

## The Schwarz Alternating Method for Multiscale Coupling in Solid Mechanics

Alejandro Mota<sup>1</sup>, Irina Tezaur<sup>1</sup>, Coleman Alleman<sup>1</sup>, Greg Phlipot<sup>2</sup>

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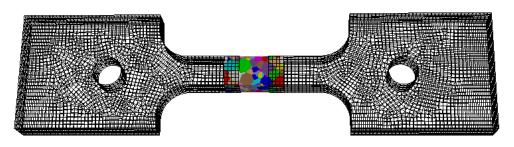


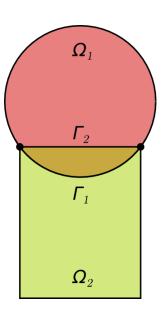
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## SAND2019-5577 C

# Outline

- 1. Motivation
- 2. Schwarz Alternating Method for Concurrent Multiscale Coupling for Quasistatics
  - Formulation
  - Implementation
  - Numerical Examples
- 3. Schwarz Alternating Method for Concurrent Multiscale Coupling for Dynamics
  - Formulation
  - Implementation
  - Numerical Examples
- 4. Summary
- 5. Future Work





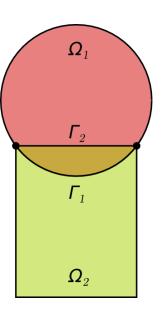


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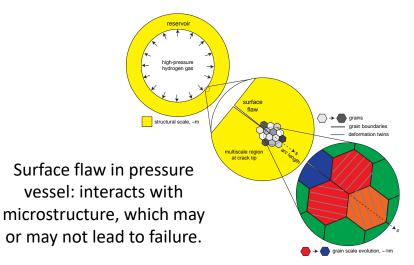
# Motivation for Concurrent Multiscale Coupling

- Large scale structural failure frequently originates from small scale phenomena such as defects, microcracks, inhomogeneities and more, which grow quickly in unstable manner.
- Failure occurs due to *tightly coupled interaction* between small scale (stress concentrations, material instabilities, cracks, etc.) and large scale (vibration, impact, high loads and other perturbations).

Concurrent multiscale methods are essential for understanding and prediction of behavior of engineering systems when a small scale failure determines the performance of the entire system.



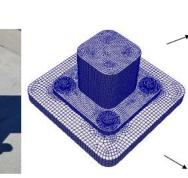
Roof failure of Boeing 737 aircraft due to fatigue cracks. From *imechanica.org* 

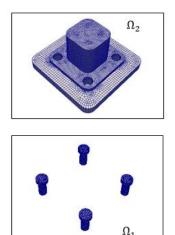


# **Requirements for Multiscale Coupling Method**

- Coupling is *concurrent* (two-way).
- *Ease of implementation* into existing massively-parallel HPC codes.
- Scalable, fast, robust (we target real engineering problems, e.g., analyses involving failure of bolted components!).
- *"Plug-and-play" framework*: simplifies task of meshing complex geometries!
  - Ability to couple regions with *different non-conformal meshes*, *different element types* and *different levels of refinement*.
  - > Ability to use *different solvers/time-integrators* in different regions.
- Coupling does not introduce *nonphysical artifacts.*
- *Theoretical* convergence properties/guarantees.







# Schwarz Alternating Method for Domain Decomposition

Proposed in 1870 by H. Schwarz for solving Laplace PDE on irregular domains.

**Crux of Method:** if the solution is known in regularly shaped domains, use those as pieces to iteratively build a solution for the more complex domain.

## **Basic Schwarz Algorithm**

Solve PDE by any method on  $\Omega_1$  w/ initial guess for Dirichlet BCs on  $\Gamma_1$ .

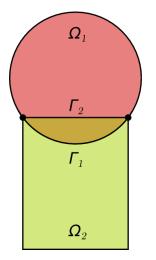
## Iterate until convergence:

Initialize:

- Solve PDE by any method (can be different than for  $\Omega_1$ ) on  $\Omega_2$  w/ Dirichlet BCs on  $\Gamma_2$  that are the values just obtained for  $\Omega_1$ .
- Solve PDE by any method (can be different than for  $\Omega_2$ ) on  $\Omega_1$  w/ Dirichlet BCs on  $\Gamma_1$  that are the values just obtained for  $\Omega_2$ .







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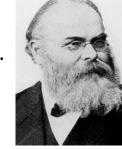
Solve PDE by any method on  $\Omega_1$  w/ initial guess for Dirichlet BCs on  $\Gamma_1$ .

## *Iterate until convergence:*

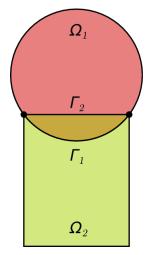
Initialize:

**Requirement for convergence**:  $\Omega_1 \cap \Omega_2 \neq \emptyset$ Solve PDE by any method (can be different than for  $\Omega_1$ ) on  $\Omega_2$  w/

- Dirichlet BCs on  $\Gamma_2$  that are the values just obtained for  $\Omega_1$ .
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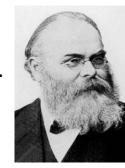
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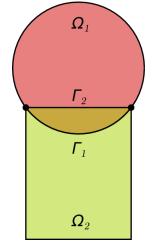
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- Solve PDE by any method (can be different than for Ω<sub>2</sub>) on Ω<sub>1</sub> w/ Dirichlet BCs on Γ<sub>1</sub> that are the values just obtained for Ω<sub>2</sub>.
- Schwarz alternating method most commonly used as a *preconditioner* for Krylov iterative methods to solve linear algebraic equations.





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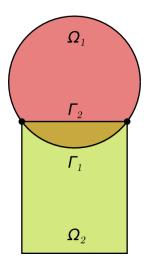
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- Schwarz alternating method most commonly used as a *preconditioner* for Krylov iterative methods to solve linear algebraic equations.

**Novel idea:** using the Schwarz alternating as a *discretization method* for solving multiscale partial differential equations (PDEs).



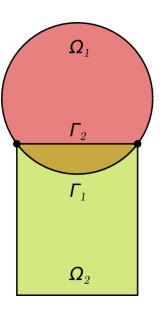




# Outline

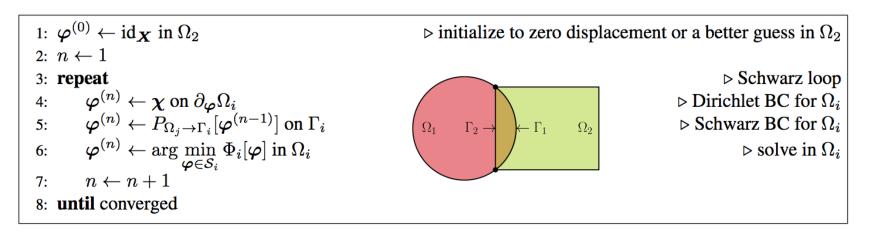
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# Schwarz Alternating Method for Multiscale Sandia Coupling in Quasistatics



## Advantages:

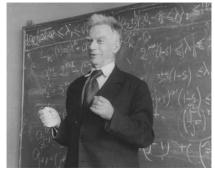
- Conceptually very simple.
- Allows the coupling of regions with *different non-conforming meshes*, *different element types*, and *different levels of refinement*.
- Information is exchanged among two or more regions, making coupling concurrent.
- Different solvers can be used for the different regions.
- *Different material models* can be coupled if they are compatible in the overlap region.
- Simplifies the task of *meshing complex geometries* for the different scales.

# **Theoretical Foundation**

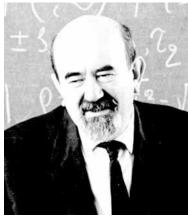
Using the Schwarz alternating as a *discretization method* for PDEs is natural idea with a sound *theoretical foundation*.

- <u>S. L. Sobolev (1936)</u>: posed Schwarz method for *linear elasticity* in variational form and *proved method's convergence* by proposing a convergent sequence of energy functionals.
- <u>S. G. Mikhlin (1951)</u>: proved convergence of Schwarz method for general linear elliptic PDEs.
- A. Mota, I. Tezaur, C. Alleman (2017)\*: derived a proof of convergence of the alternating Schwarz method for the finite deformation quasi-static nonlinear PDEs (with energy functional Φ[φ] defined below), and determined a geometric convergence rate for the finite deformation quasi-static problem.

$$\boldsymbol{\Phi}[\boldsymbol{\varphi}] = \int_{B} W(\boldsymbol{F}, \boldsymbol{Z}, T) \, dV - \int_{B} \boldsymbol{B} \cdot \boldsymbol{\varphi} \, dV - \int_{\partial_{T}B} \overline{\boldsymbol{T}} \cdot \boldsymbol{\varphi} \, dS$$
$$\nabla \cdot \boldsymbol{P} + \boldsymbol{B} = \boldsymbol{0}$$



S. L. Sobolev (1908 – 1989)



S. G. Mikhlin (1908 - 1990)

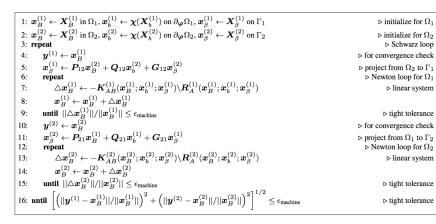


A. Mota, I. Tezaur, C. Alleman

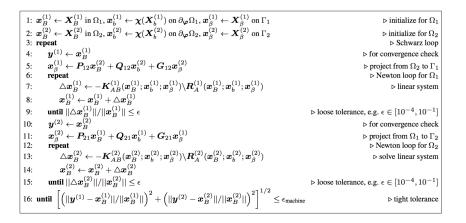
## \*A. Mota, I. Tezaur, C. Alleman. "The Schwarz Alternating Method in Solid Mechanics", CMAME 319 (2017), 19-51.

## Four Variants\* of Schwarz



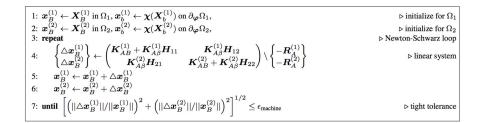


### Full Schwarz



$\triangleright$ initialize for $\Omega_1$
$\triangleright$ initialize for $\Omega_2$
▷ Newton-Schwarz loop
$\triangleright$ project from $\Omega_2$ to $\Gamma_1$
⊳ linear system
$\triangleright$ project from $\Omega_1$ to $\Gamma_2$
⊳ linear system
⊳ tight tolerance

#### Modified Schwarz



#### Monolithic Schwarz

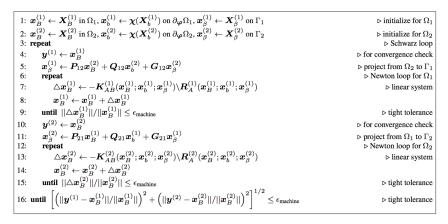
### Inexact Schwarz

\*A. Mota, I. Tezaur, C. Alleman. "The Schwarz Alternating Method in Solid Mechanics", CMAME 319 (2017), 19-51.

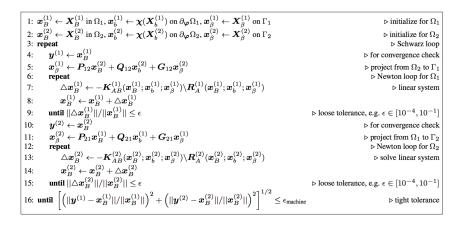
## Four Variants\* of Schwarz



## *Most performant method*: monotonic convergence, theoretical convergence guarantee.

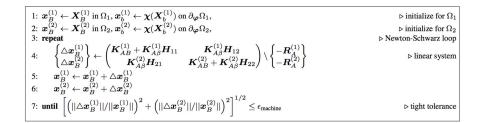


### Full Schwarz



	$\boldsymbol{X}_{b}^{(1)} \leftarrow \boldsymbol{X}_{B}^{(1)} \text{ in } \Omega_{1}, \boldsymbol{x}_{b}^{(1)} \leftarrow \boldsymbol{\chi}(\boldsymbol{X}_{b}^{(1)}) \text{ on } \partial_{\boldsymbol{\varphi}}\Omega_{1}, \boldsymbol{x}_{\beta}^{(1)} \leftarrow \boldsymbol{X}_{\beta}^{(1)} \text{ on } \Gamma_{1}$	$\triangleright$ initialize for $\Omega_1$
	$\boldsymbol{\mathcal{X}}_{B}^{(2)} \leftarrow \boldsymbol{X}_{B}^{(2)} \text{ in } \Omega_{2}, \boldsymbol{x}_{b}^{(2)} \leftarrow \boldsymbol{\chi}(\boldsymbol{X}_{b}^{(2)}) \text{ on } \partial_{\boldsymbol{\varphi}}\Omega_{2}, \boldsymbol{x}_{\beta}^{(2)} \leftarrow \boldsymbol{X}_{\beta}^{(2)} \text{ on } \Gamma_{2}$	$\triangleright$ initialize for $\Omega_2$ $\triangleright$ Newton-Schwarz loop
3: rep		b Newton-Schwarz loop
	$m{x}_eta^{(1)} \leftarrow m{P}_{12} m{x}_B^{(2)} + m{Q}_{12} m{x}_b^{(2)} + m{G}_{12} m{x}_eta^{(2)}$	$\triangleright$ project from $\Omega_2$ to $\Gamma_1$
5:		⊳ linear system
	$oldsymbol{x}_B^{(1)} \leftarrow oldsymbol{x}_B^{(1)} +  riangle oldsymbol{x}_B^{(1)}$	
	$m{x}_eta^{(2)} \leftarrow m{P}_{21} m{x}_B^{(1)} + m{Q}_{21} m{x}_b^{(1)} + m{G}_{21} m{x}_eta^{(1)}$	$\triangleright$ project from $\Omega_1$ to $\Gamma_2$
8:	$ \Delta \boldsymbol{x}_{B}^{(2)} \leftarrow -\boldsymbol{K}_{AB}^{(2)}(\boldsymbol{x}_{B}^{(2)}; \boldsymbol{x}_{b}^{(2)}; \boldsymbol{x}_{\beta}^{(2)}) \backslash \boldsymbol{R}_{A}^{(2)}(\boldsymbol{x}_{B}^{(2)}; \boldsymbol{x}_{b}^{(2)}; \boldsymbol{x}_{\beta}^{(2)}) $	⊳ linear system
	$oldsymbol{x}_B^{(2)} \leftarrow oldsymbol{x}_B^{(2)} +  riangle oldsymbol{x}_B^{(2)}$	
10: <b>unt</b>	$\mathrm{ill} \; \left[ \left(    \triangle \bm{x}_B^{(1)}    /    \bm{x}_B^{(1)}    \right)^2 + \left(    \triangle \bm{x}_B^{(2)}    /    \bm{x}_B^{(2)}    \right)^2 \right]^{1/2} \leq \epsilon_{\mathrm{machine}}$	⊳ tight tolerance

### Modified Schwarz



#### Monolithic Schwarz

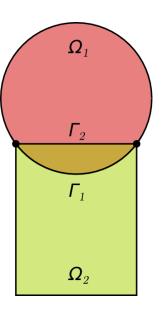
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# Implementation within Albany Code

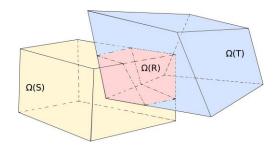
The proposed *quasistatic alternating Schwarz method* is implemented within the *LCM project* in Sandia's open-source parallel, C++, multi-physics, finite element code, *Albany*.

- Component-based design for rapid development of capabilities.
- Contains a wide variety of *constitutive models*.
- Extensive use of libraries from the open-source *Trilinos* project.
  - Use of the *Phalanx* package to decompose complex problem into simpler problems with managed dependencies.
  - Use of the *Sacado* package for *automatic differentiation*.
  - Use of *Teko* package for block preconditioning.
- Parallel implementation of Schwarz alternating method uses the Data Transfer Kit (DTK).
- All software available on *GitHub*.

https://github.com/gahansen/Albany



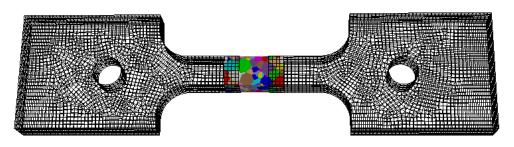
https://github.com/trilinos/trilinos

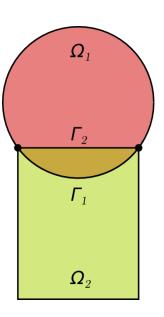


https://github.com/ORNL-CEES/DataTransferKit

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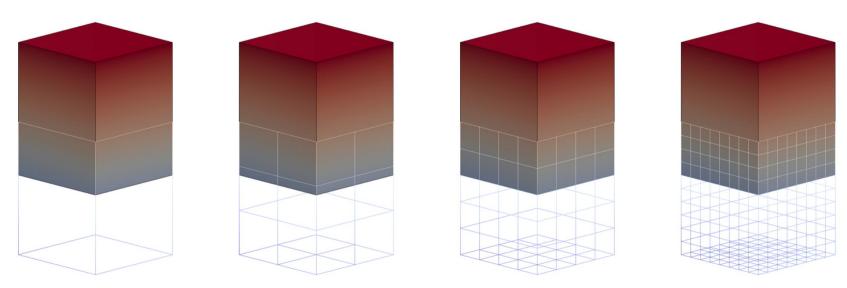
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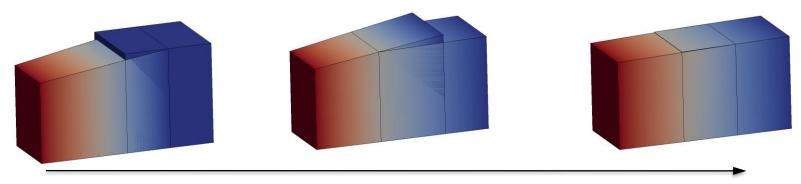
## Quasistatic Example #1: Cuboid Problem



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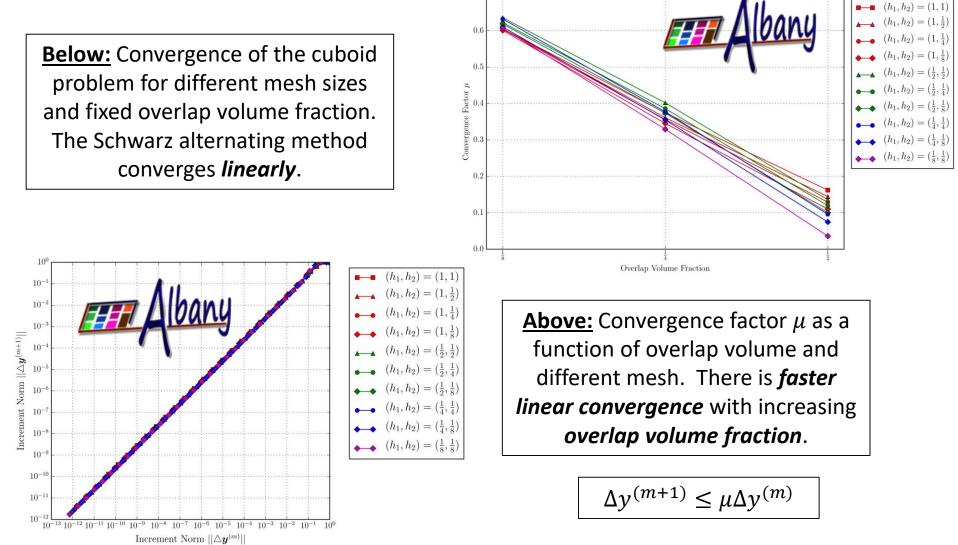
- Coupling of *two cuboids* with square base (above).
- *Neohookean*-type material model.



