(\begin{array}{c} 0 \\ \vdots \\ 0 \\ \hline G_1(t) \\ \vdots \\ G_P(t) \end{array}\right) \\
(\rho_s h)\,\dot{S} & \equiv & L & S & + & JF
\end{array} \tag{189}$$

where

$$\tilde{L}_{P\times P} \equiv \begin{pmatrix} D_{bend}\left(\nabla^4\xi_1,\xi_1\right)_{L^2}^P & 0 & \cdots \\ \vdots & \ddots & \vdots \\ 0 & \cdots & D_{bend}\left(\nabla^4\xi_P,\xi_P\right)_{L^2}^P \end{pmatrix} \tag{190}$$

By Theorem 3.1, in order for E_D to be an energy matrix with respect to D, it must satisfy

$$\frac{1}{2}\frac{d}{dt}\left(S^T E_D S\right) = S^T E_D \dot{S} = S^T [E_D D] S \leq 0 \tag{191}$$

since $\dot{S} = DS$ (recall that $D \equiv \frac{1}{\rho_s h}L$). A sufficient condition for (187) to hold is if the matrix $E_D L$ were non-positive definite (assuming $\rho_s, h \geq 0$, which they have to be to make sense as physical quantities).

Lemma 5.2.2. *The matrix*

$$E_D = \left(\begin{array}{c|c} \tilde{L}_{P\times P} & 0_{P\times P} \\ \hline 0_{P\times P} & (\rho_s h)\, I_{P\times P} \end{array}\right) \tag{192}$$

is an energy matrix for (191)*.*

Proof. It is sufficient to show that $S^T E_D L S \leq 0$ where $S = (\mathbf{b}(t) \;\; \dot{\mathbf{b}}(t))^T$. Multiplying out this quadratic form:

$$\begin{aligned}
S^T E_D L S &= \begin{pmatrix} \mathbf{b}(t)^T & \dot{\mathbf{b}}(t)^T \end{pmatrix} \left(\begin{array}{c|c} \tilde{L}_{P\times P} & 0_{P\times P} \\ \hline 0_{P\times P} & (\rho_s h) I_{P\times P} \end{array}\right) \left(\begin{array}{c|c} 0_{P\times P} & (\rho_s h) I_{P\times P} \\ \hline -\tilde{L}_{P\times P} & 0_{P\times P} \end{array}\right) \begin{pmatrix} \mathbf{b}(t) \\ \dot{\mathbf{b}}(t) \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{b}(t)^T & \dot{\mathbf{b}}(t)^T \end{pmatrix} \left(\begin{array}{c|c} 0_{P\times P} & (\rho_s h) \tilde{L}_{P\times P} \\ \hline -(\rho_s h) \tilde{L}_{P\times P} & 0_{P\times P} \end{array}\right) \begin{pmatrix} \mathbf{b}(t) \\ \dot{\mathbf{b}}(t) \end{pmatrix} \\
&= (\rho_s h) \left[-\dot{\mathbf{b}}(t)^T \tilde{L}_{P\times P} \mathbf{b}(t) + \mathbf{b}(t)^T \tilde{L}_{P\times P} \dot{\mathbf{b}}(t) \right] \\
&= 0
\end{aligned} \tag{193}$$

It follows from (193) that $\frac{1}{2}\frac{d}{dt}\left(S^T E_D S\right) = 0$, meaning E_D defined above is an energy matrix. ☐

5.2.3 The E_D Matrix in terms of Kinetic and Potential Energy

Recall the linearized, dimensional von Karman equations projected onto the POD modes $\eta_j(x,y)$ for the $\mathbf{b}(t)$ coefficients can be written as a second order system

$$(\rho_s h)\ddot{b}_k(t) + D_{bend} \sum_{l=1}^{P} b_l(t) \left(\nabla^4 \xi_l, \xi_k\right)_{L^2}^P = (p, \xi_k)_{L^2}^P \tag{194}$$

In matrix form, this system is

$$(\rho_s h) I_{P\times P} \frac{d^2\mathbf{b}}{dt^2} + \tilde{L}_{P\times P}\mathbf{b} = G \tag{195}$$

where I is the $P \times P$ identity matrix and

$$\tilde{L}_{P\times P} = \begin{pmatrix} D_{bend}\left(\nabla^4 \xi_1, \xi_1\right)_{L^2}^P & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & D_{bend}\left(\nabla^4 \xi_P, \xi_P\right)_{L^2}^P \end{pmatrix} \tag{196}$$

Compare (195) with equation (2) in [4]. In this context, $(\rho_s h) I_{P\times P}$ represents the mass matrix M and the matrix $\tilde{L}$ plays the role of the stiffness matrix K. Note that G is the vector of external forces acting on the structure. In the notation of [4], the damping matrix is zero, as (195) contains no $d\mathbf{b}/dt$ term.

