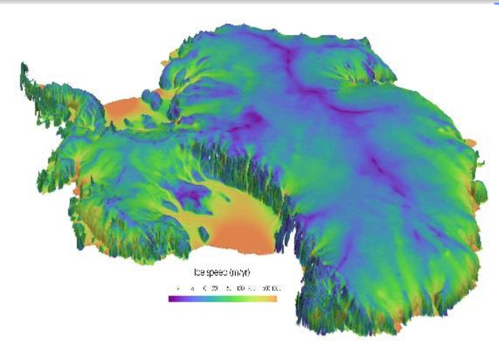
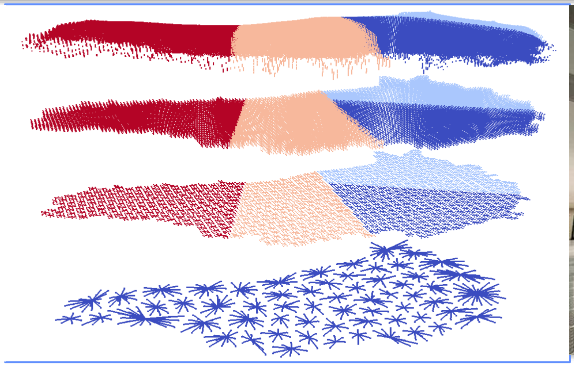




The *Albany/FELIX* Land-Ice Dynamical Core

I. Tezaur[1], A. Salinger[1], M. Perego[1], R. Tuminaro[1], J. Jakeman[1], M. Eldred[1], J. Watkins[1], S. Price[2], I. Demeshko[2]

[1] Sandia National Laboratories, Albuquerque, NM and Livermore, CA, USA.
[2] Los Alamos National Laboratory, Los Alamos, NM, USA.

AUG 2017 Livermore, CA Jan. 17-18, 2017

U.S. DEPARTMENT OF ENERGY

NNSA National Nuclear Security Administration

Sandia National Laboratories is a multi-program laboratory managed and operated by Sandia Corporation, a wholly owned subsidiary of Lockheed Martin Corporation, for the U.S. Department of Energy's National Nuclear Security Administration under contract DE-AC04-94AL85000. SAND NO. 2011-XXXXP

The PISCEES Project

* Proposal for ***follow-up funding*** under SciDaC4 in preparation.

PISCEES
SciDaC3 Application Partnership
(DOE's BER + ASCR divisions)
June 2012 – 2017*

3 land-ice dycores developed under PISCEES

FSU FELIX
FSU
Finite Element
Full Stokes Model

Albany/FELIX
SNL
Finite Element
First-Order Stokes Model

BISICLES
LBNL
Finite Volume
L1L2 Model

Increased fidelity

PISCEES: Predicting Ice Sheet Climate Evolution at Extreme Scales
FELIX: Finite Elements for Land Ice eXperiments
BISICLES: Berkeley Ice Sheet Initiative for Climate at Extreme Scales

The PISCEES Project

* Proposal for ***follow-up funding*** under SciDaC4 in preparation.

PISCEES
SciDaC3 Application Partnership
(DOE's BER + ASCR divisions)
June 2012 – 2017*

3 land-ice dycores developed under PISCEES

FSU FELIX
FSU
Finite Element
Full Stokes Model

Albany/FELIX
SNL
Finite Element
First-Order Stokes Model

BISICLES
LBNL
Finite Volume
L1L2 Model

Increased fidelity

Goal: support DOE climate missions

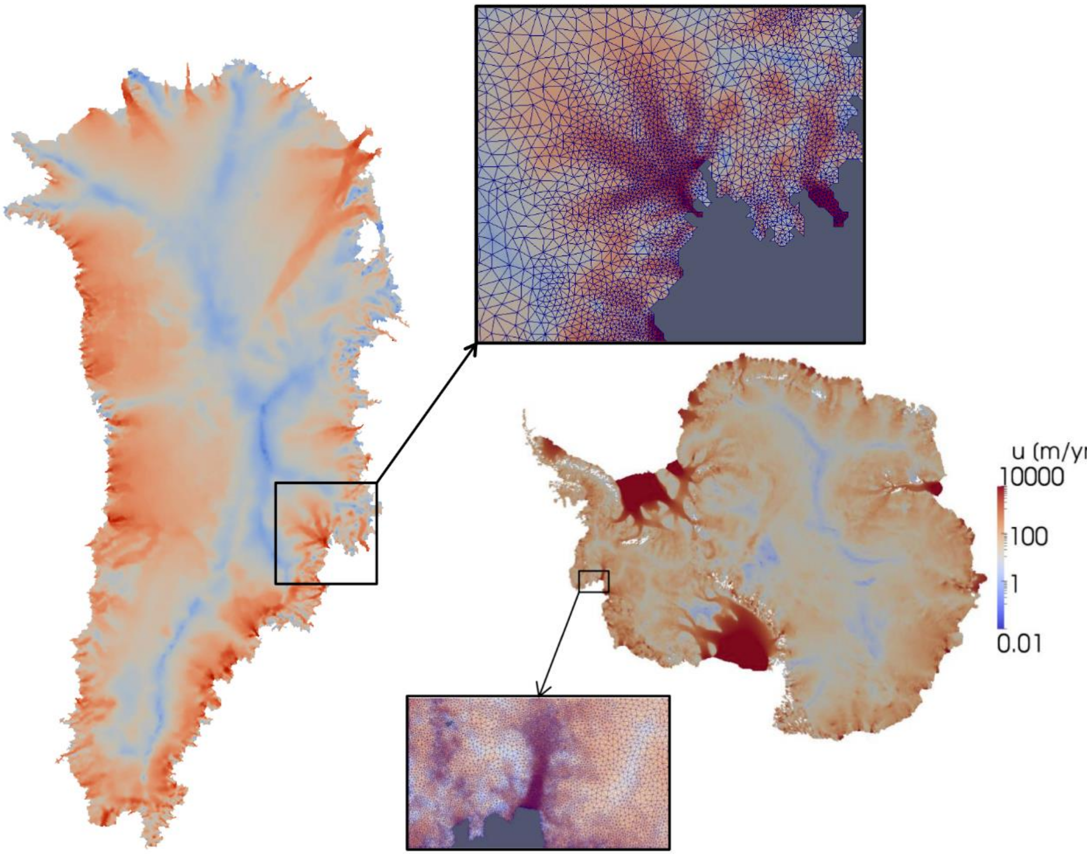
PISCEES: Predicting Ice Sheet Climate Evolution at Extreme Scales
FELIX: Finite Elements for Land Ice eXperiments
BISICLES: Berkeley Ice Sheet Initiative for Climate at Extreme Scales

Sandia's Role in PISCEES

Sandia's Role in the PISCEES Project: to **develop** and **support** a robust and scalable land ice solver based on the "First-Order" (FO) Stokes equations → *Albany/FELIX**

Requirements for *Albany/FELIX*:

- ***Unstructured grid*** finite elements.
- ***Scalable, fast*** and ***robust.***
- ***Verified*** and ***validated.***
- ***Portable*** to new/emerging architecture machines (multi-core, many-core, GPU).
- ***Advanced analysis*** capabilities: deterministic inversion, calibration, uncertainty quantification.



*Finite Elements for Land Ice eXperiments

Sandia's Role in PISCEES

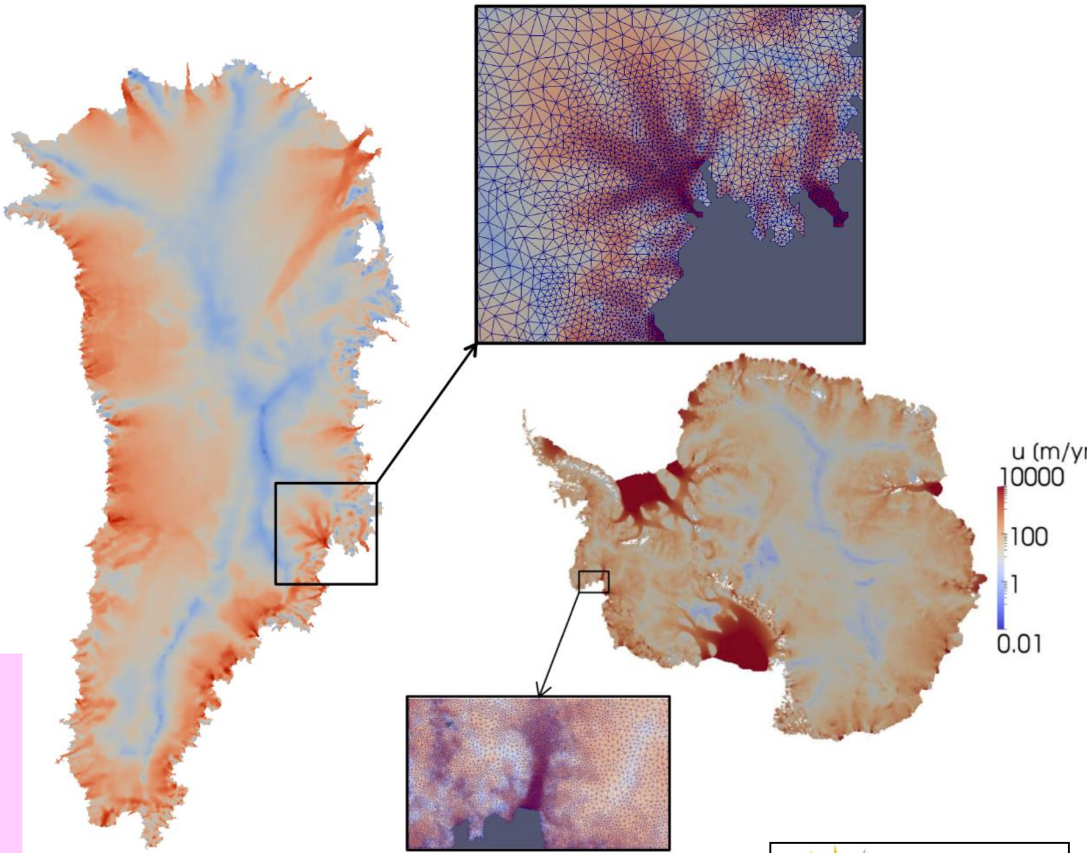
Albany/FELIX = production code!

Sandia National Laboratories

<u>Sandia's Role in the PISCEES Project</u>: to **develop** and **support** a robust and scalable land ice solver based on the "First-Order" (FO) Stokes equations → *Albany/FELIX**

<u>Requirements for *Albany/FELIX*:</u>

- ***Unstructured grid*** finite elements.
- ***Scalable, fast*** and ***robust.***
- ***Verified*** and ***validated.***
- ***Portable*** to new/emerging architecture machines (multi-core, many-core, GPU).
- ***Advanced analysis*** capabilities: deterministic inversion, calibration, uncertainty quantification.



As part of **ACME *DOE earth system model***, solver will provide actionable predictions of 21st century sea-level rise (including uncertainty).

*Finite Elements for Land Ice eXperiments

Albany

ACME Accelerated Climate Model for Energy

The First-Order Stokes Model

- Ice behaves like a very ***viscous shear-thinning fluid*** (similar to lava flow).

The First-Order Stokes Model

- Ice behaves like a very ***viscous shear-thinning fluid*** (similar to lava flow).
- ***Quasi-static*** model with ***momentum balance*** given by ***"First-Order" Stokes PDEs***: "nice" elliptic approximation* to Stokes' flow equations.

$$\begin{cases} -\nabla \cdot (2\mu \dot{\boldsymbol{\epsilon}}_1) = -\rho g \dfrac{\partial s}{\partial x} \\ -\nabla \cdot (2\mu \dot{\boldsymbol{\epsilon}}_2) = -\rho g \dfrac{\partial s}{\partial y} \end{cases}, \quad \text{in } \Omega$$

$$\dot{\boldsymbol{\epsilon}}_1{}^T = (2\dot{\epsilon}_{11} + \dot{\epsilon}_{22}, \dot{\epsilon}_{12}, \dot{\epsilon}_{13})$$
$$\dot{\boldsymbol{\epsilon}}_2{}^T = (2\dot{\epsilon}_{12}, \dot{\epsilon}_{11} + 2\dot{\epsilon}_{22}, \dot{\epsilon}_{23})$$
$$\dot{\epsilon}_{\mathrm{ij}} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

- Viscosity μ is nonlinear function given by "***Glen's law***":

$$\mu = \frac{1}{2} A(T)^{-\frac{1}{n}} \left(\frac{1}{2}\sum_{ij} \dot{\boldsymbol{\epsilon}}_{ij}{}^2\right)^{\left(\frac{1}{2n} - \frac{1}{2}\right)} \qquad (n = 3)$$

***Assumption:** aspect ratio δ is small and normals to upper/lower surfaces are almost vertical.

The First-Order Stokes Model

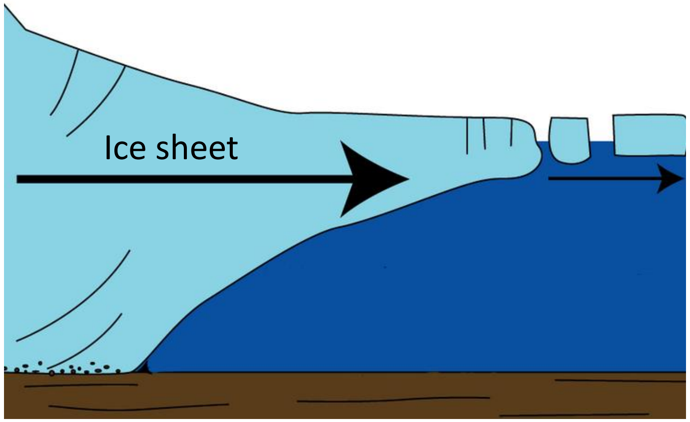
- Ice behaves like a very ***viscous shear-thinning fluid*** (similar to lava flow).
- ***Quasi-static*** model with ***momentum balance*** given by ***"First-Order" Stokes PDEs***: "nice" elliptic approximation* to Stokes' flow equations.

$$\begin{cases} -\nabla \cdot (2\mu \dot{\boldsymbol{\epsilon}}_1) = -\rho g \dfrac{\partial s}{\partial x} \\ -\nabla \cdot (2\mu \dot{\boldsymbol{\epsilon}}_2) = -\rho g \dfrac{\partial s}{\partial y} \end{cases}, \quad \text{in } \Omega$$

$$\dot{\boldsymbol{\epsilon}}_1{}^T = (2\dot{\epsilon}_{11} + \dot{\epsilon}_{22}, \dot{\epsilon}_{12}, \dot{\epsilon}_{13})$$
$$\dot{\boldsymbol{\epsilon}}_2{}^T = (2\dot{\epsilon}_{12}, \dot{\epsilon}_{11} + 2\dot{\epsilon}_{22}, \dot{\epsilon}_{23})$$
$$\dot{\epsilon}_{\mathrm{ij}} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

- Viscosity μ is nonlinear function given by "***Glen's law***":

$$\mu = \frac{1}{2} A(T)^{-\frac{1}{n}} \left(\frac{1}{2}\sum_{ij} \dot{\boldsymbol{\epsilon}}_{ij}{}^2\right)^{\left(\frac{1}{2n}-\frac{1}{2}\right)} \qquad (n = 3)$$



- Relevant boundary conditions:

***Assumption:** aspect ratio δ is small and normals to upper/lower surfaces are almost vertical.

The First-Order Stokes Model

- Ice behaves like a very ***viscous shear-thinning fluid*** (similar to lava flow).
- ***Quasi-static*** model with ***momentum balance*** given by ***"First-Order" Stokes PDEs***: "nice" elliptic approximation* to Stokes' flow equations.

