Stabilization of Projection-Based Reduced Order Models via Optimization-Based Eigenvalue Reassignment

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*Formerly **I. Kalashnikova**.

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Motivation

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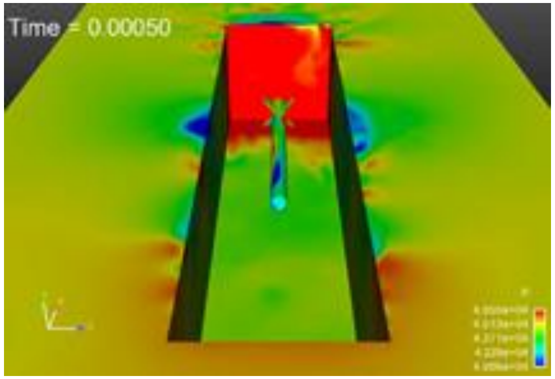

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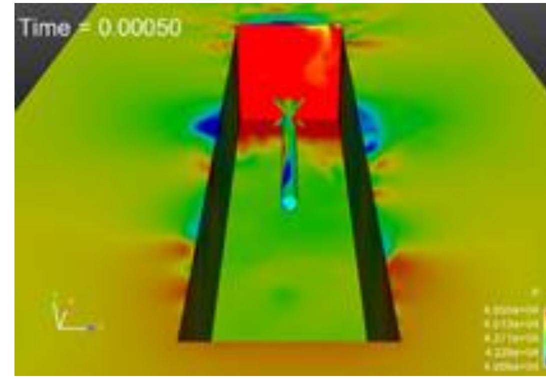

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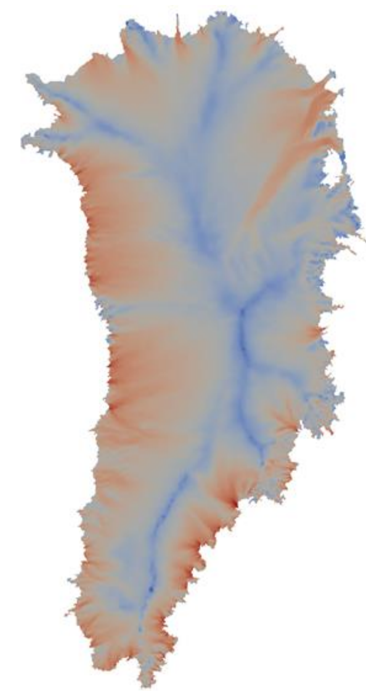
- <u>Climate modeling, e.g., ice flow simulations for sea-level rise predictions:</u> Bayesian inference tools cannot handle high-D parameter spaces, MCMC requires thousands of forward solves.



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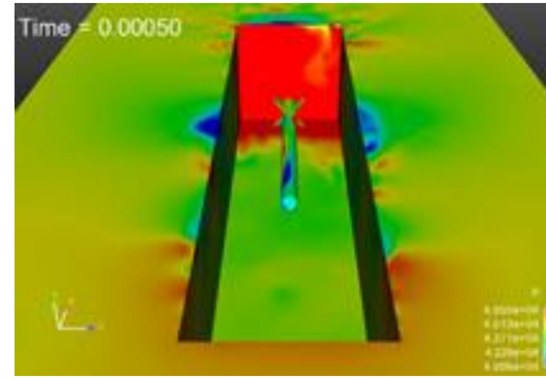

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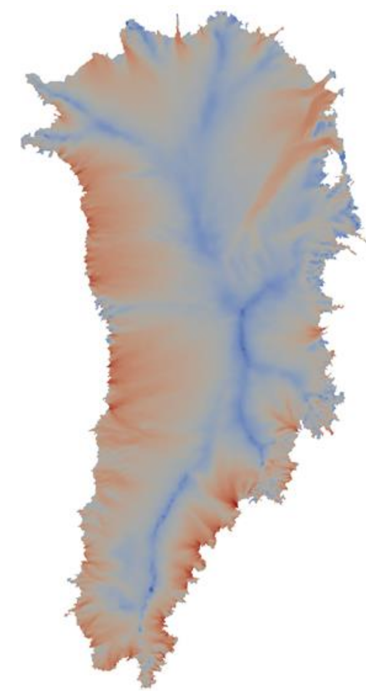

This talk presents a recent paper on ROM stabilization:

I. Kalashnikova, B.G. van Bloemen Waanders, S. Arunajatesan, M.F. Barone. "Stabilization of Projection-Based Reduced Order Models for Linear Time-Invariant Systems via Optimization-Based Eigenvalue Reassignment". *Comput. Meth. Appl. Mech. Engng.* **272** (2014) 251-270.

Proper Orthogonal Decomposition (POD)/ Galerkin Method to Model Reduction

High-Fidelity Simulations:

Snapshot 1

Snapshot 2

⋮

Snapshot K

Step 1

Modal Decomposition (POD):

$$\mathbf{x}(t) \approx \mathbf{\Phi}_M \mathbf{x}_M(t)$$

Step 2

Galerkin Projection of LTI FOM:

$$\mathbf{\Phi}_M^T \left[\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)\right]$$

- ***Snapshot matrix***: $\boldsymbol{X} = (\boldsymbol{x}^1, \ldots, \boldsymbol{x}^K) \in \mathbb{R}^{NxK}$
- ***SVD:*** $\boldsymbol{X} = \boldsymbol{U\Sigma V}^T$
- ***Truncation:*** $\boldsymbol{\Phi}_M = (\boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_M) = \boldsymbol{U}(:, 1\!:\!M)$

N = # of dofs in high-fidelity simulation
K = # of snapshots
M = # of dofs in ROM
$(M << N, M << K)$

"Small" ROM LTI System:

$$\begin{aligned} \dot{\mathbf{x}}_M(t) &= \mathbf{\Phi}_M^T \mathbf{A} \mathbf{\Phi}_M \mathbf{x}_M(t) + \mathbf{\Phi}_M^T \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}_M(t) &= \mathbf{C}\mathbf{\Phi}_M \mathbf{x}_M(t) \end{aligned}$$

