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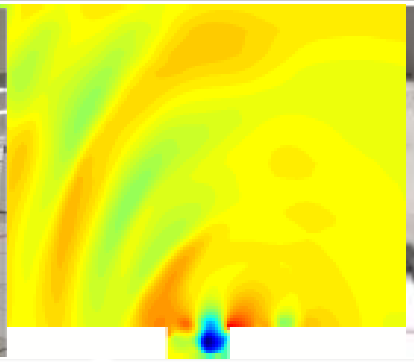

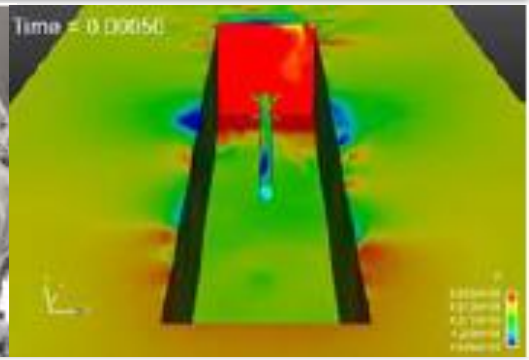

A minimal subspace rotation approach for obtaining stable & accurate low-order projection-based reduced order models for nonlinear compressible flow

Irina Tezaur[1], Maciej Balajewicz[2]

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U.S. DEPARTMENT OF ENERGY

NNSA National Nuclear Security Administration

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Outline

1. Motivation
2. Projection-based model order reduction
3. Accounting for modal truncation
   - Traditional linear eddy-viscosity approach
   - New proposed approach via subspace rotation
4. Applications
   - Low Reynolds number channel driven cavity
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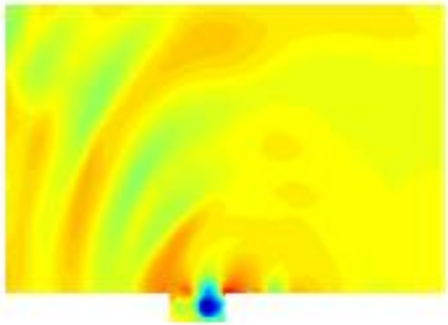

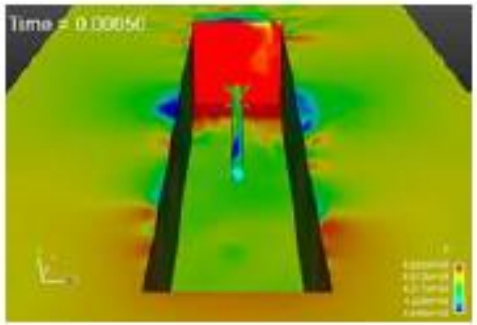

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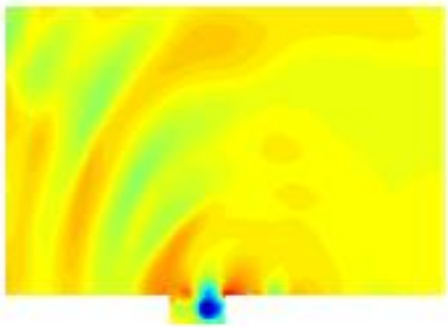

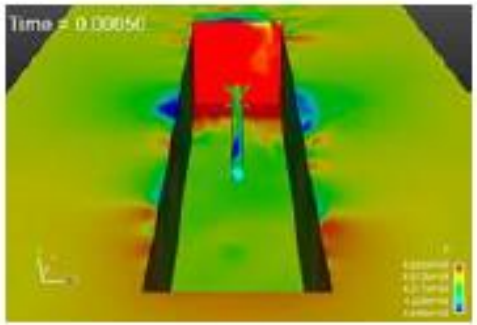

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  - ***Consistency*** (w.r.t. the continuous PDEs).
  - ***Stability:*** if full order model (FOM) is stable, ROM should be stable.
  - ***Convergence:*** requires consistency and stability.
  - ***Accuracy*** (w.r.t. FOM).
  - ***Efficiency***.
  - ***Robustness*** (w.r.t. time or parameter changes).