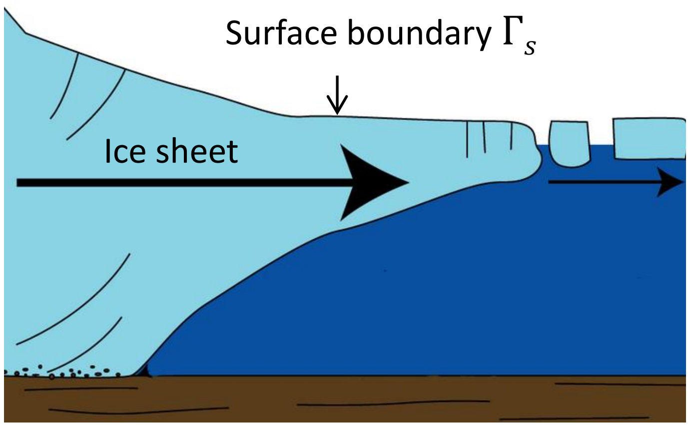
$$\begin{cases} -\nabla \cdot (2\mu \dot{\boldsymbol{\epsilon}}_1) = -\rho g \dfrac{\partial s}{\partial x} \\ -\nabla \cdot (2\mu \dot{\boldsymbol{\epsilon}}_2) = -\rho g \dfrac{\partial s}{\partial y} \end{cases}, \quad \text{in } \Omega$$

$$\dot{\boldsymbol{\epsilon}}_1{}^T = (2\dot{\epsilon}_{11} + \dot{\epsilon}_{22}, \dot{\epsilon}_{12}, \dot{\epsilon}_{13})$$
$$\dot{\boldsymbol{\epsilon}}_2{}^T = (2\dot{\epsilon}_{12}, \dot{\epsilon}_{11} + 2\dot{\epsilon}_{22}, \dot{\epsilon}_{23})$$
$$\dot{\epsilon}_{\mathrm{ij}} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

- Viscosity μ is nonlinear function given by "***Glen's law***":

$$\mu = \frac{1}{2} A(T)^{-\frac{1}{n}} \left(\frac{1}{2} \sum_{ij} \dot{\boldsymbol{\epsilon}}_{ij}{}^2 \right)^{\left(\frac{1}{2n} - \frac{1}{2}\right)} \qquad (n = 3)$$

- Relevant boundary conditions:
  - ***Stress-free BC:*** $2\mu \dot{\boldsymbol{\epsilon}}_i \cdot \boldsymbol{n} = 0$, on Γ_s



***Assumption:** aspect ratio δ is small and normals to upper/lower surfaces are almost vertical.

The First-Order Stokes Model

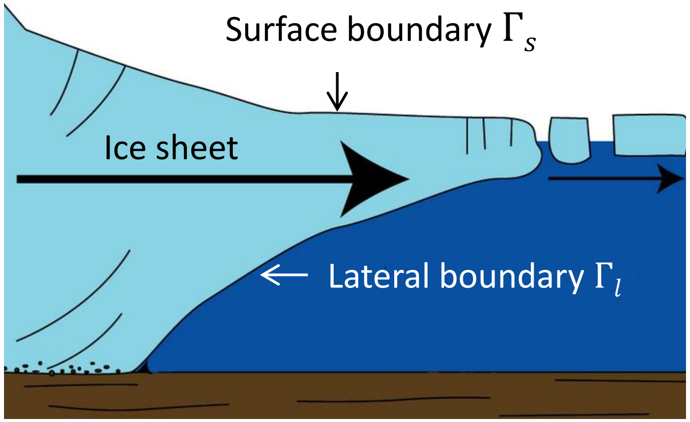
- Ice behaves like a very ***viscous shear-thinning fluid*** (similar to lava flow).
- ***Quasi-static*** model with ***momentum balance*** given by ***"First-Order" Stokes PDEs***: "nice" elliptic approximation* to Stokes' flow equations.

$$\begin{cases} -\nabla \cdot (2\mu \dot{\boldsymbol{\epsilon}}_1) = -\rho g \dfrac{\partial s}{\partial x} \\ -\nabla \cdot (2\mu \dot{\boldsymbol{\epsilon}}_2) = -\rho g \dfrac{\partial s}{\partial y} \end{cases}, \quad \text{in } \Omega$$

$$\dot{\boldsymbol{\epsilon}}_1{}^T = (2\dot{\epsilon}_{11} + \dot{\epsilon}_{22}, \dot{\epsilon}_{12}, \dot{\epsilon}_{13})$$
$$\dot{\boldsymbol{\epsilon}}_2{}^T = (2\dot{\epsilon}_{12}, \dot{\epsilon}_{11} + 2\dot{\epsilon}_{22}, \dot{\epsilon}_{23})$$
$$\dot{\epsilon}_{\mathrm{ij}} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

- Viscosity μ is nonlinear function given by "***Glen's law***":

$$\mu = \frac{1}{2} A(T)^{-\frac{1}{n}} \left(\frac{1}{2}\sum_{ij} \dot{\boldsymbol{\epsilon}}_{ij}{}^2\right)^{\left(\frac{1}{2n}-\frac{1}{2}\right)} \qquad (n = 3)$$



- Relevant boundary conditions:
  - ***Stress-free BC:*** $2\mu \dot{\boldsymbol{\epsilon}}_i \cdot \boldsymbol{n} = 0$, on Γ_s
  - ***Floating ice BC:***

$$2\mu \dot{\boldsymbol{\epsilon}}_i \cdot \boldsymbol{n} = \begin{cases} \rho g z \boldsymbol{n}, & \text{if } z > 0 \\ 0, & \text{if } z \le 0 \end{cases}, \text{ on } \Gamma_l$$

***Assumption:** aspect ratio δ is small and normals to upper/lower surfaces are almost vertical.

The First-Order Stokes Model

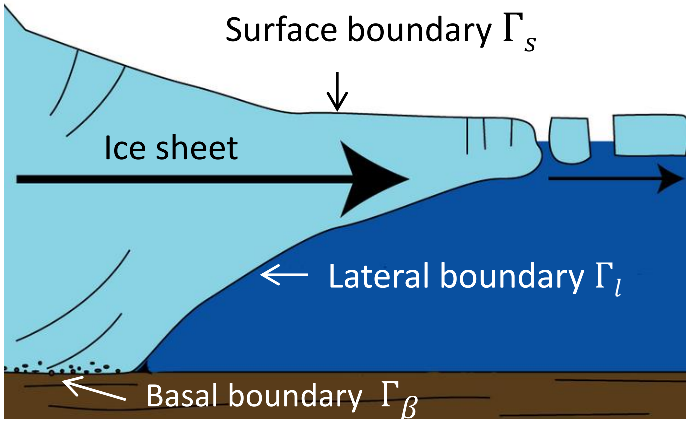
- Ice behaves like a very ***viscous shear-thinning fluid*** (similar to lava flow).
- ***Quasi-static*** model with ***momentum balance*** given by ***"First-Order" Stokes PDEs***: "nice" elliptic approximation* to Stokes' flow equations.

$$\begin{cases} -\nabla \cdot (2\mu \dot{\boldsymbol{\epsilon}}_1) = -\rho g \dfrac{\partial s}{\partial x} \\ -\nabla \cdot (2\mu \dot{\boldsymbol{\epsilon}}_2) = -\rho g \dfrac{\partial s}{\partial y} \end{cases}, \quad \text{in } \Omega$$

$$\dot{\boldsymbol{\epsilon}}_1{}^T = (2\dot{\epsilon}_{11} + \dot{\epsilon}_{22}, \dot{\epsilon}_{12}, \dot{\epsilon}_{13})$$
$$\dot{\boldsymbol{\epsilon}}_2{}^T = (2\dot{\epsilon}_{12}, \dot{\epsilon}_{11} + 2\dot{\epsilon}_{22}, \dot{\epsilon}_{23})$$
$$\dot{\epsilon}_{\mathrm{ij}} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

- Viscosity μ is nonlinear function given by "***Glen's law***":

$$\mu = \frac{1}{2} A(T)^{-\frac{1}{n}} \left(\frac{1}{2} \sum_{ij} \dot{\boldsymbol{\epsilon}}_{ij}{}^2\right)^{\left(\frac{1}{2n} - \frac{1}{2}\right)} \qquad (n = 3)$$



- Relevant boundary conditions:
  - ***Stress-free BC:*** $2\mu \dot{\boldsymbol{\epsilon}}_i \cdot \boldsymbol{n} = 0$, on Γ_s
  - ***Floating ice BC:***

$$2\mu \dot{\boldsymbol{\epsilon}}_i \cdot \boldsymbol{n} = \begin{cases} \rho g z \boldsymbol{n}, & \text{if } z > 0 \\ 0, & \text{if } z \leq 0 \end{cases}, \quad \text{on } \Gamma_l$$

  - ***Basal sliding BC:*** $2\mu \dot{\boldsymbol{\epsilon}}_i \cdot \boldsymbol{n} + \beta(x, y) u_i = 0$, on Γ_β

$\beta(x, y) = $ basal sliding coefficient

***Assumption:** aspect ratio δ is small and normals to upper/lower surfaces are almost vertical.

Thickness & Temperature Equations

- Model for ***evolution of the boundaries*** (thickness evolution equation):

$$\frac{\partial H}{\partial t} = -\nabla \cdot (\overline{\boldsymbol{u}}H) + \dot{b}$$

where $\overline{\boldsymbol{u}}$ = vertically averaged velocity, $\dot{b}$ = surface mass balance (conservation of mass).

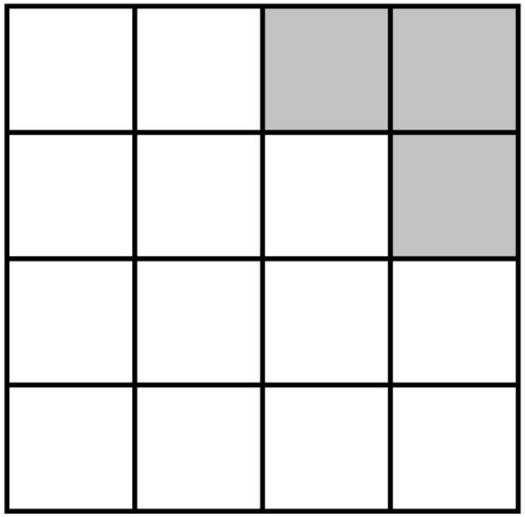
- ***Temperature equation*** (advection-diffusion):

$$\rho c \frac{\partial T}{\partial t} = \nabla \cdot (k\nabla T) - \rho c \boldsymbol{u} \cdot \nabla T + 2\dot{\boldsymbol{\epsilon}}\boldsymbol{\sigma}$$

(energy balance).

- ***Flow factor*** A in Glen's law depends on temperature T: $A = A(T)$.

- Ice sheet ***grows/retreats*** depending on thickness H.



time t_0

Ice-covered ("active") cells shaded in white ($H > H_{min}$)

Thickness & Temperature Equations

- Model for ***evolution of the boundaries*** (thickness evolution equation):

$$\frac{\partial H}{\partial t} = -\nabla \cdot (\overline{\boldsymbol{u}}H) + \dot{b}$$

where $\overline{\boldsymbol{u}}$ = vertically averaged velocity, $\dot{b}$ = surface mass balance (conservation of mass).

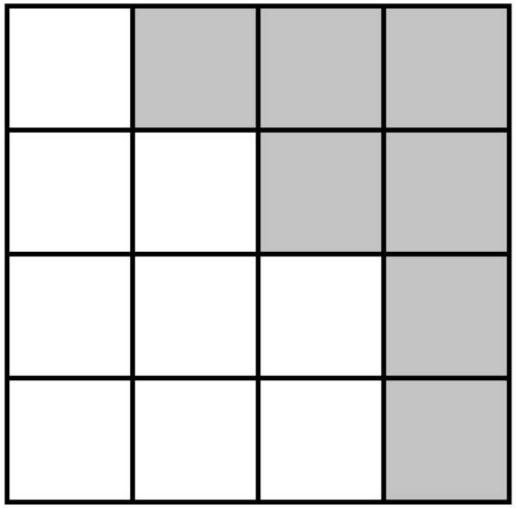
- ***Temperature equation*** (advection-diffusion):

$$\rho c \frac{\partial T}{\partial t} = \nabla \cdot (k\nabla T) - \rho c \boldsymbol{u} \cdot \nabla T + 2\dot{\boldsymbol{\epsilon}}\boldsymbol{\sigma}$$

(energy balance).

- ***Flow factor*** A in Glen's law depends on temperature T: $A = A(T)$.

- Ice sheet ***grows/retreats*** depending on thickness H.



time t_1

Ice-covered ("active") cells shaded in white ($H > H_{min}$)

Thickness & Temperature Equations

- Model for ***evolution of the boundaries*** (thickness evolution equation):

$$\frac{\partial H}{\partial t} = -\nabla \cdot (\overline{\boldsymbol{u}} H) + \dot{b}$$

where $\overline{\boldsymbol{u}}$ = vertically averaged velocity, $\dot{b}$ = surface mass balance (conservation of mass).

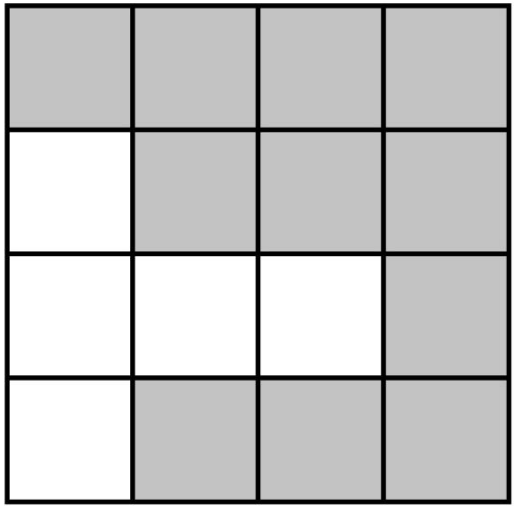
- ***Temperature equation*** (advection-diffusion):

$$\rho c \frac{\partial T}{\partial t} = \nabla \cdot (k \nabla T) - \rho c \boldsymbol{u} \cdot \nabla T + 2\dot{\boldsymbol{\epsilon}}\boldsymbol{\sigma}$$

(energy balance).

- ***Flow factor*** A in Glen's law depends on temperature T: $A = A(T)$.

- Ice sheet ***grows/retreats*** depending on thickness H.



time t_2

Ice-covered ("active") cells shaded in white ($H > H_{min}$)

Summary of Ice Sheet Equations & Codes

Sandia National Laboratories

Momentum Balance: ***First-Order Stokes*** PDEs

$$\begin{cases} -\nabla\cdot(2\mu\dot{\epsilon}_1) = -\rho g\dfrac{\partial s}{\partial x} \\ -\nabla\cdot(2\mu\dot{\epsilon}_2) = -\rho g\dfrac{\partial s}{\partial y} \end{cases}, \quad \text{in } \Omega$$

with ***Glen's law*** viscosity $\mu = \frac{1}{2}A(T)^{-\frac{1}{3}}\left(\frac{1}{2}\sum_{ij}\dot{\epsilon}_{ij}{}^2\right)^{\left(-\frac{2}{3}\right)}$.

Code:

C++

Albany

Albany/FELIX

Trilinos

Conservation of Mass: ***thickness*** evolution PDE

$$\frac{\partial h}{\partial t} = -\nabla\cdot(\bar{\boldsymbol{u}}h) + \dot{b}$$

Energy Balance: ***temperature*** advection-diffusion PDE

$$\rho c\frac{\partial T}{\partial t} = \nabla\cdot(k\nabla T) - \rho c\boldsymbol{u}\cdot\nabla T + 2\dot{\boldsymbol{\epsilon}}\boldsymbol{\sigma}$$

CESM
COMMUNITY EARTH SYSTEM MODEL

MPAS
Model for Prediction Across Scales

CISM/Albany
MPAS/Albany

Fortran

ACME
Accelerated Climate Model for Energy

CISM-Albany and *MPAS-Albany*

Sandia National Laboratories

CISM-Albany

MPAS-Albany

cism_driver

CISM (Fortran)
Thickness evolution, temperature solve, coupling to CESM

output file

Albany/FELIX (C++)
velocity solve

C++/Fortran Interface, Mesh Conversion

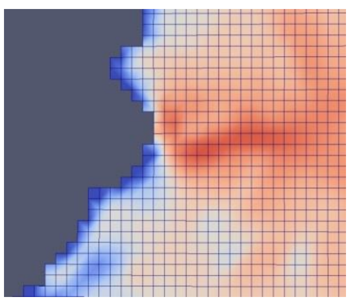
C++/Fortran Interface, Mesh Conversion

LandIce_model

MPAS Land-Ice (Fortran)
Thickness evolution, temperature solve, coupling to DOE-ESM

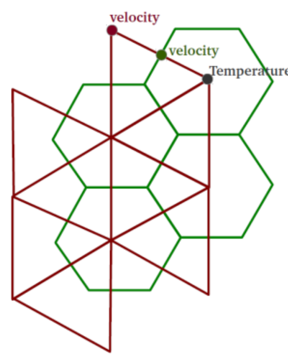
output file

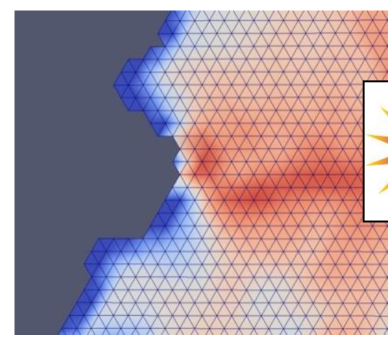
CESM
COMMUNITY EARTH SYSTEM MODEL



- Structured hexahedral meshes (rectangles extruded to hexes).

