Stabilization and fine-tuning of projection-based reduced order models for compressible flow via minimal subspace rotation on the Stiefel manifold

Maciej Balajewicz[1] Irina Tezaur[2]

[1]Aerospace Engineering
University of Illinois at Urbana-Champaign

[2]Quantitative Modeling and Analysis
Sandia National Laboratories*

*Sandia National Laboratories is a multi-program laboratory managed and operated by Sandia Corporation, a wholly owned subsidiary of Lockheed Martin Corporation, for the U.S. Department of Energy's National Nuclear Security Administration under contract DE-AC04-94AL85000.

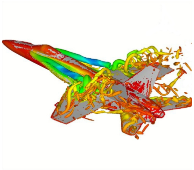
International Congress on Industrial and Applied Mathematics
(ICIAM) 2015
Beijing, China
August 10–14, 2015

Outline

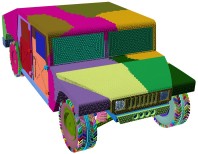
Outline

Motivation

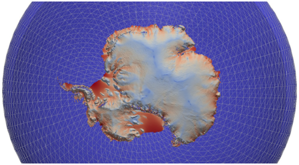
Computational models of high-dimensional systems arise in a rich variety of engineering and scientific applications



(a) CFD



(b) FEA



(c) Global climate modeling

Motivation

High-fidelity simulations remain prohibitive

- Computational costs still too high for **parametric**, **time-critical** and **many-query** applications (i.e., design, design optimization, control, UQ)
- Postprocessing/visualization difficulties

→ **There are significant scientific and engineering benefits in developing and studying low-dimensional representations of high-dimensional systems that retain physical fidelity while substantially reducing the size and cost of the computational model**

"Purpose of computing is insight, not numbers"
— Richard Hamming (1962)

Motivation

High-fidelity simulations remain prohibitive

- Computational costs still too high for **parametric**, **time-critical** and **many-query** applications (i.e., design, design optimization, control, UQ)
- Postprocessing/visualization difficulties

→ **There are significant scientific and engineering benefits in developing and studying low-dimensional representations of high-dimensional systems that retain physical fidelity while substantially reducing the size and cost of the computational model**

"Purpose of computing is insight, not numbers"
— Richard Hamming (1962)

Motivation

High-fidelity simulations remain prohibitive

- Computational costs still too high for **parametric**, **time-critical** and **many-query** applications (i.e., design, design optimization, control, UQ)
- Postprocessing/visualization difficulties

→ **There are significant scientific and engineering benefits in developing and studying low-dimensional representations of high-dimensional systems that retain physical fidelity while substantially reducing the size and cost of the computational model**

"Purpose of computing is insight, not numbers"
— Richard Hamming (1962)

Motivation

High-fidelity simulations remain prohibitive

- Computational costs still too high for **parametric**, **time-critical** and **many-query** applications (i.e., design, design optimization, control, UQ)
- Postprocessing/visualization difficulties

$\rightarrow$ **There are significant scientific and engineering benefits in developing and studying low-dimensional representations of high-dimensional systems that retain physical fidelity while substantially reducing the size and cost of the computational model**

"Purpose of computing is insight, not numbers"
— Richard Hamming (1962)

Outline

Projection-based model order reduction

Data-driven model order reduction (MOR)[1]

→ Underlying mathematical premises:

1. **Compression:** solution of governing parametric partial differential equation (PDE) or a coupled system of PDEs lies in a subspace of significantly lower dimension
2. **Off-line training:** subspace can be identified/learned off-line via training simulations and high-fidelity model can be reformulated with respect to this subspace
3. **On-line prediction:** identified parametric reduced-order models (ROMs) capable of providing new solutions at a fraction of the computational cost

[1] Not to be confused with model simplification or physics simplification

Projection-based model order reduction

Data-driven model order reduction (MOR)[1]

→ Underlying mathematical premises:

1. **Compression:** solution of governing parametric partial differential equation (PDE) or a coupled system of PDEs lies in a subspace of significantly lower dimension
2. **Off-line training:** subspace can be identified/learned off-line via training simulations and high-fidelity model can be reformulated with respect to this subspace
3. **On-line prediction:** identified parametric reduced-order models (ROMs) capable of providing new solutions at a fraction of the computational cost

[1] Not to be confused with model simplification or physics simplification

Projection-based model order reduction

Data-driven model order reduction (MOR)[1]

→ Underlying mathematical premises:

1. **Compression:** solution of governing parametric partial differential equation (PDE) or a coupled system of PDEs lies in a subspace of significantly lower dimension
2. **Off-line training:** subspace can be identified/learned off-line via training simulations and high-fidelity model can be reformulated with respect to this subspace
3. **On-line prediction:** identified parametric reduced-order models (ROMs) capable of providing new solutions at a fraction of the computational cost

[1] Not to be confused with model simplification or physics simplification

Data-driven model order reduction (MOR)[1]

→ Underlying mathematical premises:

1. **Compression:** solution of governing parametric partial differential equation (PDE) or a coupled system of PDEs lies in a subspace of significantly lower dimension
2. **Off-line training:** subspace can be identified/learned off-line via training simulations and high-fidelity model can be reformulated with respect to this subspace
3. **On-line prediction:** identified parametric reduced-order models (ROMs) capable of providing new solutions at a fraction of the computational cost

[1]Not to be confused with model simplification or physics simplification

Projection-based model order reduction

Data-driven model order reduction (MOR)[1]

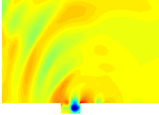
→ Underlying mathematical premises:

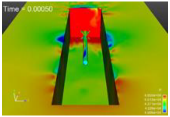
1. **Compression:** solution of governing parametric partial differential equation (PDE) or a coupled system of PDEs lies in a subspace of significantly lower dimension
2. **Off-line training:** subspace can be identified/learned off-line via training simulations and high-fidelity model can be reformulated with respect to this subspace
3. **On-line prediction:** identified parametric reduced-order models (ROMs) capable of providing new solutions at a fraction of the computational cost

[1] Not to be confused with model simplification or physics simplification

Projection-based model order reduction

Targeted application: compressible fluid flow (e.g., captive carry).



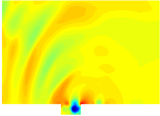


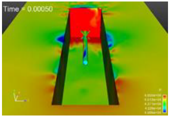
- Majority of MOR approaches in the literature for fluids are for incompressible flow.
- Some works on MOR for compressible flows:
  - **Energy-based inner products:** Rowley *et al.*, 2004 (isentropic); Barone *et al.*, 2009 (linear); Serre *et. al*, 2012 (linear); Kalashnikova *et al.*, 2014 (nonlinear).
  - **GNAT method:** Carlberg *et al.*, 2013 (nonlinear).

MOR for nonlinear, compressible fluid flows is in its infancy!

Projection-based model order reduction

Targeted application: compressible fluid flow (e.g., captive carry).



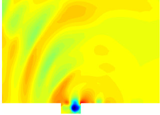


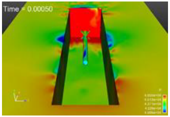
- Majority of MOR approaches in the literature for fluids are for incompressible flow.
- Some works on MOR for compressible flows:
  - **Energy-based inner products:** Rowley *et al.*, 2004 (isentropic); Barone *et al.*, 2009 (linear); Serre *et. al*, 2012 (linear); Kalashnikova *et al.*, 2014 (nonlinear).
  - **GNAT method:** Carlberg *et al.*, 2013 (nonlinear).

MOR for nonlinear, compressible fluid flows is in its infancy!

Projection-based model order reduction

Targeted application: compressible fluid flow (e.g., captive carry).



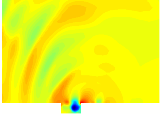


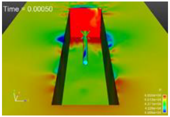
- Majority of MOR approaches in the literature for fluids are for incompressible flow.
- Some works on MOR for compressible flows:
  - **Energy-based inner products:** Rowley *et al.*, 2004 (isentropic); Barone *et al.*, 2009 (linear); Serre *et. al*, 2012 (linear); Kalashnikova *et al.*, 2014 (nonlinear).
  - **GNAT method:** Carlberg *et al.*, 2013 (nonlinear).

MOR for nonlinear, compressible fluid flows is in its infancy!

Projection-based model order reduction

Targeted application: compressible fluid flow (e.g., captive carry).



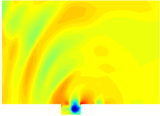


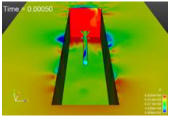
- Majority of MOR approaches in the literature for fluids are for incompressible flow.
- Some works on MOR for compressible flows:
  - **Energy-based inner products:** Rowley *et al.*, 2004 (isentropic); Barone *et al.*, 2009 (linear); Serre *et. al*, 2012 (linear); Kalashnikova *et al.*, 2014 (nonlinear).
  - **GNAT method:** Carlberg *et al.*, 2013 (nonlinear).

MOR for nonlinear, compressible fluid flows is in its infancy!