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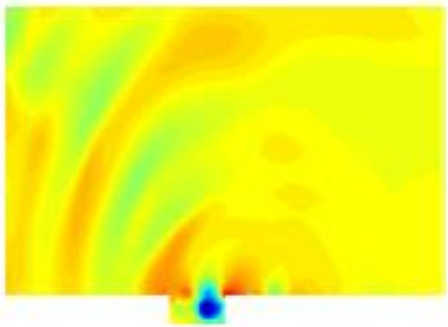

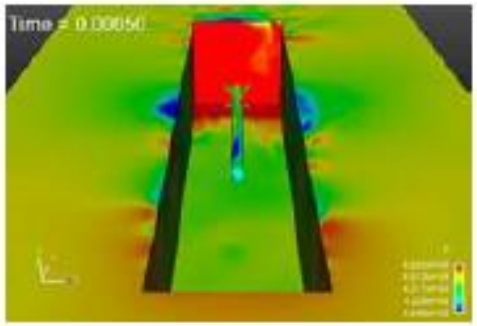

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This talk focuses on remedying "***mode truncation instability***" problem for projection-based (POD/Galerkin) compressible flow ROMs.

Projection-based model order reduction

Governing equations

- 3D compressible Navier-Stokes equations in *primitive specific volume form:*

[PDEs]

$$\zeta_{,t} + \zeta_{,j} u_j - \zeta u_{j,j} = 0$$
$$u_{i,t} + u_{i,j} u_j + \zeta p_{,i} - \frac{1}{Re} \zeta \tau_{ij,j} = 0$$
$$p_{,t} + u_j p_{,j} + \gamma u_{j,j} p - \left(\frac{\gamma}{PrRe}\right) \left(\kappa (p\zeta)_{,j}\right)_{,j} - \left(\frac{\gamma - 1}{Re}\right) u_{i,j} \tau_{ij} = 0 \quad (1)$$

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- Spectral discretization* $(\boldsymbol{q}(\boldsymbol{x}, t) \approx \sum_{i=1}^{n} a_i(t) \boldsymbol{U}_i(\boldsymbol{x}))$ + Galerkin projection applied to (1) yields a system of *n coupled quadratic ODEs*:

[ROM]

$$\frac{d\boldsymbol{a}}{dt} = \boldsymbol{C} + \boldsymbol{L}\boldsymbol{a} + \left[\boldsymbol{a}^T \boldsymbol{Q}^{(1)} \boldsymbol{a} + \boldsymbol{a}^T \boldsymbol{Q}^{(2)} \boldsymbol{a} + \cdots + \boldsymbol{a}^T \boldsymbol{Q}^{(n)} \boldsymbol{a}\right]^T \qquad (2)$$

where $\boldsymbol{C} \in \mathbb{R}^n, \boldsymbol{L} \in \mathbb{R}^{n \times n}$ and $\boldsymbol{Q}^{(i)} \in \mathbb{R}^{n \times n}$ for all $i = 1, \ldots, n$.

* Here we use a *Proper Orthogonal Decomposition (POD)* basis $\boldsymbol{U}_i(\boldsymbol{x})$.

Projection-based model order reduction

ROM limitations due to basis truncation

Projection-based MOR necessitates ***truncation.***

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- For fluid flow applications, higher-order modes are associated with energy ***dissipation*** $\Longrightarrow$ low-dimensional ROMs are often ***inaccurate*** and sometimes ***unstable***.

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Turbulence Modeling
(traditional approach)

Subspace Rotation
(our approach)

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Traditional linear eddy-viscosity approach

- Dissipative dynamics of truncated higher-order modes are modeled using an additional linear term:

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  3. Inherently a <u>*linear model*</u> → cannot be expected to perform well for all classes of problems (e.g., nonlinear).

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Illustrative example

- *Standard approach:* retain only the most energetic POD modes, i.e., $\boldsymbol{U}_1, \boldsymbol{U}_2, \boldsymbol{U}_3, \boldsymbol{U}_4, \ldots$
- *Proposed approach:* choose some higher order basis modes to increase dissipation, i.e., $\boldsymbol{U}_1, \boldsymbol{U}_2, \boldsymbol{U}_6, \boldsymbol{U}_8, \ldots$

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- *More generally:* approximate the solution using a linear superposition of $n + p$ (with $p > 0$) most energetic modes:

$$\widetilde{\boldsymbol{U}}_i = \sum_{j=1}^{n+p} X_{ij}\, \boldsymbol{U}_j, \quad i = 1, \ldots, n, \tag{3}$$

where $\boldsymbol{X} \in \mathbb{R}^{(n+p)\times n}$ is an orthonormal ($\boldsymbol{X}^T\boldsymbol{X} = \boldsymbol{I}_{n\times n}$) "rotation" matrix.