- Tetrahedral meshes (dual of hexaganonal mesh, extruded to tets).



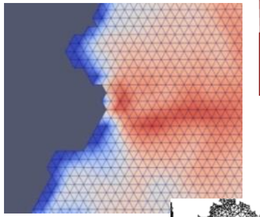


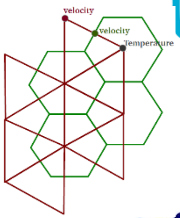
ACME
Accelerated Climate Model for Energy

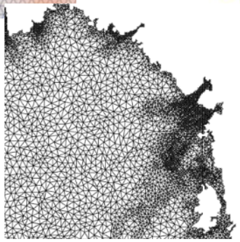
Albany/FELIX has been coupled to two land ice dycores: **C**ommunity **I**ce **S**heet **M**odel (***CISM***) and **M**odel for **P**rediction **A**cross **S**cales for Land-Ice (***MPAS***)

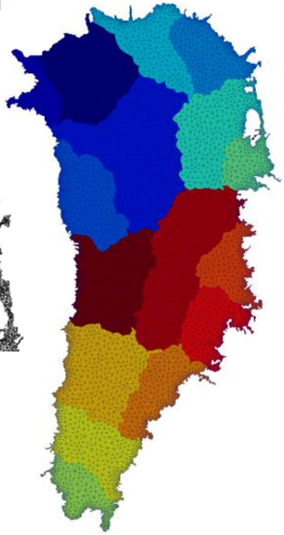
Meshes and Data

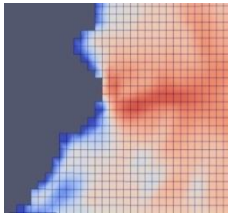
Meshes and Data











<u>Meshes:</u> can use any mesh but interested specifically in

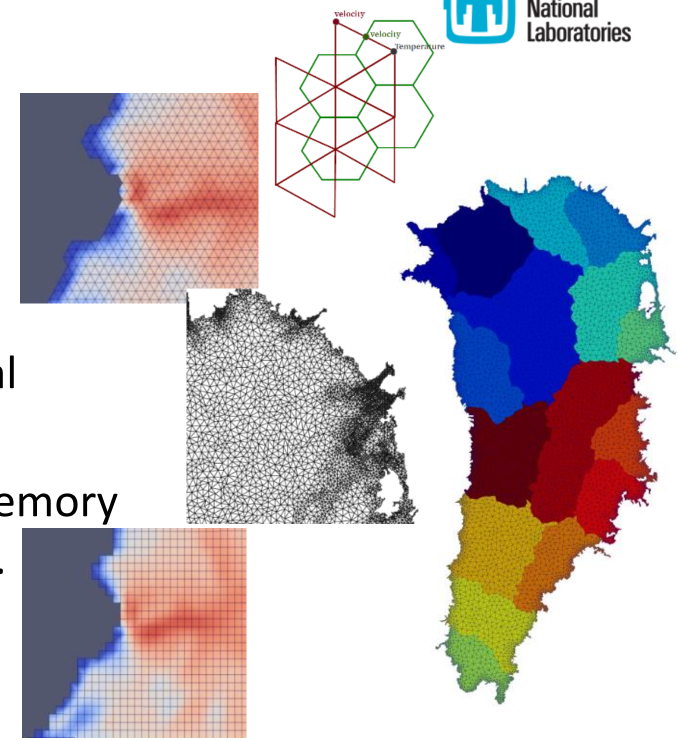
- *CISM-Albany: **structured hexahedral*** meshes
- *MPAS-Albany: **tetrahedral*** meshes (Delaunay triangle mesh = dual of hexaganonal mesh, extruded to tets).
    - ***Unstructured Delaunay triangle*** meshes w/ regional refinement based on gradient of surface velocity.
- ***Unstructured adaptively-refined*** meshes generated in memory using ***PAALS**** (Parallel Albany Adaptive Loop w/ SCOREC).

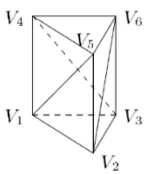
*See talks by RPI folks and Glen Hansen.

Meshes and Data

Meshes: can use any mesh but interested specifically in

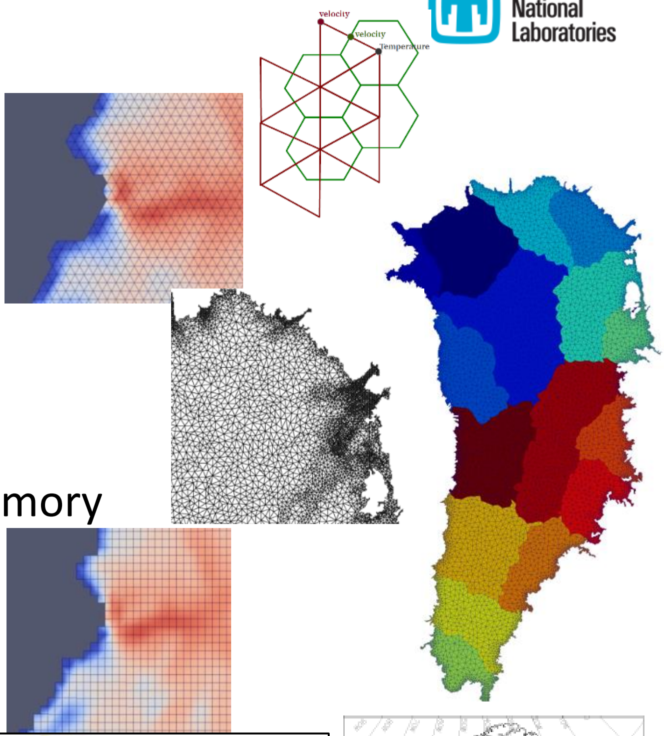
- *CISM-Albany:* ***structured hexahedral*** meshes
- *MPAS-Albany:* ***tetrahedral*** meshes (Delaunay triangle mesh = dual of hexaganonal mesh, extruded to tets).
    - ***Unstructured Delaunay triangle*** meshes w/ regional refinement based on gradient of surface velocity.
- ***Unstructured adaptively-refined*** meshes generated in memory using ***PAALS**** (Parallel Albany Adaptive Loop w/ SCOREC).





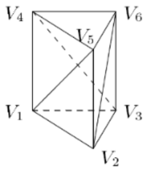
All meshes are extruded (structured) in vertical direction as tetrahedra or hexahedra.

Meshes and Data

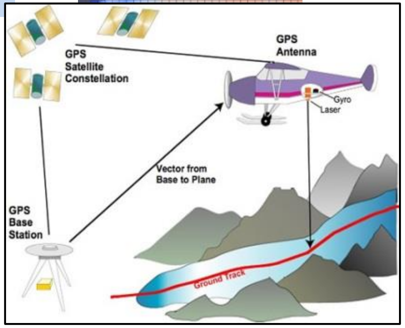


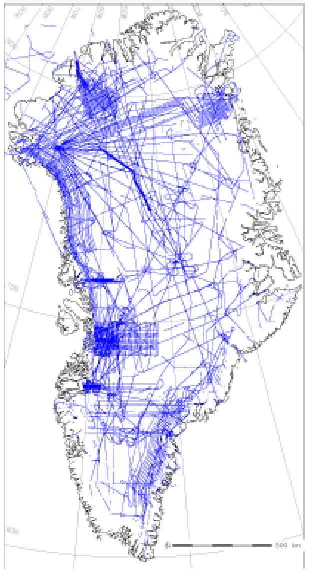
Meshes: can use any mesh but interested specifically in

- *CISM-Albany:* ***structured hexahedral*** meshes
- *MPAS-Albany:* ***tetrahedral*** meshes (Delaunay triangle mesh = dual of hexaganonal mesh, extruded to tets).
    - ***Unstructured Delaunay triangle*** meshes w/ regional refinement based on gradient of surface velocity.
- ***Unstructured adaptively-refined*** meshes generated in memory using ***PAALS**** (Parallel Albany Adaptive Loop w/ SCOREC).



All meshes are extruded (structured) in vertical direction as tetrahedra or hexahedra.





Data: needs to be imported into code to run "real" problems (Greenland, Antarctica).

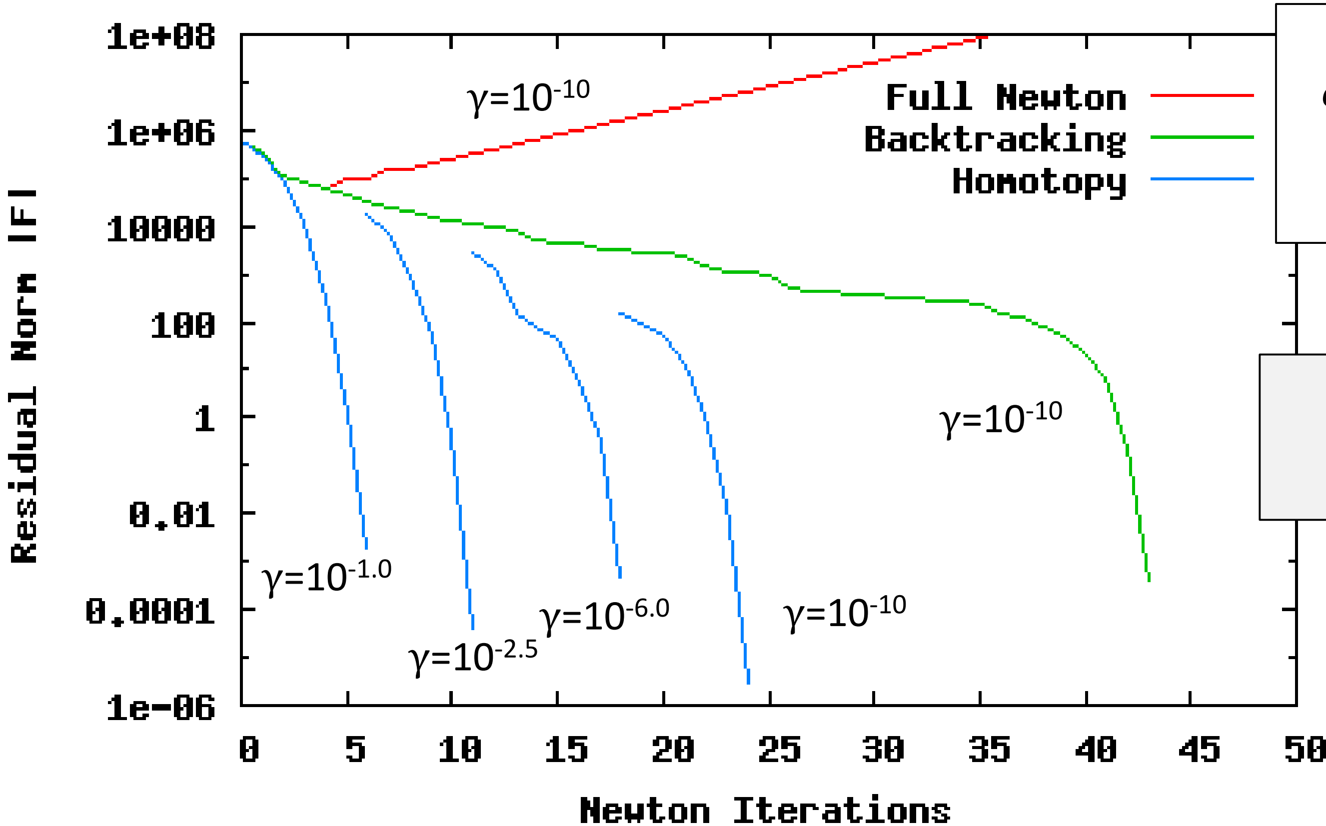
- ***Surface data*** are available from measurements (satellite infrarometry, radar, altimetry): ice extent, surface topography, surface velocity, surface mass balance.
- ***Interior ice data*** (ice thickness, basal friction) cannot be measured; estimated by solving an inverse problem.

*See talks by RPI folks and Glen Hansen.

Performance: Robustness and Scalability

Robustness of Newton's Method via Homotopy Continuation (LOCA)

Sandia National Laboratories

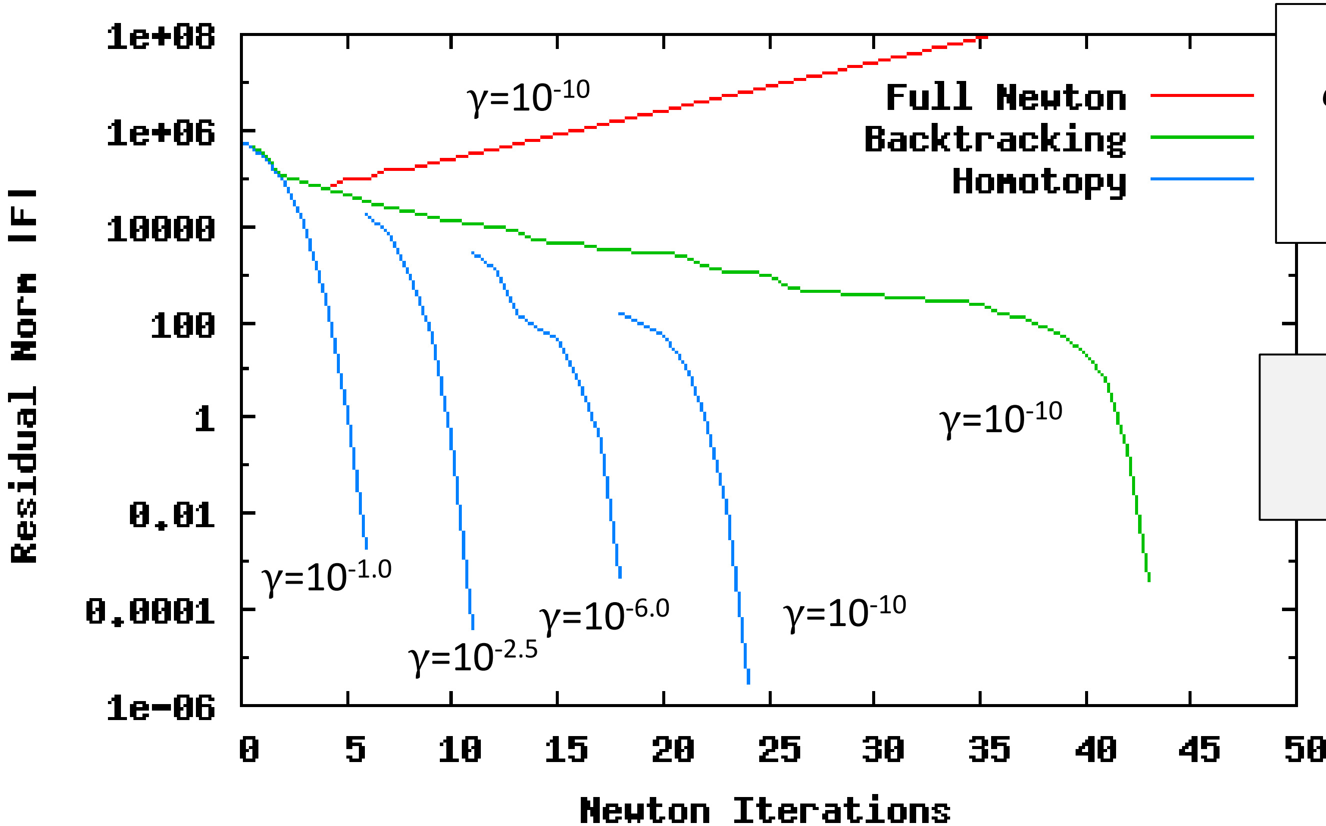


$$\dot{\boldsymbol{\epsilon}}_1{}^T = (2\dot{\epsilon}_{11} + \dot{\epsilon}_{22}, \dot{\epsilon}_{12}, \dot{\epsilon}_{13})$$
$$\dot{\boldsymbol{\epsilon}}_2{}^T = (2\dot{\epsilon}_{12}, \dot{\epsilon}_{11} + 2\dot{\epsilon}_{22}, \dot{\epsilon}_{23})$$
$$\dot{\epsilon}_{\mathrm{ij}} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

Glen's Law Viscosity:

$$\mu = \frac{1}{2} A^{-\frac{1}{3}} \left(\frac{1}{2} \sum_{ij} \dot{\boldsymbol{\epsilon}}_{ij}{}^2\right)^{-\frac{2}{3}}$$

Robustness of Newton's Method via Homotopy Continuation (LOCA)



