A minimal subspace rotation approach for stabilizing and fine-tuning projection-based reduced order models for fluid applications

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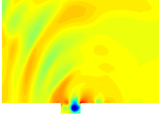
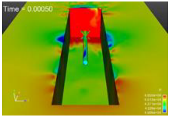
Motivation

Targeted application: compressible fluid flow (e.g., captive carry).





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- Some works on MOR for compressible flows:
  - **Energy-based inner products:** Rowley *et al.*, 2004 (isentropic); Barone *et al.*, 2007 (linear); Serre *et. al*, 2012 (linear); Kalashnikova *et al.*, 2014 (nonlinear).
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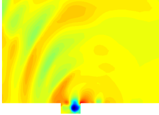

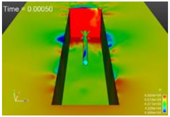

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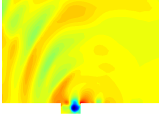

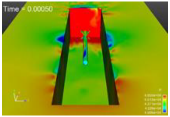

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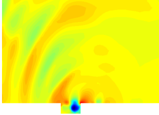

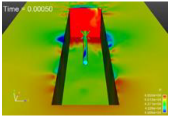

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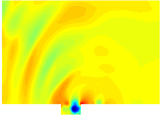

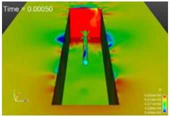

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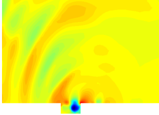

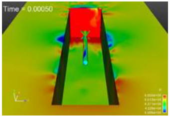

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Governing equations

- We consider the 3D compressible Navier-Stokes equations in *primitive specific volume form*:

$$\begin{aligned} \zeta_{,t} + \zeta_{,j} u_j - \zeta u_{j,j} &= 0, \\ u_{i,t} + u_{i,j} u_j + \zeta p_{,i} - \tfrac{1}{Re} \zeta \tau_{ij,j} &= 0, \\ p_{,t} + u_j p_{,j} + \gamma u_{j,j} p - \left(\tfrac{\gamma}{PrRe}\right) \left(\kappa (p\zeta)_{,j}\right)_{,j} - \left(\frac{\gamma-1}{Re}\right) u_{i,j} \tau_{ij} &= 0. \end{aligned} \tag{1}$$

- For the compressible Navier-Stokes equations (1), spectral discretization $\left(\boldsymbol{q}(\boldsymbol{x}, t) \approx \sum_{i=1}^{n} a_i(t)\, \boldsymbol{U}_i(\boldsymbol{x}) \right)$ + Galerkin projection yields a system of n coupled quadratic ODEs

$$\frac{d\boldsymbol{a}}{dt} = \boldsymbol{C} + \boldsymbol{L}\boldsymbol{a} + \left[\begin{array}{cccc} \boldsymbol{a}^{\mathrm{T}} \boldsymbol{Q}^{(1)} \boldsymbol{a} & \boldsymbol{a}^{\mathrm{T}} \boldsymbol{Q}^{(2)} \boldsymbol{a} & \cdots & \boldsymbol{a}^{\mathrm{T}} \boldsymbol{Q}^{(n)} \boldsymbol{a} \end{array}\right]^{\mathrm{T}} \tag{2}$$

where $\boldsymbol{C} \in \mathbb{R}^n$, $\boldsymbol{L} \in \mathbb{R}^{n \times n}$ and $\boldsymbol{Q}^{(i)} \in \mathbb{R}^{n \times n}$, $\forall i = 1, \ldots, n$.

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Projection-based model order reduction

Summary of technical challenges

- Projection-based MOR necessitates *truncation*.
- POD is, by definition and design, biased towards the *large, energy producing* scales of the flow (i.e., modes with large POD eigenvalues).
- Truncated/unresolved modes are negligible from a *data compression* point of view (i.e., small POD eigenvalues) but are crucial for the *dynamical equations*.
- For fluid flow applications, higher-order modes are associated with energy *dissipation* and thus, low-dimensional ROMs are often inaccurate and sometimes unstable.
- For a ROM to be stable and accurate, truncated/unresolved subspace must be accounted for (e.g., *turbulence modeling*, **subspace rotation**).