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- Once $\boldsymbol{X}$ is found, the result is a system of the form (2) with:

$$Q^{(i)}{}_{jk} \leftarrow \sum_{s,q,r=1}^{n+p} X_{si} Q^{(s)}{}_{qr} X_{qr} X_{rk}, \quad \boldsymbol{L} \leftarrow \boldsymbol{X}^T \boldsymbol{L} \boldsymbol{X}, \quad \boldsymbol{C} \leftarrow \boldsymbol{X}^T \boldsymbol{C}^*$$

Accounting for modal truncation

Minimal subspace rotation: trace minimization on Stiefel manifold

$$\begin{aligned} \text{minimize}_{\boldsymbol{X}\in\mathcal{V}_{(n+p),n}} \quad & -\operatorname{tr}\left(\boldsymbol{X}^T\boldsymbol{I}_{(n+p)\times n}\right) \\ \text{subject to} \quad & \operatorname{tr}(\boldsymbol{X}^T\boldsymbol{L}\boldsymbol{X}) = \eta \end{aligned} \tag{9}$$

- $\mathcal{V}_{(n+p),n} \in \left\{\boldsymbol{X} \in \mathbb{R}^{(n+p)\times n} : \boldsymbol{X}^T\boldsymbol{X} = \boldsymbol{I}_n, p > 0\right\}$ is the *Stiefel manifold*.
- Constraint is traditional *linear eddy-viscosity closure model ansatz* → involves overall balance between *linear energy production* and *dissipation* / vanishing of *averaged total power* ($= \operatorname{tr}(\boldsymbol{X}^T\boldsymbol{L}\boldsymbol{X}) +$ energy transfer).
  - $\eta \in \mathbb{R}$: proxy for the balance between linear energy production and energy dissipation (calculated iteratively using modal energy).
- Equation (9) is solved efficiently offline using the method of Lagrange multipliers (`Manopt MATLAB toolbox`).
- See (Balajewicz, Tezaur, Dowell, 2016) and Appendix slide for Algorithm.

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  2. Stability cannot be proven like for incompressible case.

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Channel driven cavity: low Reynolds number case

Flow over square cavity at Mach 0.6, Re = 1453.9, Pr = 0.72 ⇒ $n = 4$ ROM (91% snapshot energy).

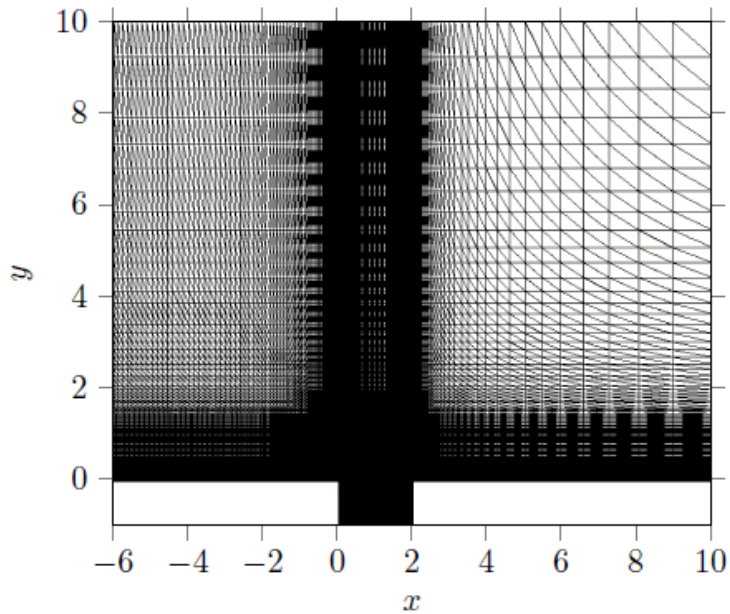

Figure 1: Domain and mesh for viscous channel driven cavity problem.