ROM = "Reduced Order Model"
FOM = "Full Order Model"
LTI = "Linear Time Invariant"

Stability Issues of POD/Galerkin ROMs

LTI Full Order Model (FOM)	**LTI Reduced Order Model (ROM)**
$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t)$ $\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t)$	$\dot{\boldsymbol{x}}_M(t) = \boldsymbol{A}_M\boldsymbol{x}_M(t) + \boldsymbol{B}_M\boldsymbol{u}(t)$ $\boldsymbol{y}_M(t) = \boldsymbol{C}_M\boldsymbol{x}_M(t)$

- ROM Linear Time-Invariant (LTI) system matrices given by:

$$\boldsymbol{A}_M = \boldsymbol{\Phi}_M{}^T\boldsymbol{A}\boldsymbol{\Phi}_M, \quad \boldsymbol{B}_M = \boldsymbol{\Phi}_M{}^T\boldsymbol{B}, \quad \boldsymbol{C}_M = \boldsymbol{C}\boldsymbol{\Phi}_M$$

Problem: $\boldsymbol{A}$ stable $\nRightarrow$ $\boldsymbol{A}_M$ stable!

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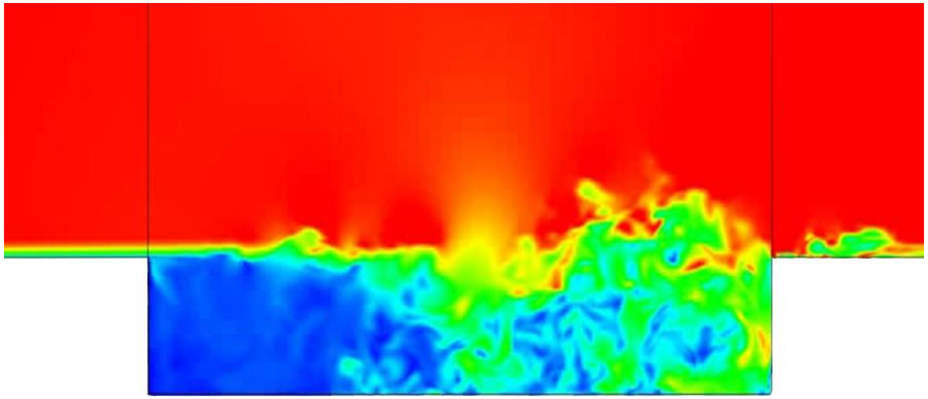
- There is no *a priori* stability guarantee for POD/Galerkin ROMs.
- Stability of a ROM is commonly evaluated *a posteriori* – **RISKY!**
- Instability of POD/Galerkin ROMs is a **real** problem in some applications…



…e.g., compressible cavity flows, high-Reynolds number flows, ...

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Stability Preserving ROM Approaches: Literature Review

Approaches for building stability-preserving POD/Galerkin ROMs found in the literature fall into **two categories:**

1. ROMs which derive ***a priori*** a stability-preserving model reduction framework (usually specific to an equation set).

- ROMs based on projection in special 'energy-based' (not L^2) inner products, e.g., Rowley *et al*. (2004), Barone & Kalashnikova *et al.* (2009), Serre *et al*. (2012).

2. ROMs which stabilize an unstable ROM through an ***a posteriori*** post-processing stabilization step applied to the algebraic ROM system.

- ROMs that require solving an optimization problem for a modified POD basis, e.g., Bond *et al.* (2008), Amsallem *et al.* (2012), Balajewicz *et al.* (2013).
- ROMs with increased numerical stability due to inclusion of 'stabilizing' terms in the ROM equations, e.g., Wang, Borggaard, Iliescu *et al.* (2012).

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Can have inconsistencies between ROM and FOM physics

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ROM Stabilization via Optimization-Based Eigenvalue Reassignment*

- Approach falls in 2nd category of stabilization methods, but ensures stabilized ROM solution deviates minimally from FOM solution.

***I. Kalashnikova**, B.G. van Bloemen Waanders, S. Arunajatesan, M.F. Barone. "Stabilization of Projection-Based Reduced Order Models for Linear Time-Invariant Systems via Optimization-Based Eigenvalue Reassignment". *Comput. Meth. Appl. Mech. Engng.* **272** (2014) 251-270.

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Problem: $\boldsymbol{A}$ stable $\nRightarrow$ $\boldsymbol{A}_M$ stable!

Goal: modify ROM system s.t. $\boldsymbol{A}_M$ is stable and discrepancy b/w ROM output $\boldsymbol{y}_M(t)$ and FOM output $\boldsymbol{y}(t)$ is minimal.

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Problem: $\boldsymbol{A}$ stable $\nRightarrow$ $\boldsymbol{A}_M$ stable!

Goal: replace unstable $\boldsymbol{A}_M$ with stable $\widetilde{\boldsymbol{A}}_M$ so discrepancy b/w ROM output $\boldsymbol{y}_M(t)$ and FOM output $\boldsymbol{y}(t)$ is minimal.

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- ***Constraints:*** $\boldsymbol{y}_M$ satisfies (1), $\widetilde{\boldsymbol{A}}_M$ stable in Lyapunov sense

$$\Longrightarrow Re\{\lambda(\widetilde{\boldsymbol{A}}_M)\} < 0$$

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ROM Stabilization via Optimization-Based Eigenvalue Reassignment (continued)

ROM Stabilization Optimization Problem
(Constrained Nonlinear Least Squares):

$$\min_{\lambda_i^u} \sum_{k=1}^{K} ||\boldsymbol{y}^k - \boldsymbol{y_M}^k||_2^2 \qquad (2)$$
$$s.t.\ Re(\lambda_i^u) < 0$$

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- λ_i^u = unstable eigenvalues of original ROM matrix $\boldsymbol{A}_M$.
- $\boldsymbol{y}^k = \boldsymbol{y}(t_k)$ = snapshot output at t_k.
- $\boldsymbol{y_M}^k = \boldsymbol{y}_M(t_k)$ = ROM output at t_k.