The mathematical definitions of kinetic and potential energy are

$$KE = \frac{1}{2}\dot{\mathbf{q}}^T M \dot{\mathbf{q}} \tag{197}$$

$$PE = \frac{1}{2}\mathbf{q}^T K \mathbf{q} \tag{198}$$

where M is the structure mass matrix, K is the structure stiffness matrix and the equation for structural displacement has the form

$$M\frac{d^2\mathbf{q}}{dt^2} + D\frac{d\mathbf{q}}{dt} + K\mathbf{q} = f^{ext} \tag{199}$$

As stated a few lines above, $M=(\rho_s h)I_{P\times P}$, $K=\tilde{L}$ and $D=0$. Note that $\rho_s h$ has the units of mass per unit area; however, the inner product $(\xi_j,\xi_j)^P_{L^2}=1$ that is implicity present in this quantity (see (160)) has units of area, meaning the units of the mass matrix are actually units of mass, as desired.

It follows that the E_D matrix has precisely the form

$$E_D=\begin{pmatrix} K & 0 \\ 0 & M \end{pmatrix} \tag{200}$$

where M represents a mass matrix and K a stiffness matrix. In this notation, the quadratic form $S^T E_D S$ decomposes into a KE and PE component:

$$\frac{1}{2}S^T E_D S=\frac{1}{2}\begin{pmatrix} \mathbf{b}^T & \dot{\mathbf{b}}^T \end{pmatrix}\begin{pmatrix} K & 0 \\ 0 & M \end{pmatrix}\begin{pmatrix} \mathbf{b} \\ \dot{\mathbf{b}} \end{pmatrix}=\frac{1}{2}\mathbf{b}^T K\mathbf{b}+\frac{1}{2}\dot{\mathbf{b}}^T M\dot{\mathbf{b}}=KE+PE \tag{201}$$

$\square$

5.3 Stability Results

Theorem 5.3.1. *Under the uniform base flow assumption ($\nabla\bar{U}\equiv 0$), $K=\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ defining the coupled system (given in* (175) *with A, B, C, D defined in* (178), (179), (180) *and* (181) *respectively) is a stable matrix if $\bar{u}=0$.*

Proof. By Theorem 3.4 in [3], K is a stable matrix if $E_A B+(E_D C)^T=0$. To simplify the notation, write

$$B=\left(\begin{array}{ccc|ccc} \left(\phi_1^5,\bar{u}\frac{\partial\xi_1}{\partial x}\right)^P_{L^2} & \cdots & \left(\phi_1^5,\bar{u}\frac{\partial\xi_P}{\partial x}\right)^P_{L^2} & -\left(\phi_1^5,\xi_1\right)^P_{L^2} & \cdots & -\left(\phi_1^5,\xi_P\right)^P_{L^2} \\ \vdots & & \vdots & \vdots & & \vdots \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \vdots & & \vdots & \vdots & & \vdots \\ \left(\phi_M^5,\bar{u}\frac{\partial\xi_M}{\partial x}\right)^P_{L^2} & \cdots & \left(\phi_M^5,\bar{u}\frac{\partial\xi_P}{\partial x}\right)^P_{L^2} & -\left(\phi_M^5,\xi_1\right)^P_{L^2} & \cdots & -\left(\phi_M^5,\xi_P\right)^P_{L^2} \end{array}\right)\equiv\left(\begin{array}{c|c} \tilde{B}_{M\times P} & -\tilde{C}^T_{P\times M} \end{array}\right) \tag{202}$$

$$C=\left(\begin{array}{ccc} & 0_{P\times M} & \\ \hline \frac{1}{\rho_s h}\left(\phi_1^5,\xi_1\right)^P_{L^2} & \cdots & \frac{1}{\rho_s h}\left(\phi_M^5,\xi_1\right)^P_{L^2} \\ \vdots & \ddots & \vdots \\ \frac{1}{\rho_s h}\left(\phi_1^5,\xi_P\right)^P_{L^2} & \cdots & \frac{1}{\rho_s h}\left(\phi_M^5,\xi_P\right)^P_{L^2} \end{array}\right)\equiv\left(\begin{array}{c} 0_{P\times M} \\ \hline \frac{1}{\rho_s h}\tilde{C}_{P\times M} \end{array}\right) \tag{203}$$