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Accounting for modal truncation

Traditional linear eddy-viscosity approach

- Dissipative dynamics of truncated higher-order modes are modeled using additional linear term

$$\frac{d\boldsymbol{a}}{dt} = \boldsymbol{C} + (\boldsymbol{L} + \boldsymbol{L}_\nu)\boldsymbol{a} + \begin{bmatrix} \boldsymbol{a}^{\mathrm{T}}\boldsymbol{Q}^{(1)}\boldsymbol{a} & \boldsymbol{a}^{\mathrm{T}}\boldsymbol{Q}^{(2)}\boldsymbol{a} & \cdots & \boldsymbol{a}^{\mathrm{T}}\boldsymbol{Q}^{(n)}\boldsymbol{a} \end{bmatrix}^{\mathrm{T}}$$

- $\boldsymbol{L}_\nu$ is designed to decrease magnitude of positive eigenvalues and increase magnitude of negative eigenvalues of $\boldsymbol{L} + \boldsymbol{L}_\nu$ (for stability).
- Disadvantages of this approach:
  1. Additional term destroys *consistency* between ROM and Navier-Stokes equations.
  2. Calibration necessary to derive optimal $\boldsymbol{L}_\nu$ and optimal value is *flow dependent*.
  3. Inherently a *linear* model $\rightarrow$ cannot be expected to perform well for all classes of problems (e.g., nonlinear).

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Proposed new approach

- Instead of modeling truncation via additional linear term, model the truncation *a priori* by "rotating" the projection subspace into a more dissipative regime.
- *Standard approach:* retain only the most energetic POD modes, i.e., $\boldsymbol{U}_1$, $\boldsymbol{U}_2$, $\boldsymbol{U}_3$, $\boldsymbol{U}_4$ ...
- *Proposed approach:* choose some higher order basis to increase dissipation, i.e., $\boldsymbol{U}_1$, $\boldsymbol{U}_2$, $\boldsymbol{U}_6$, $\boldsymbol{U}_8$, ...
- That is, approximate the solution using a linear superposition of $n+p$ (with $p>0$) most energetic modes:

$$\tilde{\boldsymbol{U}}_i = \sum_{j=1}^{n+p} X_{ji} \boldsymbol{U}_j \quad i = 1, \cdots, n, \tag{3}$$

where $\boldsymbol{X} \in \mathbb{R}^{(n+p)\times n}$ is an orthonormal ($\boldsymbol{X}^T\boldsymbol{X} = \boldsymbol{I}_{n\times n}$) "rotation" matrix.

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Goals of proposed new approach:

Find $\boldsymbol{X}$ such that

1. New modes $\widetilde{\boldsymbol{U}}$ remain good approximations of the flow $\rightarrow$ minimize the "rotation" angle, i.e. minimize $||\boldsymbol{X} - \boldsymbol{I}_{(n+p),n}||_F$.

2. New modes produce stable and accurate ROMs $\rightarrow$ ensure appropriate balance between energy production and energy dissipation.

$\rightarrow$ *Extension of earlier work for* **incompressible flow** *(Balajewicz et al., 2013).*

Once $\boldsymbol{X}$ is found, the result is system of the form (2) with

$$Q_{jk}^{(i)} \leftarrow \sum_{s,q,r=1}^{n+p} X_{si} Q_{qr}^{(s)} X_{qj} X_{rk}, \quad \boldsymbol{L} \leftarrow \boldsymbol{X}^{\mathrm{T}} \boldsymbol{L} \boldsymbol{X}, \quad \boldsymbol{C} \leftarrow \boldsymbol{X}^{T} \boldsymbol{C}^{*}. \tag{4}$$