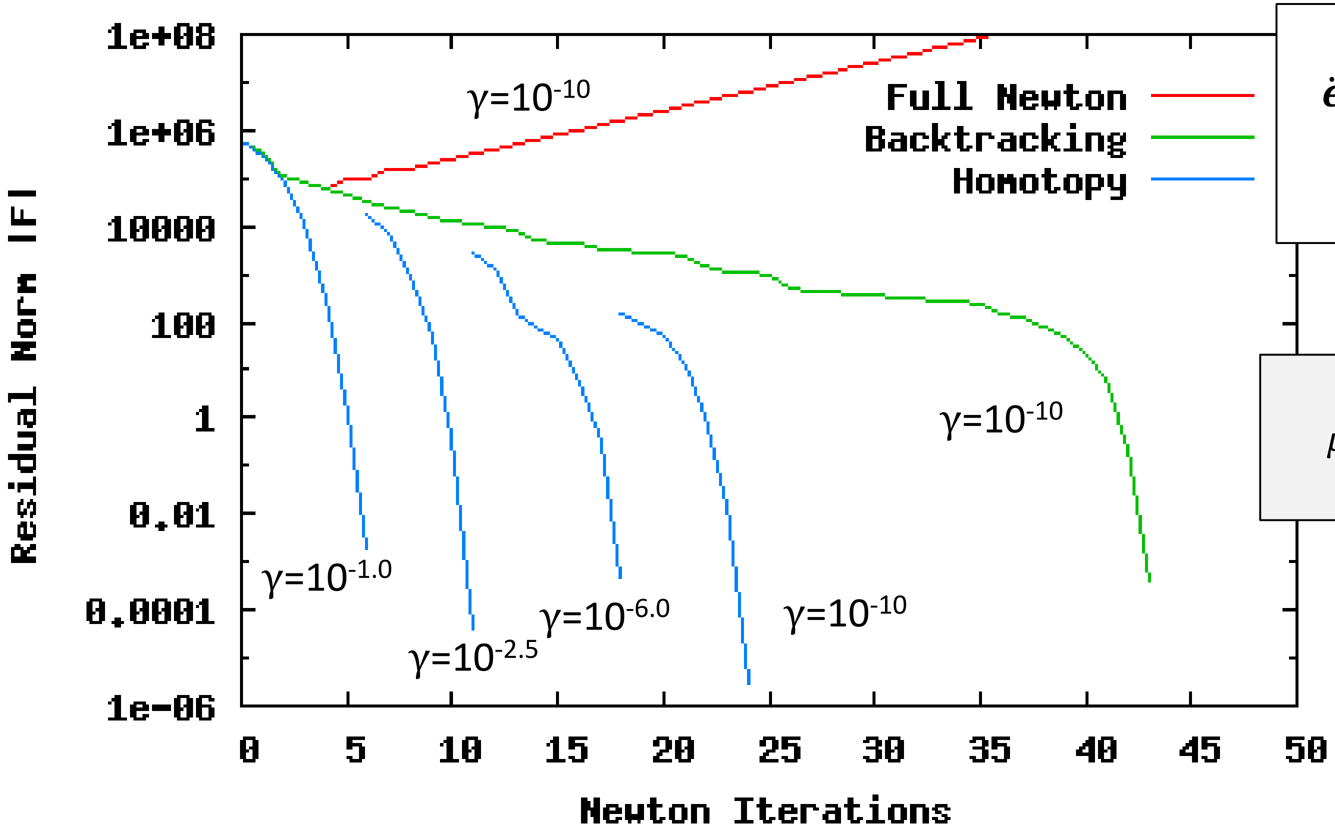
$$\dot{\boldsymbol{\epsilon}}_1{}^T = (2\dot{\epsilon}_{11} + \dot{\epsilon}_{22}, \dot{\epsilon}_{12}, \dot{\epsilon}_{13})$$
$$\dot{\boldsymbol{\epsilon}}_2{}^T = (2\dot{\epsilon}_{12}, \dot{\epsilon}_{11} + 2\dot{\epsilon}_{22}, \dot{\epsilon}_{23})$$
$$\dot{\epsilon}_{\mathrm{ij}} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

Glen's Law Viscosity:

$$\mu = \frac{1}{2} A^{-\frac{1}{3}} \left(\frac{1}{2} \sum_{ij} \dot{\boldsymbol{\epsilon}}_{ij}{}^2 + \gamma \right)^{-\frac{2}{3}}$$

γ = regularization parameter

Robustness of Newton's Method via Homotopy Continuation (LOCA)



$$\dot{\boldsymbol{\epsilon}}_1{}^T = (2\dot{\epsilon}_{11} + \dot{\epsilon}_{22}, \dot{\epsilon}_{12}, \dot{\epsilon}_{13})$$
$$\dot{\boldsymbol{\epsilon}}_2{}^T = (2\dot{\epsilon}_{12}, \dot{\epsilon}_{11} + 2\dot{\epsilon}_{22}, \dot{\epsilon}_{23})$$
$$\dot{\epsilon}_{\mathrm{ij}} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

Glen's Law Viscosity:

$$\mu = \frac{1}{2} A^{-\frac{1}{3}} \left(\frac{1}{2} \sum_{ij} \dot{\boldsymbol{\epsilon}}_{ij}{}^2 + \gamma \right)^{-\frac{2}{3}}$$

γ = regularization parameter

Newton most robust with full step + homotopy continuation of $\gamma \to 10^{-10}$: ***converges out-of-the-box***!

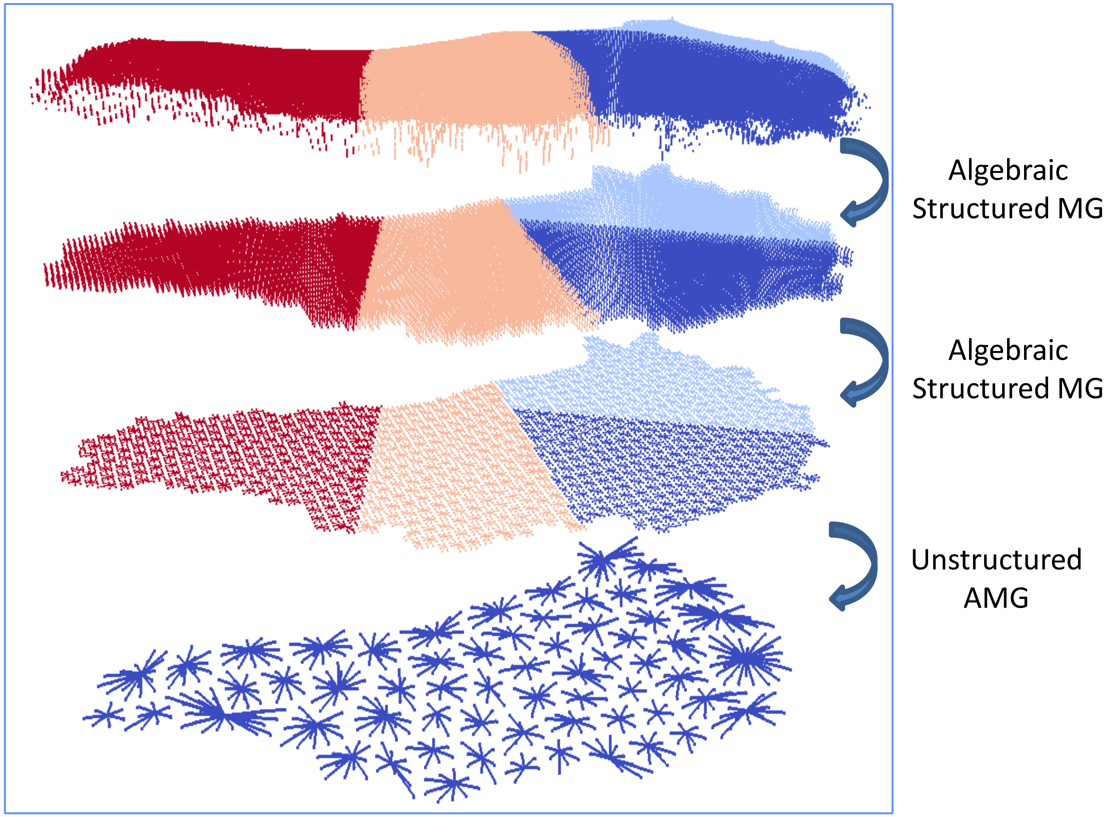
Scalability via Algebraic Multi-Grid Preconditioning with Semi-Coarsening

Bad aspect ratios ($dx \gg dz$) ruin classical AMG convergence rates!

- relatively small horizontal coupling terms, hard to smooth horizontal errors

$\Rightarrow$ Solvers (AMG and ILU) must take aspect ratios into account

New AMG solver based on aggressive **semi-coarsening** has been developed by R. Tuminaro (available in *ML/MueLu* packages of *Trilinos*)

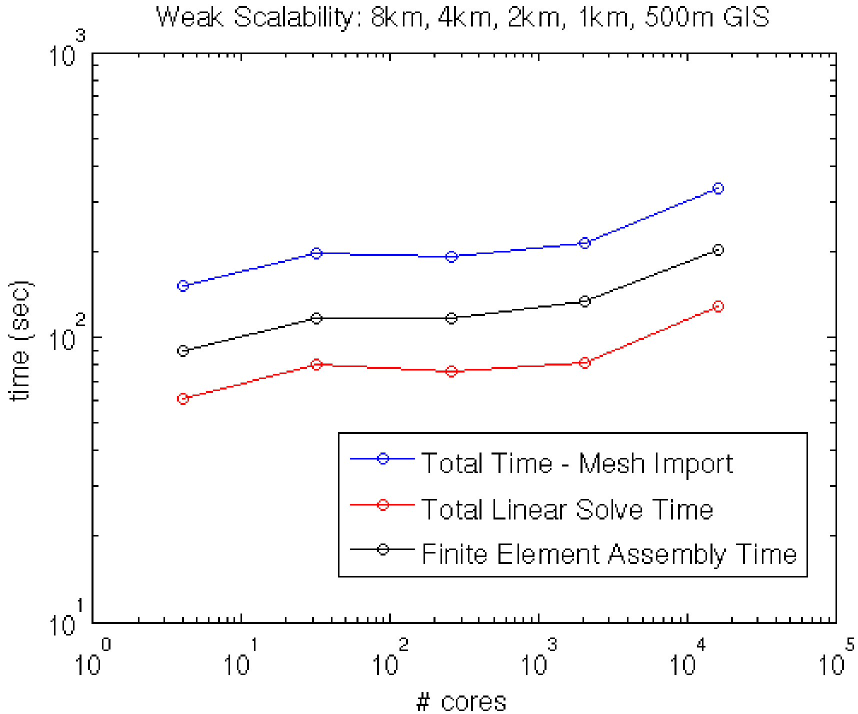


See (Tezaur *et al.*, 2015), (Tuminaro *et al.*, 2016).

Scalability studies (next slides):
New AMG preconditioner vs. ILU

Greenland Controlled Weak Scalability Study

New AMG preconditioner

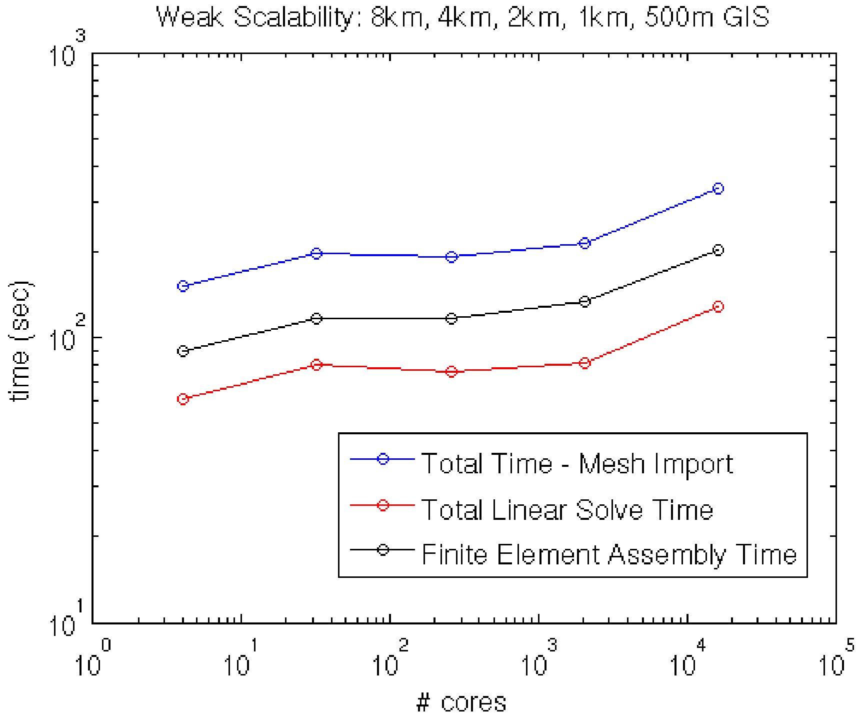


4 cores 334K dofs 8 km Greenland, 5 vertical layers	$\times\, 8^4$ scale up $\longrightarrow$	16,384 cores **1.12B dofs(!)** 0.5 km Greenland, 80 vertical layers

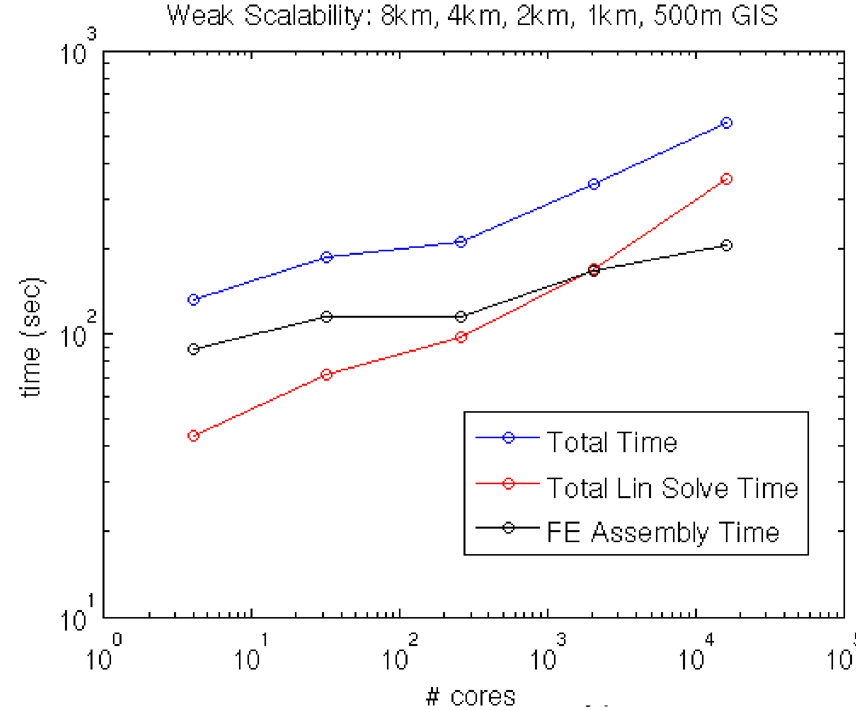
- Weak scaling study with fixed dataset, 4 mesh bisections.
- ~70-80K dofs/core.
- ***Conjugate Gradient (CG) iterative method*** for linear solves (faster convergence than GMRES).
- ***New AMG preconditioner*** developed by R. Tuminaro based on ***semi-coarsening*** (coarsening in z-direction only).
- ***Significant improvement*** in scalability with new AMG preconditioner over ILU preconditioner!

