A Stable Galerkin Reduced Order Model (ROM) for Compressible Flow

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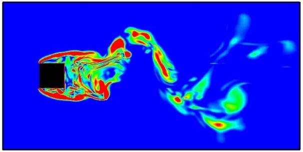

Applications in Fluid Dynamics:

- Predictive modeling across a parameter space (e.g., aeroelastic flutter analysis).
- System modeling for active flow control.
- Long-time unsteady flow analysis, e.g., fatigue of a wind turbine blade under variable wind conditions.

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Motivation for Numerical Analysis of ROMs

Use of ROMs in predictive applications raises questions about their stability & convergence.

- Projection ROM approach is an alternative discretization of the governing PDEs.
- Desired numerical properties of a ROM discretization:
  - **Consistency** (with continuous PDEs): loosely speaking, a ROM **CAN** be consistent with respect to the full simulations used to generate it.
  - **Stability**: numerical stability is **NOT** in general guaranteed *a priori* for a ROM!
  - **Convergence**: requires consistency and stability.

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This talk focuses on how to construct a Galerkin ROM that is **stable** *a priori*

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Outline

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Model Reduction Approach

High-Fidelity CFD Simulations:

- Snapshot 1
- Snapshot 2
- ⋮
- Snapshot K

Step 1 →

Fluid Modal Decomposition (POD):

$$\boxed{\mathbf{u}_M = \sum_{k=1}^{M} a_k(t)\boldsymbol{\phi}_k(\mathbf{x})}$$

Step 2 →

Galerkin Projection of Fluid PDEs:

$$\boxed{(\boldsymbol{\phi}_j, \dot{\mathbf{u}}_M + \nabla \cdot \mathbf{F}(\mathbf{u}_M)) = 0}$$

"Small" ROM ODE System:

$$\boxed{\dot{a}_k = f(a_1, \ldots, a_M)}$$

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Step 1: Constructing the Modes

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- **POD basis** $\{\boldsymbol{\phi}_i\}_{i=1}^{M}$ with $M << K$ maximizes the energy in the projection of snapshots onto span$\{\boldsymbol{\phi}_i\}$.
- **POD eigenvalue problem**:

$$\mathbf{R}\boldsymbol{\phi} = \lambda\boldsymbol{\phi}$$

where $\mathbf{R}\boldsymbol{\phi} \equiv \langle \mathbf{u}^k(\mathbf{u}^k, \boldsymbol{\phi})\rangle$.

Step 2: Galerkin Projection

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⋮

Snapshot K

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"Small" ROM ODE System:

$$\dot{a}_k = f(a_1, \ldots, a_M)$$

- Galerkin projection of ***continuous*** equations in ***continuous*** inner product onto reduced basis modes $\{\boldsymbol{\phi}_i\}_{i=1}^{M}$.

Stability Definitions

- **Practical Definition:** Numerical solution does not “blow up” in finite time.

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Linearized Compressible Navier-Stokes Equations:

$$\frac{dE}{dt} \leq 0$$

Non-increasing energy [5]

⟵ duality ⟶

Compressible Navier-Stokes Equations:

$$\frac{d}{dt}\int_{\Omega} \rho\eta d\Omega \geq 0$$

Clausius-Duhem Inequality

Non-decreasing entropy [4]

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Non-decreasing entropy [4]

- Analyzed using the **Energy Method**: Uses an equation for the evolution of numerical solution "energy" (or "entropy") to determine stability.

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3D Compressible Navier-Stokes Equations

- 3D compressible Navier-Stokes equations:

$$
\begin{aligned}
\rho\frac{Du_1}{dt} &= -\frac{\partial p}{\partial x_1} + \sum_{j=1}^{3} \frac{\partial}{\partial x_j}\left\{\mu\left(\frac{\partial u_1}{\partial x_j} + \frac{\partial u_j}{\partial x_1}\right) + \lambda\delta_{1j}\nabla\cdot\mathbf{u}\right\},\\
\rho\frac{Du_2}{dt} &= -\frac{\partial p}{\partial x_2} + \sum_{j=1}^{3} \frac{\partial}{\partial x_j}\left\{\mu\left(\frac{\partial u_2}{\partial x_j} + \frac{\partial u_j}{\partial x_2}\right) + \lambda\delta_{2j}\nabla\cdot\mathbf{u}\right\},\\
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\rho C_v\frac{DT}{dt} &= -p\nabla\cdot\mathbf{u} + \sum_{i=1}^{3}\frac{\partial}{\partial x_i}\left(\kappa\frac{\partial T}{\partial x_i}\right),\\
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\end{aligned} \tag{1}$$