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- ***<u>For general (nonlinear) systems:</u>*** (2) would have ODE constraints.
- ***<u>For LTI systems:</u>*** the solution to (1) for the ROM output at t_k can be derived analytically!

$$\boldsymbol{x}_M(t) = \exp(t\boldsymbol{A}_M)\,\boldsymbol{x}_M(0) + \int_0^t \exp\{(t-\tau)\,\boldsymbol{A}_M\}\boldsymbol{B}_M u(\tau)d\tau$$

$$\Rightarrow \boldsymbol{y}_M(t) = \boldsymbol{C}_M\left[\exp(t\boldsymbol{A}_M)\,\boldsymbol{x}_M(0) + \int_0^t \exp\{(t-\tau)\,\boldsymbol{A}_M\}\boldsymbol{B}_M u(\tau)d\tau\right]$$

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- ROM stabilization optimization problem is small: $< O(M)$.
- ROM stabilization optimization problem can be solved by standard optimization algorithms, e.g., interior point method.
    - We use `fmincon` function in MATLAB's optimization toolbox.
    - We implement ROM stabilization optimization problem in ***characteristic variables*** $\boldsymbol{z}_M(t) = \boldsymbol{S}_M{}^{-1} \boldsymbol{x}_M(t)$ where $\boldsymbol{A}_M = \boldsymbol{S}_M \boldsymbol{D}_M \boldsymbol{S}_M{}^{-1}$.

ROM Stabilization via Optimization-Based Eigenvalue Reassignment (continued)

Algorithm

- Diagonalize the ROM matrix $\boldsymbol{A}_M$: $\boldsymbol{A}_M = \boldsymbol{S}_M \boldsymbol{D}_M \boldsymbol{S}_M{}^{-1}$.
- Initialize a diagonal $M \times M$ matrix $\widetilde{\boldsymbol{D}}_M$. Set $j = 1$.
- **for** $i = 1$ to M
  - **if** $Re(D_M(i,i) < 0)$, set $\widetilde{D}_M(i,i) = D_M(i,i)$.
  - **else**, set $\widetilde{D}_M(i,i) = \lambda_j{}^u$.
- Increment $j \leftarrow j + 1$.
- Solve the optimization problem (2) for the eigenvalues $\{\lambda_j{}^u\}$ using an optimization algorithm (e.g., interior point method).
- Evaluate $\widetilde{\boldsymbol{D}}_M$ at the solution of the optimization problem (1).
- Return the stabilized ROM system, given by $\boldsymbol{A}_M \leftarrow \widetilde{\boldsymbol{A}}_M = \boldsymbol{S}_M \widetilde{\boldsymbol{D}}_M \boldsymbol{S}_M{}^{-1}$.

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- Return the stabilized ROM system, given by $\boldsymbol{A}_M \leftarrow \widetilde{\boldsymbol{A}}_M = \boldsymbol{S}_M \widetilde{\boldsymbol{D}}_M \boldsymbol{S}_M{}^{-1}$.

- Existence of solution to (2) cannot be proven in general. Regularization may help.

ROM Stabilization via Optimization-Based Eigenvalue Reassignment (continued)

Algorithm

- Diagonalize the ROM matrix $\boldsymbol{A}_M$: $\boldsymbol{A}_M = \boldsymbol{S}_M \boldsymbol{D}_M \boldsymbol{S}_M^{-1}$.
- Initialize a diagonal $M \times M$ matrix $\widetilde{\boldsymbol{D}}_M$. Set $j = 1$.
- **for** $i = 1$ to M
  - **if** $Re(D_M(i,i) < 0)$, set $\widetilde{D}_M(i,i) = D_M(i,i)$.
  - **else**, set $\widetilde{D}_M(i,i) = \lambda_j^u$.
- Increment $j \leftarrow j + 1$.
- Solve the optimization problem (2) for the eigenvalues $\{\lambda_j^u\}$ using an optimization algorithm (e.g., interior point method).
- Evaluate $\widetilde{\boldsymbol{D}}_M$ at the solution of the optimization problem (1).
- Return the stabilized ROM system, given by $\boldsymbol{A}_M \leftarrow \widetilde{\boldsymbol{A}}_M = \boldsymbol{S}_M \widetilde{\boldsymbol{D}}_M \boldsymbol{S}_M^{-1}$.

- Existence of solution to (2) cannot be proven in general. Regularization may help.
- Solution to optimization problem (2) may not be unique.
- Can solve (2) for real or complex-conjugate pair eigenvalues:
  - $\lambda_j^u \in \mathbb{R}$ s.t. constraint $\lambda_j^u < 0$.
  - $\lambda_j^u = \lambda_j^{ur} + i\,\lambda_j^{uc}$, $\lambda_{j+1}^u = \lambda_j^{ur} - i\,\lambda_j^{uc} \in \mathbb{C}$ where $\lambda_j^{ur}, \lambda_j^{uc} \in \mathbb{R}$ s.t. constraint $\lambda_j^{ur} < 0$.

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Consistency?

- One can show that $\widetilde{\boldsymbol{A}}_M$ from the algorithm on the previous slide is given by:

$$\widetilde{\boldsymbol{A}}_M = \boldsymbol{A}_M - \boldsymbol{B}_C \boldsymbol{K}_C$$

for a specific $\boldsymbol{B}_C$ and $\boldsymbol{K}_C$ (Kalashnikova *et al.* 2014).

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- Modifying system as $\boldsymbol{A}_M \leftarrow \widetilde{\boldsymbol{A}}_M$ can be viewed as adding a linear "controller" to the system:

$$\begin{aligned} \dot{\boldsymbol{x}}_M(t) &= \boldsymbol{A}_M \boldsymbol{x}_M(t) + \boldsymbol{B}_M \boldsymbol{u}(t) + \boldsymbol{B}_C \boldsymbol{u}_C(t) \\ \boldsymbol{y}_M(t) &= \boldsymbol{C}_M \boldsymbol{x}_M(t) \end{aligned}$$

where $\boldsymbol{u}_C(t) = -\boldsymbol{K}_C \boldsymbol{x}_M(t)$.

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- Approach does yield an inconsistent ROM, but one that can nonetheless be accurate.