Greenland Controlled Weak Scalability Study

New AMG preconditioner



ILU preconditioner



4 cores
334K dofs
8 km Greenland,
5 vertical layers

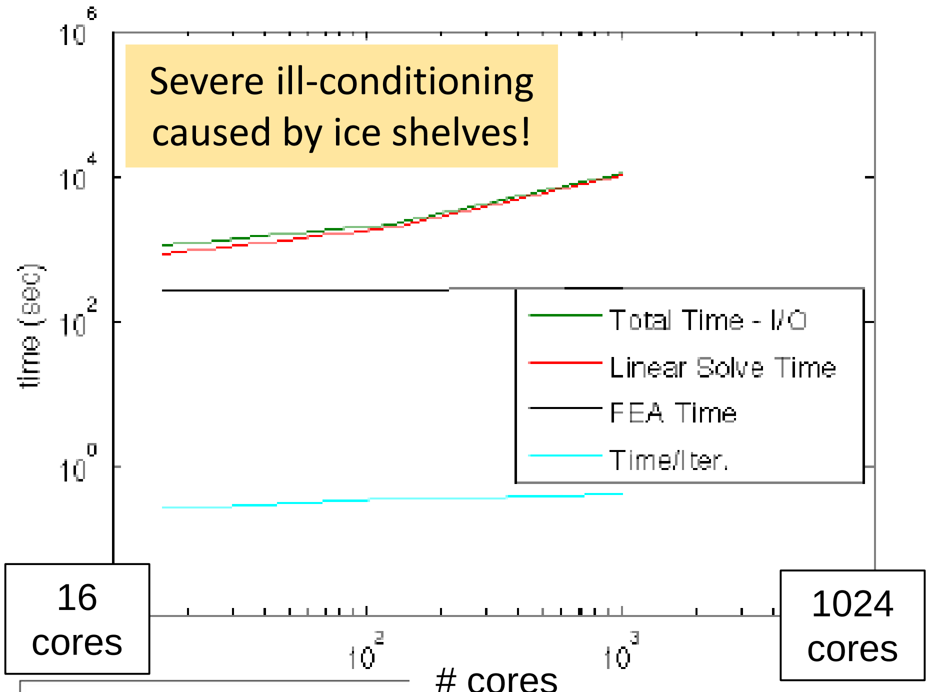
$\times\, 8^4$
scale up

16,384 cores
1.12B dofs(!)
0.5 km Greenland,
80 vertical layers

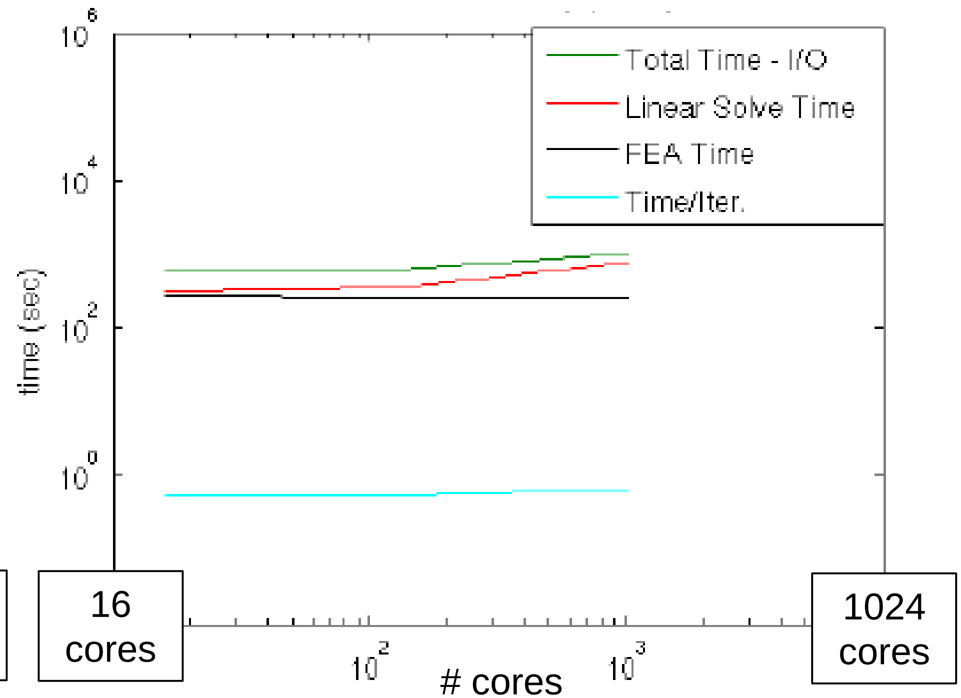
- ***Significant improvement*** in scalability with new AMG preconditioner over ILU preconditioner!

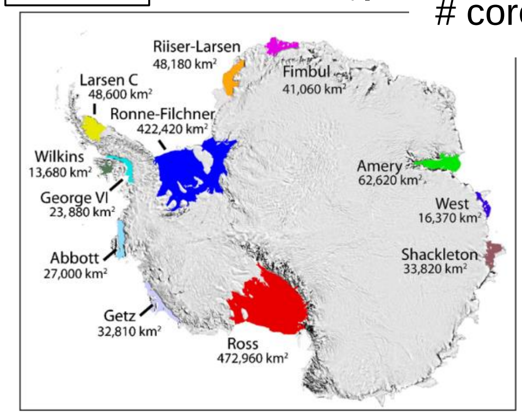
Antarctica Weak Scalability Study

ILU preconditioner



New AMG preconditioner





AMG preconditioner ***less sensitive to ill-conditioning*** caused by ***ice shelves*** than ILU (ice shelves → Green's function with modest horizontal decay → ILU is less effective).

Performance Portability

Performance Portability via *Kokkos*

We need to be able to run *Albany/FELIX* on ***new architecture machines*** (hybrid systems) and ***manycore devices*** (multi-core CPU, NVIDIA GPU, Intel Xeon Phi, etc.)

Performance Portability via *Kokkos*

We need to be able to run *Albany/FELIX* on ***new architecture machines*** (hybrid systems) and ***manycore devices*** (multi-core CPU, NVIDIA GPU, Intel Xeon Phi, etc.)

→ achieved using the ***Kokkos*** library/programming model.

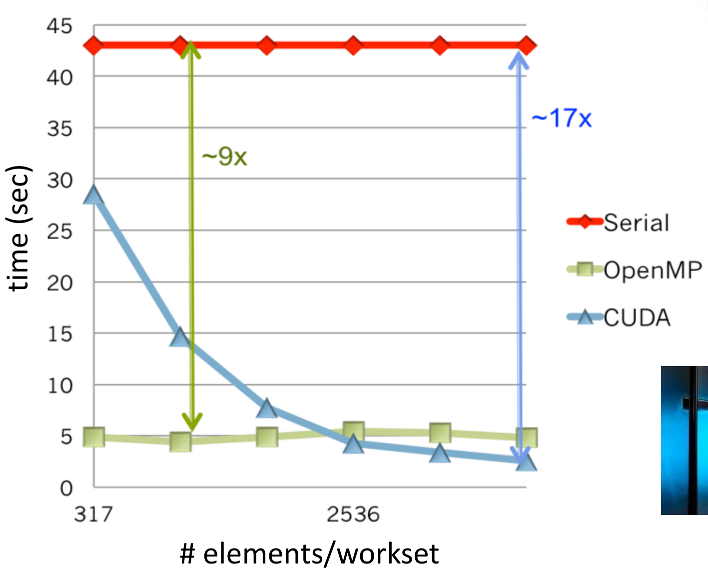
Performance Portability via *Kokkos*

We need to be able to run *Albany/FELIX* on ***new architecture machines*** (hybrid systems) and ***manycore devices*** (multi-core CPU, NVIDIA GPU, Intel Xeon Phi, etc.)

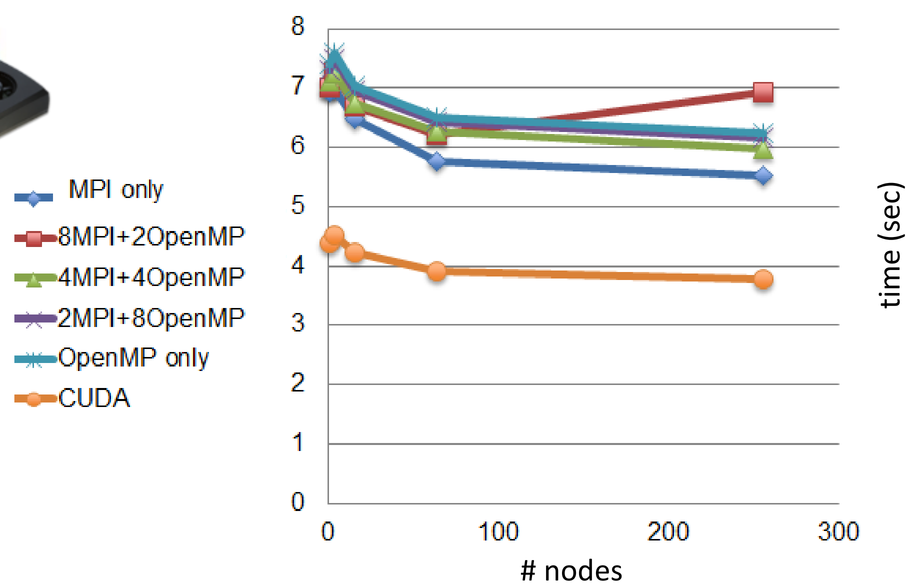
→ achieved using the ***Kokkos*** library/programming model.

- **Finite element assembly (FEA)** in *Albany/FELIX* has been rewritten using *Kokkos* functors*.

FEA Time (Greenland, on Shannon)



Compute Time (Greenland, on Titan)



*See talk by Jerry Watkins.

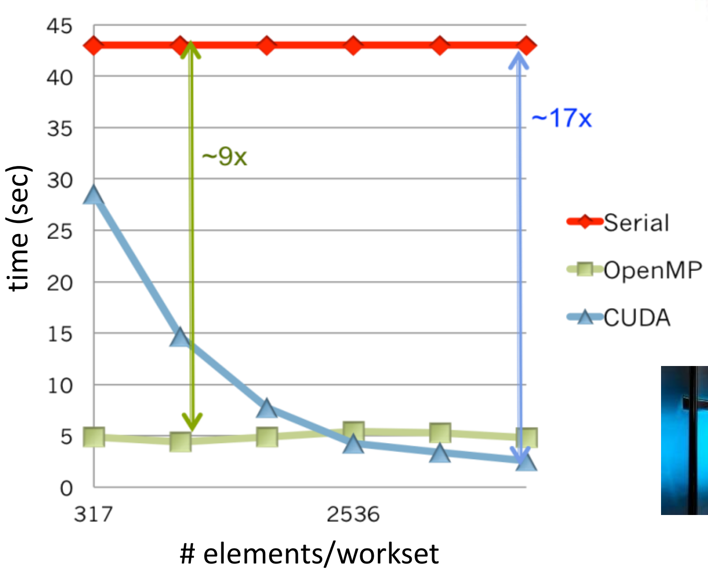
Performance Portability via *Kokkos*

We need to be able to run *Albany/FELIX* on ***new architecture machines*** (hybrid systems) and ***manycore devices*** (multi-core CPU, NVIDIA GPU, Intel Xeon Phi, etc.)

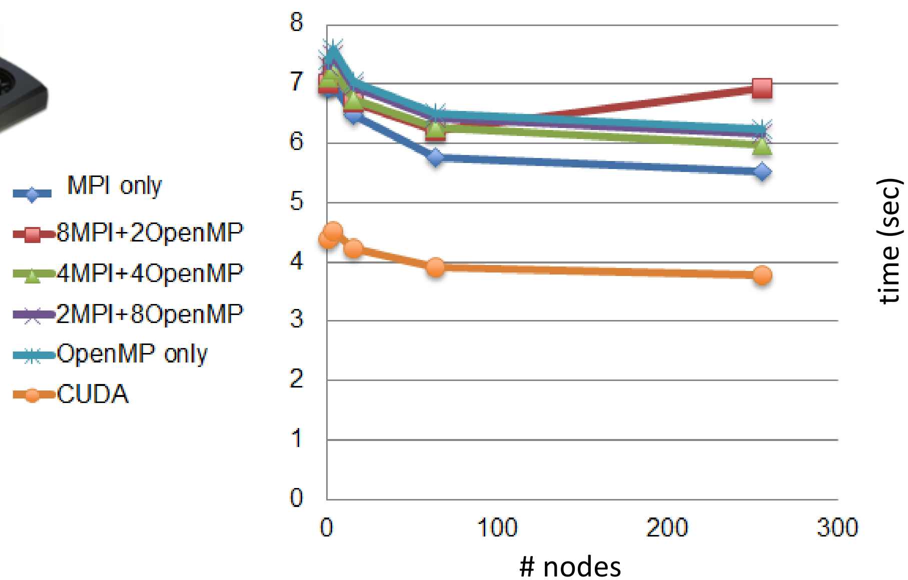
→ achieved using the ***Kokkos*** library/programming model.

- **Finite element assembly (FEA)** in *Albany/FELIX* has been rewritten using *Kokkos* functors*.
- Jerry Watkins currently working on code ***profiling*** and ***optimizations*** to get the best possible performance on ***GPUs*** and ***Intel Xeon Phis***.

FEA Time (Greenland, on Shannon)



Compute Time (Greenland, on Titan)



*See talk by Jerry Watkins.

Deterministic Inversion

Deterministic Inversion: Estimation of Ice Sheet Initial State

Objective: find ice sheet initial state that

- Matches observations (e.g., surf. vel., temp., etc.)
- Matches present-day geometry (elevation, thickness).
- Is in “equilibrium” with climate forcings (SMB).

Unknown/uncertain variables:

- Basal friction (β).
- Thickness (H).

Deterministic Inversion: Estimation of Ice Sheet Initial State

Objective: find ice sheet initial state that

- Matches observations (e.g., surf. vel., temp., etc.)
- Matches present-day geometry (elevation, thickness).
- Is in "equilibrium" with climate forcings (SMB).

Approach: invert for unknown/uncertain parameters by minimizing difference between

- Computed and measured ***surface velocity*** **($\boldsymbol{u}^{obs}$)**
- Computed divergence flux and measured ***surface mass balance (SMB)***
- Computed and ***reference thickness*** ***(H^{obs})***

Unknown/uncertain variables:

- Basal friction (β).
- Thickness (*H*).

First-Order Stokes PDE Constrained Optimization Problem:

min $J(\beta, H)$ s.t. FO Stokes PDEs where

$$J(\beta, H) = \frac{1}{2}\alpha_v \int_{\Gamma_{top}} |\boldsymbol{u} - \boldsymbol{u}^{obs}|^2 ds + \frac{1}{2}\alpha \int_{\Gamma} |div(\boldsymbol{U}H) - SMB|^2 ds + \frac{1}{2}\alpha_H \int_{\Gamma_{top}} |H - H^{obs}|^2 ds + \mathcal{R}(\beta) + \mathcal{R}(H)$$

Deterministic Inversion: Estimation of Ice Sheet Initial State

RAPID OPTIMIZATION LIBRARY

Objective: find ice sheet initial state that

- Matches observations (e.g., surf. vel., temp., etc.)
- Matches present-day geometry (elevation, thickness).
- Is in "equilibrium" with climate forcings (SMB).

Approach: invert for unknown/uncertain parameters by minimizing difference between

- Computed and measured ***surface velocity (*** $\boldsymbol{u}^{obs}$ **)**
- Computed divergence flux and measured ***surface mass balance (SMB)***
- Computed and ***reference thickness (*** H^{obs} ***)***

Software for Adjoint-Based Deterministic Inversion:*

- ***Albany/FELIX*** (FE assembly)
- ***Trilinos*** (linear/nonlinear solvers)
- ***ROL*** (gradient-based optimization)
  - Limited memory BFGS.
  - Backtrack line-search.

Unknown/uncertain variables:

- Basal friction (β).
- Thickness (*H*).

First-Order Stokes PDE Constrained Optimization Problem:

min $J(\beta, H)$ s.t. FO Stokes PDEs where

$$J(\beta,H) = \frac{1}{2}\alpha_v \int_{\Gamma_{top}} |\boldsymbol{u} - \boldsymbol{u}^{obs}|^2 ds + \frac{1}{2}\alpha \int_{\Gamma} |div(\boldsymbol{U}H) - SMB|^2 ds + \frac{1}{2}\alpha_H \int_{\Gamma_{top}} |H - H^{obs}|^2 ds + \mathcal{R}(\beta) + \mathcal{R}(H)$$

*See talk by Mauro Perego.

Deterministic Inversion: Estimation of Ice Sheet Initial State

ROL
RAPID OPTIMIZATION LIBRARY

Sandia National Laboratories

Objective: find ice sheet initial state that
- Matches observations (e.g., surf. vel., temp., etc.)
- Matches present-day geometry (elevation, thickness).
- Is in "equilibrium" with climate forcings (SMB).

Approach: invert for unknown/uncertain parameters by minimizing difference between
- Computed and measured ***surface velocity*** $(\boldsymbol{u}^{obs})$
- Computed divergence flux and measured ***surface mass balance (SMB)***
- Computed and ***reference thickness*** (H^{obs})

Software for Adjoint-Based Deterministic Inversion:*
- ***Albany/FELIX*** (FE assembly)
- ***Trilinos*** (linear/nonlinear solvers)
- ***ROL*** (gradient-based optimization)
  - Limited memory BFGS.
  - Backtrack line-search.

Unknown/uncertain variables:
- Basal friction (β).
- Thickness (H).