- ROM approach is based on local linearization of full non-linear equations (1):
  - Full non-linear equations (1) are solved to generate snapshots in high-fidelity code
  - In the ROM projection step, the equations (1) are linearized around a steady base flow and projected onto the POD modes

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    $\Rightarrow$ non-linear dynamics *are* captured in POD modes.
  - In the ROM projection step, the equations (1) are linearized around a steady base flow and projected onto the POD modes

    $\Rightarrow$ non-linear dynamics are *not* captured in ROM equations.

3D Linearized Compressible Navier-Stokes Equations

- Appropriate when a compressible fluid system can be described by viscous, small-amplitude perturbations about a steady-state mean (or base) flow.
- Linearization of full compressible Navier-Stokes equations:

$$\mathbf{q}^T(\mathbf{x},t) \equiv \begin{pmatrix} u_1 & u_2 & u_3 & T & \rho \end{pmatrix} \equiv \underbrace{\bar{\mathbf{q}}^T(\mathbf{x})}_{\text{mean}} + \underbrace{\mathbf{q}'^T(\mathbf{x},t)}_{\text{fluctuation}} \in \mathbb{R}^5$$

$$\boxed{\Rightarrow \mathbf{q}'_{,t} + \mathbf{A}_i \mathbf{q}'_{,i} - [\mathbf{K}_{ij}\mathbf{q}'_{,j}]_{,i} = \mathbf{0}}$$

where

$$\mathbf{A}_1 = \begin{pmatrix} \bar{u}_1 & 0 & 0 & R & \frac{R\bar{T}}{\bar{\rho}} \\ 0 & \bar{u}_1 & 0 & 0 & 0 \\ 0 & 0 & \bar{u}_1 & 0 & 0 \\ \bar{T}(\gamma-1) & 0 & 0 & \bar{u}_1 & 0 \\ \bar{\rho} & 0 & 0 & 0 & \bar{u}_1 \end{pmatrix}, \quad \mathbf{A}_2 = \begin{pmatrix} \bar{u}_2 & 0 & 0 & 0 & 0 \\ 0 & \bar{u}_2 & 0 & R & \frac{R\bar{T}}{\bar{\rho}} \\ 0 & 0 & \bar{u}_2 & 0 & 0 \\ 0 & \bar{T}(\gamma-1) & 0 & \bar{u}_2 & 0 \\ 0 & \bar{\rho} & 0 & 0 & \bar{u}_2 \end{pmatrix}$$

$$\mathbf{A}_3 = \begin{pmatrix} \bar{u}_3 & 0 & 0 & 0 & 0 \\ 0 & \bar{u}_3 & 0 & 0 & 0 \\ 0 & 0 & \bar{u}_3 & R & \frac{R\bar{T}}{\bar{\rho}} \\ 0 & 0 & \bar{T}(\gamma-1) & \bar{u}_3 & 0 \\ 0 & 0 & \bar{\rho} & 0 & \bar{u}_3 \end{pmatrix}, \quad \mathbf{K}_{11} \equiv \frac{1}{\bar{\rho}} \begin{pmatrix} 2\mu+\lambda & 0 & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 & 0 \\ 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & \frac{(\gamma-1)\kappa}{R} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \ldots$$

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Symmetrized Linearized Compressible Navier-Stokes Equations

Energy stability of the Galerkin ROM can be proven [1] following "symmetrization" the linearized compressible Navier-Stokes equations.

- Linearized compressible Navier-Stokes system is "symmetrizable" [5].
- Pre-multiply equations by symmetric positive definite matrix:

$$\mathbf{H} \equiv \begin{pmatrix} \bar{\rho} & 0 & 0 & 0 & 0 \\ 0 & \bar{\rho} & 0 & 0 & 0 \\ 0 & 0 & \rho & 0 & 0 \\ 0 & 0 & 0 & \frac{\bar{\rho}R}{\bar{T}(\gamma-1)} & 0 \\ 0 & 0 & 0 & 0 & \frac{R\bar{T}}{\bar{\rho}} \end{pmatrix}. \Rightarrow \boxed{\mathbf{H}\mathbf{q}'_{,t} + \mathbf{H}\mathbf{A}_i\, \mathbf{q}'_{,i} - \mathbf{H}[\mathbf{K}_{ij}\mathbf{q}'_{,i}]_{,j} = \mathbf{0}}$$

