Stabilization of Projection-Based Reduced Order Models via Optimization-Based Eigenvalue Reassignment

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Reduced order modeling is a promising tool that can enable uncertainty quantification, on-the-spot decision-making, and/or control in a number of applications. In order for a reduced order model (ROM) to be viable, it is important that the model preserve certain crucial properties of the original system, most notably stability, which is required for accuracy and convergence of the ROM. Unfortunately, popular projection-based model reduction methods such as proper orthogonal decomposition (POD) [1] lack an *a priori* stability guarantee: a ROM constructed using the POD/Galerkin method can "blow up" for certain choices of basis size even when the high-fidelity system from which the ROM was constructed is stable [2].

In this talk, an approach for stabilizing unstable ROMs through an *a posteriori* post-processing applied to the algebraic ROM system, recently developed by the authors [3], will be described. Consider a linear-time invariant (LTI) ROM system of the form:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) \qquad (1)$$
$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t),$$

and suppose the ROM system matrix $\boldsymbol{A}$ is unstable in the Lyapunov sense, i.e., $\boldsymbol{A}$ has one or more eigenvalues with a positive real part. The key idea of the proposed method is to modify the unstable eigenvalues of $\boldsymbol{A}$ by moving these eigenvalues into the stable half of the complex plane. Toward this effect, the matrix $\boldsymbol{A}$ in (1) is replaced with another matrix $\widetilde{\boldsymbol{A}}$ that is stable in the Lyapunov sense. The new matrix $\widetilde{\boldsymbol{A}}$ defining the stabilized ROM is selected to ensure that the modified ROM remains accurate. This is accomplished by formulating and solving the following constrained nonlinear least-squares optimization problem for the eigenvalues of the stabilized ROM matrix $\widetilde{\boldsymbol{A}}$ that minimizes the error in the ROM output $\boldsymbol{y}_{ROM}$ with respect to the FOM output $\boldsymbol{y}_{FOM}$ (thereby maximizing the accuracy of the stabilized ROM):

$$\min_{\lambda_i} \quad \sum_{k=1}^{K} ||\boldsymbol{y}_{ROM}(t_k) - \boldsymbol{y}_{ROM}(t_k)||_2^2, \qquad (2)$$
$$s.t. \;\; Re(\lambda_i) < 0$$

where λ_i is used to denote the eigenvalues of **A** that were unstable. The optimization problem (2) is small and inexpensive to solve using standard algorithms, e.g., the interior point method. The performance of the proposed ROM stabilization approach (including the convergence of the optimization algorithm used to solve (2)) is evaluated on several test cases for which ROMs constructed via the POD/Galerkin method suffer from instabilities. These tests reveal that the stabilization algorithm delivers ROMs that are both stable and accurate. Recent extensions of the method to nonlinear problems are described.

REFERENCES

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