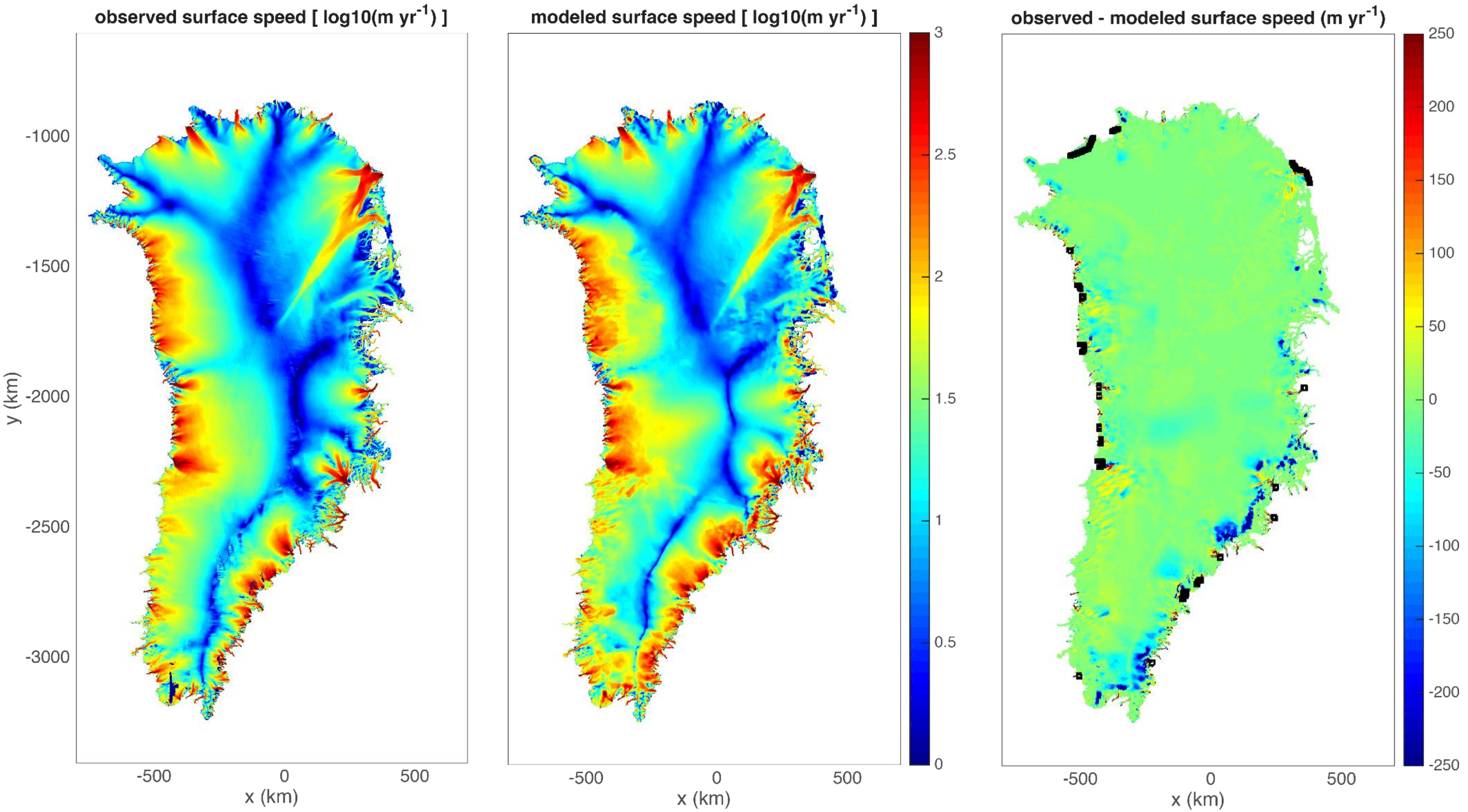
First-Order Stokes PDE Constrained Optimization Problem:

min $J(\beta, H)$ s.t. FO Stokes PDEs where

$$J(\beta,H) = \frac{1}{2}\alpha_v \int_{\Gamma_{top}} |\boldsymbol{u} - \boldsymbol{u}^{obs}|^2 ds + \frac{1}{2}\alpha \int_{\Gamma} |div(\boldsymbol{U}H) - SMB|^2 ds + \frac{1}{2}\alpha_H \int_{\Gamma_{top}} |H - H^{obs}|^2 ds + \mathcal{R}(\beta) + \mathcal{R}(H)$$

→ *significantly reduces non-physical model transients*

Deterministic Inversion: 1km Greenland Initial Condition*



*This initial condition was used for validation study, discussed later in the talk.

Uncertainty Quantification

Uncertainty Quantification Workflow

Goal: Uncertainty Quantification in 21st century sea level (QoI)

Uncertainty Quantification Workflow

Goal: Uncertainty Quantification in 21st century sea level (QoI)

- ***Sources of uncertainty:*** climate forcings (e.g., surface mass balance), basal friction (β), bedrock topography, geothermal heat flux, model parameters (e.g., Glen's law exponent).

Uncertainty Quantification Workflow

Goal: Uncertainty Quantification in 21st century sea level (QoI)

- ***Sources of uncertainty:*** climate forcings (e.g., surface mass balance), basal friction (β), bedrock topography, geothermal heat flux, model parameters (e.g., Glen's law exponent).

As a first step, we focus on effect of uncertainty in **basal friction ($\boldsymbol{\beta}$)** only.

Uncertainty Quantification Workflow

Goal: Uncertainty Quantification in 21st century sea level (QoI)

- ***Sources of uncertainty:*** climate forcings (e.g., surface mass balance), basal friction (β), bedrock topography, geothermal heat flux, model parameters (e.g., Glen's law exponent).

As a first step, we focus on effect of uncertainty in **basal friction (β)** only.

- ***3 Stage UQ Workflow Process:***

1. **Deterministic inversion:** perform adjoint-based deterministic inversion to estimate initial ice sheet state (i.e., characterize the present state of the ice sheet to be used for performing prediction runs).
2. **Bayesian calibration:** construct the posterior distribution using Markov Chain Monte Carlo (MCMC) run on an emulator of the forward model.
3. **Forward propagation:** sample the obtained distribution and perform ensemble of forward propagation runs to compute the uncertainty in the QoI.

What are the parameters that render a given set of observations?

What is the impact of uncertain parameters in the model on quantities of interest (QoI)?

Uncertainty Quantification Workflow

Sandia National Laboratories

Goal: Uncertainty Quantification in 21st century sea level (QoI)

- ***Sources of uncertainty:*** climate forcings (e.g., surface mass balance), basal friction (β), bedrock topography, geothermal heat flux, model parameters (e.g., Glen's law exponent).

As a first step, we focus on effect of uncertainty in **basal friction ($\boldsymbol{\beta}$)** only.

- ***3 Stage UQ Workflow Process:***

DAKOTA

1. **Deterministic inversion:** perform adjoint-based deterministic inversion to estimate initial ice sheet state (i.e., characterize the present state of the ice sheet to be used for performing prediction runs).
2. **Bayesian calibration:** construct the posterior distribution using Markov Chain Monte Carlo (MCMC) run on an emulator of the forward model.
3. **Forward propagation:** sample the obtained distribution and perform ensemble of forward propagation runs to compute the uncertainty in the QoI.

What are the parameters that render a given set of observations?

What is the impact of uncertain parameters in the model on quantities of interest (QoI)?

Bayesian Calibration

Difficulty in UQ: "Curse of Dimensionality"
The unknown/uncertain fields have $O(100K)$ dimensions!

Bayesian Calibration

Difficulty in UQ: "Curse of Dimensionality"
The unknown/uncertain fields have $O(100K)$ dimensions!

Approach:

Bayesian Calibration

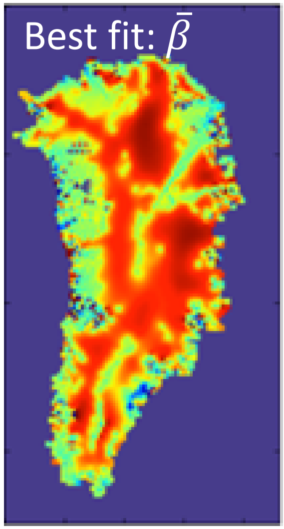
Difficulty in UQ: “Curse of Dimensionality”
The unknown/uncertain fields have $O(100K)$ dimensions!

Approach:

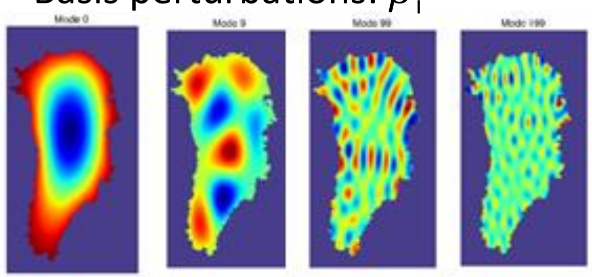
- Reduce $O(100K)$ dimensional problem to $O(10)$ dimensional problem using, e.g., ***Karhunen-Loeve Expansion (KLE), Hessian eigenvectors***, etc. [*Trilinos*].

Dimension reduction:
$\beta = \bar{\beta} + \sum_i \alpha_i \beta_i$

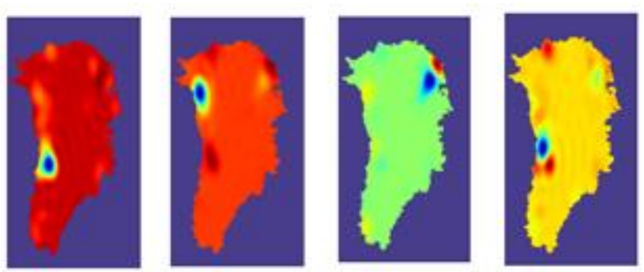
α_i: random samples
from prior distribution



Basis perturbations: β_i



Data-informed directions

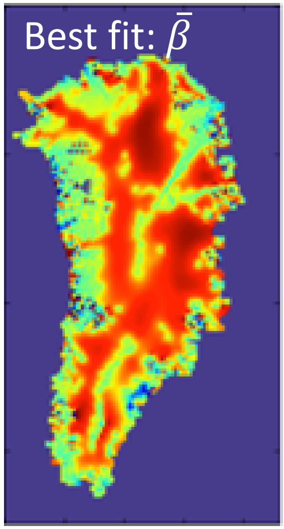


Bayesian Calibration

Difficulty in UQ: "Curse of Dimensionality"
The unknown/uncertain fields have $O(100K)$ dimensions!

Approach:

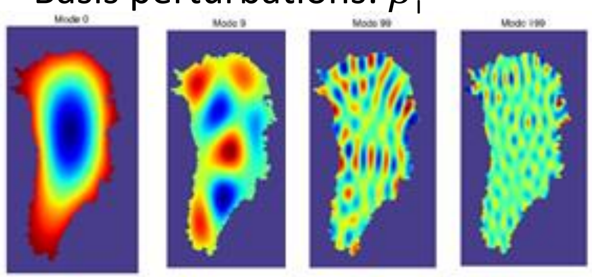
- Reduce $O(100K)$ dimensional problem to $O(10)$ dimensional problem using, e.g., ***Karhunen-Loeve Expansion (KLE), Hessian eigenvectors***, etc. [*Trilinos*].
- Form ***Polynomial Chaos Expansion (PCE)*** emulator for mismatch (over surface velocity, SMB, thickness) discrepancy [*DAKOTA*].



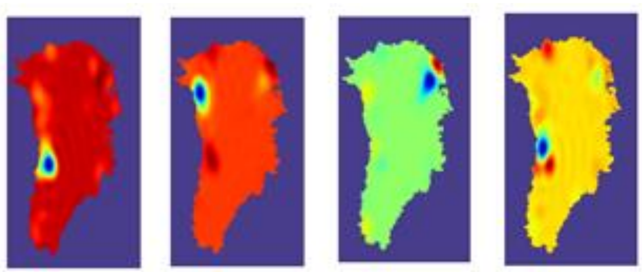
Dimension reduction:
$\beta = \bar{\beta} + \sum_i \alpha_i \beta_i$

α_i: random samples from prior distribution

Basis perturbations: β_{i}



Data-informed directions

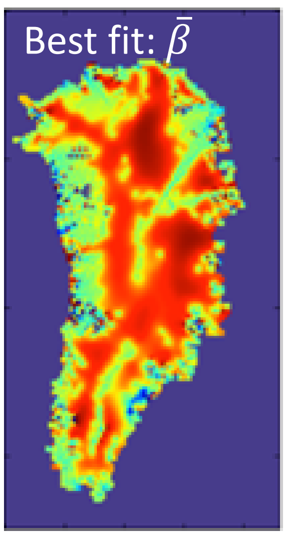


Bayesian Calibration

Difficulty in UQ: “Curse of Dimensionality”
The unknown/uncertain fields have $O(100K)$ dimensions!

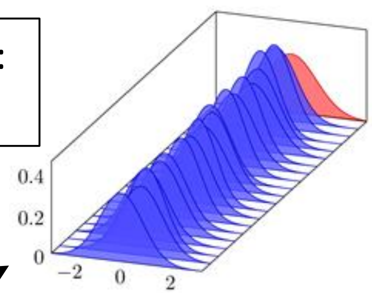
Approach:

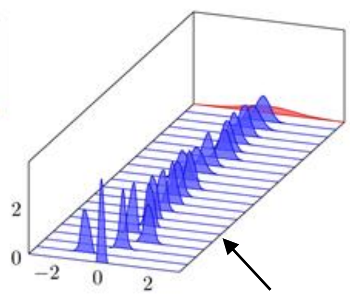
- Reduce $O(100K)$ dimensional problem to $O(10)$ dimensional problem using, e.g., ***Karhunen-Loeve Expansion (KLE), Hessian eigenvectors***, etc. [*Trilinos*].
- Form ***Polynomial Chaos Expansion (PCE)*** emulator for mismatch (over surface velocity, SMB, thickness) discrepancy [*DAKOTA*].
- ***Markov Chain Monte Carlo (MCMC)*** calibration using emulator [*QUESO*].



Dimension reduction:
$\beta = \bar{\beta} + \sum_i \alpha_i \beta_i$

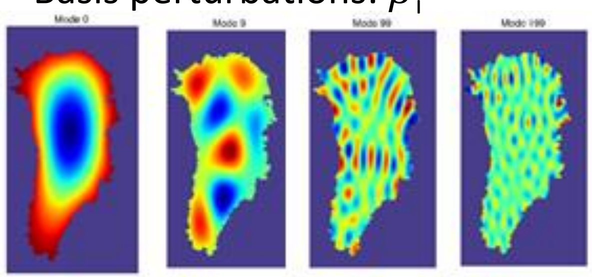
α_i: random samples from prior distribution



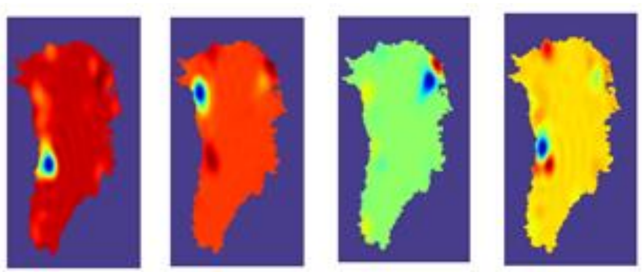


Marginal distributions of Gaussian posterior

Basis perturbations: β_i



Data-informed directions



Forward Propagation

Propagate distribution obtained in Bayesian calibration through the model to get distributions on ***total ice mass loss/gain during 21st century***

Forward Propagation

Propagate distribution obtained in Bayesian calibration through the model to get distributions on ***total ice mass loss/gain during 21st century***

Approach:

Forward Propagation

Propagate distribution obtained in Bayesian calibration through the model to get distributions on ***total ice mass loss/gain during 21st century***

Approach:

- Parameter distribution is the result of ***Bayesian calibration***.

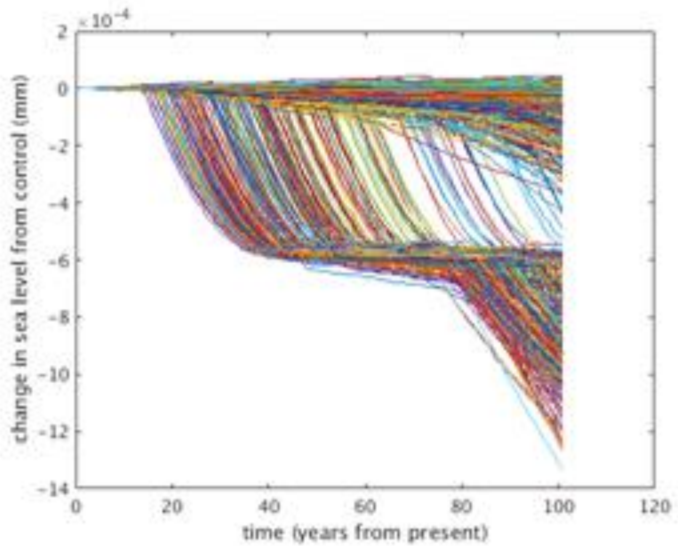
Forward Propagation

Propagate distribution obtained in Bayesian calibration through the model to get distributions on ***total ice mass loss/gain during 21st century***

Approach:

- Parameter distribution is the result of ***Bayesian calibration***.
- Run M forward *CISM/MPAS-Albany* runs each for N years w/ parameter sampled from its distribution and build ***emulator*** from these runs [*DAKOTA*].