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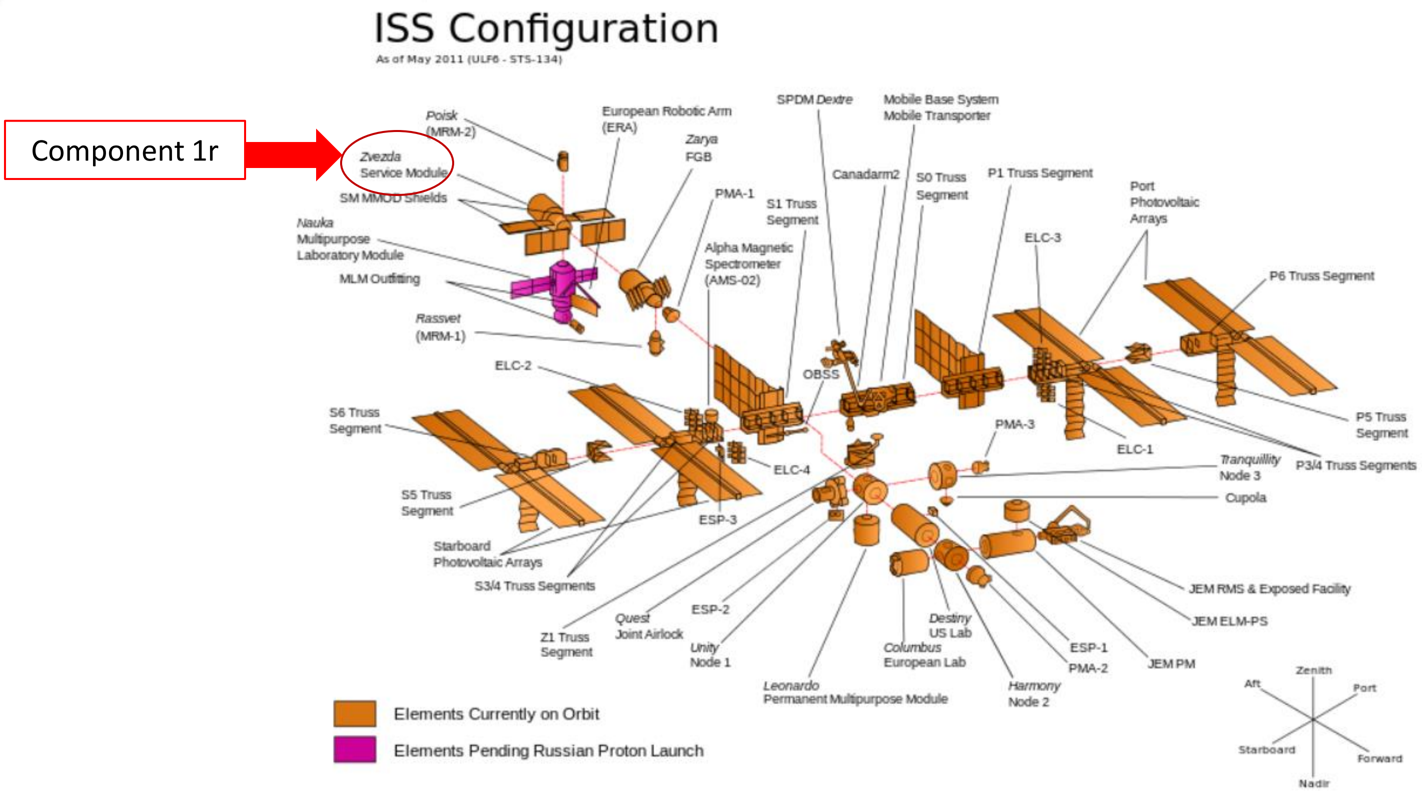
- Approach does yield an inconsistent ROM, but one that can nonetheless be accurate.

Ongoing work is to formulate ROM stabilization approaches that maintain consistency (last slide)

Numerical Results #1: International Space Station (ISS) Benchmark



- FOM: structural model of component 1r of the International Space Station (ISS).
- $\boldsymbol{A}, \boldsymbol{C}$ matrices defining FOM downloaded from NICONET ROM benchmark repository*.
- No inputs (unforced), 1 output; FOM is stable.

*NICONET ROM benchmark repository: www.icm.tu-bs.de/NICONET/benchmodred.html.

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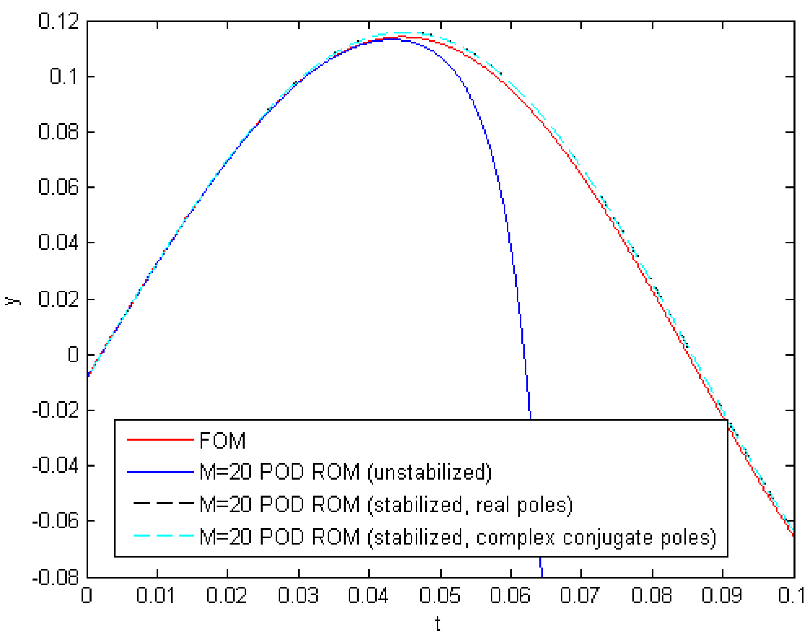
Numerical Results #1 : ISS Benchmark (continued)

- $M = 20$ POD/Galerkin ROM constructed from $K = 2000$ snapshots up to time $t = 0.1$.
- $M = 20$ POD/Galerkin ROM has 4 unstable eigenvalues: 2 real, 2 complex
  - Two options for ROM stabilization optimization problem:

    Option 1: Solve for $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R}$ s.t. the constraint $\lambda_1, \lambda_2, \lambda_3, \lambda_4 < 0$.