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Trace minimization on Stiefel manifold

$$\begin{aligned} &\underset{\boldsymbol{X}\in\mathcal{V}_{(n+p),n}}{\text{minimize}} \quad -\operatorname{tr}\left(\boldsymbol{X}^{\mathrm{T}}\boldsymbol{I}_{(n+p)\times n}\right) \\ &\text{subject to} \quad \operatorname{tr}(\boldsymbol{X}^{\mathrm{T}}\boldsymbol{L}\boldsymbol{X}) = \eta \end{aligned} \tag{5}$$

where $\eta \in \mathbb{R}$ and

$$\mathcal{V}_{(n+p),n} \in \{\boldsymbol{X} \in \mathbb{R}^{(n+p)\times n} \ : \ \boldsymbol{X}^{\mathrm{T}}\boldsymbol{X} = \boldsymbol{I}_n \ , \ p > 0\}. \tag{6}$$

- Constraint comes from property that averaged total power ($= \operatorname{tr}(\boldsymbol{X}^{\mathrm{T}}\boldsymbol{L}\boldsymbol{X})$+energy transfer) has to vanish.
- η is a proxy for the balance between energy production and energy dissipation (calculated iteratively using modal energy).
- Equation (5) is solved efficiently offline using method of Lagrange multipliers (`Manopt MATLAB toolbox`).

Accounting for modal truncation

Trace minimization on Stiefel manifold

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- Proposed approach may be interpreted as an *a priori, consistent* formulation of the eddy-viscosity turbulence modeling approach.
- Advantages of proposed approach:
  1. Retains consistency between ROM and Navier-Stokes equations $\rightarrow$ no additional turbulence terms required.
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High angle of attack laminar airfoil

- 2D flow around an inclined NACA0012 airfoil at Mach 0.7, $\mathrm{Re} = 500$, $\mathrm{Pr} = 0.72$, $\mathrm{AOA} = 20^\circ \Rightarrow n = 4$ ROM (86% snapshot energy).

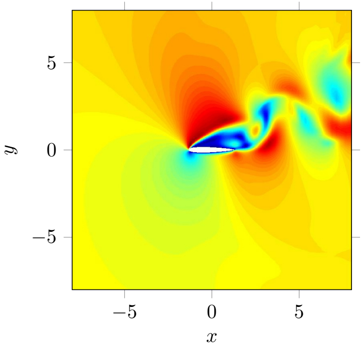


Figure 1: Contours of velocity magnitude at time of final snapshot.

High angle of attack laminar airfoil

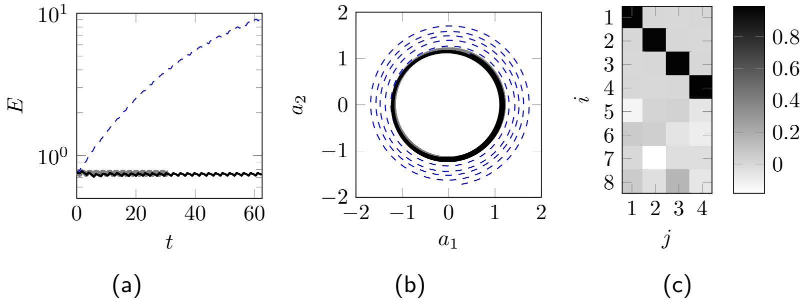


Figure 2: Nonlinear model reduction of the laminar airfoil. Evolution of modal energy (a), and phase plot of the first and second temporal basis, $a_1(t)$ and $a_2(t)$ (b); DNS (thick gray line), standard $n = 4$ ROM (dashed blue line), stabilized $n, p = 4$ ROM (solid black line). Stabilizing rotation matrix, $\boldsymbol{X}$ (c). Rotation is small: $||\boldsymbol{X} - \boldsymbol{I}_{(n+p)\times n}||_F/n = 0.083$, $\boldsymbol{X} \approx \boldsymbol{I}_{(n+p)\times n}$.