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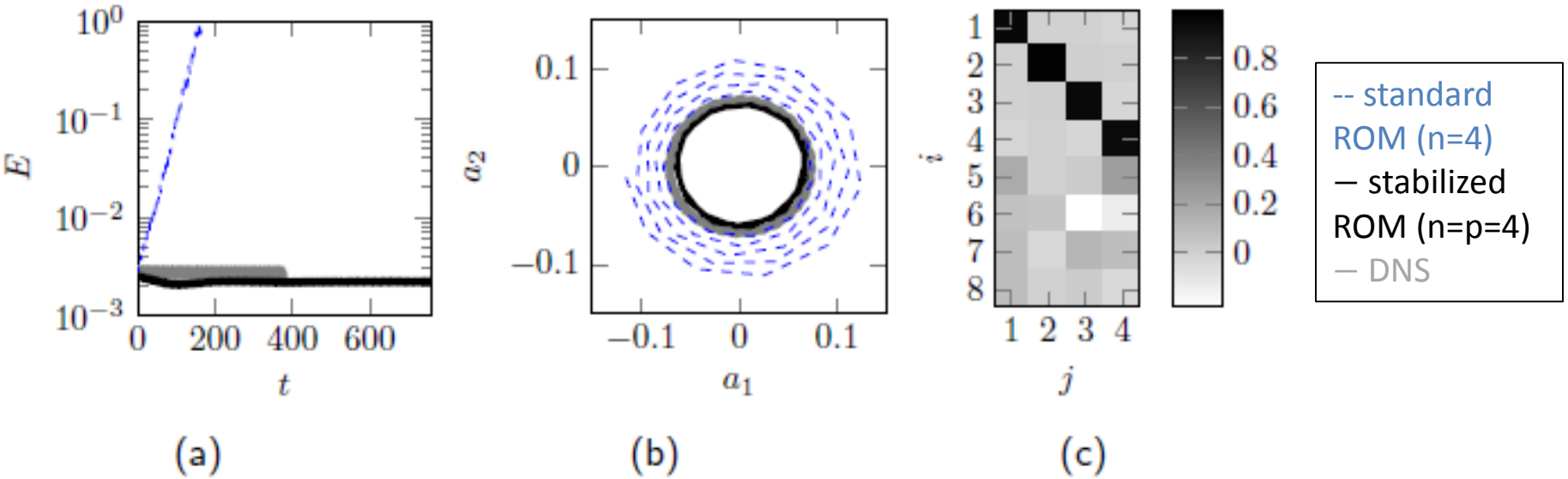

Figure 2: (a) evolution of modal energy, (b) phase plot of first and second temporal basis $a_1(t)$ and $a_2(t)$, (c) illustration of stabilizing rotation showing that rotation is small: $\frac{\|\boldsymbol{X}-\boldsymbol{I}_{(n+p),n}\|_F}{n} = 0.188, \boldsymbol{X} \approx \boldsymbol{I}_{(n+p),n}$

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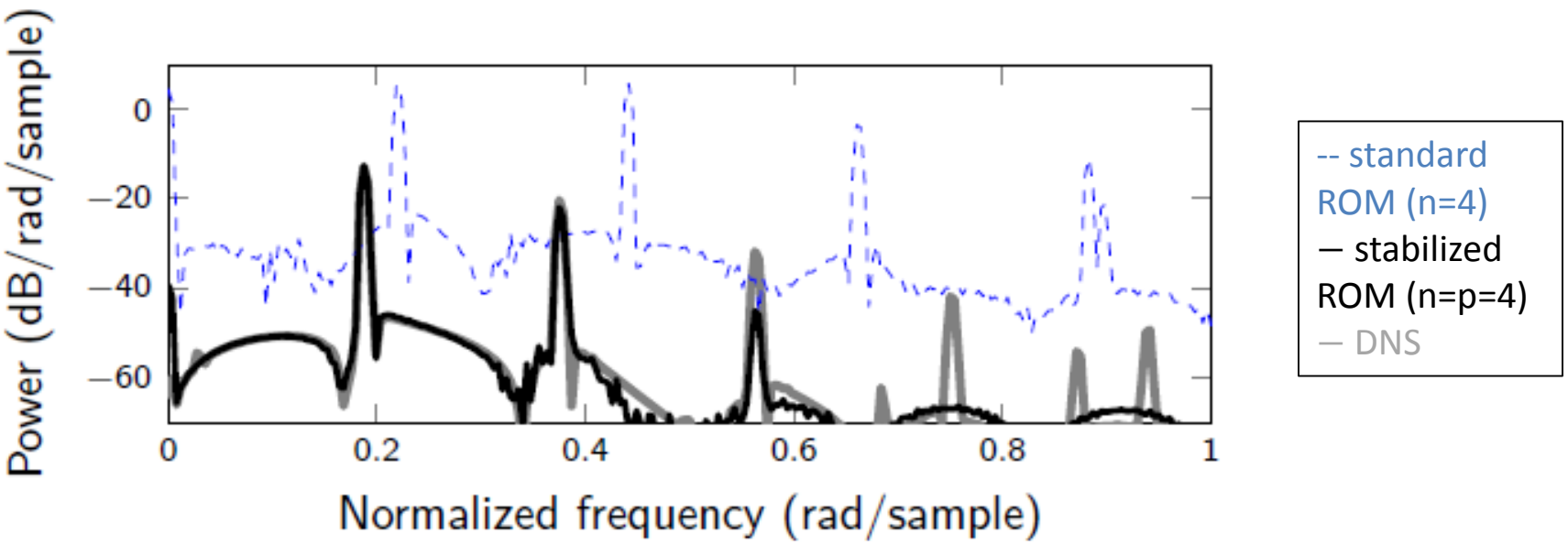

Figure 3: Pressure power spectral density (PSD) at location $\boldsymbol{x} = (2, -1)$; stabilized ROM minimizes subspace rotation.