## Schwarz Iteration

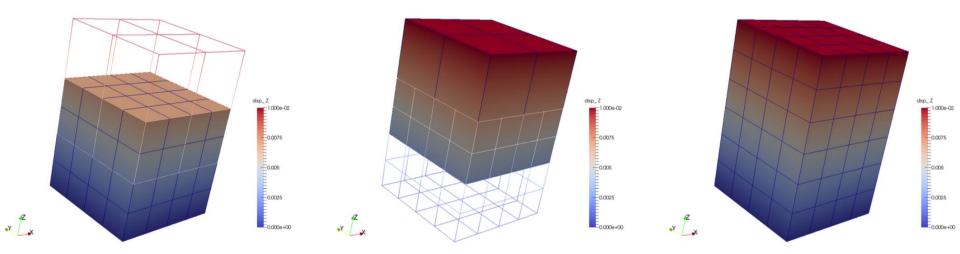
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# Cuboid Problem: Convergence with Overlap & Refinement



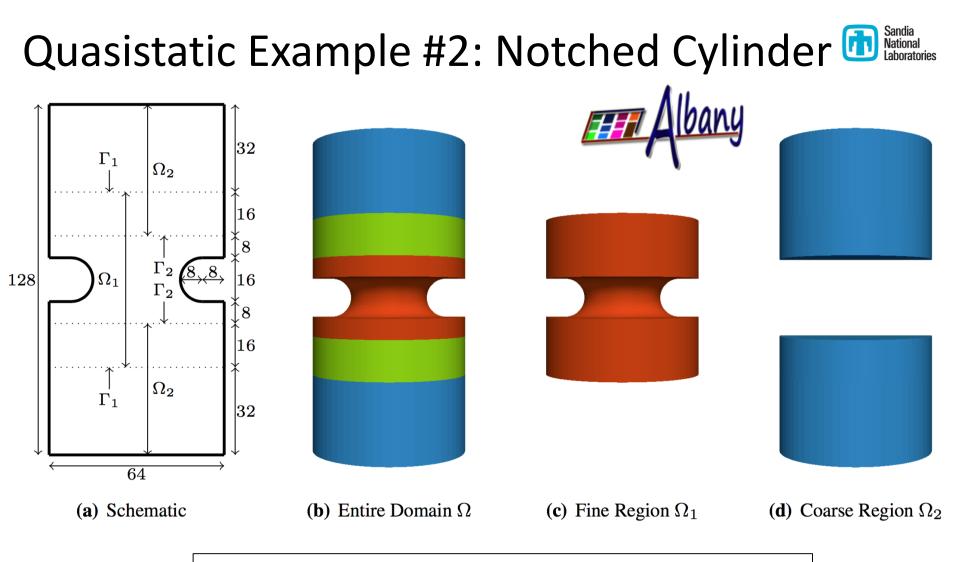
## Cuboid Problem: Schwarz Error





Subdomain	$u_3$ relative error	$\sigma_{33}$ relative error	
$\Omega_1 \ \Omega_2$	$1.24 \times 10^{-14}$ $7.30 \times 10^{-15}$	$\begin{array}{c} 2.31 \times 10^{-13} \\ 3.06 \times 10^{-13} \end{array}$	



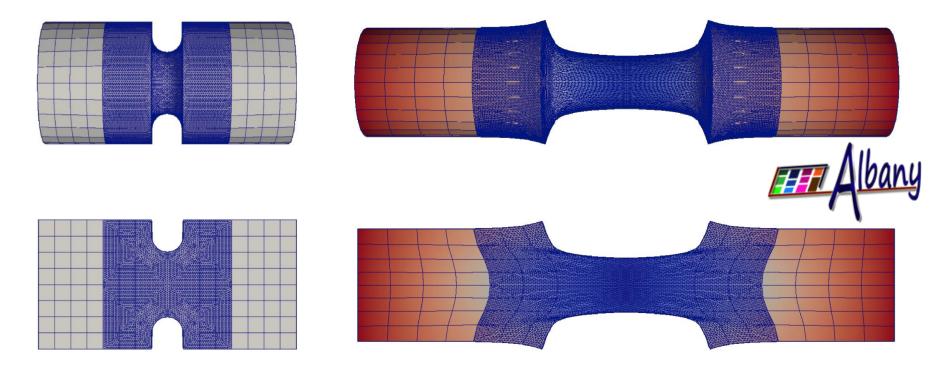


- Notched cylinder that is stretched along its axial direction.
- Domain decomposed into *two subdomains*.
- Neohookean-type material model.

# Notched Cylinder: TET-HEX Coupling

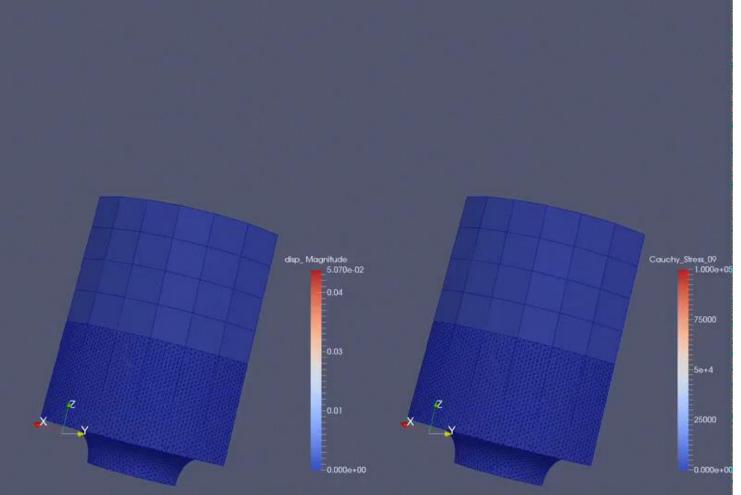


- The Schwarz alternating method is capable of coupling *different mesh topologies*.
- The notched region, where stress concentrations are expected, is *finely* meshed with *tetrahedral* elements.
- The top and bottom regions, presumably of less interest, are meshed with *coarser hexahedral* elements.



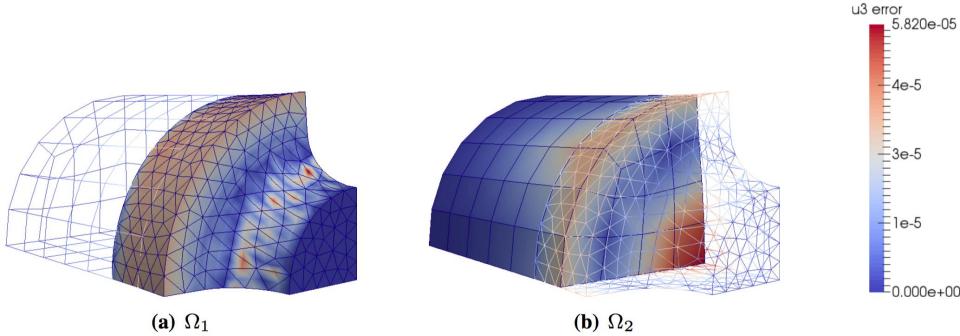
## Notched Cylinder: TET-HEX Coupling





# Notched Cylinder: Conformal TET-HEX Coupling





	$u_3$ relative error	
Absolute residual tolerance	$\Omega_1$	$\Omega_2$
$1.0  imes 10^{-14}$	$9.27  imes 10^{-3}$	$3.70 \times 10^{-3}$



# Notched Cylinder: Coupling Different Materials

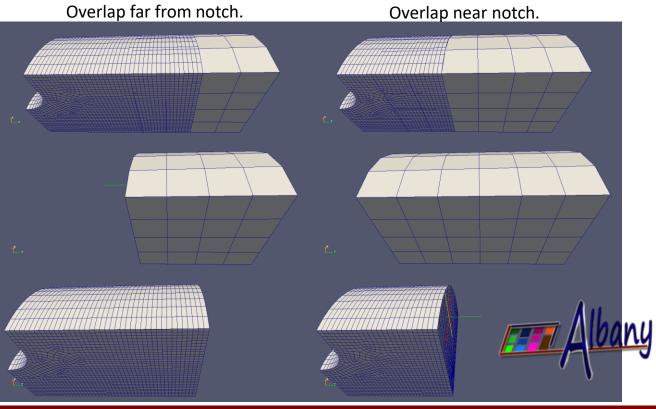
The Schwarz method is capable of coupling regions with *different material models*.

- Notched cylinder subjected to tensile load with an *elastic* and *J2 elasto-plastic* regions.
- *Coarse* region is *elastic* and *fine* region is *elasto-plastic*.
- The overlap region in the first mesh is nearer the notch, where plastic behavior is expected.

Coupled regions

Coarse, elastic region

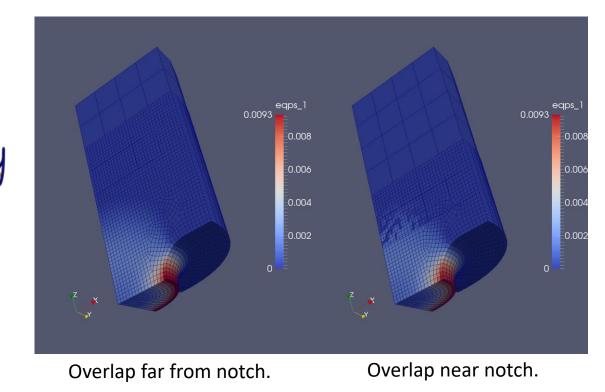
Fine, elasto-plastic region



# Notched Cylinder: Coupling Different Materials

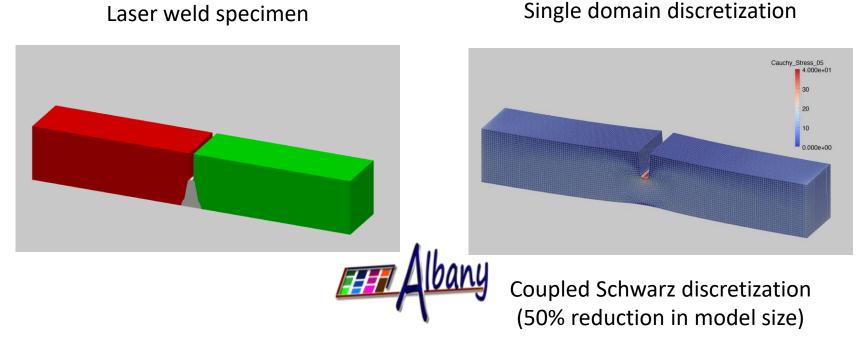
Need to be careful to do domain decomposition so that material models are *consistent* in overlap region.

- When the *overlap* region is *far from the notch*, no plastic deformation exists in it: the coarse and fine regions predict the *same behavior*.
- When the *overlap* region is *near the notch*, plastic deformation spills onto it and the two models predict different behavior, affecting convergence *adversely*.

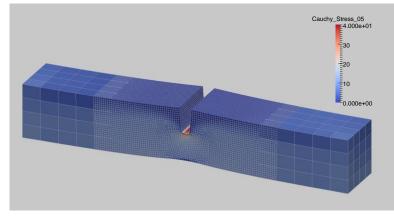


# Quasistatic Example #3: Laser Weld

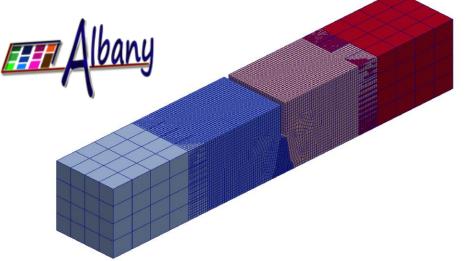




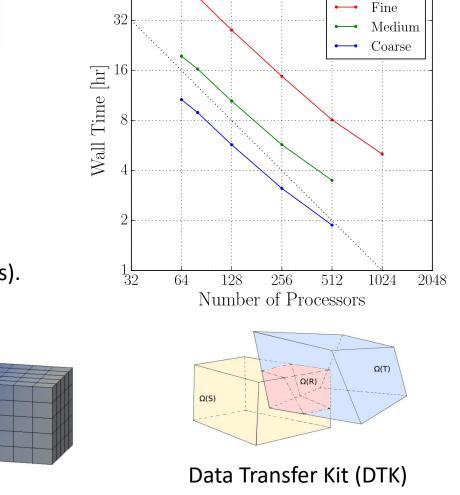
- Problem of *practical scale (~200K dofs).*
- *Isotropic elasticity* and *J2 plasticity* with linear isotropic hardening.
- *Identical parameters* for weld and base materials for proof of concept, to become independent models.



# Laser Weld: Strong Scalability of Parallel Schwarz with DTK



• *Near-ideal linear speedup* (64-1024 cores).



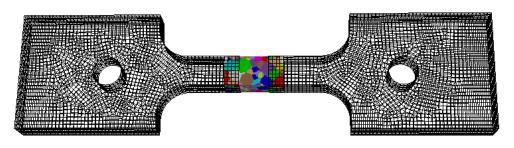
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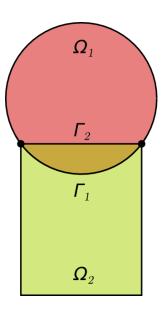
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Ideal

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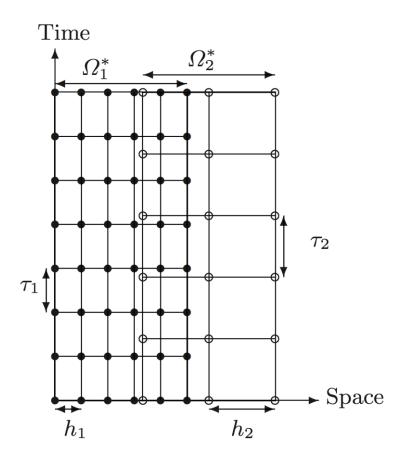






# Schwarz Alternating Method for Dynamics

 In the literature the Schwarz method is applied to dynamics by using *spacetime discretizations*.



Overlapping non-matching meshes and time steps in dynamics.

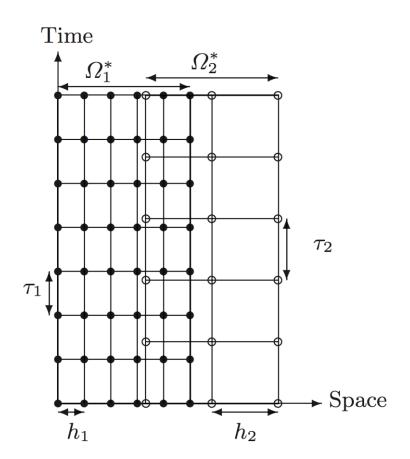


# Schwarz Alternating Method for Dynamics

 In the literature the Schwarz method is applied to dynamics by using *spacetime discretizations*.

Pro ☺: Can use *non-matching* meshes and time-steps (see right figure).

**Con** ②: *Unfeasible* given the design of our current codes and size of simulations.

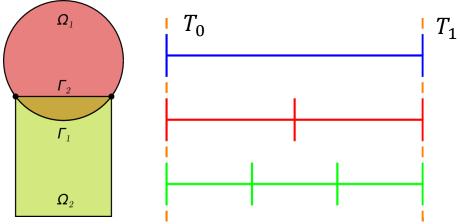


Overlapping non-matching meshes and time steps in dynamics.