Projection-based model order reduction

Targeted application: compressible fluid flow (e.g., captive carry).



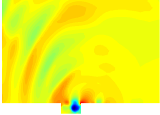
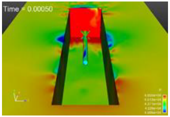


- Majority of MOR approaches in the literature for fluids are for incompressible flow.
- Some works on MOR for compressible flows:
  - **Energy-based inner products:** Rowley *et al.*, 2004 (isentropic); Barone *et al.*, 2009 (linear); Serre *et. al*, 2012 (linear); Kalashnikova *et al.*, 2014 (nonlinear).
  - **GNAT method:** Carlberg *et al.*, 2013 (nonlinear).

MOR for nonlinear, compressible fluid flows is in its infancy!

Projection-based model order reduction

Targeted application: compressible fluid flow (e.g., captive carry).

- Majority of MOR approaches in the literature for fluids are for incompressible flow.
- Some works on MOR for compressible flows:
  - **Energy-based inner products:** Rowley *et al.*, 2004 (isentropic); Barone *et al.*, 2009 (linear); Serre *et. al*, 2012 (linear); Kalashnikova *et al.*, 2014 (nonlinear).
  - **GNAT method:** Carlberg *et al.*, 2013 (nonlinear).

MOR for nonlinear, compressible fluid flows is in its infancy!

Projection-based model order reduction

Standard 3 step recipe

- Low-rank approximation of snapshot matrix $\boldsymbol{M} \in \mathbb{R}^{N \times K}$ (Proper Orthogonal Decomposition, a.k.a., POD):

$$\min_{\boldsymbol{U} \in \mathbb{R}^{N \times n}, \boldsymbol{V} \in \mathbb{R}^{n \times K}} \|\boldsymbol{M} - \boldsymbol{U}\boldsymbol{V}\|_F. \tag{1}$$

- Spectral discretization

$$\boldsymbol{u}(\boldsymbol{x}, t) \approx \sum_{i=1}^{n} a_i(t)\, \boldsymbol{U}_i(\boldsymbol{x}). \tag{2}$$

- Projection yields a small set of evolution equations for the mode coefficients a_i

$$\frac{d}{dt} a_i = f_i(\boldsymbol{a}). \tag{3}$$

Projection-based model order reduction

Standard 3 step recipe

- Low-rank approximation of snapshot matrix $\boldsymbol{M} \in \mathbb{R}^{N \times K}$ (Proper Orthogonal Decomposition, a.k.a., POD):

$$\min_{\boldsymbol{U} \in \mathbb{R}^{N \times n}, \boldsymbol{V} \in \mathbb{R}^{n \times K}} \|\boldsymbol{M} - \boldsymbol{U}\boldsymbol{V}\|_F. \tag{1}$$

- Spectral discretization

$$\boldsymbol{u}(\boldsymbol{x}, t) \approx \sum_{i=1}^{n} a_i(t)\, \boldsymbol{U}_i(\boldsymbol{x}). \tag{2}$$

- Projection yields a small set of evolution equations for the mode coefficients a_i

$$\frac{d}{dt} a_i = f_i(\boldsymbol{a}). \tag{3}$$

Projection-based model order reduction

Standard 3 step recipe

- Low-rank approximation of snapshot matrix $\boldsymbol{M} \in \mathbb{R}^{N \times K}$ (Proper Orthogonal Decomposition, a.k.a., POD):

$$\min_{\boldsymbol{U} \in \mathbb{R}^{N \times n}, \boldsymbol{V} \in \mathbb{R}^{n \times K}} \|\boldsymbol{M} - \boldsymbol{U}\boldsymbol{V}\|_F. \tag{1}$$

- Spectral discretization

$$\boldsymbol{u}(\boldsymbol{x}, t) \approx \sum_{i=1}^{n} a_i(t)\, \boldsymbol{U}_i(\boldsymbol{x}). \tag{2}$$

- Projection yields a small set of evolution equations for the mode coefficients a_i

$$\frac{d}{dt} a_i = f_i(\boldsymbol{a}). \tag{3}$$

Projection-based model order reduction

Governing equations

- We consider the 3D compressible Navier-Stokes equations in <u>*primitive specific volume form*</u>:

$$
\begin{aligned}
\zeta_{,t} + \zeta_{,j} u_j - \zeta u_{j,j} &= 0, \\
u_{i,t} + u_{i,j} u_j + \zeta p_{,i} - \frac{1}{Re} \zeta \tau_{ij,j} &= 0, \\
p_{,t} + u_j p_{,j} + \gamma u_{j,j} p - \left(\frac{\gamma}{PrRe}\right) \left(\kappa (p\zeta)_{,j}\right)_{,j} - \left(\frac{\gamma - 1}{Re}\right) u_{i,j} \tau_{ij} &= 0.
\end{aligned}
\tag{4}
$$

- For the compressible Navier-Stokes equations (4), Galerkin projection yields a system of n coupled quadratic ODEs

$$
\frac{d\boldsymbol{a}}{dt} = \boldsymbol{C} + \boldsymbol{L}\boldsymbol{a} + \begin{bmatrix} \boldsymbol{a}^{\mathrm{T}} \boldsymbol{Q}^{(1)} \boldsymbol{a} & \boldsymbol{a}^{\mathrm{T}} \boldsymbol{Q}^{(2)} \boldsymbol{a} & \cdots & \boldsymbol{a}^{\mathrm{T}} \boldsymbol{Q}^{(n)} \boldsymbol{a} \end{bmatrix}^{\mathrm{T}} \tag{5}
$$

where $\boldsymbol{C} \in \mathbb{R}^n$, $\boldsymbol{L} \in \mathbb{R}^{n \times n}$ and $\boldsymbol{Q}^{(i)} \in \mathbb{R}^{n \times n}$, $\forall i = 1, \ldots, n$.

Governing equations

- We consider the 3D compressible Navier-Stokes equations in *primitive specific volume form*:

$$
\begin{aligned}
\zeta_{,t} + \zeta_{,j} u_j - \zeta u_{j,j} &= 0, \\
u_{i,t} + u_{i,j} u_j + \zeta p_{,i} - \tfrac{1}{Re} \zeta \tau_{ij,j} &= 0, \\
p_{,t} + u_j p_{,j} + \gamma u_{j,j} p - \left(\tfrac{\gamma}{PrRe}\right) \left(\kappa (p\zeta)_{,j}\right)_{,j} - \left(\frac{\gamma - 1}{Re}\right) u_{i,j} \tau_{ij} &= 0.
\end{aligned}
\tag{4}
$$

- For the compressible Navier-Stokes equations (4), Galerkin projection yields a system of n coupled quadratic ODEs

$$
\frac{d\boldsymbol{a}}{dt} = \boldsymbol{C} + \boldsymbol{L}\boldsymbol{a} + \left[\begin{array}{cccc} \boldsymbol{a}^{\mathrm{T}} \boldsymbol{Q}^{(1)} \boldsymbol{a} & \boldsymbol{a}^{\mathrm{T}} \boldsymbol{Q}^{(2)} \boldsymbol{a} & \cdots & \boldsymbol{a}^{\mathrm{T}} \boldsymbol{Q}^{(n)} \boldsymbol{a} \end{array}\right]^{\mathrm{T}} \tag{5}
$$

where $\boldsymbol{C} \in \mathbb{R}^n$, $\boldsymbol{L} \in \mathbb{R}^{n \times n}$ and $\boldsymbol{Q}^{(i)} \in \mathbb{R}^{n \times n}$, $\forall i = 1, \ldots, n$.

Projection-based model order reduction

Summary of technical challenges

- Projection-based MOR necessitates *truncation*.
- POD is, by definition and design, biased towards the *large, energy producing* scales of the flow (i.e., modes with large POD eigenvalues).
- Truncated/unresolved modes are negligible from a *data compression* point of view (i.e., small POD eigenvalues) but are crucial for the *dynamical equations*.
- For fluid flow applications, higher-order modes are associated with energy *dissipation* and thus, low-dimensional ROMs are often inaccurate and sometimes unstable.
- For a ROM to be stable and accurate, truncated/unresolved subspace must be accounted for (e.g., *turbulence modeling*, **subspace rotation**).

Projection-based model order reduction

Summary of technical challenges

- Projection-based MOR necessitates *truncation*.
- POD is, by definition and design, biased towards the *large, energy producing* scales of the flow (i.e., modes with large POD eigenvalues).
- Truncated/unresolved modes are negligible from a *data compression* point of view (i.e., small POD eigenvalues) but are crucial for the *dynamical equations*.
- For fluid flow applications, higher-order modes are associated with energy *dissipation* and thus, low-dimensional ROMs are often inaccurate and sometimes unstable.
- For a ROM to be stable and accurate, truncated/unresolved subspace must be accounted for (e.g., *turbulence modeling*, **subspace rotation**).

