A Discontinuous Enrichment Method for the Solution of the Advection-Diffusion Equation

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Scalar advection-diffusion equation

$$\mathcal{L}u = \underbrace{-k\nabla^2 u}_{\text{diffusion}} + \underbrace{a \times \tilde{\nabla} u}_{\text{advection}} = f$$

k = diffusivity = 1

a = flow velocity

f = source term

- Describes many transport phenomena in fluid mechanics
- Usual model for the more challenging Navier-Stokes equations
- Important dimensionless parameter: Péclet number

$$\text{Pé} = \frac{\text{rate of advection}}{\text{rate of diffusion}} = \frac{l\,|a|}{k} = \text{Re} \times \begin{cases} \text{Pr (thermal diffusion)} \\ \text{Sc (mass diffusion)} \end{cases}$$

Advection-dominated regime

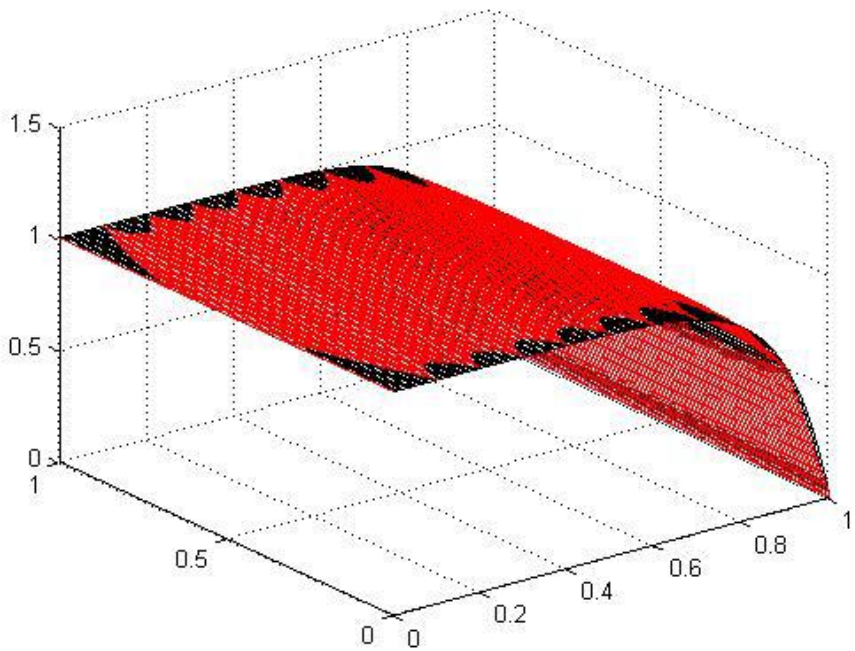


Figure 1: Galerkin FE solution
(Pé = 10 ⍰ 150, n = 10)

Classical Galerkin FEM fraught with difficulties, namely spurious oscillations, when Pé is high

<u>Approaches to handle difficulty:</u>

- Stabilized FEMs (SUPG, GLS, USFEM): add a weighted residual (numerical diffusion) to variational equation to damp out oscillations

- RFB, VMS, PUM: construct conforming spaces that incorporate knowledge of local behavior of solution

Discontinuous Enrichment Method (DEM)

Standard finite element polynomial field is "enriched" by the free-space solutions to the governing constant-coefficient homogeneous PDE not in V^P:

$$\boxed{u^h = u^P + u^E \subset H^1(\Omega)}$$

- Unlike PUM, VMS & RFB: enrichment field in DEM is not required to vanish at element boundaries

- Continuity across element boundaries is enforced weakly using Lagrange multipliers:

$$\boxed{\lambda^h = \nabla u^E \cdot \boldsymbol{n}}$$

<u>Babuška-Brezzi *inf-sup* condition:</u>

Lagrange multiplier constraint equations ≤ # enrichment equations $\Longrightarrow$ $n^\lambda \leq \dfrac{n^E}{2}$ (asymptotically)[1]

where $\begin{cases} n^\lambda = \#\text{ LMs per edge} \\ n^E = \#\text{ enrichment functions} \end{cases}$

[1] Note that this is a necessary but not a sufficient condition for generating a non-singular global interface problem.