# Schwarz Alternating Method for Dynamic

# **Multiscale Coupling**



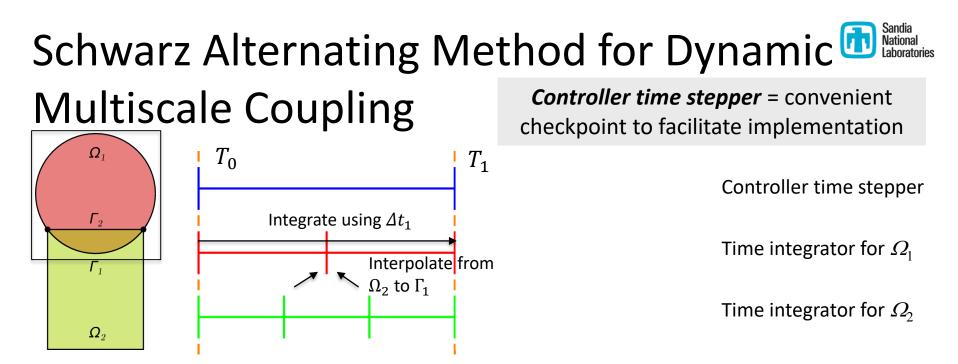
**<u>Step 0</u>**: Initialize i = 0 (controller time index).

*Controller time stepper* = convenient checkpoint to facilitate implementation

Controller time stepper

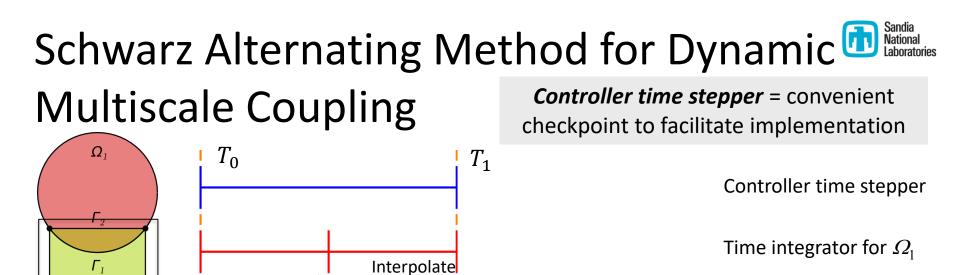
Time integrator for  $\Omega_1$ 

Time integrator for  $\varOmega_2$ 



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**<u>Step 1</u>**: Advance  $\Omega_1$  solution from time  $T_i$  to time  $T_{i+1}$  using time-stepper in  $\Omega_1$  with time-step  $\Delta t_1$ , using solution in  $\Omega_2$  interpolated to  $\Gamma_1$  at times  $T_i + n\Delta t_1$ .



Time integrator for  $\Omega_2$ 

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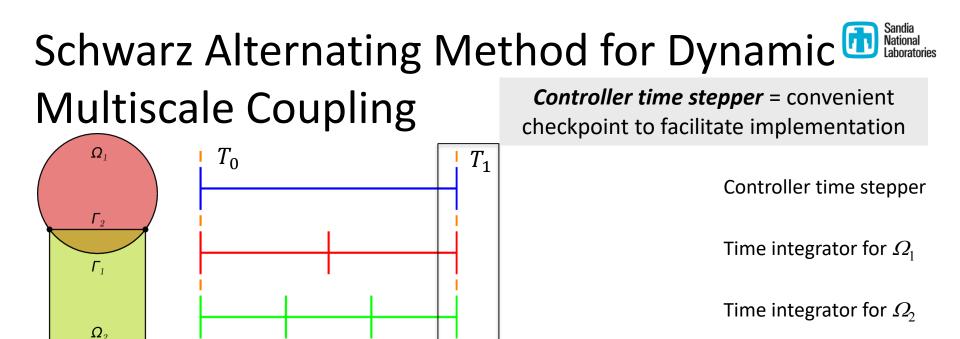
Integrate using  $\Delta t_2$ 

 $\Omega_2$ 

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from  $\Omega_1$  to  $\Gamma_2$ 

**<u>Step 2</u>**: Advance  $\Omega_2$  solution from time  $T_i$  to time  $T_{i+1}$  using time-stepper in  $\Omega_2$  with time-step  $\Delta t_2$ , using solution in  $\Omega_1$  interpolated to  $\Gamma_2$  at times  $T_i + n\Delta t_2$ .

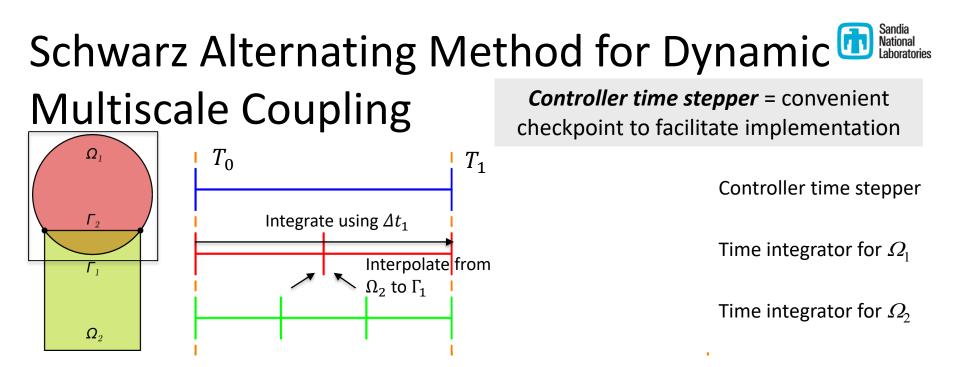


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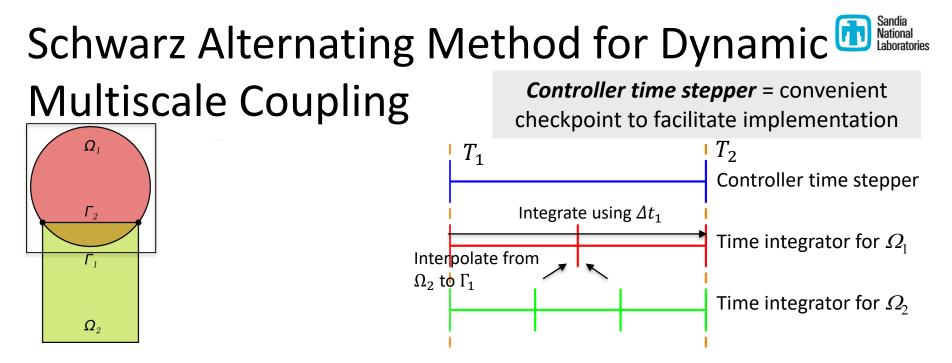
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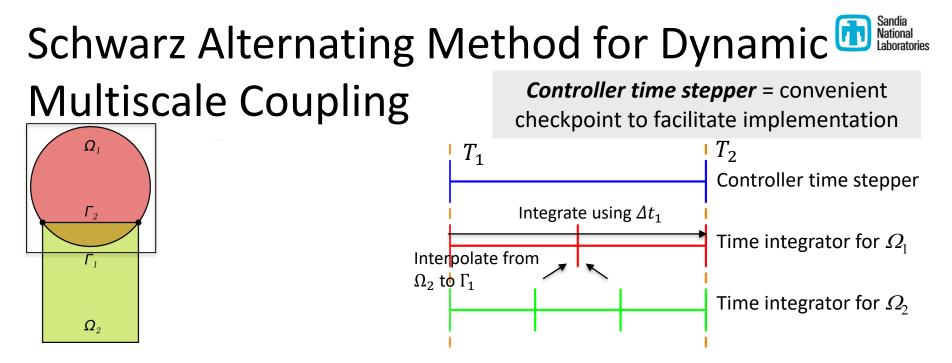
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Can use *different integrators* with *different time steps* within each domain!

# Schwarz Alternating Method for Dynamic Multiscale Coupling: Theory

For quasistatics, we derived a *proof of convergence* of the alternating Schwarz method for the *finite deformation* problem, and determined a *geometric convergence rate* [(Mota, Tezaur, Alleman, *CMAME*, 2017) and previous talk].

**Theorem 1.** Assume that the energy functional  $\Phi[\varphi]$  satisfies properties 1–5 above. Consider the Schwarz alternating method of Section 2 defined by (9)–(13) and its equivalent form (39). Then

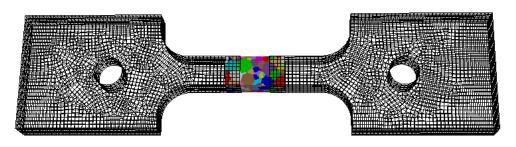
- (a)  $\Phi[\tilde{\varphi}^{(0)}] \ge \Phi[\tilde{\varphi}^{(1)}] \ge \cdots \ge \Phi[\tilde{\varphi}^{(n-1)}] \ge \Phi[\tilde{\varphi}^{(n)}] \ge \cdots \ge \Phi[\varphi]$ , where  $\varphi$  is the minimizer of  $\Phi[\varphi]$  over S.
- (b) The sequence  $\{\tilde{\varphi}^{(n)}\}\$  defined in (39) converges to the minimizer  $\varphi$  of  $\Phi[\varphi]$  in S.
- (c) The Schwarz minimum values  $\Phi[\tilde{\varphi}^{(n)}]$  converge monotonically to the minimum value  $\Phi[\varphi]$  in S starting from any initial guess  $\tilde{\varphi}^{(0)}$ .

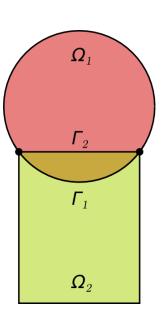
Extending these results to *dynamics* is *work in progress*.

- Quasistatic proof *extends naturally* assuming conformal meshes and the same time step is used in each Schwarz subdomain.
- Some analysis of Schwarz for evolution problems was performed in (Lions, 1988) and may be possible to *leverage*.
- Our numerical results suggest theoretical analysis is *possible*.

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  - Formulation
  - Implementation
  - Numerical Examples
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## Implementation within Albany Code

The proposed *dynamic alternating Schwarz method* is implemented within the *LCM project* in Sandia's open-source parallel, C++, multi-physics, finite element code, *Albany*.

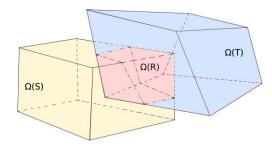
- Component-based design for rapid development of capabilities.
- Contains a wide variety of *constitutive models*.
- Extensive use of libraries from the open-source *Trilinos* project.
  - Use of the *Phalanx* package to decompose complex problem into simpler problems with managed dependencies.
  - Use of the *Sacado* package for *automatic differentiation*.
  - Use of *Tempus* package for *time-integration\**.
- Parallel implementation of Schwarz alternating method uses the Data Transfer Kit (DTK).
- All software available on *GitHub*.

Albany

https://github.com/gahansen/Albany



https://github.com/trilinos/trilinos

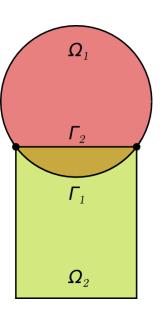


https://github.com/ORNL-CEES/DataTransferKit

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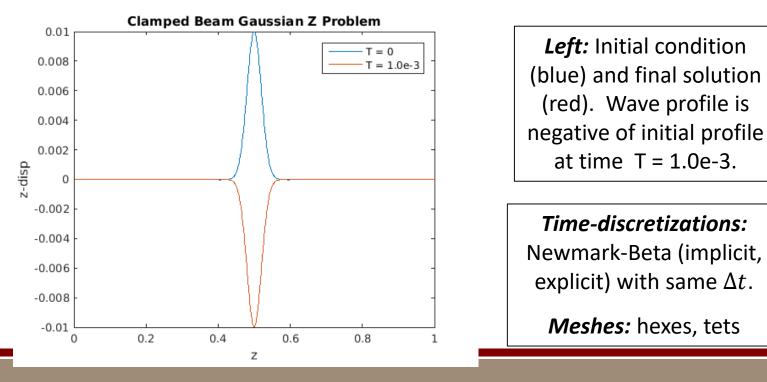




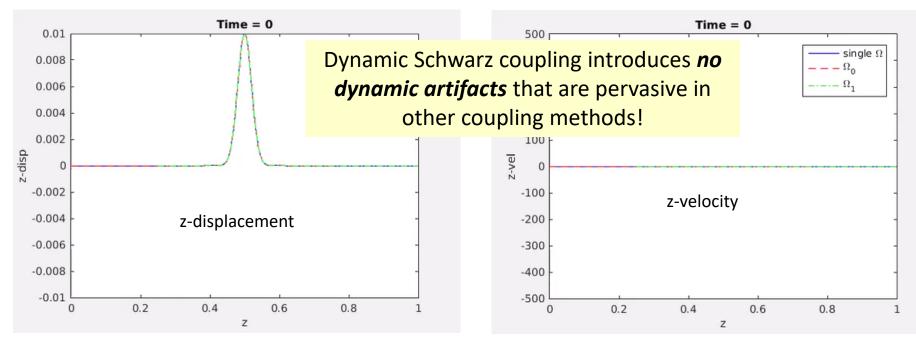


## Dynamic Example #1: Elastic Wave Propagation

- Linear elastic *clamped beam* with Gaussian initial condition for the *z*-displacement (see figures to the right and below).
- Simple problem with analytical exact solution but very *stringent test* for discretization methods.
- Test Schwarz with **2** subdomains:  $\Omega_0 = (0,0.001) \times (0.001) \times (0,0.75), \Omega_1 = (0,0.001) \times (0.001) \times (0.25,1).$



## **Elastic Wave Propagation**



<u>**Table 1:</u>** Averaged (over times + domains) relative errors in **z-displacement** (blue) and **z-velocity** (green) for several different Schwarz couplings, 50% overlap volume fraction</u>

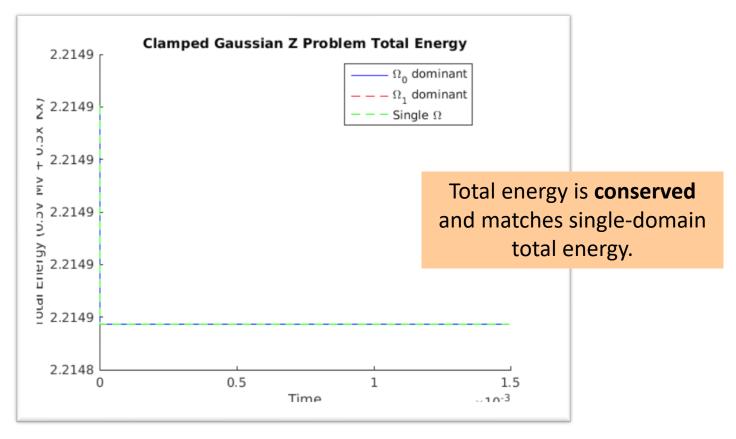
	Implicit-Implicit		Explicit(CM)-Implicit		Explicit(LM)-Implicit	
Conformal hex-hex	2.79e-3	7.32e-3	3.53e-3	8.70e-3	4.72e-3	1.19e-2
Nonconformal hex-hex	2.90e-3	7.10e-3	2.82e-3	7.29e-3	2.84e-3	7.33e-3
Tet-hex	2.79e-3	7.58e-3	3.52e-3	8.92e-3	4.72e-3	1.19e-2

#### LM = Lumped Mass, CM = Consistent Mass

## **Elastic Wave Propagation**



## **Energy Conservation**

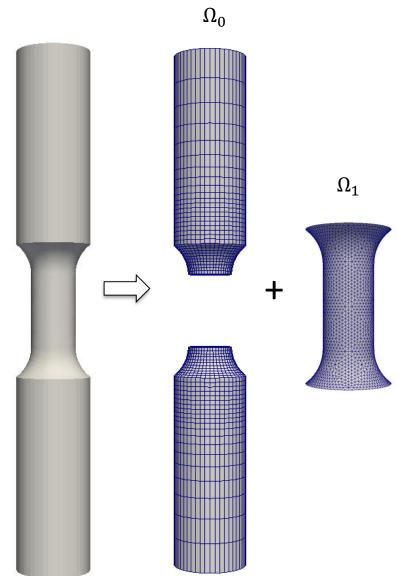


- For clamped beam problem, total energy (TE =  $0.5x^TKx + 0.5\dot{x}^TM\dot{x}$ ) should be conserved.
- Total energy is calculated in 2 ways: with most of contribution from  $\Omega_0$  and from  $\Omega_1$ .

## Example #2: Tension Specimen

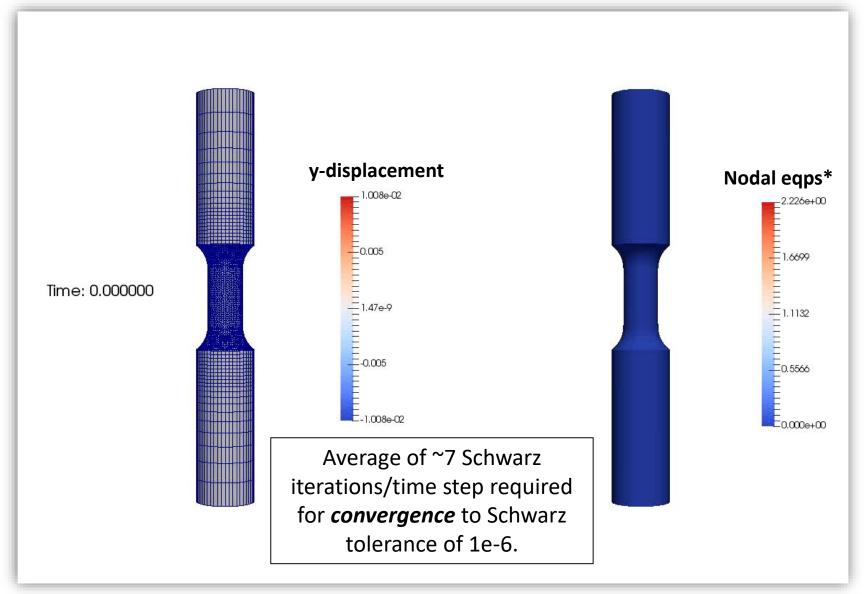


- Uniaxial aluminum cylindrical tensile specimen with *inelastic J<sub>2</sub> material model*.
- Domain decomposition into **two subdomains** (right):  $\Omega_0$  = ends,  $\Omega_1$  = gauge.
- Nonconformal hex + composite tet 10 coupling via Schwarz.
- Implicit Newmark time-integration with adaptive time-stepping algorithm employed in both subdomains.
- Slight *imperfection* introduced at center of gauge to force *necking* upon pulling in vertical direction.