Projection-based model order reduction

Summary of technical challenges

- Projection-based MOR necessitates *truncation*.
- POD is, by definition and design, biased towards the *large, energy producing* scales of the flow (i.e., modes with large POD eigenvalues).
- Truncated/unresolved modes are negligible from a *data compression* point of view (i.e., small POD eigenvalues) but are crucial for the *dynamical equations*.
- For fluid flow applications, higher-order modes are associated with energy *dissipation* and thus, low-dimensional ROMs are often inaccurate and sometimes unstable.
- For a ROM to be stable and accurate, truncated/unresolved subspace must be accounted for (e.g., *turbulence modeling*, **subspace rotation**).

Projection-based model order reduction

Summary of technical challenges

- Projection-based MOR necessitates *truncation*.
- POD is, by definition and design, biased towards the *large, energy producing* scales of the flow (i.e., modes with large POD eigenvalues).
- Truncated/unresolved modes are negligible from a *data compression* point of view (i.e., small POD eigenvalues) but are crucial for the *dynamical equations*.
- For fluid flow applications, higher-order modes are associated with energy *dissipation* and thus, low-dimensional ROMs are often inaccurate and sometimes unstable.
- For a ROM to be stable and accurate, truncated/unresolved subspace must be accounted for (e.g., *turbulence modeling*, **subspace rotation**).

Projection-based model order reduction

Summary of technical challenges

- Projection-based MOR necessitates *truncation*.
- POD is, by definition and design, biased towards the *large, energy producing* scales of the flow (i.e., modes with large POD eigenvalues).
- Truncated/unresolved modes are negligible from a *data compression* point of view (i.e., small POD eigenvalues) but are crucial for the *dynamical equations*.
- For fluid flow applications, higher-order modes are associated with energy *dissipation* and thus, low-dimensional ROMs are often inaccurate and sometimes unstable.
- For a ROM to be stable and accurate, truncated/unresolved subspace must be accounted for (e.g., *turbulence modeling*, **subspace rotation**).

Outline

Accounting for modal truncation

Traditional linear eddy-viscosity approach

- Dissipative dynamics of truncated higher-order modes are modeled using additional linear term

$$\frac{d\boldsymbol{a}}{dt} = \boldsymbol{C} + (\boldsymbol{L} + \boldsymbol{L}_\nu)\boldsymbol{a} + \begin{bmatrix} \boldsymbol{a}^{\mathrm{T}}\boldsymbol{Q}^{(1)}\boldsymbol{a} & \boldsymbol{a}^{\mathrm{T}}\boldsymbol{Q}^{(2)}\boldsymbol{a} & \cdots & \boldsymbol{a}^{\mathrm{T}}\boldsymbol{Q}^{(n)}\boldsymbol{a} \end{bmatrix}^{\mathrm{T}}$$

- $\boldsymbol{L}_\nu$ is designed to decrease magnitude of positive eigenvalues and increase magnitude of negative eigenvalues of $\boldsymbol{L} + \boldsymbol{L}_\nu$ (for stability).

- Disadvantages of this approach:
  1. Additional term destroys *consistency* between ROM and Navier-Stokes equations.
  2. Calibration necessary to derive optimal $\boldsymbol{L}_\nu$ and optimal value is *flow dependent*.
  3. Inherently a *linear* model → cannot be expected to perform well for all classes of problems (e.g., nonlinear).

Accounting for modal truncation

Traditional linear eddy-viscosity approach

- Dissipative dynamics of truncated higher-order modes are modeled using additional linear term

$$\frac{d\boldsymbol{a}}{dt} = \boldsymbol{C} + (\boldsymbol{L} + \boldsymbol{L}_\nu)\boldsymbol{a} + \begin{bmatrix} \boldsymbol{a}^{\mathrm{T}}\boldsymbol{Q}^{(1)}\boldsymbol{a} & \boldsymbol{a}^{\mathrm{T}}\boldsymbol{Q}^{(2)}\boldsymbol{a} & \cdots & \boldsymbol{a}^{\mathrm{T}}\boldsymbol{Q}^{(n)}\boldsymbol{a} \end{bmatrix}^{\mathrm{T}}$$

- $\boldsymbol{L}_\nu$ is designed to decrease magnitude of positive eigenvalues and increase magnitude of negative eigenvalues of $\boldsymbol{L} + \boldsymbol{L}_\nu$ (for stability).
- Disadvantages of this approach:
  1. Additional term destroys *consistency* between ROM and Navier-Stokes equations.
  2. Calibration necessary to derive optimal $\boldsymbol{L}_\nu$ and optimal value is *flow dependent*.
  3. Inherently a *linear* model → cannot be expected to perform well for all classes of problems (e.g., nonlinear).

Accounting for modal truncation

Traditional linear eddy-viscosity approach

- Dissipative dynamics of truncated higher-order modes are modeled using additional linear term

$$\frac{d\boldsymbol{a}}{dt} = \boldsymbol{C} + (\boldsymbol{L} + \boldsymbol{L}_\nu)\boldsymbol{a} + \begin{bmatrix} \boldsymbol{a}^{\mathrm{T}}\boldsymbol{Q}^{(1)}\boldsymbol{a} & \boldsymbol{a}^{\mathrm{T}}\boldsymbol{Q}^{(2)}\boldsymbol{a} & \cdots & \boldsymbol{a}^{\mathrm{T}}\boldsymbol{Q}^{(n)}\boldsymbol{a} \end{bmatrix}^{\mathrm{T}}$$

- $\boldsymbol{L}_\nu$ is designed to decrease magnitude of positive eigenvalues and increase magnitude of negative eigenvalues of $\boldsymbol{L} + \boldsymbol{L}_\nu$ (for stability).
- Disadvantages of this approach:
  1. Additional term destroys *consistency* between ROM and Navier-Stokes equations.
  2. Calibration necessary to derive optimal $\boldsymbol{L}_\nu$ and optimal value is *flow dependent*.
  3. Inherently a *linear* model $\rightarrow$ cannot be expected to perform well for all classes of problems (e.g., nonlinear).

Accounting for modal truncation

Traditional linear eddy-viscosity approach

- Dissipative dynamics of truncated higher-order modes are modeled using additional linear term

$$\frac{d\boldsymbol{a}}{dt} = \boldsymbol{C} + (\boldsymbol{L} + \boldsymbol{L}_\nu)\boldsymbol{a} + \begin{bmatrix} \boldsymbol{a}^{\mathrm{T}}\boldsymbol{Q}^{(1)}\boldsymbol{a} & \boldsymbol{a}^{\mathrm{T}}\boldsymbol{Q}^{(2)}\boldsymbol{a} & \cdots & \boldsymbol{a}^{\mathrm{T}}\boldsymbol{Q}^{(n)}\boldsymbol{a} \end{bmatrix}^{\mathrm{T}}$$

- $\boldsymbol{L}_\nu$ is designed to decrease magnitude of positive eigenvalues and increase magnitude of negative eigenvalues of $\boldsymbol{L} + \boldsymbol{L}_\nu$ (for stability).

- Disadvantages of this approach:
  1. Additional term destroys *consistency* between ROM and Navier-Stokes equations.
  2. Calibration necessary to derive optimal $\boldsymbol{L}_\nu$ and optimal value is *flow dependent*.
  3. Inherently a *linear* model $\rightarrow$ cannot be expected to perform well for all classes of problems (e.g., nonlinear).

Accounting for modal truncation

Traditional linear eddy-viscosity approach

- Dissipative dynamics of truncated higher-order modes are modeled using additional linear term

$$\frac{d\boldsymbol{a}}{dt} = \boldsymbol{C} + (\boldsymbol{L} + \boldsymbol{L}_\nu)\boldsymbol{a} + \begin{bmatrix} \boldsymbol{a}^{\mathrm{T}}\boldsymbol{Q}^{(1)}\boldsymbol{a} & \boldsymbol{a}^{\mathrm{T}}\boldsymbol{Q}^{(2)}\boldsymbol{a} & \cdots & \boldsymbol{a}^{\mathrm{T}}\boldsymbol{Q}^{(n)}\boldsymbol{a} \end{bmatrix}^{\mathrm{T}}$$

- $\boldsymbol{L}_\nu$ is designed to decrease magnitude of positive eigenvalues and increase magnitude of negative eigenvalues of $\boldsymbol{L} + \boldsymbol{L}_\nu$ (for stability).

- Disadvantages of this approach:
  1. Additional term destroys *consistency* between ROM and Navier-Stokes equations.
  2. Calibration necessary to derive optimal $\boldsymbol{L}_\nu$ and optimal value is *flow dependent*.
  3. Inherently a *linear* model $\rightarrow$ cannot be expected to perform well for all classes of problems (e.g., nonlinear).

Accounting for modal truncation

Traditional linear eddy-viscosity approach

- Dissipative dynamics of truncated higher-order modes are modeled using additional linear term

$$\frac{d\boldsymbol{a}}{dt} = \boldsymbol{C} + (\boldsymbol{L} + \boldsymbol{L}_\nu)\boldsymbol{a} + \begin{bmatrix} \boldsymbol{a}^{\mathrm{T}}\boldsymbol{Q}^{(1)}\boldsymbol{a} & \boldsymbol{a}^{\mathrm{T}}\boldsymbol{Q}^{(2)}\boldsymbol{a} & \cdots & \boldsymbol{a}^{\mathrm{T}}\boldsymbol{Q}^{(n)}\boldsymbol{a} \end{bmatrix}^{\mathrm{T}}$$

- $\boldsymbol{L}_\nu$ is designed to decrease magnitude of positive eigenvalues and increase magnitude of negative eigenvalues of $\boldsymbol{L} + \boldsymbol{L}_\nu$ (for stability).