High angle of attack laminar airfoil

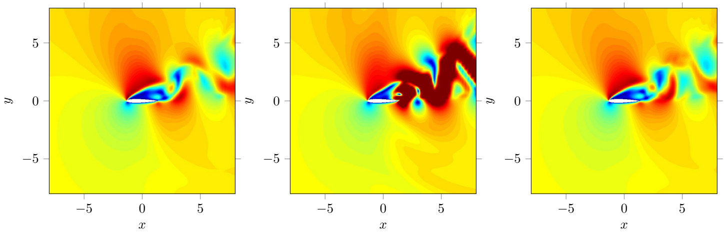


Figure 3: Snapshot of high angle of attack airfoil at final snapshot; contours of velocity magnitude. DNS (left), standard $n = 4$ ROM (middle), and stabilized $n, p = 4$ ROM (right)

Channel driven cavity: low Reynolds number case

- Flow over square cavity at Mach 0.6, $\mathrm{Re} = 1453.9$, $\mathrm{Pr} = 0.72$ $\Rightarrow n = 4$ ROM (91% snapshot energy).

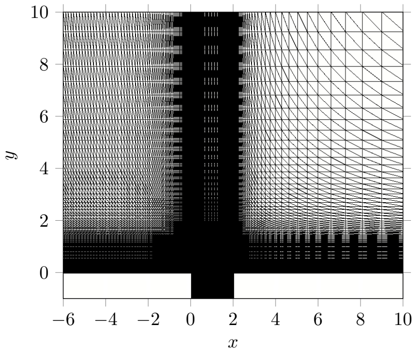
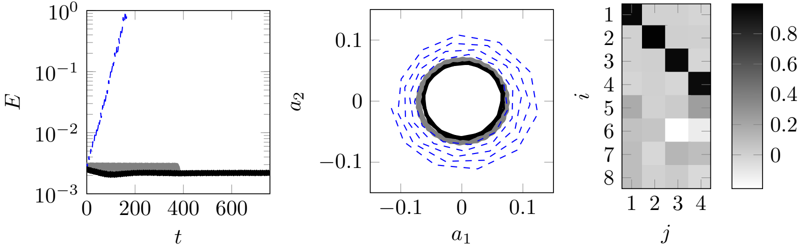


Figure 4: Domain and mesh for viscous channel driven cavity problem

Channel driven cavity: low Reynolds number case



(a) (b) (c)

Figure 5: Nonlinear model reduction of channel drive cavity at $\mathrm{Re} \approx 1500$. Evolution of modal energy (a) and phase plot of the first and second temporal basis, $a_1(t)$ and $a_2(t)$ (b); DNS (thick gray line), standard $n = 4$ ROM (dashed blue line), stabilized $n, p = 4$ ROM (solid black line). Stabilizing rotation matrix, $\boldsymbol{X}$ (c). Rotation is small: $||\boldsymbol{X} - \boldsymbol{I}_{(n+p)\times n}||_F / n = 0.118$, $\boldsymbol{X} \approx \boldsymbol{I}_{(n+p)\times n}$.

Applications

Channel driven cavity: low Reynolds number case

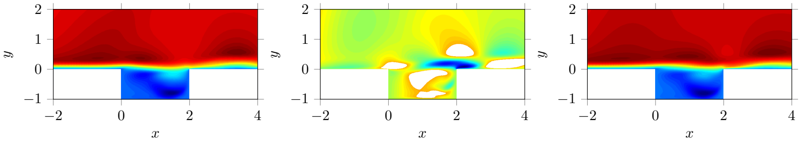


Figure 6: Snapshot of channel drive cavity $\mathrm{Re} \approx 1500$; contours of u-velocity magnitude at the final snapshot. DNS (left), standard $n = 4$ ROM (middle) and stabilized $n, p = 4$ ROM (right)

Channel driven cavity: low Reynolds number case

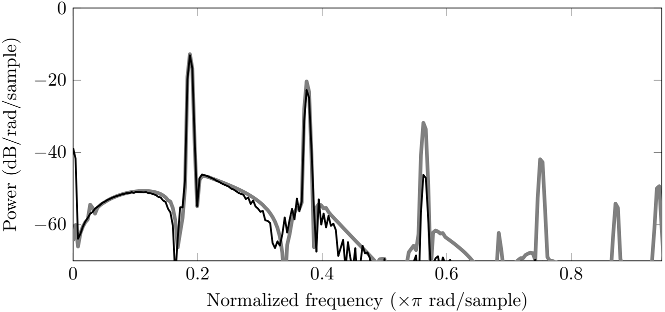


Figure 7: PSD of $p(\boldsymbol{x}, t)$ where $x = (2, -1)$ of channel drive cavity $\mathrm{Re} \approx 1500$. DNS (thick gray line), stabilized $n, p = 4$ ROM (black line)