Sea level time-history for 1000 50-year forward runs with steady state forcing



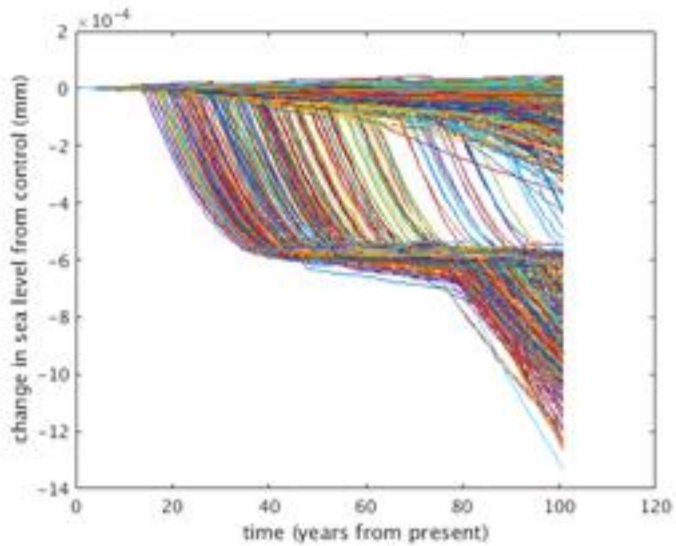
Forward Propagation

Propagate distribution obtained in Bayesian calibration through the model to get distributions on ***total ice mass loss/gain during 21st century***

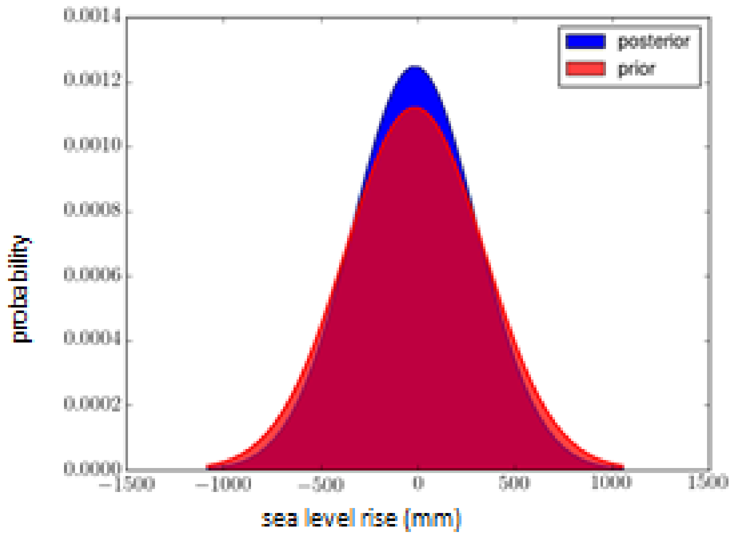
Approach:

- Parameter distribution is the result of ***Bayesian calibration***.
- Run M forward *CISM/MPAS-Albany* runs each for N years w/ parameter sampled from its distribution and build ***emulator*** from these runs [*DAKOTA*].
- Use MCMC and emulator to perform ***uncertainty propagation*** [*QUESO*].

Sea level time-history for 1000 50-year forward runs with steady state forcing



PDFs of SLR

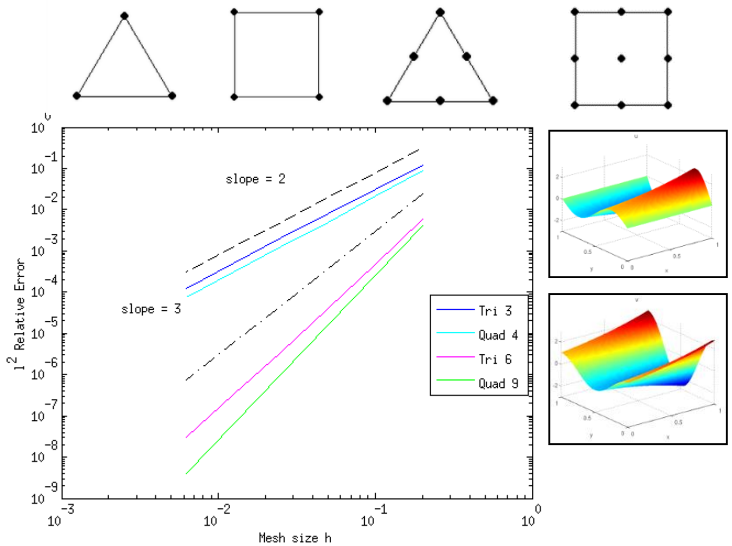


Verification and Validation

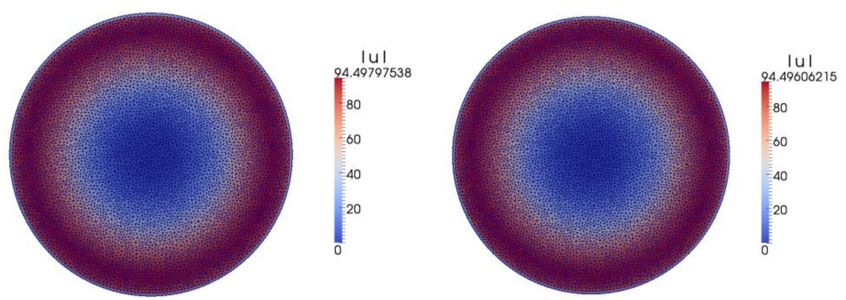
Verification

Is the code bug free?

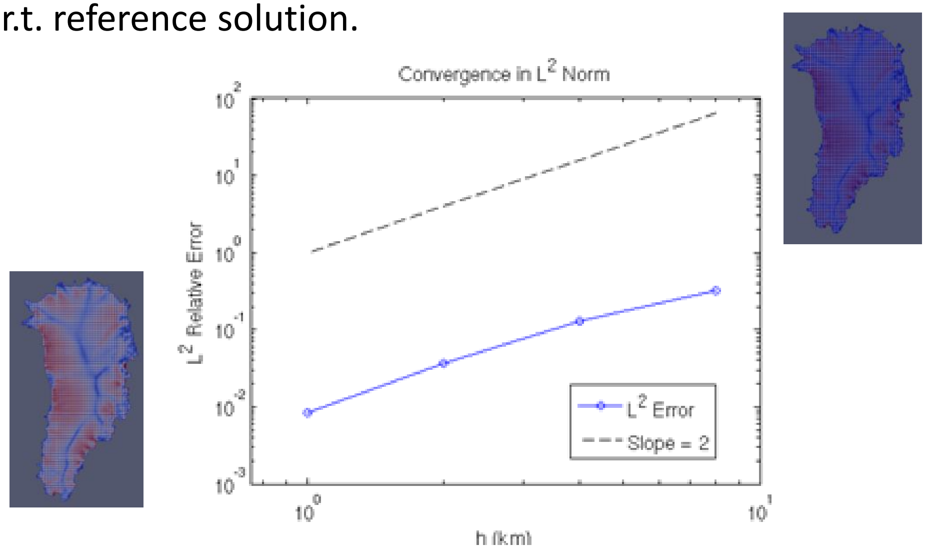
Stage 1: solution verification on 2D MMS problems.



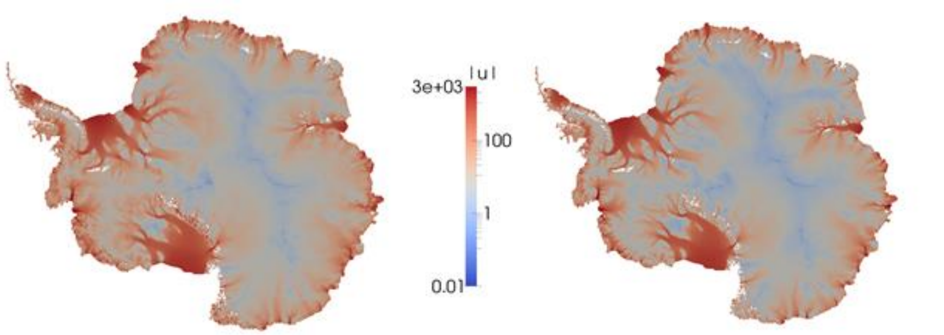
Stage 2: code-to-code comparisons on canonical ice sheet benchmarks (*Albany/FELIX* – left; *LifeV* – right).



Stage 3: full 3D mesh convergence study on Greenland w.r.t. reference solution.



Stage 4: reasonable solutions for large-scale realistic GIS & AIS problems (*Albany/FELIX* – left; reference solution – right).



Validation: Definition & Workflow

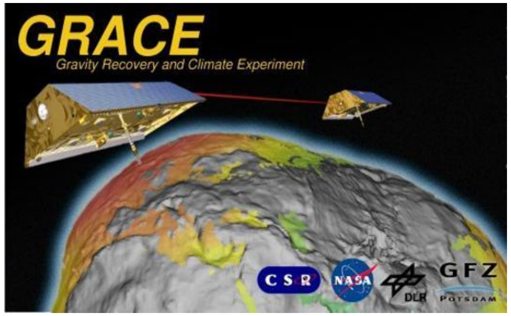
Validation: how well does model represent the real ice sheet?

Validation: Definition & Workflow

Validation: how well does model represent the real ice sheet?

- There are currently (up to) 2 decades of large-scale ***satellite*** observations of Greenland ice sheet geometry change:

ICESat1	2003 – 2009
GRACE	2002 – 201? (ongoing)



Validation Workflow (with LANL & NASA):

- Run *CISM-Albany* for period where observations exist.
- Process model output and observations for comparison.
- Evaluate model performance relative to observations.
  - ***ICESat***: ice sheet surface elevation [state comparison]
  - ***GRACE***: rate of mass change [trend comparison]

- ***Validation time periods***: 2003-2009 (IceSAT), 2003-2011 (GRACE)

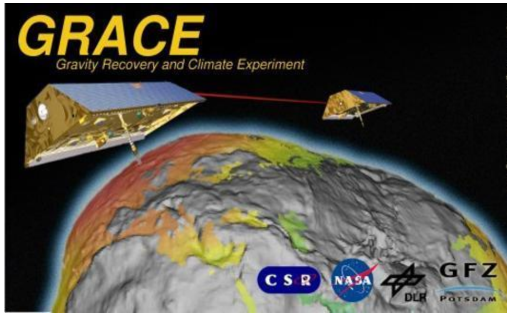
Validation: Definition & Workflow

Sandia National Laboratories

Validation: how well does model represent the real ice sheet?

- There are currently (up to) 2 decades of large-scale ***satellite*** observations of Greenland ice sheet geometry change:

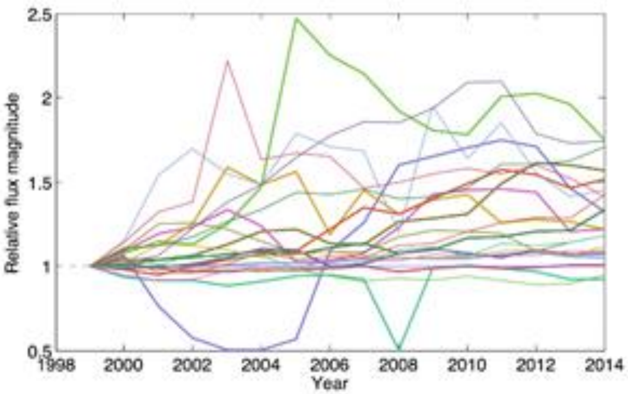
ICESat1	2003 – 2009
GRACE	2002 – 201? (ongoing)



Validation Workflow (with LANL & NASA):

- Run *CISM-Albany* for period where observations exist.
- Process model output and observations for comparison.
- Evaluate model performance relative to observations.
    - ***ICESat***: ice sheet surface elevation [state comparison]
    - ***GRACE***: rate of mass change [trend comparison]

- ***Validation time periods***: 2003-2009 (IceSAT), 2003-2011 (GRACE)
- ***Model forcing:*** monthly surface mass balance (SMB) anomalies from RACMO2 and/or outlet glacier flux-forcing (FF) at grounding line applied.



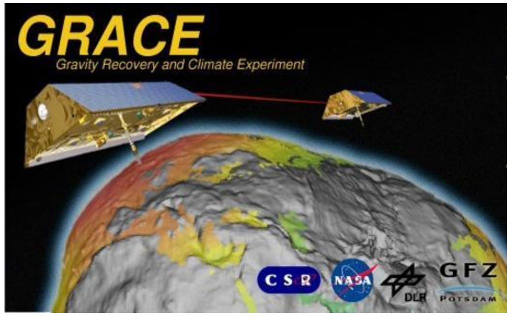
Validation: Definition & Workflow

Sandia National Laboratories

Validation: how well does model represent the real ice sheet?

- There are currently (up to) 2 decades of large-scale ***satellite*** observations of Greenland ice sheet geometry change:

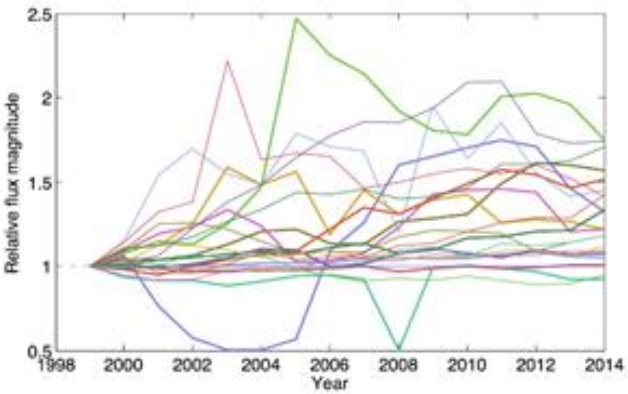
ICESat1	2003 – 2009
GRACE	2002 – 201? (ongoing)



Validation Workflow (with LANL & NASA):

- Run *CISM-Albany* for period where observations exist.
- Process model output and observations for comparison.
- Evaluate model performance relative to observations.
    - ***ICESat***: ice sheet surface elevation [state comparison]
    - ***GRACE***: rate of mass change [trend comparison]

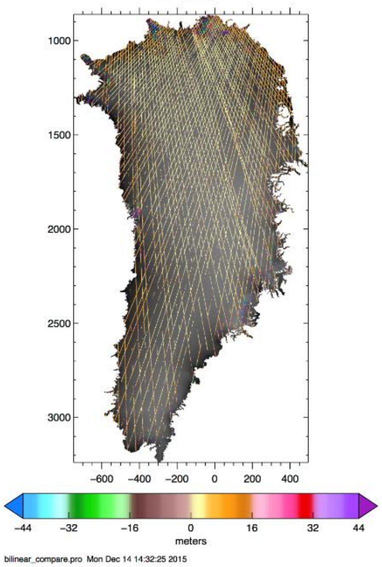
- ***Validation time periods***: 2003-2009 (IceSAT), 2003-2011 (GRACE)
- ***Model forcing:*** monthly surface mass balance (SMB) anomalies from RACMO2 and/or outlet glacier flux-forcing (FF) at grounding line applied.
- ***Initial condition*** (1km GIS) obtained through deterministic inversion (shown on earlier slide).