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Applications

Channel driven cavity: moderate Reynolds number case

Flow over square cavity at Mach 0.6, Re = 5452.1, Pr = 0.72
$\Rightarrow n = 20$ ROM (71.8% snapshot energy).

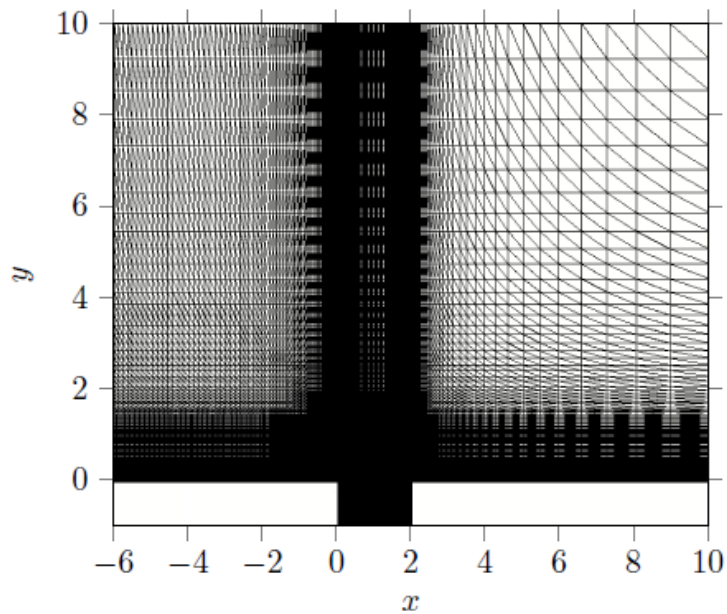

Figure 4: Domain and mesh for viscous channel driven cavity problem.

Applications

Channel driven cavity: moderate Reynolds number case

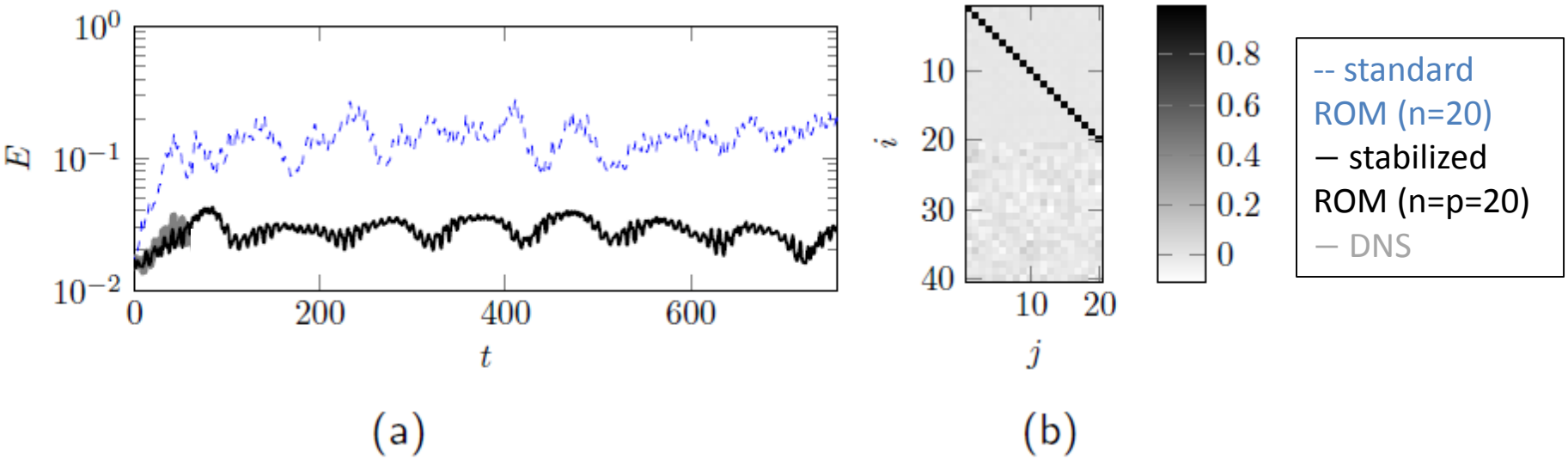

Figure 5: (a) evolution of modal energy, (b) illustration of stabilizing rotation showing that rotation is small: $\frac{\|\boldsymbol{X}-\boldsymbol{I}_{(n+p),n}\|_F}{n} = 0.038, \boldsymbol{X} \approx \boldsymbol{I}_{(n+p),n}$