Channel driven cavity: moderate Reynolds number case

- Flow over square cavity at Mach 0.6, $\mathrm{Re} = 5452.1$, $\mathrm{Pr} = 0.72$ $\Rightarrow n = 20$ ROM (71.8% snapshot energy).

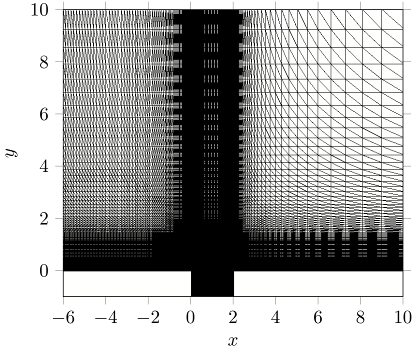
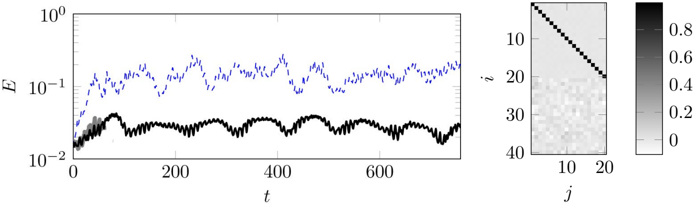


Figure 8: Domain and mesh for viscous channel driven cavity problem

Channel driven cavity: moderate Reynolds number case



(a) (b)

Figure 9: Nonlinear model reduction of channel drive cavity at $\mathrm{Re} \approx 5500$. Evolution of modal energy (a); DNS (thick gray line), standard $n = 20$ ROM (dashed blue line), stabilized $n, p = 20$ ROM (solid black line). Stabilizing rotation matrix, $\boldsymbol{X}$ (b). Rotation is small: $||\boldsymbol{X} - \boldsymbol{I}_{(n+p)\times n}||_F/n = 0.038$, $\boldsymbol{X} \approx \boldsymbol{I}_{(n+p)\times n}$.

Channel driven cavity: moderate Reynolds number case

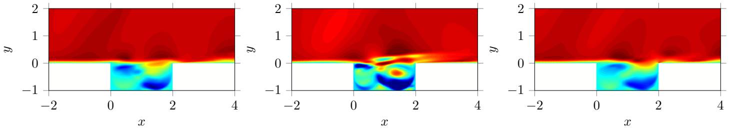


Figure 10: Snapshot of channel drive cavity $\mathrm{Re} \approx 5500$; contours of u-velocity magnitude at the final snapshot. DNS (left), standard $n = 20$ ROM (middle), and stabilized $n, p = 20$ ROM (right)

Channel driven cavity: moderate Reynolds number case

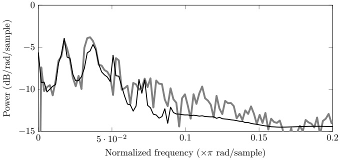


Figure 11: CPSD of $p(\boldsymbol{x}_1, t)$ and $p(\boldsymbol{x}_2, t)$ where $\boldsymbol{x}_1 = (2, -0.5)$ and $\boldsymbol{x}_2 = (0, -0.5)$ of channel driven cavity at $\mathrm{Re} \approx 5500$. DNS (thick gray line), stabilized $n, p = 20$ ROM (black line)

- Power and phase lag at the fundamental frequency, and first two super harmonics are predicted accurately using the fine-tuned ROM.
- Phase lag at these three frequencies as predicted by the CFD and the stabilized ROM is identified by red squares and blue triangles, respectively.