- $\mathbf{H}$ is called the "symmetrizer" of the system:
  - The convective flux matrices $\mathbf{H}\mathbf{A}_i$ are all symmetric.
  - The following augmented viscosity matrix

$$\mathbf{K}^S \equiv \begin{pmatrix} \mathbf{H}\mathbf{K}_{11} & \mathbf{H}\mathbf{K}_{12} & \mathbf{H}\mathbf{K}_{13} \\ \mathbf{H}\mathbf{K}_{21} & \mathbf{H}\mathbf{K}_{22} & \mathbf{H}\mathbf{K}_{23} \\ \mathbf{H}\mathbf{K}_{31} & \mathbf{H}\mathbf{K}_{32} & \mathbf{H}\mathbf{K}_{33} \end{pmatrix},$$

is symmetric positive semi-definite.

Symmetry Inner Product & A Stable Galerkin ROM

- Define the "symmetry" inner product and "symmetry" norm:

$$(\mathbf{q}'^{(1)}, \mathbf{q}'^{(2)})_{(\mathbf{H},\Omega)} \equiv \int_\Omega [\mathbf{q}'^{(1)}]^T \mathbf{H} \mathbf{q}'^{(2)} d\Omega, \qquad ||\mathbf{q}'||_{(\mathbf{H},\Omega)} \equiv (\mathbf{q}', \mathbf{q}')_{(\mathbf{H},\Omega)} \tag{2}$$

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- Galerkin approximation $\mathbf{q}'_M = \sum_{i=1}^M a_k(t)\phi_k(\mathbf{x})$ satisfies the same energy expression as the solutions to the continuous equations:

$$||\mathbf{q}'_M(\mathbf{x}, t)||_{(\mathbf{H},\Omega)} \leq e^{\beta t} ||\mathbf{q}'_M(\mathbf{x}, 0)||_{(\mathbf{H},\Omega)}$$

where β is an upper bound on the eigenvalues of the matrix $\mathbf{B} \equiv \mathbf{H}^{-T/2} \frac{\partial(\mathbf{H}\mathbf{A}_i)}{\partial x_i} \mathbf{H}^{-1/2}$.

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Practical Implication:
Symmetry inner product ensures Galerkin projection step of the ROM is stable for **any** basis!

Steps to Obtain a Stable Compressible Fluid ROM

- Galerkin-project the equations in the symmetry inner product (2):

$$\left(\phi_k, \frac{\partial \mathbf{q}'_M}{\partial t}\right)_{(\mathbf{H},\Omega)} + \left(\phi_k, \mathbf{A}_i \frac{\partial \mathbf{q}'_M}{\partial x_i}\right)_{(\mathbf{H},\Omega)} + \left(\phi_k, \frac{\partial}{\partial x_j}\left[\mathbf{K}_{ij} \frac{\partial \mathbf{q}'_M}{\partial x_i}\right]\right)_{(\mathbf{H},\Omega)} = 0 \quad (3)$$

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- Integrate viscous term in (3) by parts and apply boundary conditions:

$$\left(\phi_k, \frac{\partial \mathbf{q}'_M}{\partial t}\right)_{(\mathbf{H},\Omega)} = \int_\Omega \left[\phi_k^T \mathbf{H}\mathbf{A}_i \mathbf{q}'_{M,i} - \phi_{k,j}^T \mathbf{H}\mathbf{K}_{ij}\mathbf{q}'_{M,i}\right] d\Omega - \int_{\partial\Omega} \phi_k^T \mathbf{H}\mathbf{K}_{ij} n_j \mathbf{q}'_{M,i} \, dS$$

Insert boundary conditions into boundary integrals (weak implementation)

* Energy stability is maintained if the boundary conditions are such that $\int_{\partial\Omega} \phi_k^T \mathbf{H}\mathbf{K}_{ij} n_j \mathbf{q}'_{M,i} dS \geq 0$.

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- Substitute modal decomposition $\mathbf{q}'_M = \sum_k a_k(t)\phi_k(\mathbf{x})$ to obtain an $M \times M$ linear dynamical system of the form $\dot{\mathbf{a}} = \mathbf{C}\mathbf{a}$

Implementation

- **Stability-Preserving Discrete Implementation of ROM:**
  - ROM is implemented in a C++ code that uses distributed vector and matrix data structures and parallel eigensolvers from the Trilinos project [8].
  - POD modes defined using piecewise smooth finite elements.
  - Gauss quadrature rules of sufficient accuracy are used to compute exactly inner products with the help of `libmesh` library.