## **Tension Specimen**



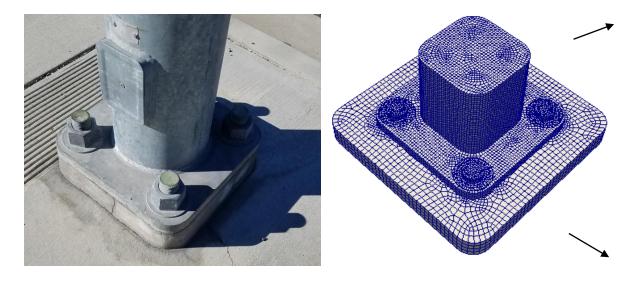


### \*Nodal eqps = equivalent plastic strain computed via weighted volume average.

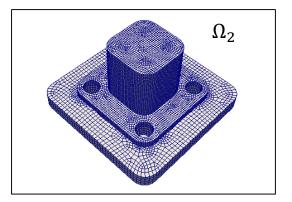
## Example #3: Bolted Joint Problem

### Problem of *practical scale*.

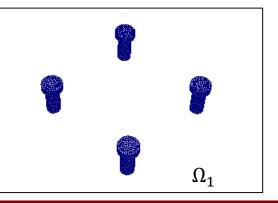
• Schwarz solution compared to single-domain solution on composite tet 10 mesh.



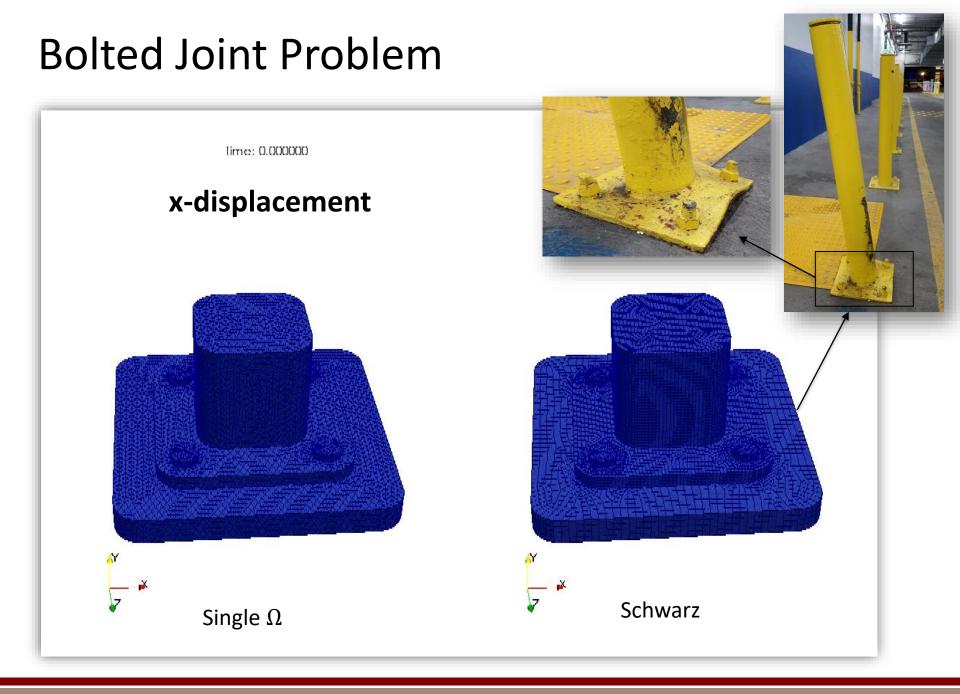
- $\Omega_1 = \text{bolts}$  (composite tet 10),  $\Omega_2 = \text{parts}$  (hex).
- Inelastic J<sub>2</sub> material model in both subdomains.
  - $\Omega_1$ : steel
  - $\Omega_2$ : steel component, aluminum (bottom) plate



- BC: x-disp = 0.02 at T = 1.0e-3 on top of parts.
- Run until T = 5.0e-4 w/ dt = 1e-5 + implicit Newmark with analytic mass matrix for composite tet 10s.

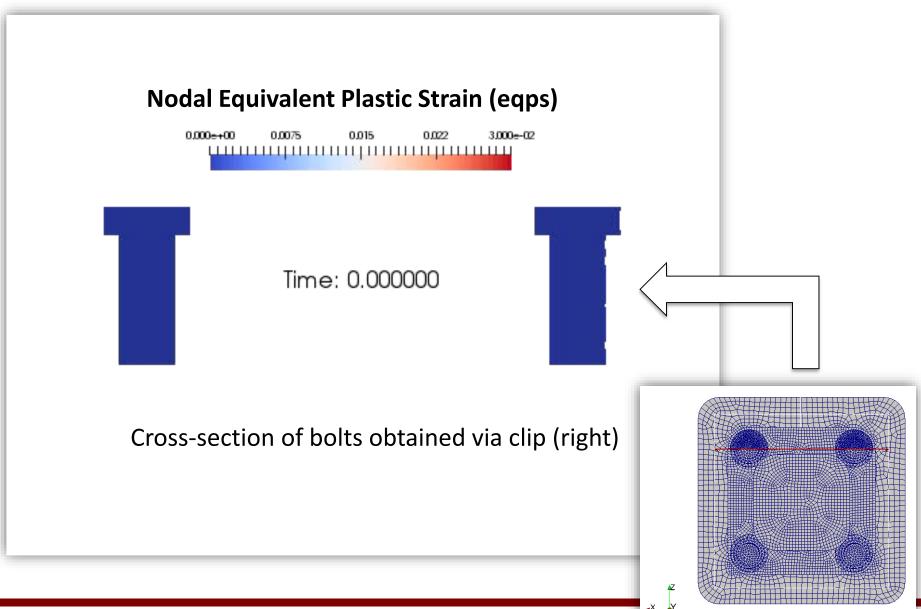






## **Bolted Joint Problem**





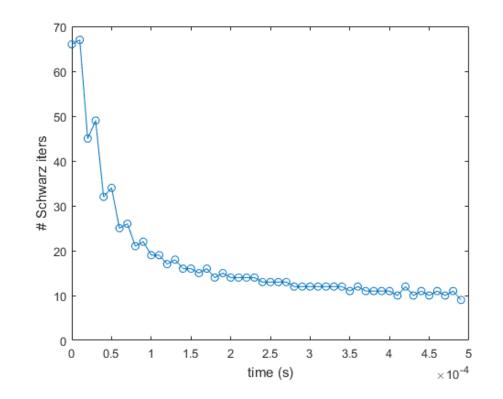
## **Bolted Joint Problem**



## **Some Performance Results**

Schwarz / solver settings

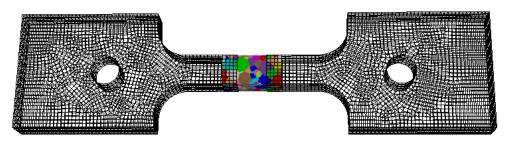
- Relatively loose Schwarz tolerances were used:
  - Relative Tolerance: 1.0e-3.
  - Absolute Tolerance: 1.0e-4.
- Newton tolerance on NormF: 1e-8
- Linear solver tolerance: 1e-5
- MueLu preconditioner

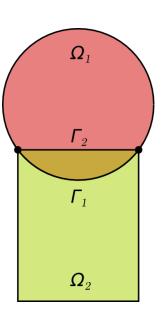


- *Top right plot:* # Schwarz iterations for each time step.
  - After start-up, # Schwarz iterations / time step is ~9-10. This is not bad given how small is the size of the overlap region for this problem.

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## Summary



The **alternating Schwarz** coupling method has been developed/implemented for **concurrent multiscale quasistatic & dynamic modeling** in Sandia's Albany/LCM code.

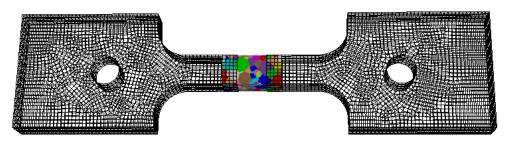
⊙ Coupling is *concurrent* (two-way).

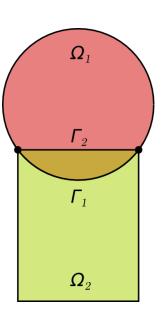


- ③ *Ease of implementation* into existing massively-parallel HPC codes.
- Scalable, fast, robust (we target real engineering problems, e.g., analyses involving failure of bolted components!).
- ③ *"Plug-and-play" framework*: simplifies task of meshing complex geometries!
  - Oblight to couple regions with *different non-conformal meshes*, *different element types* and *different levels of refinement*.
  - ③ Ability to use *different solvers/time-integrators* in different regions.
- ⓒ Coupling does not introduce *nonphysical artifacts.*
- Theoretical convergence properties/guarantees (③ for quasistatics).

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# Ongoing/Future Work

- Develop *theory* for dynamic alternating Schwarz formulation.
- *Journal article* on dynamic Schwarz formulation.
- Extension of Albany/LCM dynamic Schwarz implementation to allow for *different time steps* in different subdomains.
- Apply dynamic Schwarz to problem of interest to *production*.
- Implement alternating Schwarz method in Sandia *production codes* (Sierra Solid Mechanics).
- Development of a *multi-physics coupling framework* based on variational formulations and the Schwarz alternating method.



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## Appendix. Previous Work

Comput Mech (2014) 54:803-820 DOI 10.1007/s00466-014-1034-0

#### ORIGINAL PAPER

#### A multiscale overlapped coupling formulation for large-deformation strain localization

WaiChing Sun · Alejandro Mota

Received: 18 September 2013 / Accepted: 7 April 2014 / Published online: 3 May 2014 © Springer-Verlag Berlin Heidelberg 2014

Abstract We generalize the multiscale overlapped domain framework to couple multiple rate-independent standard dissipative material models in the finite deformation regime across different length scales. We show that a fully coupled multiscale incremental boundary-value problem can be recast as the stationary point that optimizes the partitioned incremental work of a three-field energy functional. We also establish inf-sup tests to examine the numerical stability issues that arise from enforcing weak compatibility in the three-field formulation. We also devise a new block solver for the domain coupling problem and demonstrate the performance of the formulation with one-dimensional numerical examples. These simulations indicate that it is sufficient to introduce a localization limiter in a confined region of interest to regularize the partial differential equation if loss of ellipticity occurs.

strain localization may lead to the eventual failure of materials, this phenomenon is of significant importance to modern engineering applications.

The objective of this work is to introduce concurrent coupling between sub-scale and macro-scale simulations for inelastic materials that are prone to strain localization. Since it is not feasible to conduct sub-scale simulations on macroscopic problems, we use the domain coupling method such that computational resources can be efficiently allocated to regions of interest [14,23,24,30]. To the best of our knowledge, this is the first work focusing on utilizing the domain coupling method to model strain localization in inelastic materials undergoing large deformation.

Nevertheless, modeling strain localization with the conventional finite element method may lead to spurious meshdependent results due to the loss of ellipticity at the onset of strain localization [31]. To circumvent the loss of mate-

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*Three-field* multiscale coupling formulation with compatibility enforced weakly using *Lagrange multipliers*.

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## Appendix. Full Schwarz Method

**Classical** algorithm originally proposed by Schwarz with **outer Schwarz loop** and **inner Newton loop**, each converged to a **tight tolerance** ( $\epsilon_{machine}$ ).

1:  $\boldsymbol{x}_{B}^{(1)} \leftarrow \boldsymbol{X}_{B}^{(1)}$  in  $\Omega_{1}, \boldsymbol{x}_{b}^{(1)} \leftarrow \boldsymbol{\chi}(\boldsymbol{X}_{b}^{(1)})$  on  $\partial_{\boldsymbol{\varphi}}\Omega_{1}, \boldsymbol{x}_{\beta}^{(1)} \leftarrow \boldsymbol{X}_{\beta}^{(1)}$  on  $\Gamma_{1}$  $\triangleright$  initialize for  $\Omega_1$ 2:  $\boldsymbol{x}_{B}^{(2)} \leftarrow \boldsymbol{X}_{B}^{(2)}$  in  $\Omega_{2}, \boldsymbol{x}_{b}^{(2)} \leftarrow \boldsymbol{\chi}(\boldsymbol{X}_{b}^{(2)})$  on  $\partial_{\boldsymbol{\varphi}}\Omega_{2}, \boldsymbol{x}_{\beta}^{(2)} \leftarrow \boldsymbol{X}_{\beta}^{(2)}$  on  $\Gamma_{2}$  $\triangleright$  initialize for  $\Omega_2$ 3: repeat ▷ Schwarz loop  $oldsymbol{u}^{(1)} \leftarrow oldsymbol{x}_{\mathrm{D}}^{(1)}$ 4:  $\triangleright$  for convergence check  $m{x}_eta^{(1)} \leftarrow m{P}_{12} m{x}_B^{(2)} + m{Q}_{12} m{x}_b^{(2)} + m{G}_{12} m{x}_eta^{(2)}$ 5:  $\triangleright$  project from  $\Omega_2$  to  $\Gamma_1$  $\triangleright$  Newton loop for  $\Omega_1$ 6:  $\triangle \boldsymbol{x}_{B}^{(1)} \leftarrow -\boldsymbol{K}_{AB}^{(1)}(\boldsymbol{x}_{B}^{(1)}; \boldsymbol{x}_{b}^{(1)}; \boldsymbol{x}_{\beta}^{(1)}) \backslash \boldsymbol{R}_{A}^{(1)}(\boldsymbol{x}_{B}^{(1)}; \boldsymbol{x}_{b}^{(1)}; \boldsymbol{x}_{\beta}^{(1)})$ 7:  $\triangleright$  linear system  $oldsymbol{x}_B^{(1)} \leftarrow oldsymbol{x}_B^{(1)} + riangle oldsymbol{x}_B^{(1)}$ 8: until  $|| \triangle \boldsymbol{x}_{B}^{(1)} || / || \boldsymbol{x}_{B}^{(1)} || \leq \epsilon_{\text{machine}}$ 9:  $\triangleright$  tight tolerance  $oldsymbol{y}^{(2)} \leftarrow oldsymbol{x}^{(2)}_B$ 10:  $\triangleright$  for convergence check  $oldsymbol{x}_eta^{(2)} \leftarrow oldsymbol{P}_{21}oldsymbol{x}_B^{(1)} + oldsymbol{Q}_{21}oldsymbol{x}_b^{(1)} + oldsymbol{G}_{21}oldsymbol{x}_eta^{(1)}$ 11:  $\triangleright$  project from  $\Omega_1$  to  $\Gamma_2$ 12:  $\triangleright$  Newton loop for  $\Omega_2$ repeat  $\triangle \bm{x}_B^{(2)} \leftarrow -\bm{K}_{AB}^{(2)}(\bm{x}_B^{(2)};\bm{x}_b^{(2)};\bm{x}_\beta^{(2)}) \backslash \bm{R}_A^{(2)}(\bm{x}_B^{(2)};\bm{x}_b^{(2)};\bm{x}_\beta^{(2)})$ 13:  $\triangleright$  linear system  $oldsymbol{x}_{\mathrm{D}}^{(2)} \leftarrow oldsymbol{x}_{\mathrm{D}}^{(2)} + riangle oldsymbol{x}_{\mathrm{D}}^{(2)}$ 14: until  $|| \triangle \boldsymbol{x}_{B}^{(2)} || / || \boldsymbol{x}_{B}^{(2)} || \leq \epsilon_{\text{machine}}$ 15:  $\triangleright$  tight tolerance 16: until  $\left[\left(||\boldsymbol{y}^{(1)} - \boldsymbol{x}_{B}^{(1)}||/||\boldsymbol{x}_{B}^{(1)}||\right)^{2} + \left(||\boldsymbol{y}^{(2)} - \boldsymbol{x}_{B}^{(2)}||/||\boldsymbol{x}_{B}^{(2)}||\right)^{2}\right]^{1/2} \leq \epsilon_{\text{machine}}$  $\triangleright$  tight tolerance



## Appendix. Inexact Schwarz Method

*Classical* algorithm originally proposed by Schwarz with *outer Schwarz loop* and *inner Newton loop*, with Newton step converged to a *loose tolerance*.