Applications

Channel driven cavity: moderate Reynolds number case

— stabilized ROM (n=p=20)
— DNS

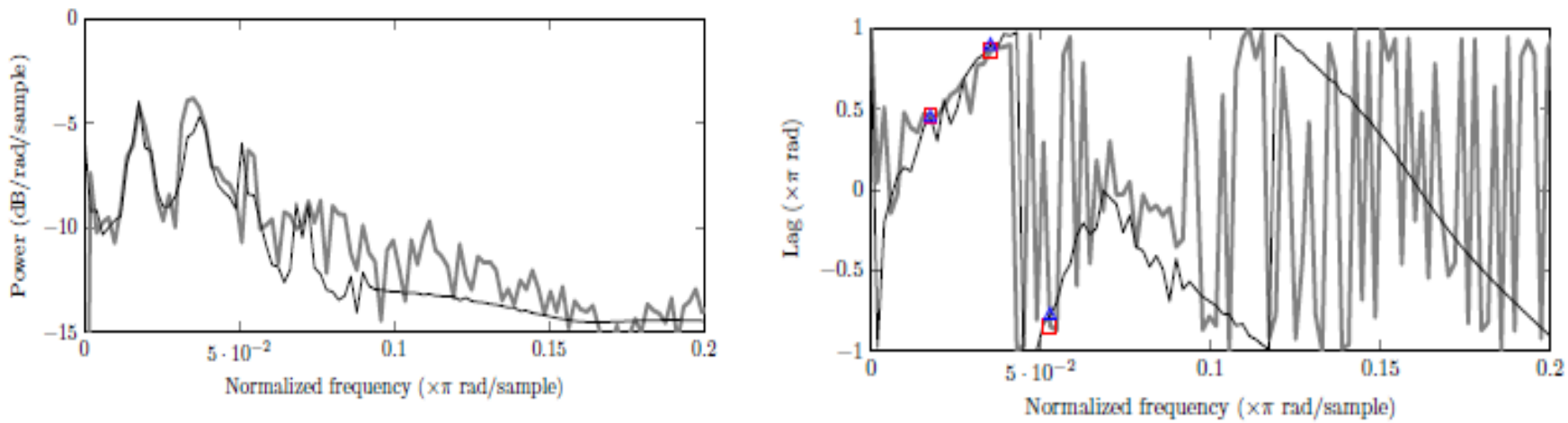

Figure 6: Pressure cross PSD of of $p(\boldsymbol{x}_1, t)$ and $p(\boldsymbol{x}_2, t)$ where $\boldsymbol{x}_1 = (2, -0.5)$, $\boldsymbol{x}_2 = (0, -0.5)$

Power and phase lag at fundamental frequency, and first two super harmonics are predicted accurately using the fine-tuned ROM (Δ = stabilized ROM, □ = DNS)

Outline

1. Motivation
2. Projection-based model order reduction
3. Accounting for modal truncation
   - Traditional linear eddy-viscosity approach
   - New proposed approach via subspace rotation
4. Applications
   - Low Reynolds number channel driven cavity
   - Moderate Reynolds number channel driven cavity
5. **Summary**
6. Future work
7. References

Summary

- We have developed a non-intrusive approach for *stabilizing* and *fine-tuning* projection-based ROMs for compressible flows.
- The standard POD modes are *"rotated"* into a more dissipative regime to account for the dynamics in the higher order modes truncated by the standard POD method.
- The new approach is *consistent* and does not require the addition of empirical turbulence model terms unlike traditional approaches.
- Mathematically, the approach is formulated as a *quadratic matrix program* on the Stiefel manifold.
- The constrained minimization problem is solved *offline* and *small* enough to be solved in MATLAB.
- The method is demonstrated on several compressible flow problems and shown to deliver *stable* and *accurate* ROMs.