- Disadvantages of this approach:
  1. Additional term destroys *consistency* between ROM and Navier-Stokes equations.
  2. Calibration necessary to derive optimal $\boldsymbol{L}_\nu$ and optimal value is *flow dependent*.
  3. Inherently a *linear* model $\rightarrow$ cannot be expected to perform well for all classes of problems (e.g., nonlinear).

Accounting for modal truncation

Proposed new approach

- Instead of modeling truncation via additional linear term, model the truncation *a priori* by "rotating" the projection subspace into a more dissipative regime.
- *Standard approach:* retain only the most energetic POD modes, i.e., $\boldsymbol{U}_1$, $\boldsymbol{U}_2$, $\boldsymbol{U}_3$, $\boldsymbol{U}_4$ ...
- *Proposed approach:* choose some higher order basis to increase dissipation, i.e., $\boldsymbol{U}_1$, $\boldsymbol{U}_2$, $\boldsymbol{U}_6$, $\boldsymbol{U}_8$, ...
- That is, approximate the solution using a linear superposition of $n + p$ (with $p > 0$) most energetic modes:

$$\tilde{\boldsymbol{U}}_i = \sum_{j=1}^{n+p} X_{ji} \boldsymbol{U}_j \quad i = 1, \cdots, n, \tag{6}$$

where $\boldsymbol{X} \in \mathbb{R}^{(n+p)\times n}$ is an orthonormal ($\boldsymbol{X}^T \boldsymbol{X} = \boldsymbol{I}_{n\times n}$) "rotation" matrix.

Accounting for modal truncation

Proposed new approach

- Instead of modeling truncation via additional linear term, model the truncation *a priori* by "rotating" the projection subspace into a more dissipative regime.
- *Standard approach:* retain only the most energetic POD modes, i.e., $\boldsymbol{U}_1$, $\boldsymbol{U}_2$, $\boldsymbol{U}_3$, $\boldsymbol{U}_4$ ...
- *Proposed approach:* choose some higher order basis to increase dissipation, i.e., $\boldsymbol{U}_1$, $\boldsymbol{U}_2$, $\boldsymbol{U}_6$, $\boldsymbol{U}_8$, ...
- That is, approximate the solution using a linear superposition of $n+p$ (with $p>0$) most energetic modes:

$$\tilde{\boldsymbol{U}}_i = \sum_{j=1}^{n+p} X_{ji} \boldsymbol{U}_j \quad i = 1, \cdots, n, \tag{6}$$

where $\boldsymbol{X} \in \mathbb{R}^{(n+p)\times n}$ is an orthonormal ($\boldsymbol{X}^T\boldsymbol{X} = \boldsymbol{I}_{n\times n}$) "rotation" matrix.

Accounting for modal truncation

Proposed new approach

- Instead of modeling truncation via additional linear term, model the truncation *a priori* by "rotating" the projection subspace into a more dissipative regime.
- *Standard approach:* retain only the most energetic POD modes, i.e., $\boldsymbol{U}_1$, $\boldsymbol{U}_2$, $\boldsymbol{U}_3$, $\boldsymbol{U}_4$ ...
- *Proposed approach:* choose some higher order basis to increase dissipation, i.e., $\boldsymbol{U}_1$, $\boldsymbol{U}_2$, $\boldsymbol{U}_6$, $\boldsymbol{U}_8$, ...
- That is, approximate the solution using a linear superposition of $n+p$ (with $p>0$) most energetic modes:

$$\tilde{\boldsymbol{U}}_i = \sum_{j=1}^{n+p} X_{ji} \boldsymbol{U}_j \quad i = 1, \cdots, n, \tag{6}$$

where $\boldsymbol{X} \in \mathbb{R}^{(n+p)\times n}$ is an orthonormal ($\boldsymbol{X}^T\boldsymbol{X} = \boldsymbol{I}_{n\times n}$) "rotation" matrix.

Accounting for modal truncation

Proposed new approach

- Instead of modeling truncation via additional linear term, model the truncation *a priori* by "rotating" the projection subspace into a more dissipative regime.
- *Standard approach:* retain only the most energetic POD modes, i.e., $\boldsymbol{U}_1$, $\boldsymbol{U}_2$, $\boldsymbol{U}_3$, $\boldsymbol{U}_4$ ...
- *Proposed approach:* choose some higher order basis to increase dissipation, i.e., $\boldsymbol{U}_1$, $\boldsymbol{U}_2$, $\boldsymbol{U}_6$, $\boldsymbol{U}_8$, ...
- That is, approximate the solution using a linear superposition of $n+p$ (with $p>0$) most energetic modes:

$$\tilde{\boldsymbol{U}}_i = \sum_{j=1}^{n+p} X_{ji} \boldsymbol{U}_j \quad i = 1, \cdots, n, \tag{6}$$

where $\boldsymbol{X} \in \mathbb{R}^{(n+p)\times n}$ is an orthonormal ($\boldsymbol{X}^T \boldsymbol{X} = \boldsymbol{I}_{n\times n}$) "rotation" matrix.

Accounting for modal truncation

Goals of proposed new approach:

Find $\boldsymbol{X}$ such that

1. New modes $\widetilde{\boldsymbol{U}}$ remain good approximations of the flow $\rightarrow$ minimize the "rotation" angle, i.e. minimize $||\boldsymbol{X} - \boldsymbol{I}_{(n+p),n}||_F$.

2. New modes produce stable and accurate ROMs $\rightarrow$ ensure appropriate balance between energy production and energy dissipation.

$\rightarrow$ *Extension of earlier work for* **incompressible flow** *(Balajewicz et al., 2013) where it was shown that new POD modal basis is guaranteed to respect the power balance equation for the resolved turbulent kinetic energy and gives a stable ROM.*

Accounting for modal truncation

Goals of proposed new approach:

Find $\boldsymbol{X}$ such that

1. New modes $\widetilde{\boldsymbol{U}}$ remain good approximations of the flow $\rightarrow$ minimize the "rotation" angle, i.e. minimize $||\boldsymbol{X} - \boldsymbol{I}_{(n+p),n}||_F$.
2. New modes produce stable and accurate ROMs $\rightarrow$ ensure appropriate balance between energy production and energy dissipation.

$\rightarrow$ *Extension of earlier work for* **incompressible flow** *(Balajewicz et al., 2013) where it was shown that new POD modal basis is guaranteed to respect the power balance equation for the resolved turbulent kinetic energy and gives a stable ROM.*

Accounting for modal truncation

Goals of proposed new approach:

Find $\boldsymbol{X}$ such that

1. New modes $\widetilde{\boldsymbol{U}}$ remain good approximations of the flow $\rightarrow$ minimize the "rotation" angle, i.e. minimize $||\boldsymbol{X} - \boldsymbol{I}_{(n+p),n}||_F$.

2. New modes produce stable and accurate ROMs $\rightarrow$ ensure appropriate balance between energy production and energy dissipation.

$\rightarrow$ *Extension of earlier work for* **incompressible flow** *(Balajewicz et al., 2013) where it was shown that new POD modal basis is guaranteed to respect the power balance equation for the resolved turbulent kinetic energy and gives a stable ROM.*

Accounting for modal truncation

Trace minimization on Stiefel manifold

$$\begin{aligned} \underset{\boldsymbol{X} \in \mathcal{V}_{(n+p),n}}{\text{minimize}} \quad & -\operatorname{tr}\left(\boldsymbol{X}^{\mathrm{T}} \boldsymbol{I}_{(n+p)\times n}\right) \\ \text{subject to} \quad & \operatorname{tr}(\boldsymbol{X}^{\mathrm{T}} \boldsymbol{L} \boldsymbol{X}) = \eta \end{aligned} \tag{7}$$

where $\eta \in \mathbb{R}$ and

$$\mathcal{V}_{(n+p),n} \in \{\boldsymbol{X} \in \mathbb{R}^{(n+p)\times n} \ : \ \boldsymbol{X}^{\mathrm{T}}\boldsymbol{X} = \boldsymbol{I}_n \ , \ p > 0\}. \tag{8}$$

- η is a proxy for the balance between energy production and energy dissipation (calculated iteratively using modal energy).
- Equation (7) is solved efficiently offline using method of Lagrange multipliers (`Manopt MATLAB toolbox`).
- Result is system of the form (5) with

$$Q_{jk}^{(i)} \leftarrow \sum_{s,q,r=1}^{n+p} X_{si} Q_{qr}^{(s)} X_{qj} X_{rk}, \quad \boldsymbol{L} \leftarrow \boldsymbol{X}^{\mathrm{T}} \boldsymbol{L} \boldsymbol{X}, \quad \boldsymbol{C} \leftarrow \boldsymbol{X}^{T} \boldsymbol{C}^{*}.$$