## Appendix. Monolithic Schwarz Method



Combines Schwarz and Newton loop into *since Newton-Schwarz loop*, with *elimination of Schwarz boundary DOFs,* and tight convergence tolerance.

$$\begin{array}{ll} 1: \ \boldsymbol{x}_{B}^{(1)} \leftarrow \boldsymbol{X}_{B}^{(1)} \text{ in } \Omega_{1}, \ \boldsymbol{x}_{b}^{(1)} \leftarrow \boldsymbol{\chi}(\boldsymbol{X}_{b}^{(1)}) \text{ on } \partial_{\boldsymbol{\varphi}} \Omega_{1}, & \triangleright \text{ initialize for } \Omega_{1} \\ 2: \ \boldsymbol{x}_{B}^{(2)} \leftarrow \boldsymbol{X}_{B}^{(2)} \text{ in } \Omega_{2}, \ \boldsymbol{x}_{b}^{(2)} \leftarrow \boldsymbol{\chi}(\boldsymbol{X}_{b}^{(2)}) \text{ on } \partial_{\boldsymbol{\varphi}} \Omega_{2}, & \triangleright \text{ initialize for } \Omega_{2} \\ 3: \ \mathbf{repeat} & \triangleright \text{ Newton-Schwarz loop} \\ 4: \quad \left\{ \begin{array}{c} \Delta \boldsymbol{x}_{B}^{(1)} \\ \Delta \boldsymbol{x}_{B}^{(2)} \end{array} \right\} \leftarrow \left( \begin{array}{c} \boldsymbol{K}_{AB}^{(1)} + \boldsymbol{K}_{A\beta}^{(1)} \boldsymbol{H}_{11} & \boldsymbol{K}_{A\beta}^{(1)} \boldsymbol{H}_{12} \\ \boldsymbol{K}_{A\beta}^{(2)} \boldsymbol{H}_{21} & \boldsymbol{K}_{AB}^{(2)} + \boldsymbol{K}_{A\beta}^{(2)} \boldsymbol{H}_{22} \end{array} \right) \setminus \left\{ \begin{array}{c} -\boldsymbol{R}_{A}^{(1)} \\ -\boldsymbol{R}_{A}^{(2)} \end{array} \right\} & \triangleright \text{ linear system} \\ 5: \quad \boldsymbol{x}_{B}^{(1)} \leftarrow \boldsymbol{x}_{B}^{(1)} + \Delta \boldsymbol{x}_{B}^{(1)} \\ 6: \quad \boldsymbol{x}_{B}^{(2)} \leftarrow \boldsymbol{x}_{B}^{(2)} + \Delta \boldsymbol{x}_{B}^{(2)} \\ 7: \ \text{ until } \left[ \left( ||\Delta \boldsymbol{x}_{B}^{(1)}||/||\boldsymbol{x}_{B}^{(1)}|| \right)^{2} + \left( ||\Delta \boldsymbol{x}_{B}^{(2)}||/||\boldsymbol{x}_{B}^{(2)}|| \right)^{2} \right]^{1/2} \leq \epsilon_{\text{machine}} \qquad \triangleright \text{ tight tolerance} \end{array}$$

### Advantages:

By-passes Schwarz loop.

### **Disadvantages:**

• Off-diagonal coupling terms  $\rightarrow$  block linear solver is needed.

## Appendix. Modified Schwarz Method



Combines Schwarz and Newton loop into *since Newton-Schwarz loop*, with *Schwarz boundaries* at *Dirichlet boundaries* and tight convergence tolerance.

1: $\boldsymbol{x}_B^{(1)} \leftarrow \boldsymbol{X}_B^{(1)}$ in $\Omega_1, \boldsymbol{x}_b^{(1)} \leftarrow \boldsymbol{\chi}(\boldsymbol{X}_b^{(1)})$ on $\partial_{\boldsymbol{\varphi}}\Omega_1, \boldsymbol{x}_{\beta}^{(1)} \leftarrow \boldsymbol{X}_{\beta}^{(1)}$ on $\Gamma_1$ $\triangleright$ initialize for $\Omega$	21
2: $\boldsymbol{x}_{B}^{(2)} \leftarrow \boldsymbol{X}_{B}^{(2)}$ in $\Omega_{2}, \boldsymbol{x}_{b}^{(2)} \leftarrow \boldsymbol{\chi}(\boldsymbol{X}_{b}^{(2)})$ on $\partial_{\boldsymbol{\varphi}}\Omega_{2}, \boldsymbol{x}_{\beta}^{(2)} \leftarrow \boldsymbol{X}_{\beta}^{(2)}$ on $\Gamma_{2}$ 3: <b>repeat</b> $\triangleright$ Newton-Schwarz loc	~
4: $\boldsymbol{x}_{\beta}^{(1)} \leftarrow \boldsymbol{P}_{12}\boldsymbol{x}_{\beta}^{(2)} + \boldsymbol{Q}_{12}\boldsymbol{x}_{b}^{(2)} + \boldsymbol{G}_{12}\boldsymbol{x}_{\beta}^{(2)}$ $\triangleright$ project from $\Omega_2$ to I	<b>^</b>
5: $\Delta \boldsymbol{x}_{B}^{(1)} \leftarrow -\boldsymbol{K}_{AB}^{(1)}(\boldsymbol{x}_{B}^{(1)};\boldsymbol{x}_{b}^{(1)};\boldsymbol{x}_{\beta}^{(1)}) \setminus \boldsymbol{R}_{A}^{(1)}(\boldsymbol{x}_{B}^{(1)};\boldsymbol{x}_{b}^{(1)};\boldsymbol{x}_{\beta}^{(1)})$ $\triangleright$ linear system in the system of the system	m
6: $oldsymbol{x}_B^{(1)} \leftarrow oldsymbol{x}_B^{(1)} +  riangle oldsymbol{x}_B^{(1)}$	
7: $\boldsymbol{x}_{\beta}^{(2)} \leftarrow \boldsymbol{P}_{21}\boldsymbol{x}_{B}^{(1)} + \boldsymbol{Q}_{21}\boldsymbol{x}_{b}^{(1)} + \boldsymbol{G}_{21}\boldsymbol{x}_{\beta}^{(1)}$ $\triangleright$ project from $\Omega_{1}$ to I	$\mathbf{\overline{2}}$
8: $\Delta \boldsymbol{x}_{B}^{(2)} \leftarrow -\boldsymbol{K}_{AB}^{(2)}(\boldsymbol{x}_{B}^{(2)};\boldsymbol{x}_{b}^{(2)};\boldsymbol{x}_{\beta}^{(2)}) \setminus \boldsymbol{R}_{A}^{(2)}(\boldsymbol{x}_{B}^{(2)};\boldsymbol{x}_{b}^{(2)};\boldsymbol{x}_{\beta}^{(2)})$ $\triangleright$ linear system in the system of the system	m
9: $oldsymbol{x}_B^{(2)} \leftarrow oldsymbol{x}_B^{(2)} +  riangle oldsymbol{x}_B^{(2)}$	
10: <b>until</b> $\left[ \left(    \triangle \boldsymbol{x}_B^{(1)}    /    \boldsymbol{x}_B^{(1)}    \right)^2 + \left(    \triangle \boldsymbol{x}_B^{(2)}    /    \boldsymbol{x}_B^{(2)}    \right)^2 \right]^{1/2} \le \epsilon_{\text{machine}}$ $\triangleright$ tight tolerand	ce

### Advantages:

- By-passes Schwarz loop.
- No diagonal coupling (conventional linear solver can be used in each subdomain).

*Least-intrusive variant*: by-passes Schwarz iteration, no need for block solver.

## Appendix. Convergence Proof

and



#### A. Moto, I. Tezaut, C. Alleman Schwarz Alternating Method in Solid Mechanics

2 Formulation of the Schwarz Alternating Method

We start by defining the standard finite deformation variational formulation to establish notation before presenting the formulation of the coupling method.

#### 2.1 Variational Formulation on a Single Domain

A. Mota, I. Tezant, C. Alleman

2.1 Variations formations on a single to omain factor is a single to omain factor is the operator. For a single to obtain the single of the solution of the single of the solution of th

 $\Phi[\varphi] := \int_{\Omega} A(F, Z) dV - \int_{\Omega} RB \cdot \varphi dV - \int_{\Omega \cap \Omega} T \cdot \varphi dS,$ (1)

in which A(F, Z) is the Helmholtz free-energy density and Z is a collection of internal variables. The weak form of the problem is obtained by minimizing the energy functional  $\Phi(\omega)$  over the Stobler's space  $W_1^1(0)$ that is comprised of all functions that are square-integrable and have square-integrable in derivatives. Define  $S := \{\varphi \in W_2^1(\Omega) : \varphi = \chi \text{ on } \partial_\varphi \Omega\}$ (2)

 $\mathcal{V} := \{ \xi \in W_2^1(\Omega) : \xi = 0 \text{ on } \partial_{\varphi} \Omega \}$ 

(3) where ξ ∈ V is a test function. The potential energy is minimized if and only if Φ[φ] ≤ Φ[φ + cξ] for all ξ ∈ V and ε ∈ ℝ. It is straightforward to show that the minimum of Φ[φ] is the mapping φ ∈ S that satisfies

 $D\Phi[\varphi](\xi) = \int_{\Omega} \mathbf{P} : \text{Grad} \, \xi \, dV - \int_{\Omega} R\mathbf{B} \cdot \xi \, dV - \int_{\partial \neq \Omega} \mathbf{T} \cdot \xi \, dS = 0,$  (4) where  $P = \partial A/\partial F$  denotes the first Piela Kirchhoff stress. The Euler-Lagrange equation corresponding to the variational statement (4) is

Div P + RB = 0, in  $\Omega$ ,

PN = T, on  $\partial_T \Omega$ ,  $\varphi = \chi$ , on  $\partial_{\varphi} \Omega$ .

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(5)

2.2 Coupling Two or More Subdomains via the Schwarz Alternating Method In this section, we describe the Solwarz adversarial method for coupling multiple or exclusions determine of the section of t

 $n \in \mathbb{N}^{0} = \{0, 1, 2, ...\}, \quad i = 2 - n + 2 \lfloor \frac{n}{2} \rfloor \in \{1, 2\}, \quad j = n + 1 - 2 \lfloor \frac{n}{2} \rfloor \in \{1, 2\},$  (6) 5

	onvergence of the Schwarz Alternating Meth rmation Inelastic Problem	od for the
	a proof of Theorem 1. The proof relies on several properties, pre- rties 1–5 enumerated in Section 3 hold.	sented below as
	vivity of $\Phi[\varphi]$ , it follows from the Lax-Milgram theorem that a uniq sts, i.e., the minimization of $\Phi[\varphi]$ is well-posed.	ue minimizer to
	npacchia theorem, the minimization of $\Phi[\varphi]$ in $S$ is equivalent to find	ling $\varphi \in S$ such
hat	$(\Phi'[\varphi], \xi - \varphi) \ge 0$	(51)
for all $\xi \in S$ .		
Remark 3 Recall that	the strict convexity property of $\Phi[\varphi]$ can be written as	
	$\Phi[\psi_2] - \Phi[\psi_1] - (\Phi'[\psi_1], \psi_2 - \psi_1) \ge 0,$	(52)
$(\psi_1, \psi_2 \in S. \text{ From (36)})$ in $\alpha_R > 0$ such that $\forall \psi_1$	remark that if $\Phi[\varphi]$ is strictly convex over $S \forall R \in \mathbb{R}$ such that $R < \psi_2 \in K_R$ we have	∞, we can find
	$\Phi[\psi_2] - \Phi[\psi_1] - (\Phi'[\psi_1], \psi_2 - \psi_1) \ge \alpha_R   \psi_2 - \psi_1  ^2.$	(53)
Remark 4 By propert $\omega : K_R \rightarrow K_R$ , such that	; 5, the uniform continuity of $\Psi'[\varphi],$ there exists a modulus of continu	uity $\omega > 0$ , with
· · · ·	$  \Phi'(\psi_1) - \Phi'(\psi_2)   \le \omega(  \psi_1 - \psi_2  ),$	(54)
$\phi_1, \phi_2 \in \mathcal{K}_R$ . By defin	ition, $\omega(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$ .	
Remark 5. It was show such that	vn in [35] that in the case $\Omega_1 \cap \Omega_2 \neq \emptyset$ , $\forall \varphi \in S$ , there exist $\zeta_1 \in$	$S_1$ and $\zeta_2 \in S_2$
	$\varphi = \zeta_1 + \zeta_2$ ,	(55)

#### and $\max(||\zeta_1||, ||\zeta_2||) \le C_0||\varphi||,$ (56) Remark 6 Note that (39) can be written as $(\Phi'[\hat{\varphi}^{(n)}], \boldsymbol{\xi}^{(i)}) = 0, \quad \text{ for } \hat{\varphi}^{(n)} \in \hat{S}_n, \forall \boldsymbol{\xi}^{(i)} \in S_i,$ (57)

for  $i \in \{1, 2\}$  and  $n \in \{0, 1, 2, ...\}$  (secall from (6) the relation between i and n). This is due to the uniqueness of the solution to each minimization problem over  $\hat{S}_n$  and the definition of  $\hat{\varphi}^{(n)}$  as the minimizer of  $\Phi[\varphi]$  over

Remark 7 Let  $\tilde{\varphi}^{(n)} \in \tilde{S}_n$ , and let  $\xi \in S$ . By Remark 5, there exist  $\zeta_1 \in S_1$  and  $\zeta_2 \in S_2$  such that  $(\Phi^{i}[\hat{\omega}^{(n)}], \ell) = (\Phi^{i}[\hat{\omega}^{(n)}], \ell_{1} + \ell_{2}),$ 

(58) 34

Schwarz Alternating Method in Solid Mechanics A. Mota, I. Tezaur, C. Alleman Γ2 3 Figure 1: Two autoomains (3) and (3) and the corresponding boundaries (1) and (1) used by the Schwarz alternating method

that is i = 1 and j = 2 if n is odd, and i = 2 and j = 1 if n is even. Introduce the following definitions for each set determine it • Closure: 00 = 10/ 000 Dirichlet boundary: @ III := @ III III: Neumann boundary: @ III := @ III II. Schwarz boundary: Γ<sub>1</sub> := @Q \ IQ. Note that with these definitions we guarantee that  $@ \varpi \land @ \varpi \simeq ;, @ \varpi \land \Gamma_{i} = ;$  and  $@ \varpi \land \Gamma_{i} = ;$ . Now define the space  $S_i := \{ ' \ 2 \ W_2^1(\mathbb{R}) : ' = \chi \text{ on } \otimes \mathbb{R}_i, ' = P_{\mathbb{R}_i / \Gamma_i}('(\mathbb{R}_i)) \text{ on } \Gamma_i \}$ , (7)  $V_i := \{ \leftarrow 2 \ W_2^1(\infty) : \leftarrow = 0 \text{ on } \circledast \otimes f \ \Gamma_i \ ,$  $\begin{array}{c} & \\ & {}^{(n)} = \begin{array}{c} & 8 \\ & {}^{<} \operatorname{id}_{X} , & \text{for } n = 0; \\ & {}_{:} \arg \min_{2 \leq 0} \Phi_{i}['], & \text{for } n > 0; \end{array}$ (9) where  $id_X$  is the identity map that maps X onto itself (i.e. zero displacement), and  $\Phi_{i}['] := A(F,Z) dV - RB \cdot ' dV - T \cdot ' dS.$ (10) A better guess, if available, may be used to initialize '  $^{(0)}$  on  $\mathbb{R}_2$  rather than the identity map id<sub>X</sub>. The minimization of the functional (10) leads to a variational formulation of the form (4)–(5) for each subdomain z Z  $D\Phi_i[r^{(n)}](e^{j(i)}) = \underset{\infty}{P} : Grad e^{j(i)} dV - \underset{\infty}{RB} \cdot e^{j(i)} dV - \underset{\widetilde{W}}{T} \cdot e^{j(i)} dS = 0,$  (11)