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Future work

- Application to *higher Reynolds number* problems.
- Extension of the proposed approach to problems with *generic nonlinearities*, where the ROM involves some form of hyper-reduction (e.g., DEIM, gappy POD).
- Extension of the method to *minimal-residual-based* nonlinear ROMs.
- Extension of the method to *predictive applications*, e.g., problems with varying Reynolds number and/or Mach number.
- Selecting different *goal-oriented* objectives and constraints in our optimization problem:

$$\begin{aligned} &\text{minimize}_{\boldsymbol{X}\in\mathcal{V}_{(n+p),n}} \quad f(\boldsymbol{X}) \\ &\text{subject to} \quad g(\boldsymbol{X},\boldsymbol{L}) = 0 \end{aligned}$$

e.g.,

- Maximize parametric robustness:

$$f = \sum_{i=1}^{k} \beta_i \|\boldsymbol{U}^*(\mu_i)\boldsymbol{X} - \boldsymbol{U}^*(\mu_i)\|_F.$$

- ODE constraints: $g = \|\boldsymbol{a}(t) - \boldsymbol{a}^*(t)\|$.

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References

Balajewicz, M. & Dowell, E. 2012 Stabilization of projection-based reduced order models of the Navier-Stokes equation. *Nonlinear Dynamics* **70** (2), 1619–1632.

Balajewicz, M. & Dowell, E. & Noack, B. 2013 Low-dimensional modelling of high-Reynolds-number shear flows incorporating constraints from the Navier-Stokes equation. *Journal of Fluid Mechanics* **729**, 285–308.

Balajewicz, M., Tezaur, I. & Dowell, E. 2016 Minimal subspace rotation on the Stiefel manifold for stabilization and enhancement of projection-based reduced order models for the compressible Navier-Stokes equations", *J. Comput. Phys.* **321** 224–241.

Barone, M., Kalashnikova, I., Segalman, D. & Thornquist, H. 2009 Stable Galerkin reduced order models for linearized compressible flow. *J. Computat. Phys.* **228** (6), 1932–1946.

Carlberg, K., Farhat, C., Cortial, J. & Amsallem, D. 2013 The GNAT method for nonlinear model reduction: effective implementation and application to computational uid dynamics and turbulent flows. *J. Computat. Phys.* **242** 623–647.

Kalashnikova, I., Arunajatesan, S., Barone, M., van Bloemen Waanders, B. & Fike, J. 2014 Reduced order modeling for prediction and control of large-scale systems. *Sandia Tech. Report*.

Rowley, C., Colonius, T. & Murray, R. 2004 Model reduction for compressible ows using pod and galerkin projection. *Physica D: Nonlinear Phenomena* **189** (1) 115–129.

Serre, G., Lafon, P., Gloerfelt, X. & Bailly, C. 2012 Reliable reduced-order models for timedependent linearized euler equations. *J. Computat. Phys.* **231** (15) 5176–5194.

Aubry, N., Holmes, P., Lumley, J. L., & Stone E. 1988 The dynamics of coherent structures in the wall region of a turbulent boundary layer. *J. Fluid Mech.* **192** (115) 115–173.

Osth, J., Noack, B. R., Krajnovic, C., Barros, D., & Boree, J. 2014 On the need for a nonlinear subscale turbulence term in POD models as exemplified for a high Reynolds number flow over an Ahmed body. *J. Fluid Mech.* **747** 518–544.

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8. **Appendix**

Appendix: Accounting for modal truncation

Stabilization algorithm: returns stabilizing rotation matrix $\boldsymbol{X}$.

Inputs: Initial guess $\eta^{(0)} = \mathrm{tr}(L(1:n, 1:n))$ ($\boldsymbol{X} = \boldsymbol{I}_{(n+p)\times n}$), ROM size n and $p \geq 1$, ROM matrices associated with the first $n+p$ most energetic POD modes, convergence tolerance TOL, maximum number of iterations k_{max}.

for $k = 0, \cdots, k_{max}$

Solve constrained optimization problem on Stiefel manifold:

$$\begin{aligned} \underset{\boldsymbol{X}^{(k)} \in \mathcal{V}_{(n+p),n}}{\text{minimize}} \quad & -\mathrm{tr}\left(\boldsymbol{X}^{(k)\mathrm{T}} \boldsymbol{I}_{(n+p)\times n}\right) \\ \text{subject to} \quad & \mathrm{tr}(\boldsymbol{X}^{(k)\mathrm{T}} \boldsymbol{L} \boldsymbol{X}^{(k)}) = \eta^{(k)}. \end{aligned}$$

Construct new Galerkin matrices using (4).

Integrate numerically new Galerkin system.

Calculate "modal energy" $E(t)^{(k)} = \sum_i^n (a(t)_i^{(k)})^2$.