In this notation, the matrix sum of interest becomes

$$\begin{aligned} E_A B+(E_D C)^T &= E_A B+C^T E_D^T \\ &= I\left(\begin{array}{c|c} \tilde{B}_{M\times P} & -\tilde{C}^T_{P\times M} \end{array}\right)+\left(\begin{array}{c|c} 0_{M\times P} & \frac{1}{\rho_s h}\tilde{C}^T_{P\times M} \end{array}\right)\left(\begin{array}{c|c} \tilde{L}_{P\times P} & 0_{P\times P} \\ \hline 0_{P\times P} & (\rho_s h)I_{P\times P} \end{array}\right) \\ &= \left(\begin{array}{c|c} \tilde{B}_{M\times P} & -\tilde{C}^T_{P\times M} \end{array}\right)+\left(\begin{array}{c|c} 0_{M\times P} & \tilde{C}^T_{P\times M} \end{array}\right) \\ &= \left(\begin{array}{c|c} \tilde{B}_{M\times P} & 0_{P\times M} \end{array}\right) \end{aligned} \tag{204}$$

(204) is the zero matrix if $\tilde{B}_{M\times P} = 0$ where

$$\tilde{B}_{M\times P} = \begin{pmatrix} \left(\phi_1^5, \bar{u}\frac{\partial \xi_1}{\partial x}\right)_{L^2}^P & \cdots & \left(\phi_1^5, \bar{u}\frac{\partial \xi_P}{\partial x}\right)_{L^2}^P \\ \vdots & & \vdots \\ \vdots & \ddots & \vdots \\ \vdots & & \vdots \\ \left(\phi_M^5, \bar{u}\frac{\partial \xi_M}{\partial x}\right)_{L^2}^P & \cdots & \left(\phi_M^5, \bar{u}\frac{\partial \xi_P}{\partial x}\right)_{L^2}^P \end{pmatrix} \tag{205}$$

Clearly this holds if $\bar{u} = 0$, so $\bar{u} = 0 \Rightarrow E_A B + (E_D C)^T = 0$. By Theorem 3.4 in [3], K is stable under the hypotheses of the claim.

□

Note that the system *can* be stable for non-zero $\bar{u}$; it is just not guaranteed to remain stable. In the case where $\bar{u} = 0$, the structure cannot extract energy from the mean flow, as occurs in flutter. For $\bar{u} \neq 0$, an aero-elastic analysis proceeds by determining the conditions under which the eigenvalues of K will have positive real part. For supersonic flow, it turns out that once $\bar{u}$ exceeds a certain threshold (the flutter speed), the system becomes linearly unstable.

6 Current and Future Work: Staggered Time Integration Scheme for Coupled Linearized System

Having studied the stability of the coupled linear system (138), the next step is to derive a staggered time-integration scheme for this system. We are considering a 2nd order accurate explicit/implicit (fluid/structure) scheme similar to the 1st order accurate explicit/implicit schemes discussed in [3]. The scheme currently being considered involves 2nd order Runge-Kutta (RK-2) time integration for the fluid field and an implicit, Crank-Nicholson scheme for the structure side. For example, the scheme might have the form:

1. Start with some initial guesses F^0 and S^0.
2. Advance the fluid system forward in time using the RK-2 explicit scheme:

$$\frac{F^* - F^n}{\Delta t} = AF^n + BS^n \tag{206}$$

$$\frac{F^{**} - F^*}{\Delta t} = AF^* + BS^n \tag{207}$$

$$\overline{F^{n+1}} = \frac{1}{2}(F^* + F^{**}) \tag{208}$$

3. Correct the updated fluid field to enforce stability and 2nd order accuracy:

$$F^{n+1} = \overline{F^{n+1}} + \left[F^{n+1^c}\right] \tag{209}$$

4. Compute $\overline{S^{n+1}}$ using the trapezoidal family of schemes for a chosen $\alpha \in [0,1]$ (note that the scheme will be implicit for $\alpha \in [1/2,1]$).

$$\frac{\overline{S^{n+1}} - S^n}{\Delta t} = CF^{n+1} + DS^{n+\alpha} \tag{210}$$

where

$$S^{n+\alpha} = (1-\alpha)S^n + \alpha\overline{S^{n+1}} \tag{211}$$

5. Correct the updated structure field to enforce stability and 2nd order accuracy:

$$S^{n+1} = \overline{S^{n+1}} + \left[S^{n+1^c}\right] \tag{212}$$

One might also try some variations of the scheme above with “tighter” coupling of the fluid and structure fields. Proving stability of such a scheme is likely to be far more difficult. Current work is focussed around deriving the corrector terms $\left[F^{n+1^c}\right]$ and $\left[S^{n+1^c}\right]$ to enforce 2nd order accuracy and stability of schemes such as the one given above.

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