ROM code is potentially compatible with any CFD code that can output a mesh and snapshot data stored at the nodes of this mesh.

- **High-fidelity CFD Code: SIGMA CFD**
  - Sandia in-house finite volume flow solver derived from LESLIE3D [7], a LES flow solver originally developed in the Computational Combustion Laboratory at Georgia Tech.
  - Solves the turbulent compressible flow equations using an explicit 2-4 MacCormack scheme.
  - A hybrid scheme coupling the MacCormack scheme to flux difference splitting schemes is employed to capture shocks.

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Inviscid Pulse in a Uniform Base Flow

- Uniform base flow:

$$\bar{p} = 101,325 \text{ Pa}$$
$$\bar{T} = 300 \text{ K}$$
$$\bar{\rho} = \frac{\bar{p}}{R\bar{T}} = 1.17 \text{ kg/m}^3$$
$$\bar{u}_1 = \bar{u}_2 = \bar{u}_3 = 0.0 \text{ m/s}$$
$$\bar{c} = 348.0 \text{ m/s}.$$

- Domain $\Omega = (-1,1) \times (-1,1) \times (-1,-0.9)$ initialized with pressure pulse:

$$p'(\mathbf{x};0) = 141.9e^{-10(x^2+y^2)},$$
$$\rho'(\mathbf{x};0) = \frac{p'(\mathbf{x};0)}{R\bar{T}},$$
$$T'(\mathbf{x};0) = 0,$$
$$u_1'(\mathbf{x};0) = u_2'(\mathbf{x};0) = u_3'(\mathbf{x};0) = 0.$$

- Slip wall boundary conditions applied on all 6 boundaries of Ω.
- High-fidelity CFD simulation run on 3362 node mesh until time $T = 0.01$ seconds.
- 200 snapshots (saved every 5×10^{-5} seconds), used to construct 20 mode POD bases.

Time History of ROM Modal Amplitudes

Figure 1: 20 Mode Symmetry ROM

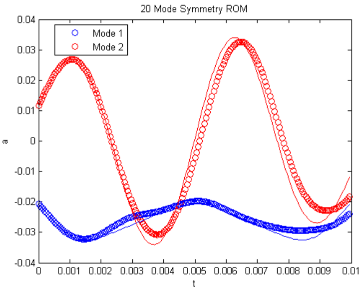
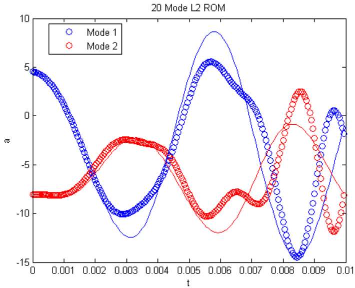


Figure 2: 20 Mode L^2 ROM



- Figures show:
  - $\circ$: t vs. $a_i(t)$ (ROM coefficients).
  - $-$: t vs. $(\mathbf{q}'_{CFD}(\mathbf{x}, t), \boldsymbol{\phi}_i(\mathbf{x}))_{(\mathbf{H},\Omega)}$ (projection of snapshots onto modes).
- Good agreement between the symmetry ROM and the full simulation for all times.
- Oscillations in the L^2 ROM modal amplitudes observed for $t > 0.008$ seconds suggest the presence of an instability in the L^2 ROM.

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20 Mode ROM vs. High-Fidelity Pressure Solutions

Symmetry ROM

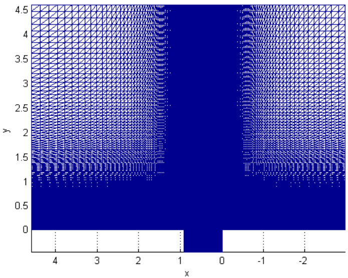
L^2 ROM

CFD

- Good qualitative agreement between the high-fidelity solution and the symmetry ROM solution.
- In contrast: L^2 ROM solution blows up by $t = 7.95 \times 10^{-3}$ seconds.

Laminar Viscous Cavity Problem (Case L2 in [9])

- Free stream pressure = 25 Pa, free stream temperature = 300 K, free stream velocity = 208.8 m/s, $\mu = 1.846 \times 10^{-5}$ kg/(m·s) and $\kappa = 2.587 \times 10^{-2}$ m^2/s.
- Flow initialized to:
  - Zero velocity, free stream pressure, and temperature inside cavity.
  - Free stream conditions, and allowed to evolve, in region above the cavity.