Perform linear fit of temporal data $E(t)^{(k)} \approx c_1^{(k)} t + c_0^{(k)}$, where $c_1^{(k)}$ =energy growth.

Calculate ϵ such that $c_1^{(k)}(\epsilon) = 0$ (no energy growth) using root-finding algorithm.

Perform update $\eta^{(k+1)} = \eta^{(k)} + \epsilon$.

if $||c_1^{(k)}|| < TOL$

$\boldsymbol{X} := \boldsymbol{X}^{(k)}$.

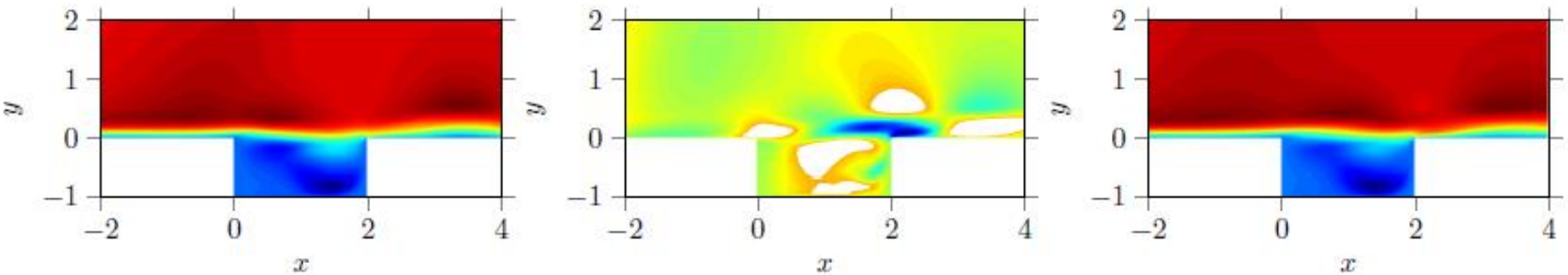
terminate the algorithm.

end

end

Applications

Channel driven cavity: low Reynolds number case



DNS

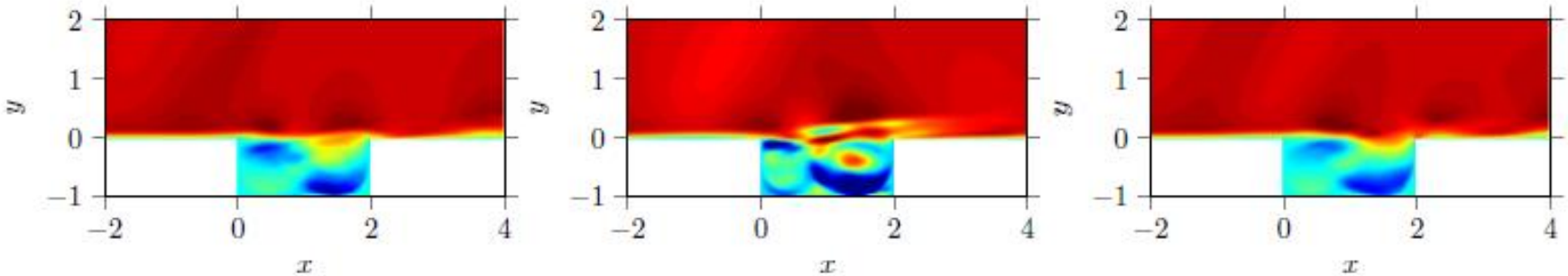
Standard ROM ($n = 4$)

Stabilized ROM ($n = p = 4$)

Figure 7: Channel driven cavity Re $\approx$ 1500 contours of u-velocity at time of final snapshot.

Applications

Channel driven cavity: moderate Reynolds number case



DNS

Standard ROM
($n = 20$)

Stabilized ROM
($n = p = 20$)

Figure 8: Channel driven cavity Re ≈ 5500 contours of u-velocity at time of final snapshot.

Applications

CPU times (CPU-hours) for offline and online computations

	Procedure	Low Re Cavity	Moderate Re Cavity
	FOM # of DOF	288,250	243,750
offline	Time-integration of FOM	72 hrs	179 hrs
offline	Basis construction (size $n+p$ ROM)	0.88 hrs	3.44 hrs
offline	Galerkin projection (size $n+p$ ROM)	5.44 hrs	14.8 hrs
offline	Stabilization	14 sec	170 sec
	ROM # of DOF	4	20
online	Time-integration of ROM	0.16 sec	0.83 sec
	Online computational speed-up	1.6e6	7.8e5

- Stabilization is *fast* (O(sec) or O(min)).
- Significant *online computational speed-up*!