Applications

CPU times (CPU-hours) for off-line and on-line computations

	Numerical Experiment		
Procedure	Airfoil	Cavity, Low-Re	Cavity, Moderate-Re
FOM # of DOF	360,000	288,250	243,750
Time-integration of FOM	7.8 hrs	72 hrs	179 hrs
Basis construction (size $n + p$ ROM)	0.16 hrs	0.88 hrs	3.44 hrs
Galerkin projection (size $n + p$ ROM)	0.74 hrs	5.44 hrs	14.8 hrs
Stabilization	28 sec	14 sec	170 sec
ROM # of DOF	4	4	20
Time-integration of ROM	0.31 sec	0.16 sec	0.83 sec
Online computational speed-up	9.1×10^4	1.6×10^6	7.8×10^5

Summary

- We have developed a non-intrusive approach for stabilizing and fine-tuning projection-based ROMs for compressible flows.
- The standard POD modes are “rotated” into a more dissipative regime to account for the dynamics in higher order modes truncated by the standard POD method.
- The new method is consistent and does not require the addition of empirical turbulence model terms unlike traditional approaches.
- Mathematically, the approach is formulated as a quadratic matrix program on the Stiefel manifold.
- This constrained minimization problem is solved offline and small enough to be solved in MATLAB.
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Future work

- Extension of the proposed approach to problems with generic non-linearities, where the ROM involves some form of hyper-reduction (e.g., DEIM, gappy POD).
- Extension of the method to predictive applications, e.g., problems with varying Reynolds number and geometry.
- Selecting different objectives and constraints in our optimization problem:

$$\begin{aligned} &\underset{\boldsymbol{X} \in \mathcal{V}_{(n+p),n}}{\text{minimize}} && f(\boldsymbol{X}) \\ &\text{subject to} && g(\boldsymbol{X}, \boldsymbol{L}) \end{aligned} \tag{7}$$

  e.g.,
  - Maximize parametric robustness: $f = \sum_{i=1}^{k} \beta_i || \boldsymbol{U}^*(\mu_i)\boldsymbol{X} - \boldsymbol{U}^*(\mu_i)||_F$.
  - ODE constraints: $g = ||\boldsymbol{a}(t) - \boldsymbol{a}^*(t)||$.

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Thank you!

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Appendix: Stabilization algorithm: returns stabilizing rotation matrix $\boldsymbol{X}$.

Inputs: Initial guess $\eta^{(0)} = \text{tr}(L(1:n, 1:n))$ ($\boldsymbol{X} = \boldsymbol{I}_{(n+p)\times n}$), ROM size n and $p \geq 1$, ROM matrices associated with the first $n+p$ most energetic POD modes, convergence tolerance TOL, maximum number of iterations k_{max}.

for $k = 0, \cdots, k_{max}$
    Solve constrained optimization problem on Stiefel manifold:

$$\begin{aligned} \underset{\boldsymbol{X}^{(k)} \in \mathcal{V}_{(n+p),n}}{\text{minimize}} \quad & -\operatorname{tr}\left(\boldsymbol{X}^{(k)\mathrm{T}} \boldsymbol{I}_{(n+p)\times n}\right) \\ \text{subject to} \quad & \operatorname{tr}(\boldsymbol{X}^{(k)\mathrm{T}} \boldsymbol{L} \boldsymbol{X}^{(k)}) = \eta^{(k)}. \end{aligned}$$

    Construct new Galerkin matrices using (4).
    Integrate numerically new Galerkin system.
    Calculate "modal energy" $E(t)^{(k)} = \sum_i^n (a(t)_i^{(k)})^2$.
    Perform linear fit of temporal data $E(t)^{(k)} \approx c_1^{(k)} t + c_0^{(k)}$, where $c_1^{(k)}$ =energy growth.
    Calculate ϵ such that $c_1^{(k)}(\epsilon) = 0$ (no energy growth) using root-finding algorithm.
    Perform update $\eta^{(k+1)} = \eta^{(k)} + \epsilon$.
    if $||c_1^{(k)}|| < TOL$
        $\boldsymbol{X} := \boldsymbol{X}^{(k)}$.
        terminate the algorithm.
    end
end