Accounting for modal truncation

Trace minimization on Stiefel manifold

$$\begin{aligned} \underset{\boldsymbol{X}\in\mathcal{V}_{(n+p),n}}{\text{minimize}} \quad & -\operatorname{tr}\left(\boldsymbol{X}^{\mathrm{T}}\boldsymbol{I}_{(n+p)\times n}\right) \\ \text{subject to} \quad & \operatorname{tr}(\boldsymbol{X}^{\mathrm{T}}\boldsymbol{L}\boldsymbol{X}) = \eta \end{aligned} \tag{7}$$

where $\eta \in \mathbb{R}$ and

$$\mathcal{V}_{(n+p),n} \in \{\boldsymbol{X} \in \mathbb{R}^{(n+p)\times n} \ : \ \boldsymbol{X}^{\mathrm{T}}\boldsymbol{X} = \boldsymbol{I}_n \ , \ p > 0\}. \tag{8}$$

- η is a proxy for the balance between energy production and energy dissipation (calculated iteratively using modal energy).
- Equation (7) is solved efficiently offline using method of Lagrange multipliers (`Manopt MATLAB toolbox`).
- Result is system of the form (5) with

$$Q_{jk}^{(i)} \leftarrow \sum_{s,q,r=1}^{n+p} X_{si} Q_{qr}^{(s)} X_{qj} X_{rk}, \quad \boldsymbol{L} \leftarrow \boldsymbol{X}^{\mathrm{T}}\boldsymbol{L}\boldsymbol{X}, \quad \boldsymbol{C} \leftarrow \boldsymbol{X}^{T}\boldsymbol{C}^{*}.$$

Trace minimization on Stiefel manifold

$$\begin{aligned} \underset{\boldsymbol{X}\in\mathcal{V}_{(n+p),n}}{\text{minimize}} \quad & -\operatorname{tr}\left(\boldsymbol{X}^{\mathrm{T}}\boldsymbol{I}_{(n+p)\times n}\right) \\ \text{subject to} \quad & \operatorname{tr}(\boldsymbol{X}^{\mathrm{T}}\boldsymbol{L}\boldsymbol{X}) = \eta \end{aligned} \tag{7}$$

where $\eta \in \mathbb{R}$ and

$$\mathcal{V}_{(n+p),n} \in \{\boldsymbol{X} \in \mathbb{R}^{(n+p)\times n} \ : \ \boldsymbol{X}^{\mathrm{T}}\boldsymbol{X} = \boldsymbol{I}_n \ , \ p > 0\}. \tag{8}$$

- η is a proxy for the balance between energy production and energy dissipation (calculated iteratively using modal energy).
- Equation (7) is solved efficiently offline using method of Lagrange multipliers (`Manopt MATLAB toolbox`).
- Result is system of the form (5) with

$$Q_{jk}^{(i)} \leftarrow \sum_{s,q,r=1}^{n+p} X_{si} Q_{qr}^{(s)} X_{qj} X_{rk}, \quad \boldsymbol{L} \leftarrow \boldsymbol{X}^{\mathrm{T}}\boldsymbol{L}\boldsymbol{X}, \quad \boldsymbol{C} \leftarrow \boldsymbol{X}^{T}\boldsymbol{C}^{*}.$$

Accounting for modal truncation

Trace minimization on Stiefel manifold

$$\begin{aligned} \underset{\boldsymbol{X} \in \mathcal{V}_{(n+p),n}}{\text{minimize}} \quad & -\operatorname{tr}\left(\boldsymbol{X}^{\mathrm{T}} \boldsymbol{I}_{(n+p)\times n}\right) \\ \text{subject to} \quad & \operatorname{tr}(\boldsymbol{X}^{\mathrm{T}} \boldsymbol{L} \boldsymbol{X}) = \eta \end{aligned} \tag{7}$$

where $\eta \in \mathbb{R}$ and

$$\mathcal{V}_{(n+p),n} \in \{\boldsymbol{X} \in \mathbb{R}^{(n+p)\times n} \ : \ \boldsymbol{X}^{\mathrm{T}}\boldsymbol{X} = \boldsymbol{I}_n \ , \ p > 0\}. \tag{8}$$

- η is a proxy for the balance between energy production and energy dissipation (calculated iteratively using modal energy).
- Equation (7) is solved efficiently offline using method of Lagrange multipliers (`Manopt MATLAB toolbox`).
- Result is system of the form (5) with

$$Q^{(i)}_{jk} \leftarrow \sum_{s,q,r=1}^{n+p} X_{si} Q^{(s)}_{qr} X_{qj} X_{rk}, \quad \boldsymbol{L} \leftarrow \boldsymbol{X}^{\mathrm{T}} \boldsymbol{L} \boldsymbol{X}, \quad \boldsymbol{C} \leftarrow \boldsymbol{X}^{T} \boldsymbol{C}^{*}.$$

Accounting for modal truncation

Stabilization algorithm: returns stabilizing rotation matrix $\boldsymbol{X}$.

Inputs: Initial guess $\eta^{(0)} = \mathrm{tr}(L(1:n, 1:n))$ ($\boldsymbol{X} = \boldsymbol{I}_{(n+p)\times n}$), ROM size n and $p \geq 1$, ROM matrices associated with the first $n+p$ most energetic POD modes, convergence tolerance TOL, maximum number of iterations k_{max}.

for $k = 0, \cdots, k_{max}$

Solve constrained optimization problem on Stiefel manifold:

$$\begin{aligned} \underset{\boldsymbol{X}^{(k)} \in \mathcal{V}_{(n+p),n}}{\text{minimize}} \quad & -\mathrm{tr}\left(\boldsymbol{X}^{(k)\mathrm{T}} \boldsymbol{I}_{(n+p)\times n}\right) \\ \text{subject to} \quad & \mathrm{tr}(\boldsymbol{X}^{(k)\mathrm{T}} \boldsymbol{L} \boldsymbol{X}^{(k)}) = \eta^{(k)}. \end{aligned}$$

Construct new Galerkin matrices using (9).

Integrate numerically new Galerkin system.

Calculate "modal energy" $E(t)^{(k)} = \sum_i^n (a(t)_i^{(k)})^2$.

Perform linear fit of temporal data $E(t)^{(k)} \approx c_1^{(k)} t + c_0^{(k)}$, where $c_1^{(k)}$ =energy growth.

Calculate ϵ such that $c_1^{(k)}(\epsilon) = 0$ (no energy growth) using root-finding algorithm.

Perform update $\eta^{(k+1)} = \eta^{(k)} + \epsilon$.

if $||c_1^{(k)}|| < TOL$

$\boldsymbol{X} := \boldsymbol{X}^{(k)}$.

terminate the algorithm.

end

end

Accounting for modal truncation

Remarks

- Proposed approach may be interpreted as an *a priori, consistent* formulation of the eddy-viscosity turbulence modeling approach.
- Advantages of proposed approach:
  1. Retains *consistency* between ROM and Navier-Stokes equations $\rightarrow$ no additional turbulence terms required.
  2. Inherently a *nonlinear* model $\rightarrow$ should be expected to outperform linear models.
  3. Works with *any* basis and Petrov-Galerkin projection.
- Disadvantages of proposed approach:
  1. Off-line calibration of a free parameter η is required.
  2. Stability cannot be proven like for incompressible case.
  3. Existence/uniqueness of solution to (7) is *not* guaranteed; general rules-of-thumbs are available in (Balajewicz *et al.*, 2015).

Accounting for modal truncation

Remarks

- Proposed approach may be interpreted as an *a priori, consistent* formulation of the eddy-viscosity turbulence modeling approach.

- Advantages of proposed approach:
  1. Retains *consistency* between ROM and Navier-Stokes equations → no additional turbulence terms required.
  2. Inherently a *nonlinear* model → should be expected to outperform linear models.
  3. Works with *any* basis and Petrov-Galerkin projection.

- Disadvantages of proposed approach:
  1. Off-line calibration of a free parameter η is required.
  2. Stability cannot be proven like for incompressible case.
  3. Existence/uniqueness of solution to (7) is *not* guaranteed; general rules-of-thumbs are available in (Balajewicz *et al.*, 2015).

Accounting for modal truncation

Remarks

- Proposed approach may be interpreted as an *a priori, consistent* formulation of the eddy-viscosity turbulence modeling approach.
- Advantages of proposed approach:
  1. Retains *consistency* between ROM and Navier-Stokes equations $\rightarrow$ no additional turbulence terms required.
  2. Inherently a *nonlinear* model $\rightarrow$ should be expected to outperform linear models.
  3. Works with *any* basis and Petrov-Galerkin projection.
- Disadvantages of proposed approach:
  1. Off-line calibration of a free parameter η is required.
  2. Stability cannot be proven like for incompressible case.
  3. Existence/uniqueness of solution to (7) is *not* guaranteed; general rules-of-thumbs are available in (Balajewicz *et al.*, 2015).

Accounting for modal truncation

Remarks

- Proposed approach may be interpreted as an *a priori, consistent* formulation of the eddy-viscosity turbulence modeling approach.