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By (57), $(\Phi'[\phi^{(w-1)}], \zeta_2) = 0$ . Hence,		
$(\Phi'[\phi^{(\alpha)}], \zeta_1 + \zeta_2) = (\Phi'[\phi^{(\alpha)}], \zeta_1 + \zeta_2) - (\Phi'[\phi^{(\alpha-1)}])$	$], \zeta_2) = \langle \Phi'[\tilde{\varphi}^{(\alpha)}], \zeta_2 \rangle - \langle \Phi'[\tilde{\varphi}^{(\alpha-1)}], \zeta_2 \rangle,$ (5)	9)
since $(\Phi'[\phi^{(n)}], \zeta_1) = 0$ , also by (57). By the Cauchy-Schw	arz inequality,	
$(\Phi'[\phi^{(n)}], \zeta_2) - (\Phi'[\phi^{(n-1)}], \zeta_2) = (\Phi'[\phi^{(n)}] - \Phi'[\phi^{(n-1)}], \zeta_2)$	$  , \zeta_2) \le   \Phi'[\phi^{(n)}] - \Phi'[\phi^{(n-1)}]   \cdot   \zeta_2  .$ (6)	0)
Again using (57) and also (58) in (60) leads to		
$(\Phi'[\bar{\varphi}^{(n)}] - \Phi'[\bar{\varphi}^{(n-1)}], \zeta_2) = (\Phi'[\bar{\varphi}^{(n)}], \xi)$	$\leq   \Phi'[\tilde{\varphi}^{(u)}] - \Phi'[\tilde{\varphi}^{(u-1)}]   \cdot   \zeta_2  ,$ (6)	1)
and substituting (56) into (61) we finally obtain that		
$\langle \Phi' [\hat{\varphi}^{(n)}], \xi \rangle \le C_0   \Phi' [\hat{\varphi}^{(n)}] $	$-\Phi'[\hat{\varphi}^{(n-1)}]  \cdot  \xi  ,$ (6)	2)
$\forall \boldsymbol{\xi} \in S.$		
Remark 8 For part (d) of Theorem 1, recall the definition	1 of geometric convergence:	
$E_{n+1} \le CE$	in, (6	3)
$\forall n \in \{0,1,2,\dots\}$ for some $C > 0,$ where		
$E_n :=   \phi^{(n+1)} - \phi^{(n+1)}  $	$ \phi^{(n)}  $ . (6	4)
Remark 9 Recall from the definition of continuity that it there exists a constant $K \ge 0$ such that	$\Phi'[\varphi]$ is Lipshitz continuous at $\bar{\varphi}^{(a)}$ near $\varphi$ , th	20
$\frac{  \Phi'(\phi^{(\alpha)}) - \Phi'(\varphi) }{  \phi^{(\alpha)} - \varphi  }$	$\underline{  } \leq K$ . (6	5)
Considering that $\Phi'[\phi]=0$ since $\phi$ is the minimizer of $\Phi[\phi]$	p], (65) is equivalent to	
$  \Phi'[\phi^{(n)}]   \le K   \phi $	$ ^{(6)} - \varphi  .$ (6)	6)
Proof of Theorem 1		
$\begin{split} & Proof of(a). \mbox{ Let } \ddot{\varphi}^{(1)} = \arg\min_{\psi\in \vec{S}_1} \Phi[\psi]. \mbox{ By (40)}, \\ & \varphi[\psi^{(1)}] > \Phi[\vec{\varphi}^{(1)}]. \mbox{ But this is a contradiction,} \\ & \mbox{ that } \Phi[\vec{\varphi}^{(1)}] < \Phi[\vec{\varphi}^{(2)}] \mbox{ where } \vec{\varphi}^{(2)} = \arg\min_{\psi\in \vec{S}_1} \Phi[\varphi] \end{split}$	since we can take $\hat{\varphi}^* = \hat{\varphi}^{(1)}$ . Hence, it cannot	
$\Phi(\hat{\varphi}^{(n)}) \le \Phi(q)$	$\bar{\rho}^{(n-1)}$ ] (6	7)
for $n \in \{1, 2, 3,\}$ . Now let $\varphi$ be the minimizer of $\Phi$ mique. Hence $\Phi[\varphi] \leq \Phi[\hat{\varphi}^{(n)}]$ for all $n \in \{1, 2, 3,\}$ .		is 0

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1: $\boldsymbol{x}_{D}^{(1)} \leftarrow \boldsymbol{X}_{D}^{(1)} \inf \Omega_{1}, \boldsymbol{x}_{b}^{(1)} \leftarrow \boldsymbol{\chi}(\boldsymbol{X}_{b}^{(1)}) \operatorname{on} \partial_{\boldsymbol{\theta}} \Omega_{1},$ 2: $\boldsymbol{x}_{D}^{(2)} \leftarrow \boldsymbol{X}_{D}^{(2)} \inf \Omega_{1}, \boldsymbol{x}_{b}^{(2)} \leftarrow \boldsymbol{\chi}(\boldsymbol{X}_{b}^{(2)}) \operatorname{on} \partial_{\boldsymbol{\theta}} \Omega_{2}.$	► initialize for f
	⇒ initialize for Ω ⇒ Newton-Schwarz lee
$\mathbf{\hat{e}} = \begin{bmatrix} \Delta \mathbf{x}_{B}^{(1)} \\ \Delta \mathbf{x}_{B}^{(2)} \end{bmatrix} \leftarrow \begin{pmatrix} \mathbf{K}_{AB}^{(1)} + \mathbf{K}_{AB}^{(1)} \mathbf{H}_{11} & \mathbf{K}_{AB}^{(1)} \mathbf{H}_{12} \\ \mathbf{K}_{AB}^{(2)} \mathbf{H}_{21} & \mathbf{K}_{AB}^{(2)} + \mathbf{K}_{AB}^{(2)} \mathbf{H}_{22} \end{pmatrix} \setminus \begin{cases} -\mathbf{R}_{A}^{(1)} \\ -\mathbf{R}_{A}^{(1)} \end{cases} = -\mathbf{R}_{A}^{(1)} \end{bmatrix}$	> linear syste
5: $\mathbf{x}_{B}^{(i)} \leftarrow \mathbf{x}_{B}^{(i)} + \Delta \mathbf{x}_{B}^{(i)}$ 6: $\mathbf{x}_{B}^{(i)} \leftarrow \mathbf{x}_{B}^{(i)} + \Delta \mathbf{x}_{B}^{(i)}$	
7: until $\left[\left(  \triangle x_B^{(1)}  /  x_B^{(1)}  \right)^2 + \left(  \triangle x_B^{(2)}  /  x_B^{(2)}  \right)^2\right]^{1/2} \le \epsilon_{matrix}$	> tight toleran

[15, 34, 4]. Although we do not provide here formal convergence proofs for the remaining variants of the Schwarz method, we offer some numerical results illustrating their convergence in Section 4. Consider the energy functional  $\delta[\phi]$  defined in (1). We will denote by ( $\phi$ , the usual  $L^2$  inner product over  $\Omega$ , that is,  $(\psi_1, \psi_2) := \int_{\omega} \psi_1 \cdot \psi_2 \, dV,$ (35) for  $\psi_1, \psi_2 \in W_2^1(\Omega)$ , with corresponding nerm  $\|\cdot\|_{\infty}^{-1}$  proof of the convergence of the Schwarz alternating method requires that the functional  $\Phi[\varphi]$  satisfy the following properties over the space S defined in (2): 1.  $\Phi[\varphi]$  is coercive. 2.  $\Phi[\varphi]$  is Fréchet differentiable, with  $\Phi'[\varphi]$  denoting its Fréchet derivative. 3.  $\Phi[\varphi]$  is strictly convex. 4.  $\Phi[\varphi]$  is lower semi-contin 5.  $\Phi'[\varphi]$  is uniformly continuous on  $K_B$ , where  $\mathcal{K}_R := \{ \varphi \in S : \Phi[\varphi] < R, R \in \mathbb{R}, R < \infty \}$ . (36) It can be shown that the energy functional  $\vartheta(\varphi)$  defined in (1) is strictly convex in S (property 3) provided that the Helmholtz free-energy density  $\mathcal{A}(F, Z)$  is a quasi-convex function of F [26]. Properties 1, 2, 4 and 5 follow from the strict convexity of  $\vartheta(\varphi)$ . Next, then we conditional sets of spaces  $\hat{S}_{\alpha} := \left\{ \varphi \in S : \varphi = P_{\Omega_{\ell} \to \Gamma_{1}}[\varphi^{(\alpha-1)}(\Omega_{\ell})] \text{ on } \Gamma_{\ell}, \varphi = \varphi^{(\alpha-1)} \text{ on } \Omega \setminus \Omega_{\ell} \right\},$ (37) and (38)

 $\tilde{V}_i := \{ \xi \in S : \xi = 0 \text{ in } \Omega \setminus \Omega_i \},\$ where i = 1 and j = 2 if n is odd, and i = 2 and j = 1 if n is even for  $n \in \{1, 2, ...\}$  as given by (6) and with the function  $q^{(0)} \in S$  an initial gauss. Note that the spaces  $S_n$  in (37) are extensions of the spaces  $S_n$  in (7) to the entities domain 10. With this notation in place, the solution to be -nd problem  $(9) \leftarrow (13)$  and be recarded as (12) + (12 $\bar{\varphi}^{(n)} = \begin{cases} \mathrm{id}_{\boldsymbol{X}}, & \text{for } n = 0; \\ \arg\min_{\boldsymbol{w} \in \mathcal{N}_{n}} \Phi[\varphi], & \text{for } n > 0. \end{cases}$ (39)

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Proof of (b). By (a) $\Phi[\phi^{(n)}] \rightarrow l$ as $n \rightarrow \infty$ for bound	ir some $l \in \mathbb{R}$ . Now, combining (51) and (53), we have the
$\Phi[\hat{\varphi}^{(n)}] - \Phi[\hat{\varphi}^{(n+1)}] \geq \Phi[\hat{\varphi}^{(n)}] - \Phi[\hat{\varphi}^{(n+1)}]$	$-\left(\Phi'[\tilde{\varphi}^{(n+1)}], \tilde{\varphi}^{(n)} - \tilde{\varphi}^{(n+1)}\right) \ge \alpha_B   \tilde{\varphi}^{(n)} - \tilde{\varphi}^{(n+1)}  ^2$
for all $n \in \{1, 2, 3,\}$ . Since $\Phi[\hat{\varphi}^{(n)}] \rightarrow l$ as i	$\tau \rightarrow \infty$ , it follows that $\Phi[\tilde{\varphi}^{(n)}] - \Phi[\tilde{\varphi}^{(n+1)}] \rightarrow 0$ as $n \rightarrow c$
From (68), we have that	$ (n) - \hat{\varphi}^{(n+1)}  ^2 = 0,$ (69)
from which we can conclude that $\tilde{\varphi}^{(n)} = \tilde{\varphi}^{(n+1)}$	
$  \varphi - \tilde{\varphi}^{(a)}  ^2 \le \frac{1}{\alpha_R} \left\{ \Phi[\varphi] \right\}$	$[-\Phi[\hat{\varphi}^{(n)}] - (\Phi'[\hat{\varphi}^{(n)}], \varphi - \hat{\varphi}^{(n)}))\}.$ (7)
Since $\varphi$ is the minimum of $\Phi[\varphi]$ , by (a) we have	we that $\Phi[\varphi] \leq \Phi[\varphi^{(n)}]$ . It follows that
$\Phi[\varphi] - \Phi[\hat{\varphi}^{(n)}] - \left(\Phi'[\hat{\varphi}^{(n)}], \varphi - \hat{\varphi}^{(n)}\right) \le$	$-\left(\Phi'[\hat{\varphi}^{(n)}], \varphi - \bar{\varphi}^{(n)}\right) = \left(\Phi'[\hat{\varphi}^{(n)}], \hat{\varphi}^{(n)} - \varphi\right).$ (7)
Subsituting (71) into (70) we have	
$  \varphi - \tilde{\varphi}^{(n)}  ^2 \le$	$\frac{1}{\alpha \nu} \left( \Psi'[\hat{\varphi}^{(n)}], \hat{\varphi}^{(n)} - \varphi \right).  (7)$
Now by (62) (Remark 7),	
$\left(\Phi'[\hat{\varphi}^{(n)}], \hat{\varphi}^{(n)} - \varphi\right) \le C_0$	$  \Phi'[\tilde{\varphi}^{(n)}] - \Phi'[\tilde{\varphi}^{(n-1)}]   \cdot   \tilde{\varphi}^{(n)} - \varphi  .$ (7)
Substituting (73) into (72) leads to	
$  \bar{\varphi}^{(n)} - \varphi   \le \frac{1}{2}$	$\frac{\overline{\beta}_{0}}{\eta_{R}}[ \Phi'[\bar{\varphi}^{(n)}] - \Phi'[\bar{\varphi}^{(n-1)}] ].  (7)$
Applying the uniform continuity assumption (5-	4), we obtain
$  \hat{arphi}^{(n)} - arphi   \le$	$\frac{C_0}{\alpha_B}\omega\left(\ \tilde{\varphi}^{(n)} - \tilde{\varphi}^{(n-1)}\ \right).  (7)$
By (69), $  \hat{\varphi}^{(n)} - \hat{\varphi}^{(n-1)}   \rightarrow 0$ as $n \rightarrow \infty$ . $n \rightarrow \infty$ .	From this we obtain the result, namely that $\hat{arphi}^{(v)}  ightarrow arphi$ :
Proof of (c). This follows immediately from (a)	and (b).
Proof of (d). By (b), for large enough n, there-	
$  \hat{\varphi}^{(n)} - \varphi  ^2$	$\leq C_1   \hat{\varphi}^{(n+1)} - \hat{\varphi}^{(n)}  ^2$ . (7)
Let us choose $C_1$ such that $C_1 > \alpha_R/K$ , when (68) with (76) leads to	e K is the Lipshitz continuity constant in (66). Combinin
1 (ac.z(n); ac.z(n+1))).	$\geq   \tilde{\varphi}^{(n+1)} - \tilde{\varphi}^{(n)}  ^2 \geq \frac{1}{C_c}   \tilde{\varphi}^{(n)} - \varphi  ^2.$ (7)

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#### A. Mota, I. Teşaur, C. Alleman Schwarz Alternating Method in Solid Mechanics

Remark that [50]  $\hat{\mathcal{S}}_n = \hat{\varphi}^{(n-1)} + \hat{\mathcal{V}}_i \quad \text{for} \quad \hat{\varphi}^{(n-1)} \in \hat{\mathcal{S}}_{n-1} \Rightarrow \hat{\varphi}^{(n-1)} \in \hat{\mathcal{S}}_n.$ (40) **Theorem 1.** Assume that the energy functional  $\Phi(\varphi)$  ratisfies properties 1–5 above. Consider the Schwarz alternating method of Section 2 defined by (9)–(13) and its equivalent form (39). Then  $(a) \ \Phi[\hat{\varphi}^{(0)}] \geq \Phi[\hat{\varphi}^{(1)}] \geq \cdots \geq \Phi[\hat{\varphi}^{(n-1)}] \geq \Phi[\hat{\varphi}^{(n)}] \geq \cdots \geq \Phi[\varphi], \text{ where } \varphi \text{ is the minimizer of } \Phi[\varphi]$ (b) the sequence  $\{\tilde{\varphi}^{(n)}\}$  defined in (39) converges to the minimizer  $\varphi$  of  $\Phi[\varphi]$  in S.

(c) the Schwarz minimum values  $\Phi[\vec{\varphi}^{(n)}]$  converge monotonically to the minimum value  $\Phi[\varphi]$  in S starting from any initial guess  $\vec{\varphi}^{(0)}$ .

(d) if Φ<sup>†</sup>(φ) is Lipschit; continuous in a neighborhood of φ, then the sequence {φ<sup>(ψ)</sup>} converges geometrically to the minimizer φ<sup>2</sup>. Proof. See Appendix A. п

Finally, while most of works cited above present their analysis for the specific case of two subdomains extension to multiple subdomains is in general straightforward. The case of multiple subdomains is considered specifically in Liesen [35], Baler [4], and Li Shan and Faran [34].

#### 4 Numerical Examples

It is easili, we present material examples of the behavior of the Schwarz alaynating method for sur-indress of the section of the section of the schwarz alaynating method for sur-indre approximate ALANY finite choices tool (12). Not, we discuss the enter measures and discuss the surgestion of the section of the schwarz and the schwarz and the schwarz and Schwarz alawarz alaynating. The schwarz and the schwarz and the schwarz and schwarz alaynatic schwarz and schwarz a

#### 4.1 Implementation

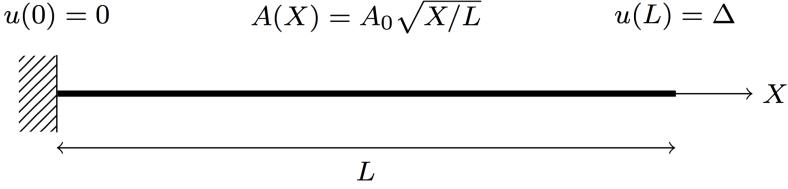
The four variants of the Schwarz alternating method described in Section 2.4 have been implemented in a one dimensional MATLAB code. The objective is to determine the convergence behavior, efficiency, and performance of each variant. This code has been optimized both in terms of memory usage and execution

eed. In addition, the Modified variant of the Schwarz alternating method described in Section 2.4 has been uplemented in ALBANY, an open-source multiphysics research platform developed mainly at Sandis National <sup>3</sup>See Remark 9 in the Appendix for a definition of geometric corresponce. 15

A. Mota, I. Tezaut, C. Alleman	Schwarz Alternating Method in Solid Mechanics
Using the identity $\Phi[\tilde{\varphi}^{(\alpha)}] - \Phi[\tilde{\varphi}^{(\alpha+1)}] = (\Phi[\tilde{\varphi}^{(\alpha)}] - \Phi[that \\ (\Phi[\tilde{\varphi}^{(\alpha)}] - \Phi[\varphi]) - (\Phi[\tilde{\varphi}^{(\alpha+1)}] - the \Phi[\tilde{\varphi}^{(\alpha)}] - \Phi[\varphi])$	
Substituting $\psi_1 = \varphi^{(n)}$ and $\psi_2 = \varphi$ into (53) and rearrang	sing, we obtain
$\left(\Phi'[\phi^{(n)}], \varphi - \phi^{(n)}\right) \le \left(\Phi'[\phi^{(n)}], \varphi - \phi^{(n)}\right)$	$+ \alpha_R    \varphi - \hat{\varphi}^{(n)}    \le \Phi  \varphi  - \Phi [\hat{\varphi}^{(n)}]$ (79)
since $\alpha_B \geq 0.$ Now, by the Cauchy-Schwarz inequality follo of $\Phi'[\varphi]$ (66) we can write	word by the application of the Lipshitz continuity
$\left(\Phi'   \tilde{\varphi}^{(n)} \rangle, \varphi - \tilde{\varphi}^{(n)} \right) \le   \Phi'   \tilde{\varphi}^{(n)}     \cdot   $	$\varphi - \tilde{\varphi}^{(n)}    \le K   \varphi - \tilde{\varphi}^{(n)}  ^2.$ (80)
Hence, from (79), $\Phi[{\cal Q}^{(n)}] - \Phi[{\cal Q}] \leq K   $	$ \dot{\varphi}^{(n)} - \varphi  ^2$ . (81)
Moreover, by (53) since $\Phi'[\varphi] = 0$ , $\Phi[\bar{\varphi}^{(n)}] - \Phi[\varphi] \ge \alpha_0   $	$ \tilde{\varphi}^{(u)} - \varphi  ^2$ . (82)
Using (81) and (82) we obtain	F FI . (00)
$\left(\Phi[\tilde{\varphi}^{(n)}] - \Phi[\varphi]\right) - \left(\Phi[\tilde{\varphi}^{(n+1)}] - \Phi[\varphi]\right) \le I$	$V   \hat{\varphi}^{(n)} - \varphi  ^2 - \alpha_R   \hat{\varphi}^{(n+1)} - \varphi  ^2.$ (83)
Combining (83) and (78) leads to	
$\frac{\alpha_R}{C_1}    \dot{\varphi}^{(n)} - \varphi   ^2 \leq \left( \Phi[\dot{\varphi}^{(n)}] - \Phi[\varphi] \right) - \left( \Phi[\dot{\varphi}^{(n+1)}] - \dot{\varphi}^{(n+1)} \right) = 0$	$\mathbb{P}[\varphi] \le K   \hat{\varphi}^{(n)} - \varphi  ^2 - \alpha_R   \hat{\varphi}^{(n+1)} - \varphi  ^2.$ (84)
or $  \phi^{(n+1)} - \phi   \le B  $	$ \hat{\varphi}^{(n)} - \varphi  $ (85)
with $B := \sqrt{\frac{K}{\alpha_{H}}}$	$\frac{1}{C_1}$ . (86)
and $B \in \mathbb{R}$ as we chose $C_1 > \alpha_R/K$ . Furthermore, since t the minimizer $\varphi$ of $\Phi[\varphi]$ by (b) and (c), it follows that $B$	
can be recast as $  \bar{\varphi}^{(\alpha+1)} - \bar{\varphi}^{(\alpha)}   \leq C   \bar{\varphi}$	$\phi^{(n)} = \phi^{(n-1)}   $ (87)
whereupon the claim is proven.	0
B Analytic Solution for Linear-Elasti	ic Singular Bar
As reference, herein we provide the solution of the singu equilibrium equation is	lar bar of Section 4.3 for linear elasticity. The
$P=\sigma(X)A(X)={\rm const.}, \sigma(X)=E\epsilon(X), e(J$	$X$ := $u'(X)$ , $A(X) = A_0 \left(\frac{X}{L}\right)^{\frac{1}{2}}$ , (88)
37	

# Appendix. Foulk's Singular Bar

- **1D proof of concept** problem:
  - **1D bar** with area proportional to square root of length.
  - Strong *singularity* on left end of bar.
  - Simple *hyperelestic* material model with no damage.
  - MATLAB implementation.