- High-fidelity CFD simulation was run on 343,408 node mesh until time $T = 0.2$ seconds.
- 101 snapshots were saved (every 2×10^{-3} seconds), to construct 30 mode POD bases.

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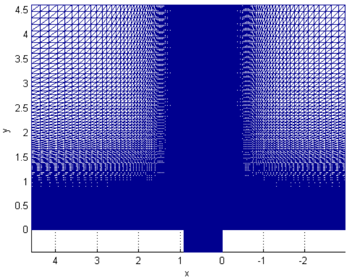
  - Zero velocity, free stream pressure, and temperature inside cavity.
  - Free stream conditions, and allowed to evolve, in region above the cavity.



- High-fidelity CFD simulation was run on 343,408 node mesh until time $T = 0.2$ seconds.
- 101 snapshots were saved (every 2×10^{-3} seconds), to construct 30 mode POD bases.

Inherently non-linear problem!
High-fidelity solution obtained by solving full *non-linear* Navier-Stokes equations.

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Expected ROM Performance

ROM based on Navier-Stokes equations
linearized around snapshot mean.

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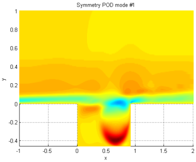
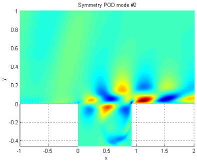
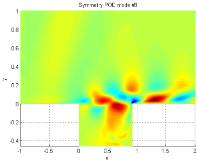
Non-linear dynamics of flow
are captured in
POD reduced basis modes.



Mode 1 (52.2% energy)



Mode 2 (15.5% energy)



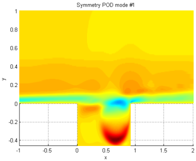
Mode 3 (13.8% energy)

Expected ROM Performance

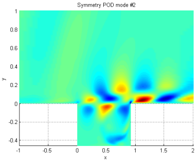
ROM based on Navier-Stokes equations
linearized around snapshot mean.

Non-linear dynamics of flow
are captured in
POD reduced basis modes.

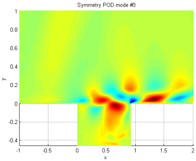
Non-linear dynamics of the flow
are *not* captured in equations
projected onto POD modes.



Mode 1 (52.2% energy)



Mode 2 (15.5% energy)



Mode 3 (13.8% energy)

Expected ROM Performance (continued)

- As shear layer separates from the leading edge of the cavity, instabilities develop and grow non-linearly to form vortices convecting down the shear layer.

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Expected ROM Performance (continued)

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- Further downstream, vortices impinge on the aft wall giving rise to linear and non-linear pressure waves that are propagated upstream through the free stream and the cavity.

The linear waves (expected in this low Re number regime) should be accurately captured by the ROM.

30 Mode ROM vs. High-Fidelity Velocity Solutions

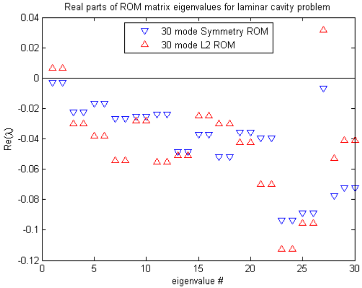
CFD

L^2 ROM

Symmetry ROM

- Reasonable qualitative agreement between ROM and high-fidelity solutions.
- ROMs do not capture in full detail inherently non-linear vortical structures present in the high-fidelity solution.

Stability of 30 Mode ROMs



- Figure plots real part of each eigenvalue of the 30×30 ROM dynamical system matrix $\mathbf{C}$ for the 30 mode symmetry and L^2 ROMs.
- 30 mode symmetry ROM is stable, whereas stability of L^2 ROM is not guaranteed.

Future Work

- Incorporate into the ROM equations non-linear terms in a stability-preserving and computationally tractable way.
- Study of the predictive capabilities of the proposed ROM for long-time simulations.
- Explore robustness of ROM with respect to parameter changes (reduced basis interpolation techniques [6]).
- Investigate the viability of the POD basis for non-linear problems: are there "better" bases to employ? (stability result is **basis independent**!)
- Targeted application: cavity flows in captive carry environments.
- Reduced order modeling for closed-loop control of large-scale systems.

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Thank you! Questions?

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