    Option 2: Solve for $\lambda_1 + \lambda_2 i, \lambda_1 - \lambda_2 i \in \mathbb{C}, \lambda_3, \lambda_4 \in \mathbb{R}$ s.t. the constraint $\lambda_1, \lambda_3, \lambda_4 < 0$.
- Initial guess for $\mathtt{fmincon}$ interior point method: $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = -1$.



ROM	$\frac{\sqrt{\sum_{k=1}^{K} \|\|\boldsymbol{y}^k - \boldsymbol{y}_M{}^k\|\|_2{}^2}}{\sqrt{\sum_{k=1}^{K} \|\|\boldsymbol{y}_k\|\|_2{}^2}}$
Unstabilized POD	1737.8
Optimization Stabilized POD (Real Poles)	0.0259
Optimization Stabilized POD (Complex-Conjugate Poles)	0.0252

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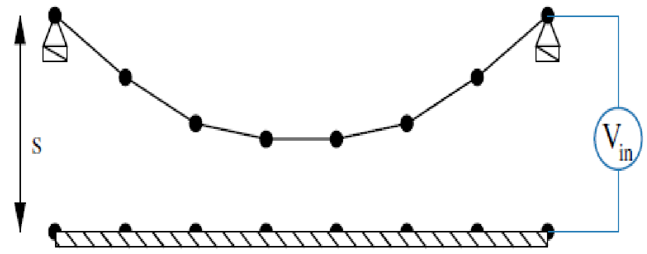
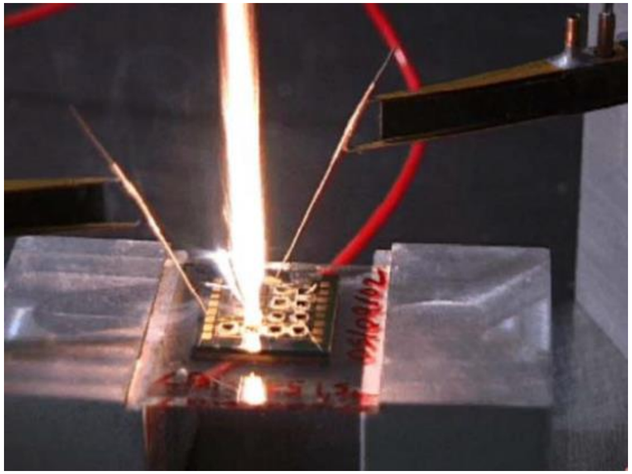
Numerical Results #2: Electrostatically Actuated Beam Benchmark

- FOM = 1D model of electrostatically actuated beam.
- Application of model: microelectromechanical systems (MEMS) devices such as electromechanical radio frequency (RF) filters.
- 1 input corresponding to periodic on/off switching, 1 output, initial condition $\boldsymbol{x}(0) = \boldsymbol{0}_N$.
- Second order linear semi-discrete system of the form:

$$\boldsymbol{M}\ddot{\boldsymbol{x}}(t) + \boldsymbol{E}\dot{\boldsymbol{x}}(t) + \boldsymbol{K}\boldsymbol{x}(t) = \boldsymbol{B}\boldsymbol{u}(t)$$
$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t)$$

- Matrices $\boldsymbol{M}, \boldsymbol{E}, \boldsymbol{K}, \boldsymbol{B}, \boldsymbol{C}$ specifying the problem downloaded from the Oberwolfach ROM repository*.
- 2nd order linear system re-written as 1st order LTI system for purpose of analysis/model reduction.





- FOM is stable.

* Oberwolfach ROM benchmark repository: http://simulation.uni-freiburg.de/downloads/benchmark.

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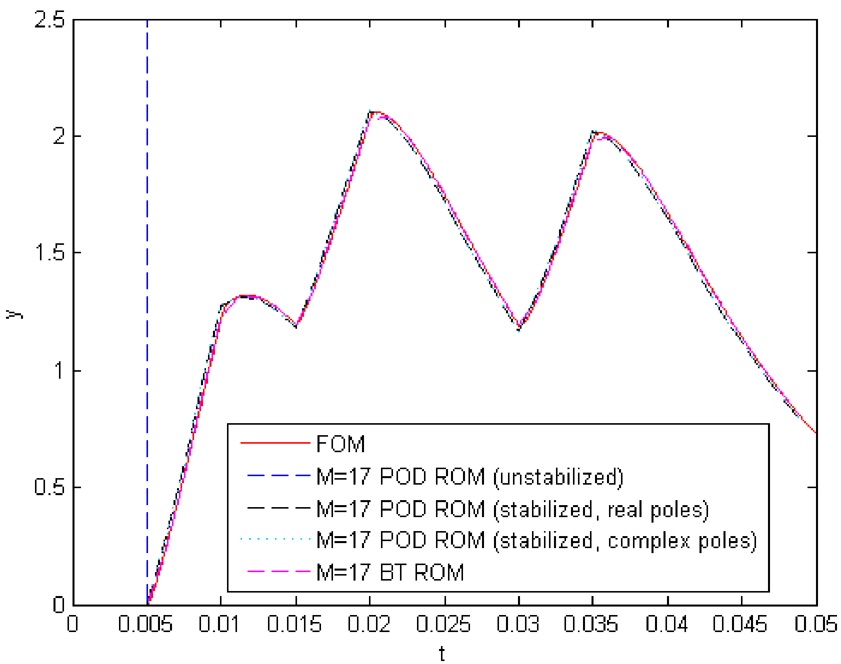
Numerical Results #2: Electrostatically Actuated Beam Benchmark (continued)

- $M = 17$ POD/Galerkin ROM constructed from $K = 1000$ snapshots up to time $t = 0.05$.
- $M = 17$ POD/Galerkin ROM has 4 unstable eigenvalues (all real).
  - Two options for ROM stabilization optimization problem:

    Option 1: Solve for $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R}$ s.t. the constraint $\lambda_1, \lambda_2, \lambda_3, \lambda_4 < 0$.