- Advantages of proposed approach:
  1. Retains *consistency* between ROM and Navier-Stokes equations $\rightarrow$ no additional turbulence terms required.
  2. Inherently a *nonlinear* model $\rightarrow$ should be expected to outperform linear models.
  3. Works with *any* basis and Petrov-Galerkin projection.

- Disadvantages of proposed approach:
  1. Off-line calibration of a free parameter η is required.
  2. Stability cannot be proven like for incompressible case.
  3. Existence/uniqueness of solution to (7) is *not* guaranteed; general rules-of-thumbs are available in (Balajewicz *et al.*, 2015).

Accounting for modal truncation

Remarks

- Proposed approach may be interpreted as an *a priori, consistent* formulation of the eddy-viscosity turbulence modeling approach.
- Advantages of proposed approach:
  1. Retains *consistency* between ROM and Navier-Stokes equations $\rightarrow$ no additional turbulence terms required.
  2. Inherently a *nonlinear* model $\rightarrow$ should be expected to outperform linear models.
  3. Works with *any* basis and Petrov-Galerkin projection.
- Disadvantages of proposed approach:
  1. Off-line calibration of a free parameter η is required.
  2. Stability cannot be proven like for incompressible case.
  3. Existence/uniqueness of solution to (7) is *not* guaranteed; general rules-of-thumbs are available in (Balajewicz *et al.*, 2015).

Accounting for modal truncation

Remarks

- Proposed approach may be interpreted as an *a priori, consistent* formulation of the eddy-viscosity turbulence modeling approach.
- Advantages of proposed approach:
  1. Retains *consistency* between ROM and Navier-Stokes equations $\rightarrow$ no additional turbulence terms required.
  2. Inherently a *nonlinear* model $\rightarrow$ should be expected to outperform linear models.
  3. Works with *any* basis and Petrov-Galerkin projection.
- Disadvantages of proposed approach:
  1. Off-line calibration of a free parameter η is required.
  2. Stability cannot be proven like for incompressible case.
  3. Existence/uniqueness of solution to (7) is *not* guaranteed; general rules-of-thumbs are available in (Balajewicz *et al.*, 2015).

Accounting for modal truncation

Remarks

- Proposed approach may be interpreted as an *a priori, consistent* formulation of the eddy-viscosity turbulence modeling approach.
- Advantages of proposed approach:
  1. Retains *consistency* between ROM and Navier-Stokes equations $\rightarrow$ no additional turbulence terms required.
  2. Inherently a *nonlinear* model $\rightarrow$ should be expected to outperform linear models.
  3. Works with *any* basis and Petrov-Galerkin projection.
- Disadvantages of proposed approach:
  1. Off-line calibration of a free parameter η is required.
  2. Stability cannot be proven like for incompressible case.
  3. Existence/uniqueness of solution to (7) is *not* guaranteed; general rules-of-thumbs are available in (Balajewicz *et al.*, 2015).

Accounting for modal truncation

Remarks

- Proposed approach may be interpreted as an *a priori, consistent* formulation of the eddy-viscosity turbulence modeling approach.
- Advantages of proposed approach:
  1. Retains *consistency* between ROM and Navier-Stokes equations $\rightarrow$ no additional turbulence terms required.
  2. Inherently a *nonlinear* model $\rightarrow$ should be expected to outperform linear models.
  3. Works with *any* basis and Petrov-Galerkin projection.
- Disadvantages of proposed approach:
  1. Off-line calibration of a free parameter η is required.
  2. Stability cannot be proven like for incompressible case.
  3. Existence/uniqueness of solution to (7) is *not* guaranteed; general rules-of-thumbs are available in (Balajewicz *et al.*, 2015).

Accounting for modal truncation

Remarks

- Proposed approach may be interpreted as an *a priori, consistent* formulation of the eddy-viscosity turbulence modeling approach.
- Advantages of proposed approach:
  1. Retains *consistency* between ROM and Navier-Stokes equations $\rightarrow$ no additional turbulence terms required.
  2. Inherently a *nonlinear* model $\rightarrow$ should be expected to outperform linear models.
  3. Works with *any* basis and Petrov-Galerkin projection.
- Disadvantages of proposed approach:
  1. Off-line calibration of a free parameter η is required.
  2. Stability cannot be proven like for incompressible case.
  3. Existence/uniqueness of solution to (7) is *not* guaranteed; general rules-of-thumbs are available in (Balajewicz *et al.*, 2015).

Outline

High angle of attack laminar airfoil

- 2D flow around an inclined NACA0012 airfoil at Mach 0.7, $\mathrm{Re} = 500$, $\mathrm{Pr} = 0.72$, $\mathrm{AOA} = 20^\circ$, 1.25M DOFs $\Rightarrow n = 4$ ROM (86% snapshot energy).

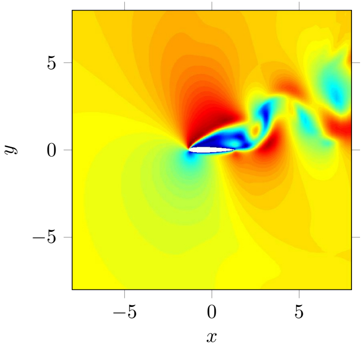


Figure 1: Contours of velocity magnitude at time of final snapshot.

High angle of attack laminar airfoil

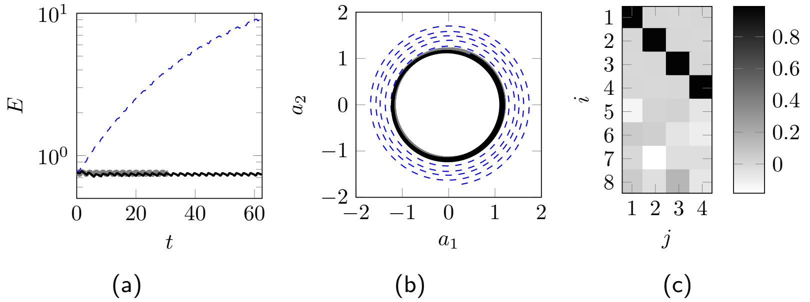


Figure 2: Nonlinear model reduction of the laminar airfoil. Evolution of modal energy (a), and phase plot of the first and second temporal basis, $a_1(t)$ and $a_2(t)$ (b); DNS (thick gray line), standard $n = 4$ ROM (dashed blue line), fine-tuned $n, p = 4$ ROM (solid black line). Stabilizing rotation matrix, $\boldsymbol{X}$ (c). Rotation is small: $||\boldsymbol{X} - \boldsymbol{I}_{(n+p)\times n}||_F/n = 0.083$, $\boldsymbol{X} \approx \boldsymbol{I}_{(n+p)\times n}$.

High angle of attack laminar airfoil

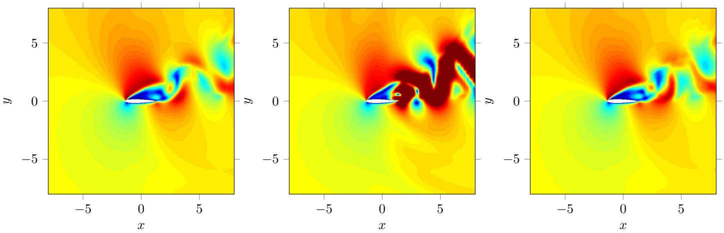


Figure 3: Snapshot of high angle of attack airfoil at final snapshot; contours of velocity magnitude. DNS (left), standard $n = 4$ ROM (middle), and fine-tuned $n, p = 4$ ROM (right)

Applications

Channel driven cavity: low Reynolds number case

- Flow over square cavity at Mach 0.6, $\mathrm{Re} = 1453.9$, $\mathrm{Pr} = 0.72$, 500K DOFs $\Rightarrow$ $n = 4$ ROM (91% snapshot energy).

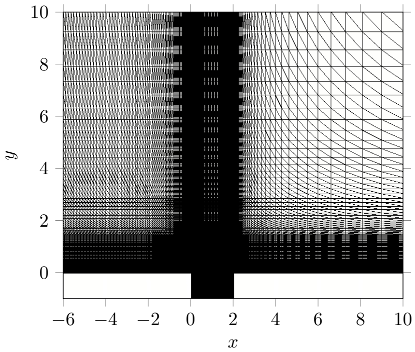


Figure 4: Domain and mesh for viscous channel driven cavity problem

Channel driven cavity: low Reynolds number case

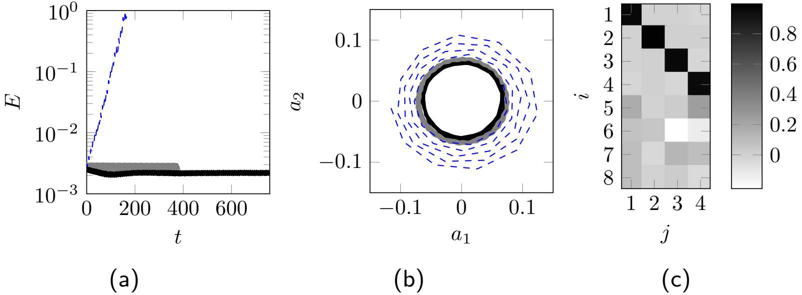


Figure 5: Nonlinear model reduction of channel drive cavity at $\mathrm{Re} \approx 1500$. Evolution of modal energy (a) and phase plot of the first and second temporal basis, $a_1(t)$ and $a_2(t)$ (b); DNS (thick gray line), standard $n = 4$ ROM (dashed blue line), fine-tuned $n, p = 4$ ROM (solid black line). Stabilizing rotation matrix, $\boldsymbol{X}$ (c). Rotation is small: $||\boldsymbol{X} - \boldsymbol{I}_{(n+p)\times n}||_F/n = 0.118$, $\boldsymbol{X} \approx \boldsymbol{I}_{(n+p)\times n}$.