Validation Results

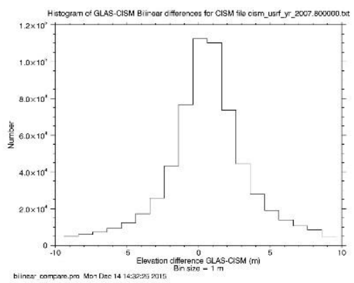
Surface Elevation Comparison [IceSAT]

Oct. 07 bilinear differences

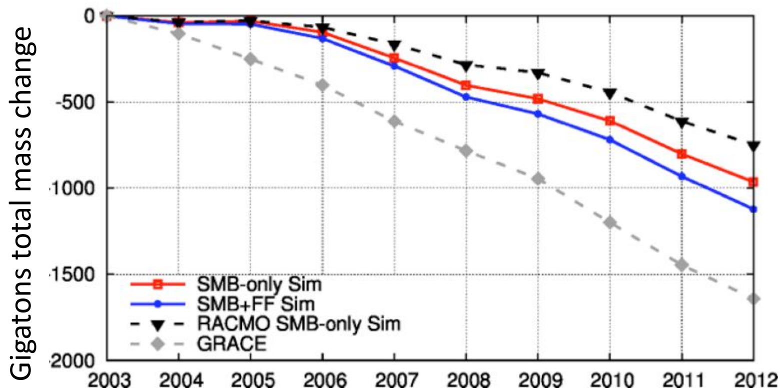


Surface elevation predictions (states) agree well with ***GLAS*** (***Geoscience Laser Altimeter System aboard ICESat***): ***mean differences*** are ***<1 m***

Histogram of Oct. 07 bilinear differences



Whole Ice Sheet Mass Trends [GRACE]

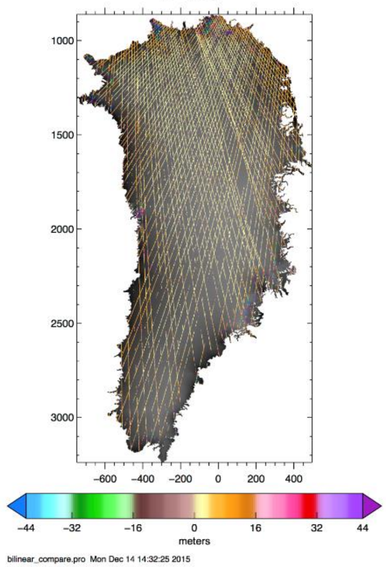


CISM-Albany results (blue, red lines) ***closer to observations*** (gray line) than older “idealized” simulations (black line).

Validation Results

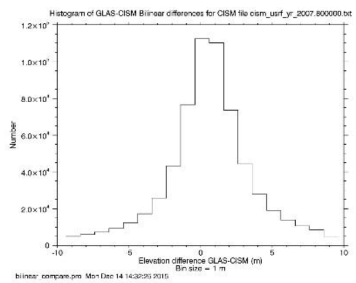
Surface Elevation Comparison [IceSAT]

Oct. 07 bilinear differences

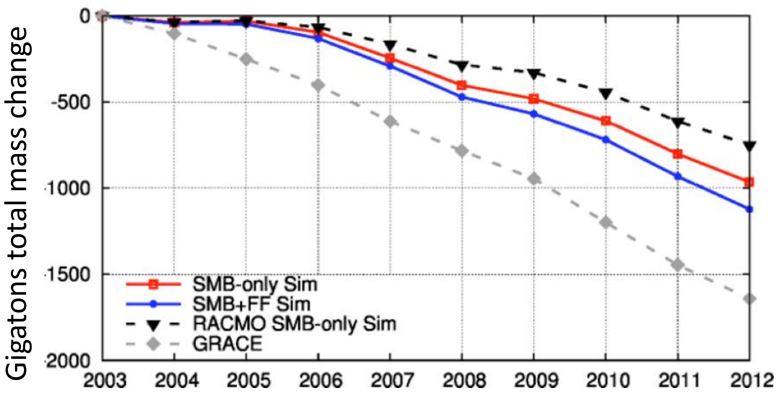


Surface elevation predictions (states) agree well with ***GLAS*** (***Geoscience Laser Altimeter System aboard ICESat***): ***mean differences*** are ***<1 m***

Histogram of Oct. 07 bilinear differences



Whole Ice Sheet Mass Trends [GRACE]



CISM-Albany results (blue, red lines) ***closer to observations*** (gray line) than older "idealized" simulations (black line).

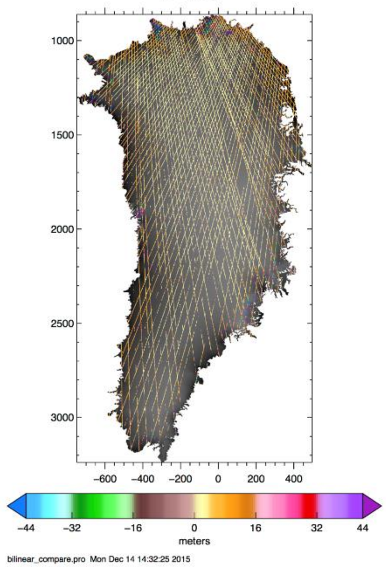
Main Takeaway from Validation Study

Current generation ice sheet models, ***<u>when appropriately forced</u>***, show ***skill*** at mimicking ice sheet observations

Validation Results

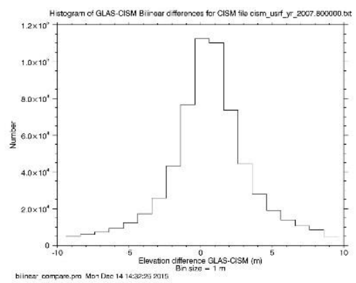
Surface Elevation Comparison [IceSAT]

Oct. 07 bilinear differences

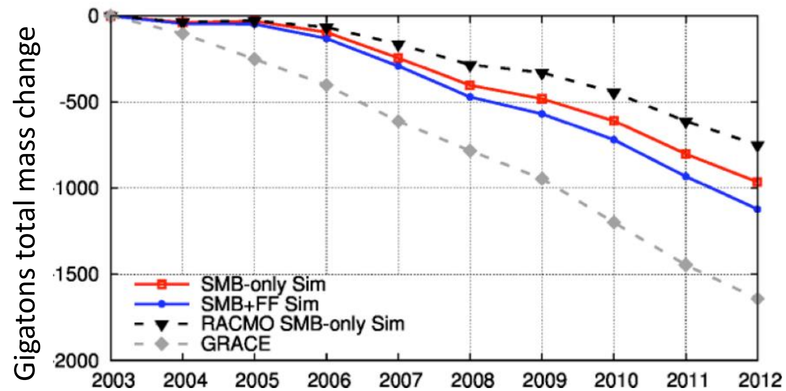


Surface elevation predictions (states) agree well with ***GLAS*** (***Geoscience Laser Altimeter System aboard ICESat***): ***mean differences*** are ***<1 m***

Histogram of Oct. 07 bilinear differences



Whole Ice Sheet Mass Trends [GRACE]



CISM-Albany results (blue, red lines) ***closer to observations*** (gray line) than older "idealized" simulations (black line).

Main Takeaway from Validation Study

Current generation ice sheet models, ***when appropriately forced***, show ***skill*** at mimicking ice sheet observations

- ***Clear improvement over a decade ago:*** SLR projections from ice sheet models were not included in the IPCC's AR4 b/c models could not explain observed ice dynamical behaviors.

Cool Movie of Validation Results

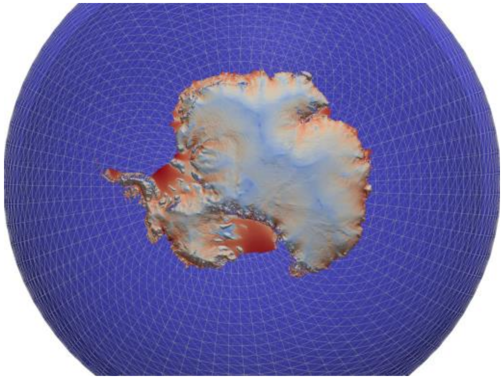
Video acknowledgement: B. Carvey (SNL)

Summary and Future Work

Summary:

- We have developed the ***Albany/FELIX*** land-ice solver, which is:
    - ***Scalable***, ***fast***, ***robust***.
    - Coupled to *CISM* and *MPAS* codes for ***dynamic*** runs and integration into ESMs.
    - ***Verified*** and ***validated***.
    - ***Portable*** to new and emerging architecture machines.
    - Equipped with ***advanced analysis capabilities*** (deterministic inversion, UQ).
    - A ***production code*** in *Albany*.

Albany/FELIX work not covered in this talk:



- ***Semi-implicit*** FO Stokes-thickness coupling methods.
- ***Temperature solver*** in *Albany/FELIX*.
- More sophisticated ***basal hydrology models***.
- FO Stokes model on ***spherical grids*** via stereographic projection.

Ongoing/future work:

- ***Science runs*** using *CISM-Albany* and *MPAS-Albany*.
- Code optimizations for ***new architecture machines*** (GPUs, Intel Xeon Phis).
- Improving ***UQ*** workflow / algorithms, towards paper.
- ***Proposal*** for follow-up funding (SciDaC4).
- Delivering code to climate community and ***coupling to ESMs***.

Funding/Acknowledgements

Support for this work was provided through Scientific Discovery through Advanced Computing (**SciDAC**) projects funded by the U.S. Department of Energy, Office of Science (**OSCR**), Advanced Scientific Computing Research and Biological and Environmental Research (**BER**) → **PISCEES SciDAC Application Partnership.**

PISCEES team members: *K. Evans, M. Gunzburger, M. Hoffman, C. Jackson, P. Jones, W. Lipscomb, M. Perego, S. Price, A. Salinger, I. Tezaur, R. Tuminaro, P. Worley.*

Trilinos/DAKOTA collaborators*: M. Eldred, J. Jakeman, E. Phipps, L. Swiler.*

Computing resources: *NERSC, OLCF.*

References

[1] M.A. Heroux *et al.* "An overview of the Trilinos project." *ACM Trans. Math. Softw.* **31**(3) (2005).

[2] A. Salinger, *et al.* "Albany: Using Agile Components to Develop a Flexible, Generic Multiphysics Analysis Code", *Int. J. Multiscale Comput. Engng*. 14(4) (2016) 415-438.

[3] **I. Tezaur**, M. Perego, A. Salinger, R. Tuminaro, S. Price. "*Albany/FELIX*: A Parallel, Scalable and Robust Finite Element Higher-Order Stokes Ice Sheet Solver Built for Advanced Analysis", *Geosci. Model Develop.* 8 (2015) 1-24.

[4] **I. Tezaur**, R. Tuminaro, M. Perego, A. Salinger, S. Price. "On the scalability of the *Albany/FELIX* first-order Stokes approximation ice sheet solver for large-scale simulations of the Greenland and Antarctic ice sheets", *MSESM/ICCS15*, Reykjavik, Iceland (June 2014).

[5] R.S. Tuminaro, **I. Tezaur**, M. Perego, A.G. Salinger. "A Hybrid Operator Dependent Multi-Grid/Algebraic Multi-Grid Approach: Application to Ice Sheet Modeling", *SIAM J. Sci. Comput.* 38(5) (2016) C504-C532.

[6] S. Price, M. Hoffman, J. Bonin, T. Neumann, I. Howat, J. Guerber, **I. Tezaur**, J. Saba, J. Lanaerts, D. Chambers, W. Lipscomb, M. Perego, A. Salinger, R. Tuminaro. "An ice sheet model validation framework for the Greenland ice sheet", *Geoscientific Model Development (GMD)* (in press).

References (cont'd)

[7] S. Shannon, *et al.* "Enhanced basal lubrication and the contribution of the Greenland ice sheet to future sea-level rise", *P. Natl. Acad. Sci.,* 110 (2013) 14156-14161.

[8] P. Fretwell, *et al.* "BEDMAP2: Improved ice bed, surface, and thickness datasets for Antarctica", *The Cryosphere* 7(1) (2013) 375-393.

[9] F. Pattyn. "Antarctic subglacial conditions inferred from a hybrid ice sheet/ice stream model", *Earth and Planetary Science Letters 295 (2010).*

[10] M. Perego, S. Price, G. Stadler. "Optimal Initial Conditions for Coupling Ice Sheet Models to Earth System Models", *J. Geophys. Res.* 119 (2014) 1894-1917.

[11] J.H. van Angelen, M.R. Van Den Broeke, B. Wouters, J.T.M. Lenaerts, "Contemporary (1960–2012) Evolution of the Climate and Surface Mass Balance of the Greenland Ice Sheet", *Surveys in Geophysics*, 35, 1155–1174, 2013.

[12] E.M. Enderlin, I.M. Howat, S. Jeong, M.J. Noh, J.H. Angelen, J. H., M.R. Broeke, "An improved mass budget for the Greenland ice sheet", *Geophysical Research Letters*, 41, 866–872, 2014.

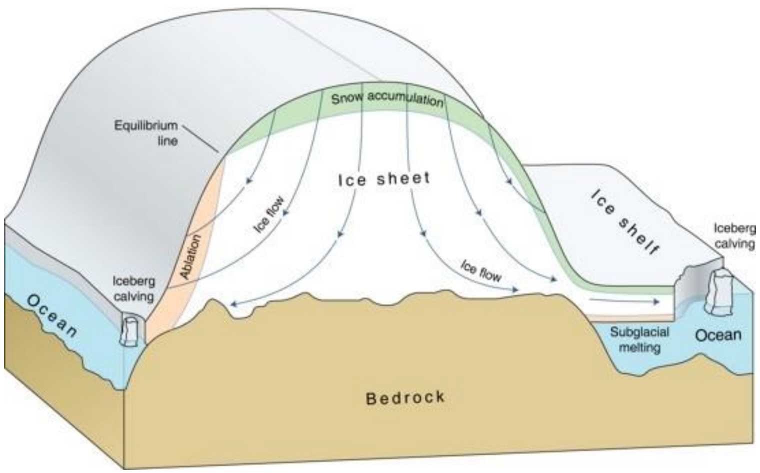
[13] I. Velicogna, J. Wahr, "Time-variable gravity observations of ice sheet mass balance: Precision and limitations of the GRACE satelliteData", Geophysical Research Letters, 2013.

Stokes Ice Flow Equations

Ice behaves like a very viscous shear-thinning fluid (similar to lava flow) and is modeled using nonlinear incompressible Stokes' equations.

- **Nonlinear incompressible Stokes' ice flow equations** (momentum balance):

$$\begin{cases} -\nabla \cdot \boldsymbol{\sigma} = \rho \boldsymbol{g} \\ -\nabla \cdot u = 0 \end{cases}, \quad \text{in } \Omega$$



with

$$\boldsymbol{\sigma} = 2\mu\,\dot{\boldsymbol{\epsilon}} - p\boldsymbol{I}, \qquad \epsilon_{\mathrm{ij}} = \frac{1}{2}\left(\frac{\partial ui}{\partial xj} + \frac{\partial uj}{\partial xi}\right)$$

and nonlinear ***"Glen's law"*** viscosity

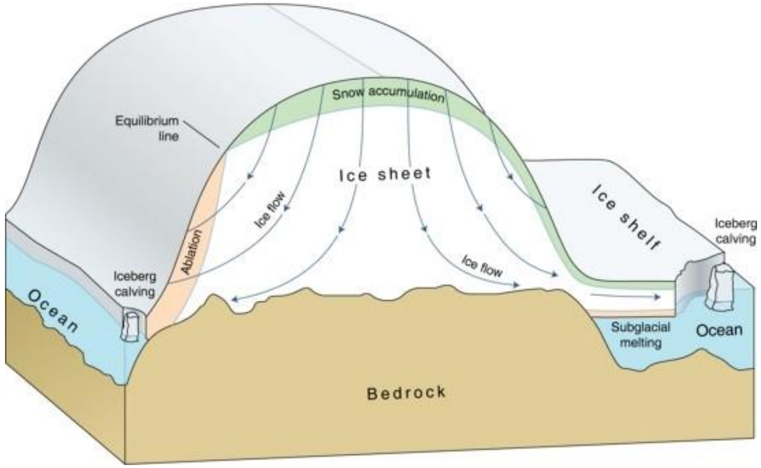
$$\mu = \frac{1}{2} A(T)^{-\frac{1}{n}} \left(\frac{1}{2}\sum_{ij} \boldsymbol{\epsilon}_{ij}{}^{2}\right)^{\left(\frac{1}{2n} - \frac{1}{2}\right)}, \quad n = 3.$$

Stokes Ice Flow Equations

Ice behaves like a very viscous shear-thinning fluid (similar to lava flow) and is modeled using nonlinear incompressible Stokes' equations.

- **Nonlinear incompressible Stokes' ice flow equations** (momentum balance):

$$\begin{cases} -\nabla \cdot \boldsymbol{\sigma} = \rho \boldsymbol{g} \\ -\nabla \cdot u = 0 \end{cases}, \quad \text{in } \Omega$$



with

$$\boldsymbol{\sigma} = 2\mu\, \dot{\boldsymbol{\epsilon}} - p\boldsymbol{I}, \qquad \epsilon_{\mathrm{ij}} = \frac{1}{2}\left(\frac{\partial ui}{\partial xj} + \frac{\partial uj}{\partial xi}\right)$$

and nonlinear ***"Glen's law"*** viscosity

$$\mu = \frac{1}{2} A(T)^{-\frac{1}{n}} \left(\frac{1}{2} \sum_{ij} \boldsymbol{\epsilon}_{ij}{}^{2}\right)^{\left(\frac{1}{2n} - \frac{1}{2}\right)}, \quad n = 3.$$

$\boxed{\rightarrow \text{“nasty” saddle point problem}}$