    Option 2: Solve for $\lambda_1 + \lambda_2 i, \lambda_1 - \lambda_2 i, \lambda_3 + \lambda_4 i, \lambda_3 - \lambda_4 i \in \mathbb{C}$ s.t. the constraint $\lambda_1, \lambda_3 < 0$.
- Initial guess for `fmincon` interior point method: $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = -1$.



ROM	$\frac{\sqrt{\sum_{k=1}^{K} \|\|\boldsymbol{y}^k - \boldsymbol{y}_M{}^k\|\|_2{}^2}}{\sqrt{\sum_{k=1}^{K} \|\|\boldsymbol{y}_k\|\|_2{}^2}}$
Unstabilized POD	NaN
Optimization Stabilized POD (Real Poles)	0.0194
Optimization Stabilized POD (Complex-Conjugate Poles)	0.0205
Balanced Truncation	$1.37e-6$

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Ongoing Work: ROM Stabilization for Nonlinear Problems (with M. Balajewicz)

Stabilization & enhancement of projection-based ROMs via minimal subspace rotation on the Stiefel manifold

- Development of ROM stabilization approach for nonlinear systems of the form:

$$\dot{\boldsymbol{a}}(t) = \boldsymbol{C} + \boldsymbol{L}\boldsymbol{a}(t) + \left[\boldsymbol{a}(t)^T \boldsymbol{Q}^{(1)} \boldsymbol{a}(t) \ldots \boldsymbol{a}(t)^T \boldsymbol{Q}^{(n)} \boldsymbol{a}(t)\right]^T$$

(e.g., ς-form of compressible Navier-Stokes equations).

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$$\min_{\boldsymbol{X}} f(\boldsymbol{X}) \quad s.t.\ g(\boldsymbol{X},\boldsymbol{L})$$

$f(\boldsymbol{X})$ = goal-oriented objective, e.g., $||\boldsymbol{X} - \boldsymbol{I}_{n+p,n}||_F$

$g(\boldsymbol{X},\boldsymbol{L})$ = constraints, e.g., $\eta_1 < tr(\boldsymbol{L}) < \eta_2$, $||\boldsymbol{a}(t) - \boldsymbol{a}^*(t)|| < \eta$

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- ***Paper under review:*** M. Balajewicz, I. Tezaur, E. Dowell, "Minimal subspace rotation on the Stiefel manifold for stabilization and enhancement of projection-based ROMs for the compressible Navier-Stokes equations", submitted to *CMAME.*
- ***Upcoming talk at ICIAM 2015***: August 2015, Beijing, China.

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Summary & Acknowledgements

(www.sandia.gov/~ikalash)

- A ROM stabilization approach that modifies *a posteriori* an unstable ROM LTI system by changing the system's unstable eigenvalues is proposed.
- In the proposed stabilization algorithm, a constrained nonlinear least squares optimization problem for the ROM eigenvalues is formulated to minimize error in ROM output.
- Excellent performance of the proposed algorithm is evaluated on two benchmarks.
- Stay tuned for extensions to nonlinear problems!

I. Kalashnikova, B.G. van Bloemen Waanders, S. Arunajatesan, M.F. Barone. "Stabilization of Projection-Based Reduced Order Models for Linear Time-Invariant Systems via Optimization-Based Eigenvalue Reassignment". *Comput. Meth. Appl. Mech. Engng.* **272** (2014) 251-270.

- This work was funded by Laboratories' Directed Research and Development (LDRD) Program at Sandia National Laboratories.
- Special thanks to
  - Prof. Lou Cattafesta (Florida State University)
  - Prof. Karen Willcox (MIT)

for useful discussions that led to some of the ideas presented here.

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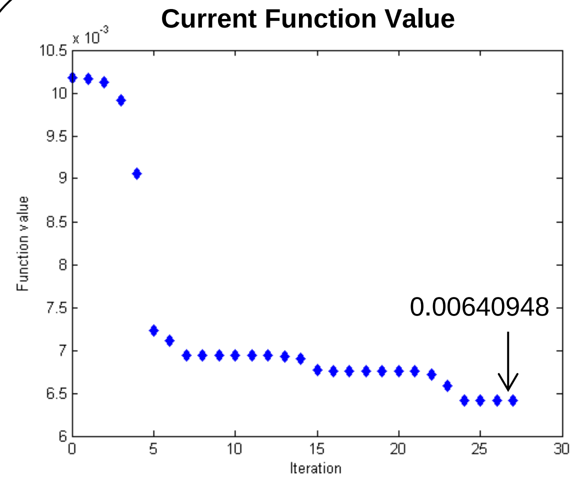
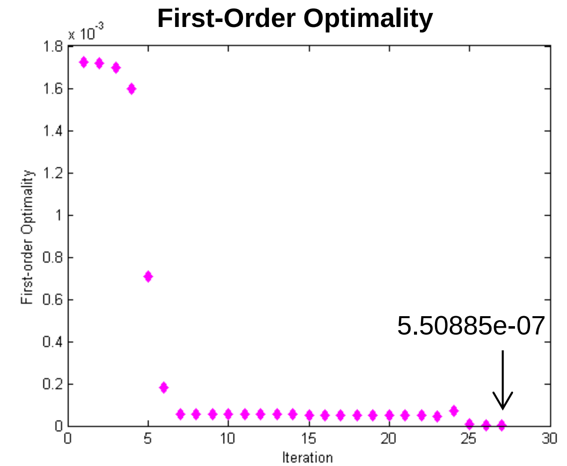
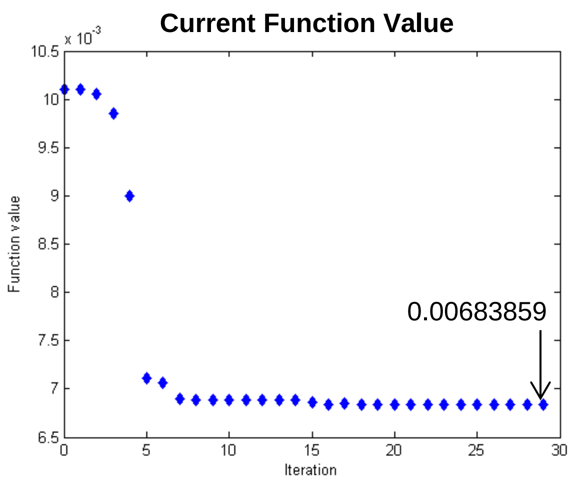
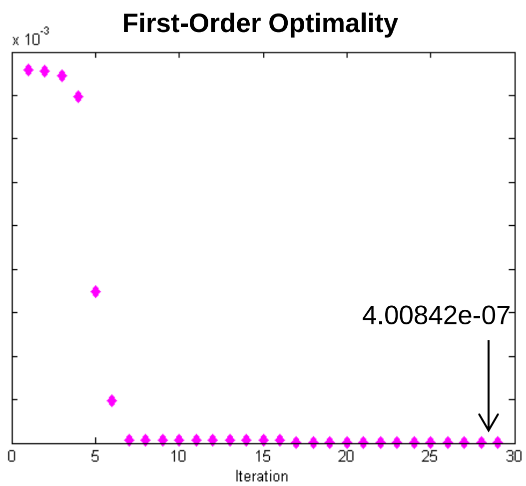
Appendix: ISS Benchmark (`fmincon` performance)