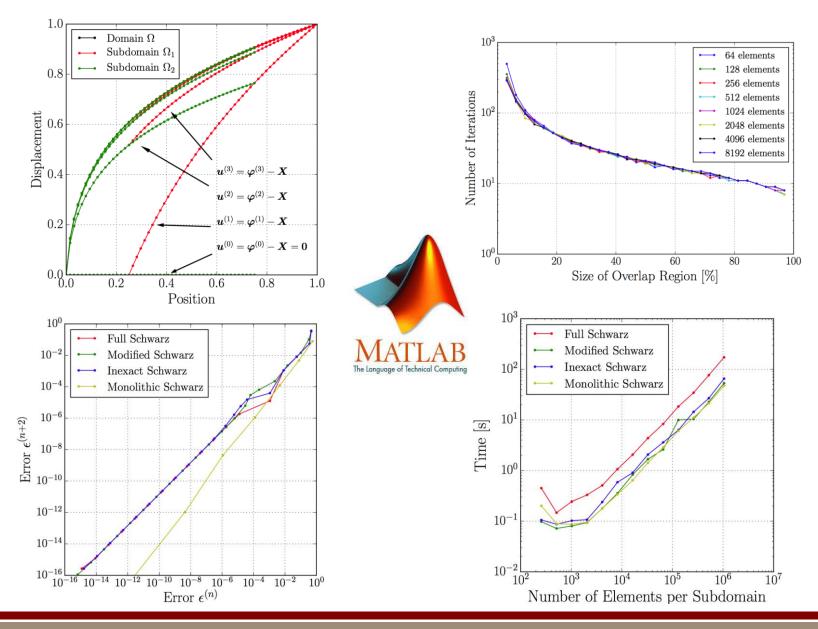


- Problem goals:
  - Explore *viability* of *4 variants* of the Schwarz alternating method.
  - Test *convergence* and compare with literature (Evans, 1986).
    - Expect *faster convergence* in *fewer iterations* with *increased overlap*.

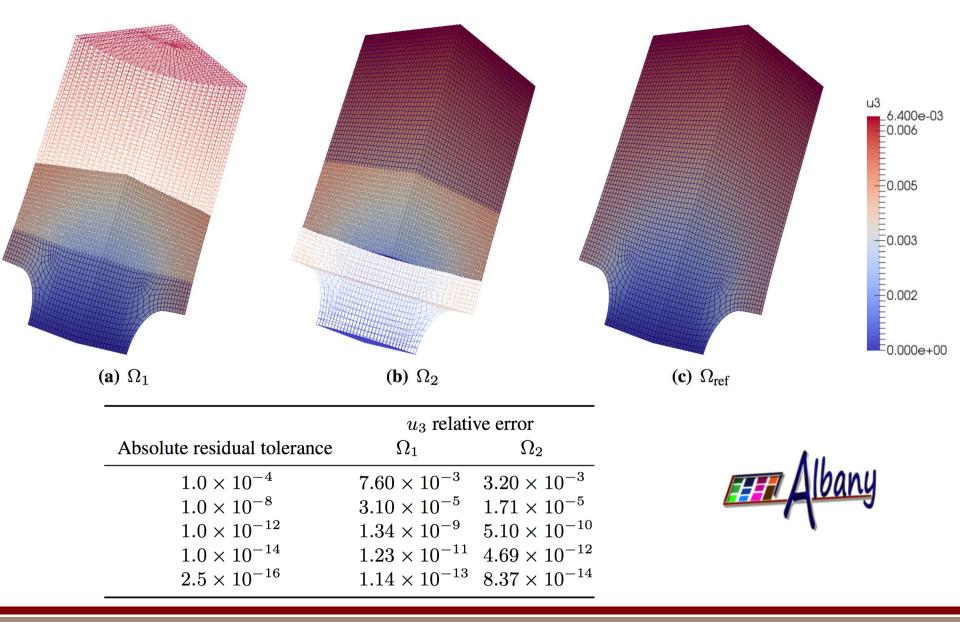




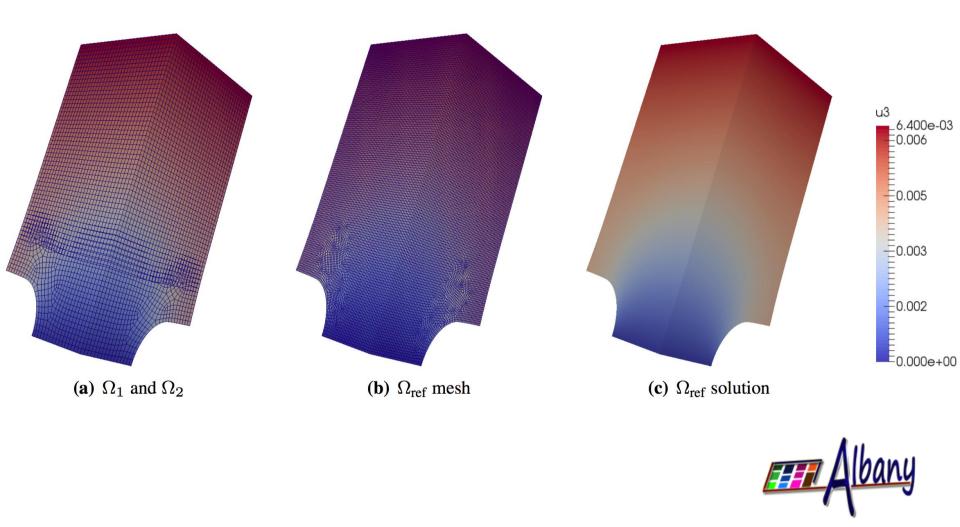
## Appendix. Singular Bar and Schwarz Varia



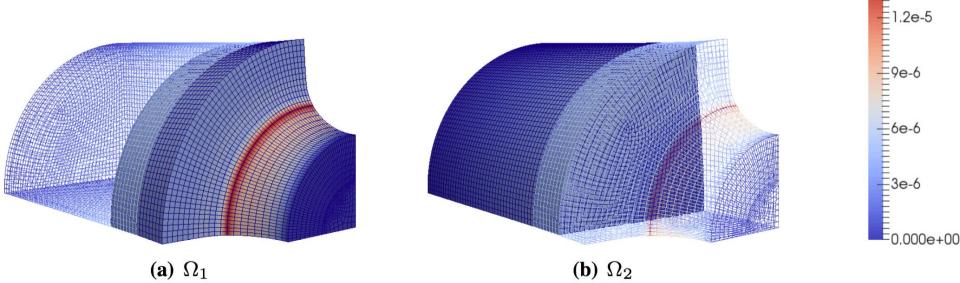
## Appendix. Notched Cylinder: HEX-HEX Coupling



# Appendix. Notched Cylinder: Nonconform



# Appendix. Notched Cylinder: Nonconform

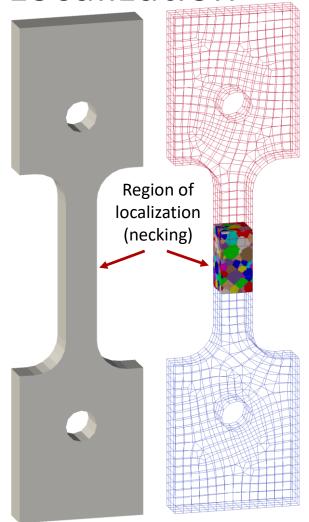


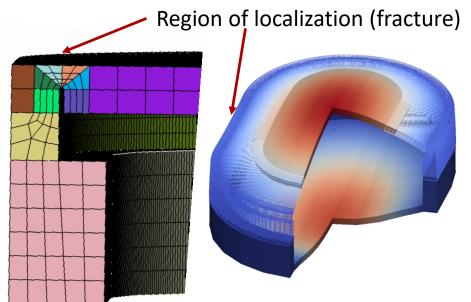
	$u_3$ relative error		
Absolute residual tolerance	$\Omega_1$	$\Omega_2$	
$1.0 \times 10^{-8}$	$1.31 \times 10^{-3}$	$4.45 \times 10^{-4}$	
$1.0 \times 10^{-12}$	$1.30 \times 10^{-3}$	$4.43 \times 10^{-4}$	
$1.0  imes 10^{-14}$	$1.30  imes 10^{-3}$	$4.43 \times 10^{-4}$	
$2.5 \times 10^{-16}$	$1.30 \times 10^{-3}$	$4.43 \times 10^{-4}$	



# Appendix. Multiscale Modeling of Localization





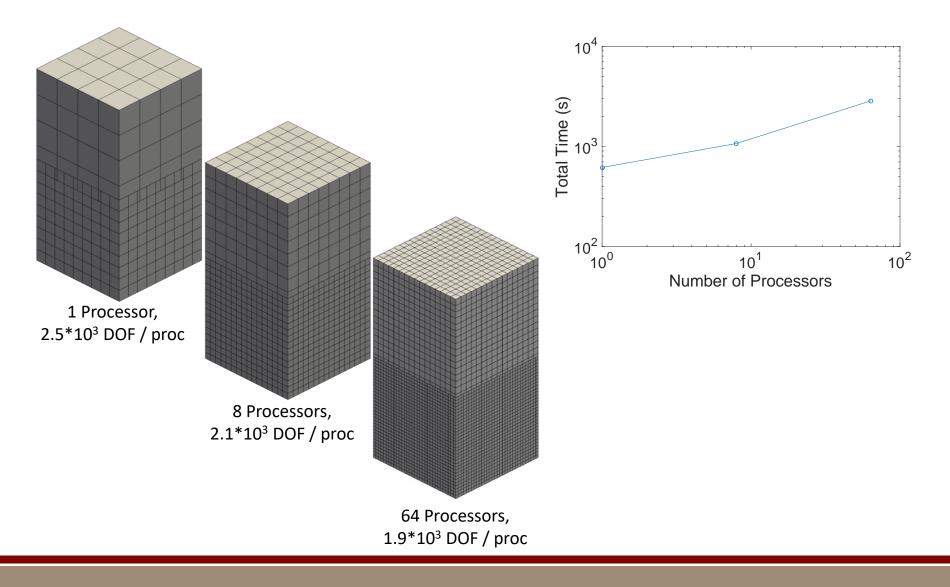


Strain localization can cause *localized necking* (left) and ultimately *fracture* (above).

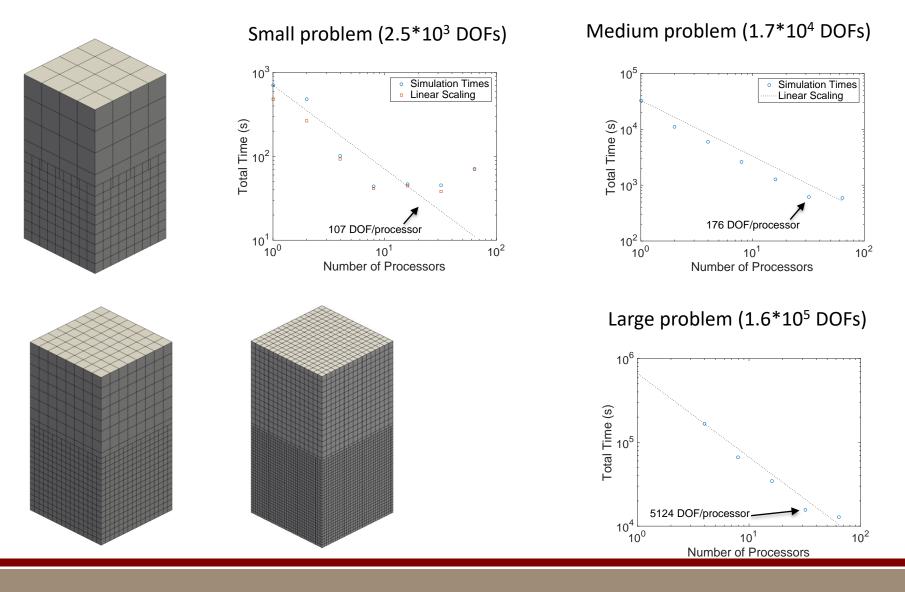
### Goals:

- Connect *physical length scales* to *engineering scale models*.
- Investigate importance of *microstructural detail.*
- Develop bridging technologies for *spatial multiscale/ multiphysics*.

# Appendix. Parallelization via DTK: Weak Scaling on Cubes Problem

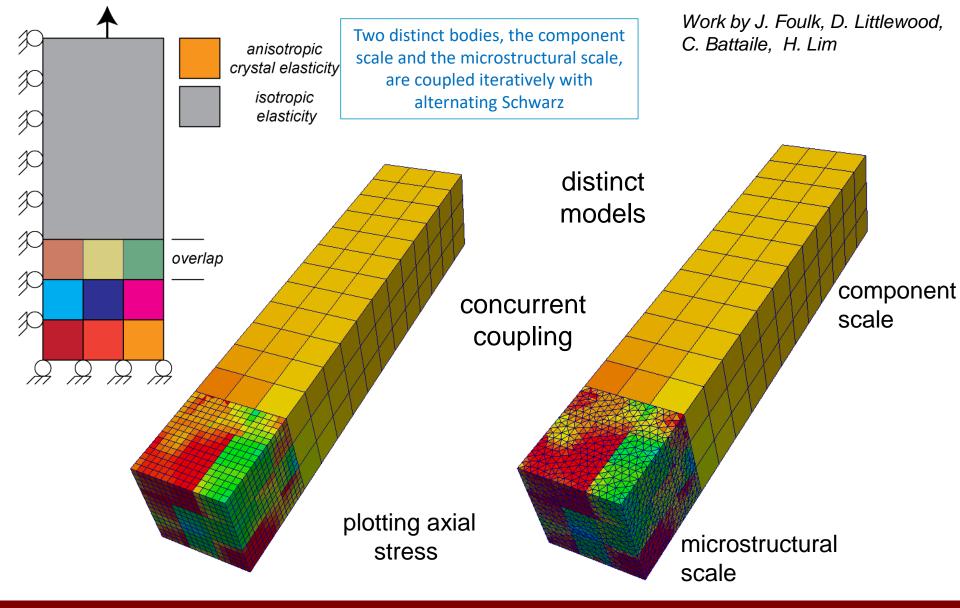


# Appendix. Parallelization via DTK: Strong Scaling on Cubes Problem



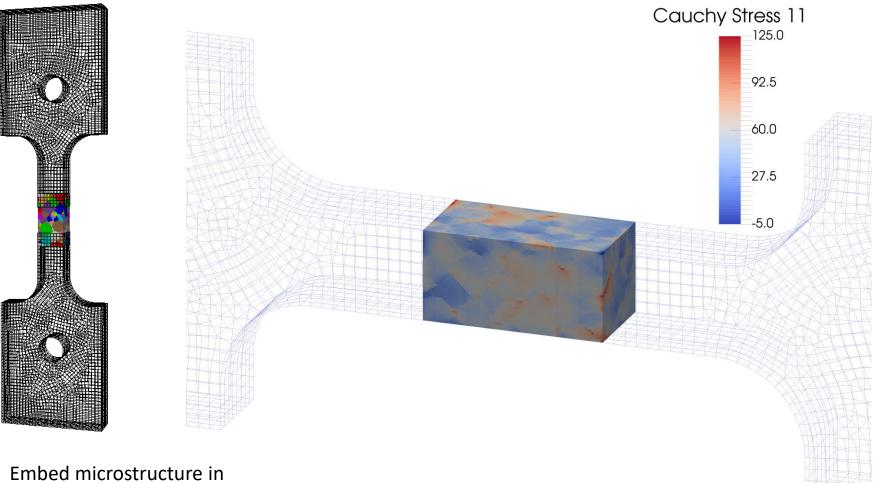
## Appendix. Rubiks Cube Problem





## Appendix. Tensile Bar





ASTM tensile geometry

# Appendix. Tensile Bar: Meso-Macroscale

Mesoscale

SPARKS-generated microstructure (F. Abdeljawad)

cubic elastic constant :  $C_{11} = 204.6$  GPa cubic elastic constant :  $C_{12} = 137.7$  GPa cubic elastic constant :  $C_{44} = 126.2$  GPa