Channel driven cavity: low Reynolds number case

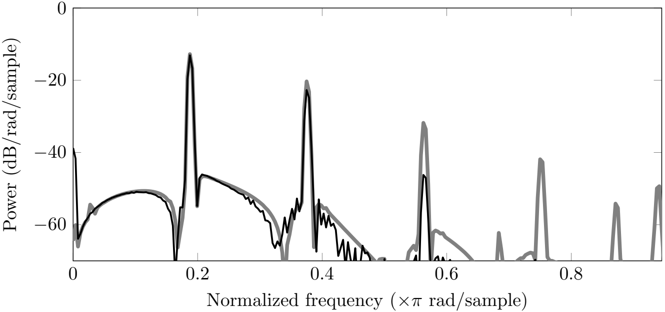


Figure 6: PSD of $p(\boldsymbol{x}, t)$ where $x = (2, -1)$ of channel drive cavity $\mathrm{Re} \approx 1500$. DNS (thick gray line), fine-tuned $n, p = 4$ ROM (black line)

Channel driven cavity: low Reynolds number case

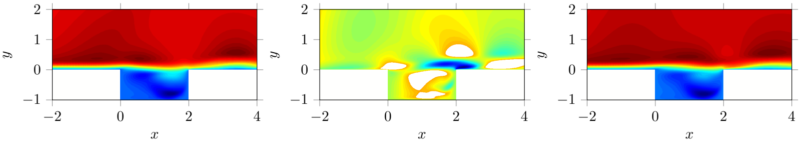


Figure 7: Snapshot of channel drive cavity $\mathrm{Re} \approx 1500$; contours of u-velocity magnitude at the final snapshot. DNS (left), standard $n = 4$ ROM (middle) and fine-tuned $n, p = 4$ ROM (right)

Channel driven cavity: moderate Reynolds number case

- Flow over square cavity at Mach 0.6, $\mathrm{Re} = 5452.1$, $\mathrm{Pr} = 0.72$ $\Rightarrow n = 20$ ROM (71% snapshot energy).

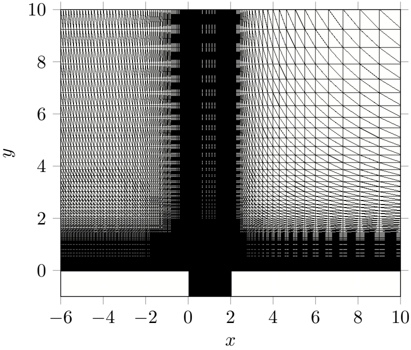
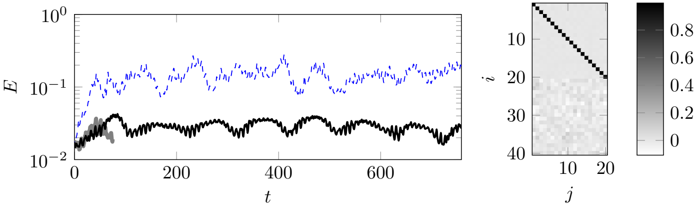


Figure 8: Domain and mesh for viscous channel driven cavity problem

Channel driven cavity: moderate Reynolds number case



(a) (b)

Figure 9: Nonlinear model reduction of channel drive cavity at $\mathrm{Re} \approx 5500$. Evolution of modal energy (a); DNS (thick gray line), standard $n = 20$ ROM (dashed blue line), fine-tuned $n, p = 20$ ROM (solid black line). Stabilizing rotation matrix, $\boldsymbol{X}$ (b). Rotation is small: $||\boldsymbol{X} - \boldsymbol{I}_{(n+p)\times n}||_F/n = 0.038$, $\boldsymbol{X} \approx \boldsymbol{I}_{(n+p)\times n}$.

Channel driven cavity: moderate Reynolds number case

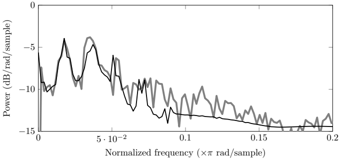


Figure 10: CPSD of $p(\boldsymbol{x}_1, t)$ and $p(\boldsymbol{x}_2, t)$ where $\boldsymbol{x}_1 = (2, -0.5)$ and $\boldsymbol{x}_2 = (0, -0.5)$ of channel driven cavity at $\mathrm{Re} \approx 5500$. DNS (thick gray line), fine-tuned $n, p = 20$ ROM (black line)

- Power and phase lag at the fundamental frequency, and first two super harmonics are predicted accurately using the fine-tuned ROM.
- Phase lag at these three frequencies as predicted by the CFD and the fine-tuned ROM is identified by red squares and blue triangles, respectively.

Channel driven cavity: moderate Reynolds number case

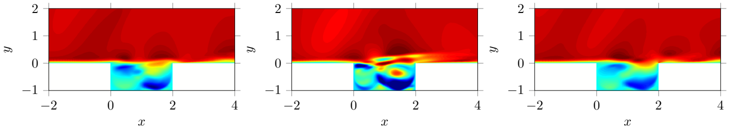


Figure 11: Snapshot of channel drive cavity $\mathrm{Re} \approx 5500$; contours of u-velocity magnitude at the final snapshot. DNS (left), standard $n = 20$ ROM (middle), and fine-tuned $n, p = 20$ ROM (right)

Channel driven cavity: moderate Reynolds number case

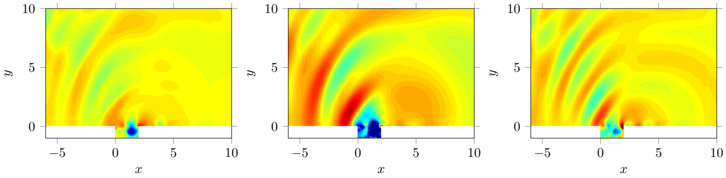


Figure 12: Snapshot of channel driven cavity $\mathrm{Re} \approx 5500$; contours of pressure at the final snapshot. DNS (left), standard $n = 20$ ROM (middle), and fine-tuned $n, p = 20$ ROM (right)

Outline

1. Motivation
2. Projection-based model order reduction
3. Accounting for modal truncation
   3.1 Traditional linear eddy-viscosity approach
   3.2 New proposed approach via subspace rotation
4. Applications
   4.1 High-angle of attack airfoil
   4.2 Low Reynolds number channel driven cavity
   4.3 Higher Reynolds number channel driven cavity
5. Conclusions and future work

Conclusions

- We have developed a non-intrusive approach for *stabilizing* and *fine-tuning* projection-based ROMs for compressible flows.
- The standard POD modes are *rotated* into a more dissipative regime to account for the dynamics in higher order modes truncated by the standard POD method.
- The new method is *consistent* and does not require addition of empirical turbulence model terms.
- Mathematically, the approach is formulated as a *quadratic matrix program* on the Stiefel manifold.
- This constrained minimization problem is solved *offline* and *small* enough to be solved in MATLAB.
- The method is demonstrated to deliver more *stable* and *accurate* ROMs on several applications.

Conclusions

- We have developed a non-intrusive approach for *stabilizing* and *fine-tuning* projection-based ROMs for compressible flows.
- The standard POD modes are *rotated* into a more dissipative regime to account for the dynamics in higher order modes truncated by the standard POD method.
- The new method is *consistent* and does not require addition of empirical turbulence model terms.
- Mathematically, the approach is formulated as a *quadratic matrix program* on the Stiefel manifold.
- This constrained minimization problem is solved *offline* and *small* enough to be solved in MATLAB.
- The method is demonstrated to deliver more *stable* and *accurate* ROMs on several applications.

Conclusions

- We have developed a non-intrusive approach for *stabilizing* and *fine-tuning* projection-based ROMs for compressible flows.
- The standard POD modes are *rotated* into a more dissipative regime to account for the dynamics in higher order modes truncated by the standard POD method.
- The new method is *consistent* and does not require addition of empirical turbulence model terms.
- Mathematically, the approach is formulated as a *quadratic matrix program* on the Stiefel manifold.
- This constrained minimization problem is solved *offline* and *small* enough to be solved in MATLAB.
- The method is demonstrated to deliver more *stable* and *accurate* ROMs on several applications.