	Real Poles	Complex-Conjugate Poles
# upper bound constraints	4	3
# iterations	29	27
# function evaluations	30	30
$\lvert \nabla L \rvert$ at convergence (1st order optimality)	4.00e-7	5.51e-7









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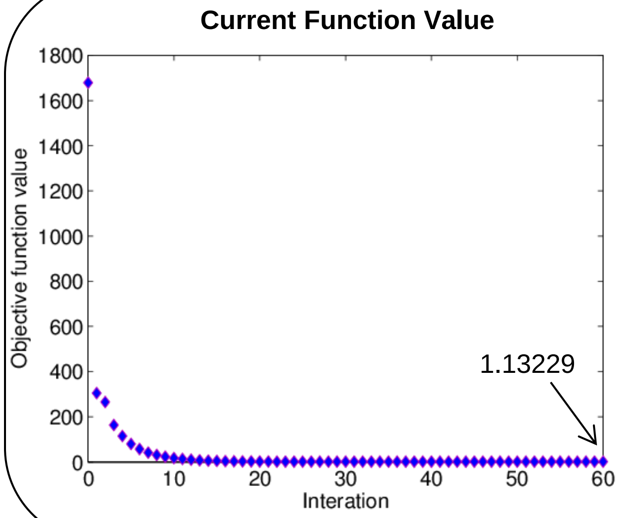
Appendix: ISS Benchmark (CPU Times)

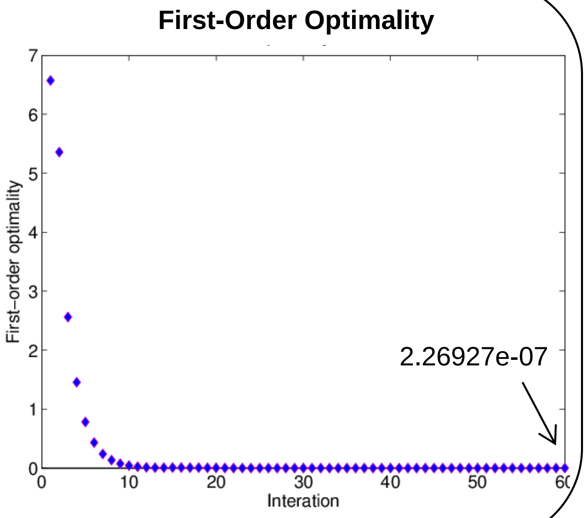
Model	Operations	CPU time (sec)
FOM	Time-Integration	1.71e2
ROM – offline stage	Snapshot collection (FOM time-integration)	1.71e2
	Loading of matrices/snapshots	6.99e-2
	POD	6.20
	Projection	8.18e-3
	Optimization	2.28e1
ROM – online stage	Time-integration	3.77

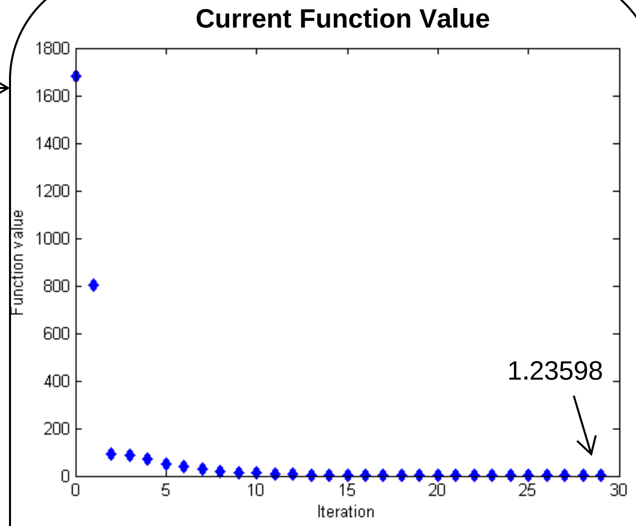
- To offset total pre-process time of ROM (time required to run FOM to collect snapshots, calculate the POD basis, perform the Galerkin projection, and solve the optimization problem (1)), the ROM would need to be run 53 times.
- Solution of optimization problem is very fast: takes < 1 minute to complete.