 $\frac{1}{2} = \frac{1}{2} = \frac{1}$ 

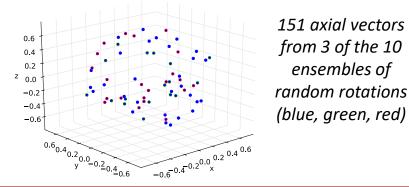
- reference shear rate :  $\dot{\gamma}_0 = 1.0 \text{ 1/s}$ 
  - rate sensitivity factor : m = 20
- hardening rate parameter :  $\dot{g}_0 = 2.0 \times 10^4 \text{ 1/s}$

initial hardness :  $g_0 = 90 \text{ MPa}$ 

saturation hardness :  $g_s = 202 \text{ MPa}$ 

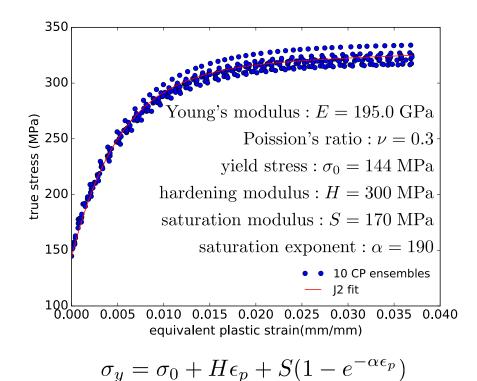
saturation exponent :  $\omega = 0.01$ 

#### Fix microstructure, investigate ensembles



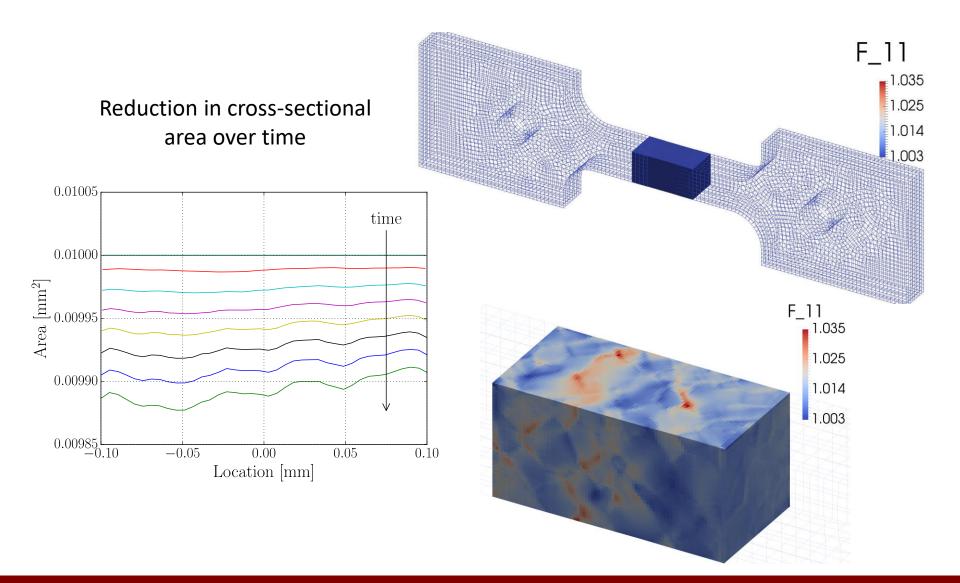


Load microstructural ensembles in uniaxial stress
 Fit flow curves with a macroscale J<sub>2</sub> plasticity model

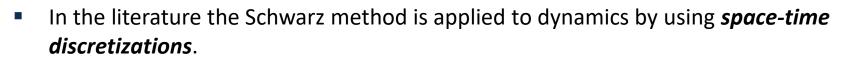


## Appendix. Tensile Bar: Results

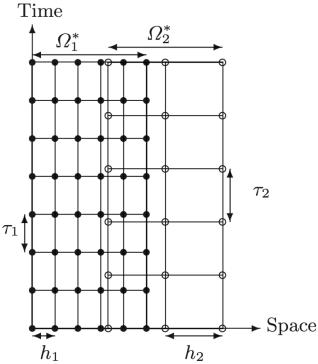




# Appendix. Schwarz Alternating Method for Dynamics



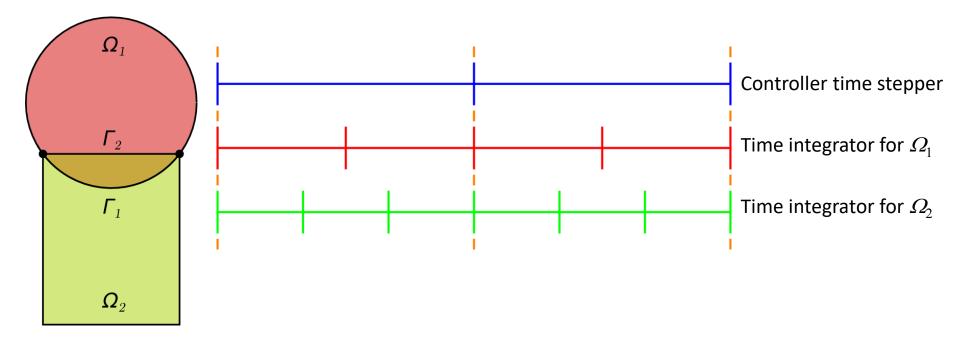
This was deemed *unfeasible* given the design of our current codes and size of simulations.



Overlapping non-matching meshes and time steps in dynamics.

# Appendix. A Schwarz-like Time Integrator Distance

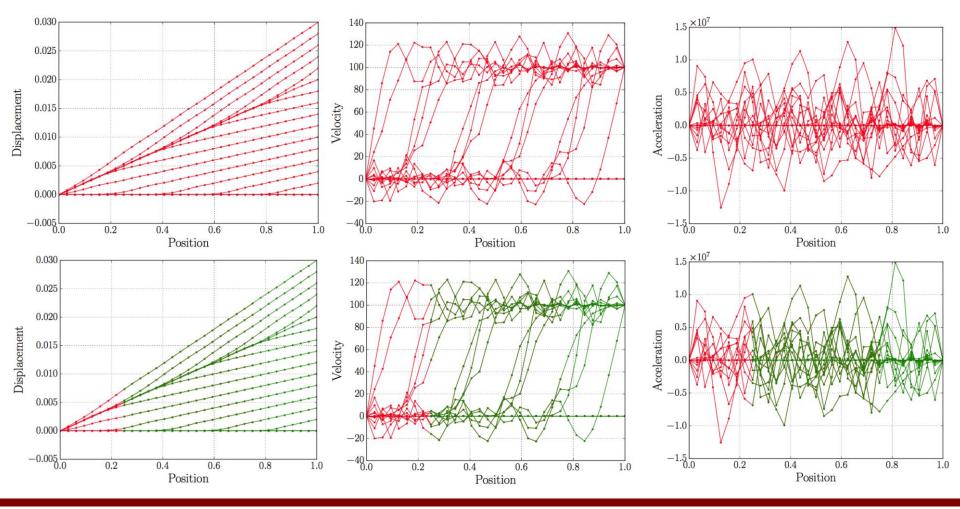
- We developed an *extension of Schwarz coupling* to *dynamics* using a governing time stepping algorithm that controls time integrators within each domain.
- Can use *different integrators* with *different time steps* within each domain.
- 1D results show *smooth coupling without numerical artifacts* such as spurious wave reflections at boundaries of coupled domains.

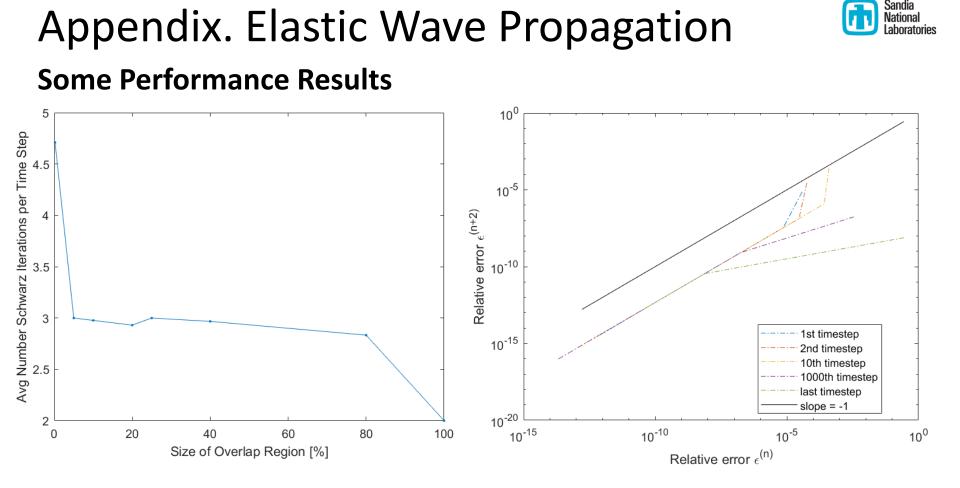


# Appendix. Dynamic Singular Bar



- Inelasticity masks problems by introducing *energy dissipation*.
- Schwarz does not introduce numerical artifacts.
- Can couple domains with *different time integration schemes* (*Explicit-Implicit* below).

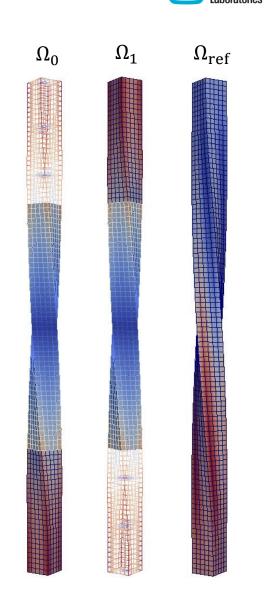




- Left figure shows *# of iterations* as a function of *overlap region size* for 2 subdomains. The method does not converge for 0% overlap. If the overlap is 100% then the single-domain solution is recovered for each of the subdomains.
- Right figure shows *linear convergence rate* of dynamic Schwarz implementation (for small overlap fraction of 0.2%).

# Appendix. Torsion

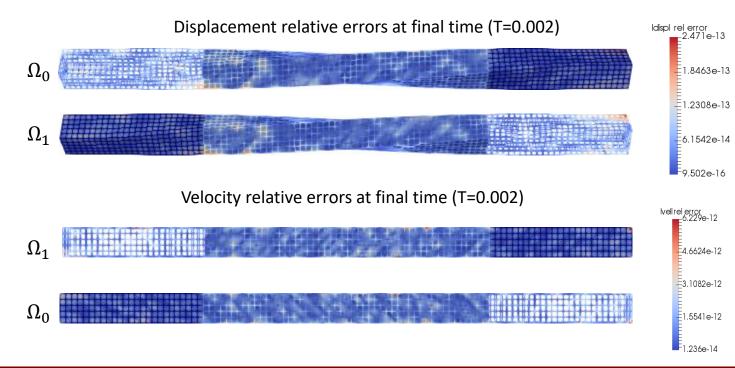
- Nonlinear elastic bar (Neohookean material model) subjected to a high degree of *torsion*.
- The *domain* is  $\Omega = (-0.025, 0.025) \times (-0.025, 0.025) \times (-0.5, 0.5).$
- We evaluate *dynamic Schwarz* with 2 subdomains:  $\Omega_0 = (-0.025, 0.025) \times (-0.025, 0.025) \times (-0.5, 0.25), \Omega_1 = (-0.025, 0.025) \times (-0.025, 0.025) \times (-0.25, 0.5).$
- **Time-discretizations:** Newmark-Beta (implicit, explicit) with same  $\Delta t$ .
- *Meshes:* hexes, composite tet 10s.



# Appendix. Torsion

#### **Conformal Hex + Hex Coupling**

- Each subdomain discretized using **uniform hex mesh** with  $\Delta x_i = 0.01$ , and advanced in time using implicit Newmark-Beta scheme with  $\Delta t = 1e-6$ .
- Results compared to single-domain solution on mesh conformal with Schwarz domain meshes.



Schwarz and single-domain results agree to almost *machine-precision*!

Sandia National Laboratories

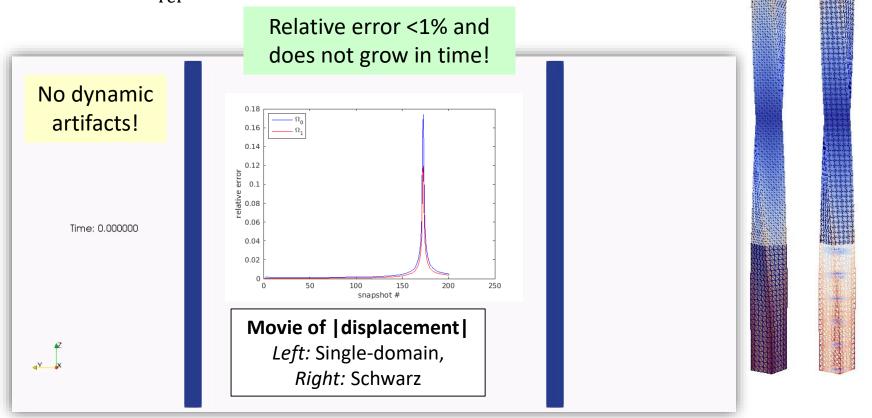
 $\Omega_{ref}$ 

 $\Omega_0$ 

# Appendix. Torsion

### Hex + Composite Tet 10 Coupling

- Coupling of composite tet 10s + explicit Newmark with consistent mass in  $\Omega_0$  with hexes + implicit Newmark in  $\Omega_1$ .
- Reference solution is computed on fine hex mesh + implicit Newmark  $\Omega_{ref}$



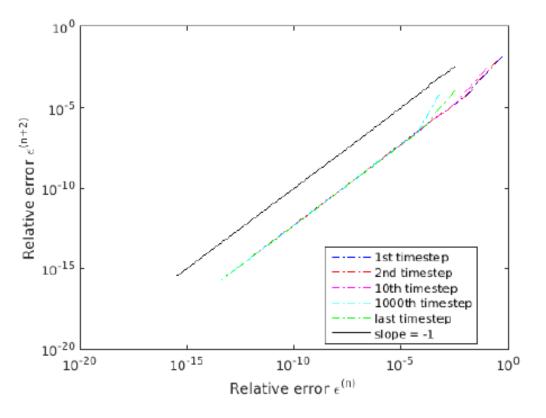
 $\Omega_1$ 

 $\Omega_0$ 

 $\Omega_{\rm ref}$ 



### Appendix. Torsion Some Performance Results



 Convergence behavior of the dynamic Schwarz algorithm for the torsion problem for small overlap volume fraction (2%) in which each subdomain is discretized using a hexahedral mesh. The plot shows that a *linear convergence rate* is achieved.

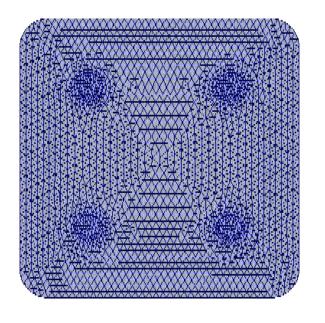
## Appendix. Bolted Joint Problem

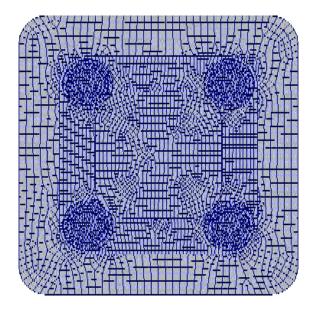
#### y-displacement

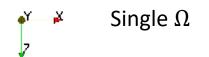
Time: 0.000000

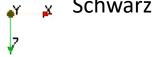
disp Y

-5.131++03 -0.002 0.00014 0.003 5.989+03 







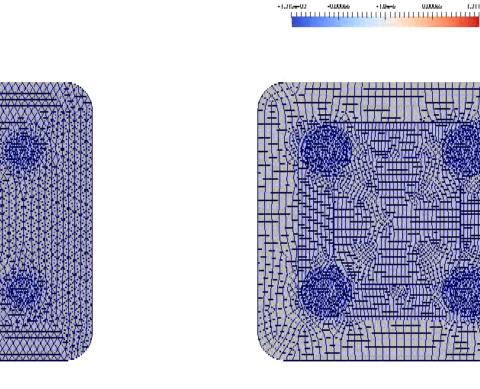


Schwarz

## Appendix. Bolted Joint Problem

#### z-displacement

Time: 0.000000







Schwarz

disp /

1.211+-02