Conclusions

- We have developed a non-intrusive approach for *stabilizing* and *fine-tuning* projection-based ROMs for compressible flows.
- The standard POD modes are *rotated* into a more dissipative regime to account for the dynamics in higher order modes truncated by the standard POD method.
- The new method is *consistent* and does not require addition of empirical turbulence model terms.
- Mathematically, the approach is formulated as a *quadratic matrix program* on the Stiefel manifold.
- This constrained minimization problem is solved *offline* and *small* enough to be solved in MATLAB.
- The method is demonstrated to deliver more *stable* and *accurate* ROMs on several applications.

Conclusions

- We have developed a non-intrusive approach for *stabilizing* and *fine-tuning* projection-based ROMs for compressible flows.
- The standard POD modes are *rotated* into a more dissipative regime to account for the dynamics in higher order modes truncated by the standard POD method.
- The new method is *consistent* and does not require addition of empirical turbulence model terms.
- Mathematically, the approach is formulated as a *quadratic matrix program* on the Stiefel manifold.
- This constrained minimization problem is solved *offline* and *small* enough to be solved in MATLAB.
- The method is demonstrated to deliver more *stable* and *accurate* ROMs on several applications.

Conclusions

- We have developed a non-intrusive approach for *stabilizing* and *fine-tuning* projection-based ROMs for compressible flows.
- The standard POD modes are *rotated* into a more dissipative regime to account for the dynamics in higher order modes truncated by the standard POD method.
- The new method is *consistent* and does not require addition of empirical turbulence model terms.
- Mathematically, the approach is formulated as a *quadratic matrix program* on the Stiefel manifold.
- This constrained minimization problem is solved *offline* and *small* enough to be solved in MATLAB.
- The method is demonstrated to deliver more *stable* and *accurate* ROMs on several applications.

Future work

- Extension of the proposed approach to problems with *generic non-linearities*, where the ROM involves some form of hyper-reduction (e.g., DEIM, gappy POD).

- Extension of the method to *predictive applications*, e.g., problems with varying Reynolds number and geometry.

- Selecting different *goal-oriented* objectives and constraints in our optimization problem:

$$\begin{aligned} \underset{\boldsymbol{X}\in\mathcal{V}_{(n+p),n}}{\text{minimize}} \quad & f(\boldsymbol{X}) \\ \text{subject to} \quad & g(\boldsymbol{X},\boldsymbol{L}) \end{aligned} \tag{10}$$

  e.g.,

  - Maximize *parametric robustness*: $f = \sum_{i=1}^{k} \beta_i ||\boldsymbol{U}^*(\mu_i)\boldsymbol{X} - \boldsymbol{U}^*(\mu_i)||_F$.
  - *ODE constraints*: $g = ||\boldsymbol{a}(t) - \boldsymbol{a}^*(t)|| < \epsilon$.

Future work

- Extension of the proposed approach to problems with *generic non-linearities*, where the ROM involves some form of hyper-reduction (e.g., DEIM, gappy POD).
- Extension of the method to *predictive applications*, e.g., problems with varying Reynolds number and geometry.
- Selecting different *goal-oriented* objectives and constraints in our optimization problem:

$$\begin{aligned} \underset{\boldsymbol{X}\in\mathcal{V}_{(n+p),n}}{\text{minimize}} \quad & f(\boldsymbol{X}) \\ \text{subject to} \quad & g(\boldsymbol{X}, \boldsymbol{L}) \end{aligned} \tag{10}$$

e.g.,

  - Maximize *parametric robustness*: $f = \sum_{i=1}^{k} \beta_i ||\boldsymbol{U}^*(\mu_i)\boldsymbol{X} - \boldsymbol{U}^*(\mu_i)||_F$.
  - *ODE constraints*: $g = ||\boldsymbol{a}(t) - \boldsymbol{a}^*(t)|| < \epsilon$.

Future work

- Extension of the proposed approach to problems with *generic non-linearities*, where the ROM involves some form of hyper-reduction (e.g., DEIM, gappy POD).
- Extension of the method to *predictive applications*, e.g., problems with varying Reynolds number and geometry.
- Selecting different *goal-oriented* objectives and constraints in our optimization problem:

$$\begin{aligned} \underset{\boldsymbol{X} \in \mathcal{V}_{(n+p),n}}{\text{minimize}} \quad & f(\boldsymbol{X}) \\ \text{subject to} \quad & g(\boldsymbol{X}, \boldsymbol{L}) \end{aligned} \tag{10}$$

e.g.,

  - Maximize *parametric robustness*: $f = \sum_{i=1}^{k} \beta_i \| \boldsymbol{U}^*(\mu_i)\boldsymbol{X} - \boldsymbol{U}^*(\mu_i)\|_F$.
  - *ODE constraints*: $g = \|\boldsymbol{a}(t) - \boldsymbol{a}^*(t)\| < \epsilon$.

Future work

- Extension of the proposed approach to problems with *generic non-linearities*, where the ROM involves some form of hyper-reduction (e.g., DEIM, gappy POD).
- Extension of the method to *predictive applications*, e.g., problems with varying Reynolds number and geometry.
- Selecting different *goal-oriented* objectives and constraints in our optimization problem:

$$\begin{aligned} \underset{\boldsymbol{X} \in \mathcal{V}_{(n+p),n}}{\text{minimize}} \quad & f(\boldsymbol{X}) \\ \text{subject to} \quad & g(\boldsymbol{X}, \boldsymbol{L}) \end{aligned} \tag{10}$$

  e.g.,
  - Maximize *parametric robustness*: $f = \sum_{i=1}^{k} \beta_i || \boldsymbol{U}^*(\mu_i)\boldsymbol{X} - \boldsymbol{U}^*(\mu_i)||_F$.
  - *ODE constraints*: $g = ||\boldsymbol{a}(t) - \boldsymbol{a}^*(t)|| < \epsilon$.

Future work

- Extension of the proposed approach to problems with *generic non-linearities*, where the ROM involves some form of hyper-reduction (e.g., DEIM, gappy POD).
- Extension of the method to *predictive applications*, e.g., problems with varying Reynolds number and geometry.
- Selecting different *goal-oriented* objectives and constraints in our optimization problem:

$$\begin{aligned} \underset{\boldsymbol{X}\in\mathcal{V}_{(n+p),n}}{\text{minimize}} \quad & f(\boldsymbol{X}) \\ \text{subject to} \quad & g(\boldsymbol{X}, \boldsymbol{L}) \end{aligned} \tag{10}$$

  e.g.,
  - Maximize *parametric robustness*: $f = \sum_{i=1}^{k} \beta_i ||\boldsymbol{U}^*(\mu_i)\boldsymbol{X} - \boldsymbol{U}^*(\mu_i)||_F$.
  - *ODE constraints*: $g = ||\boldsymbol{a}(t) - \boldsymbol{a}^*(t)|| < \epsilon$.

Thank you!

References

AUBRY, N., HOLMES, P., LUMLEY, J. L., & STONE E. 1988 The dynamics of coherent structures in the wall region of a turbulent boundary layer. *J. Fluid Mech.* **192** (115) 115–173.

BALAJEWICZ, M. & DOWELL, E. 2012 Stabilization of projection-based reduced order models of the Navier-Stokes equation. *Nonlinear Dynamics* **70** (2), 1619–1632.

BALAJEWICZ, M. & DOWELL, E. & NOACK, B. 2013 Low-dimensional modelling of high-Reynolds-number shear flows incorporating constraints from the Navier-Stokes equation. *Journal of Fluid Mechanics* **729**, 285–308.

BALAJEWICZ, M., TEZAUR, I. & DOWELL, E. 2015 Minimal subspace rotation on the Stiefel manifold for stabilization and enhancement of projection-based reduced order models for the compressible Navier-Stokes equations. ArXiV: http://arxiv.org/abs/1504.06661.

BARONE, M., KALASHNIKOVA, I., SEGALMAN, D. & THORNQUIST, H. 2009 Stable Galerkin reduced order models for linearized compressible flow. *J. Computat. Phys.* **228** (6), 1932–1946.

CARLBERG, K., FARHAT, C., CORTIAL, J. & AMSALLEM, D. 2013 The GNAT method for nonlinear model reduction: effective implementation and application to computational uid dynamics and turbulent flows. *J. Computat. Phys.* **242** 623–647.

KALASHNIKOVA, I., ARUNAJATESAN, S., BARONE, M., VAN BLOEMEN WAANDERS, B. & FIKE, J. 2014 Reduced order modeling for prediction and control of large-scale systems. *Sandia Tech. Report*.

OSTH, J., NOACK, B. R., KRAJNOVIC, C., BARROS, D., & BOREE, J. 2014 On the need for a nonlinear subscale turbulence term in POD models as exemplified for a high Reynolds number flow over an Ahmed body. *J. Fluid Mech.* **747** 518–544.

ROWLEY, C., COLONIUS, T. & MURRAY, R. 2004 Model reduction for compressible ows using pod and galerkin projection. *Physica D: Nonlinear Phenomena* **189** (1) 115–129.

SERRE, G., LAFON, P., GLOERFELT, X. & BAILLY, C. 2012 Reliable reduced-order models for timedependent linearized euler equations. *J. Computat. Phys.* **231** (15) 5176–5194.