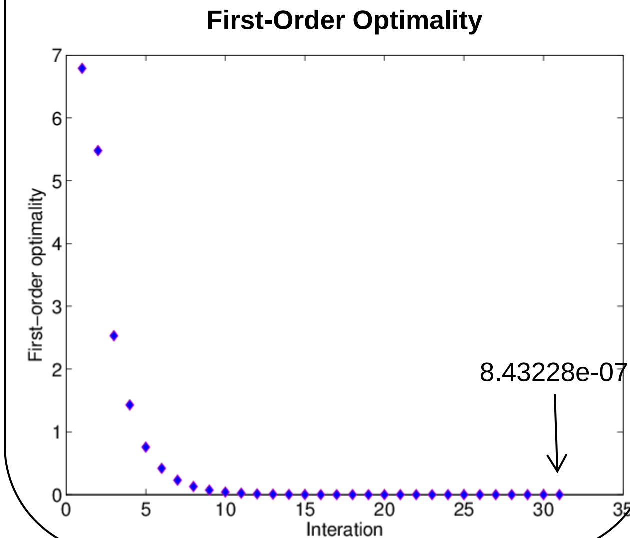
Appendix: Electrostatically Actuated Beam Benchmark (`fmincon` performance)

	Real Poles	Complex-Conjugate Poles
# upper bound constraints	4	2
# iterations	60	31
# function evaluations	64	32
$\lvert\nabla L\rvert$ at convergence (1st order optimality)	2.27e-7	8.43e-7







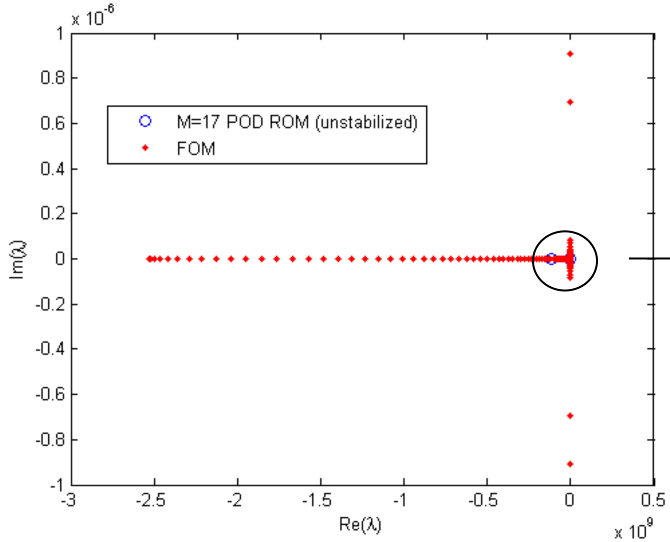


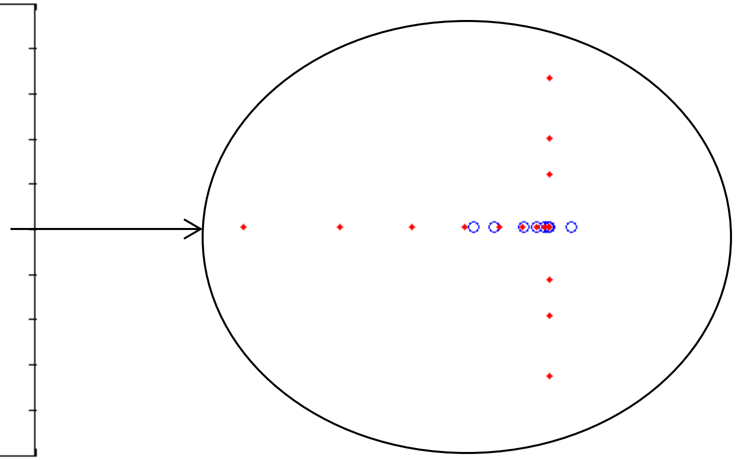
Appendix: Electrostatically Actuated Beam Benchmark (CPU Times)

Model	Operations	CPU time (sec)
FOM	Time-Integration	7.10e4
ROM – offline stage	Snapshot collection (FOM time-integration)	7.10e4
	Loading of matrices/snapshots	5.17
	POD	1.09e1
	Projection	2.55e1
	Optimization	8.79e1
ROM – online stage	Time-integration	6.78

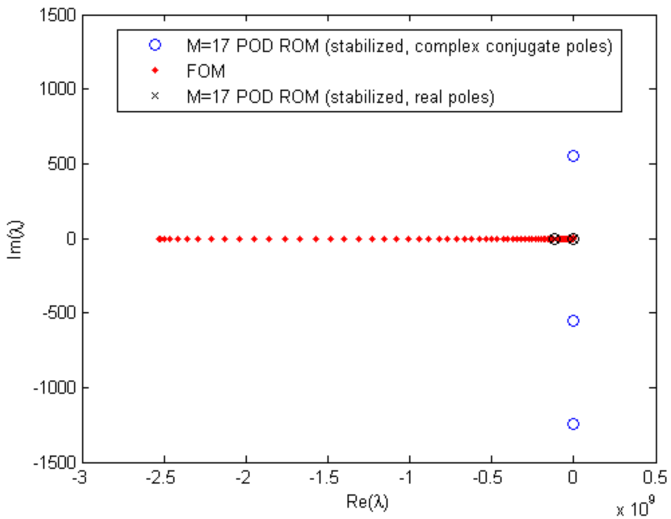
- To offset total pre-process time of ROM (time required to run FOM to collect snapshots, calculate the POD basis, perform the Galerkin projection, and solve the optimization problem (1)), the ROM would need to be run 1e4 times (due to large CPU time of FOM).
- Solution of optimization problem is very fast: takes ~1.5 minute to complete.

Appendix: Electrostatically Actuated Beam Benchmark (Eigenvalues)





Unstable Eigenvalues
$\lambda_6 = 16{,}053$
$\lambda_{12} = 48.985$
$\lambda_{14} = 12.650$
$\lambda_{17} = 0.05202$



Stabilized Eigenvalues (Real)	Stabilized Eigenvalues (Complex Conjugates)
$\lambda_6 = -7{,}043{,}505$	$\lambda_6 = -106{,}976 + 551.77i$
$\lambda_{12} = -35.364$	$\lambda_{12} = -106{,}976 - 551.77$
$\lambda_{14} = -153{,}033$	$\lambda_{14} = -2954.1 - 1244.7i$
$\lambda_{17} = -99{,}175$	$\lambda_{17} = -2954.1 + 1244.7i$

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