Implementation

Global matrix equation (if polynomial field u^p is included):

$$\begin{pmatrix} \mathbf{K}^{\mathbf{PP}} & \mathbf{K}^{\mathbf{PE}} & \mathbf{K}^{\mathbf{PC}} \\ \mathbf{K}^{\mathbf{EP}} & \mathbf{K}^{\mathbf{EE}} & \mathbf{K}^{\mathbf{EC}} \\ \mathbf{K}^{\mathbf{CP}} & \mathbf{K}^{\mathbf{CE}} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{u}^{\mathbf{P}} \\ \mathbf{u}^{\mathbf{E}} \\ \lambda \end{pmatrix} = \begin{pmatrix} \mathbf{F}^{\mathbf{P}} \\ \mathbf{F}^{\mathbf{E}} \\ \mathbf{F}^{\mathbf{C}} \end{pmatrix} \Leftarrow \begin{cases} \mathbf{K}^{\mathbf{PP}} \leftrightarrow a(v^P, u^P) \\ \mathbf{K}^{\mathbf{PE}} \leftrightarrow a(v^P, u^E) \\ \mathbf{K}^{\mathbf{PC}} \leftrightarrow (v^P, \lambda^h) \\ \mathbf{K}^{\mathbf{EE}} \leftrightarrow a(v^E, u^E) \\ \mathbf{K}^{\mathbf{EC}} \leftrightarrow (v^E, \lambda^h) \\ \vdots \end{cases}$$

where $a(v,u) = (\nabla v, \nabla u)_\Omega + (v, \boldsymbol{a} \cdot \nabla u)$

(*asymmetric* bilinear operator)

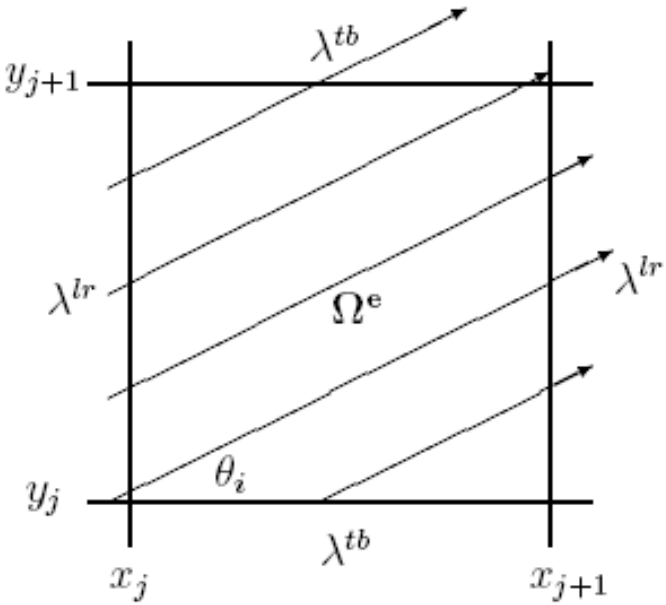
Static condensation: due to discontinuous nature of v^E, enrichment dofs can be eliminated at element level; e.g., if polynomial field is not included,

$$-\mathbf{k}^{\mathbf{CE}}(\mathbf{k}^{\mathbf{EE}})^{-1}\mathbf{k}^{\mathbf{EC}}\lambda = \mathbf{f}^{\mathbf{C}} - \mathbf{f}^{\mathbf{E}} \Rightarrow \mathbf{k}^{\mathbf{EE}}\mathbf{u}^{\mathbf{E}} = \mathbf{f}^{\mathbf{E}} - \mathbf{k}^{\mathbf{EC}}\lambda$$

(post-processing within each element)

*static condensation makes the DEM more computationally efficient than the PUM, for which # dofs = # of enrichment equations

Enrichment basis for 2D advection-diffusion



θ_i = flow direction

$\boldsymbol{a}^T = (a_1 \quad a_2)$

$$u^E = \sum_{i=1}^{n^E} u_i e^{\left(\frac{a_1}{2}+\|\boldsymbol{a}\|\cos\theta_i\right)(x-x_r)} e^{\left(\frac{a_2}{2}+\|\boldsymbol{a}\|\sin\theta_i\right)(y-y_r)}$$

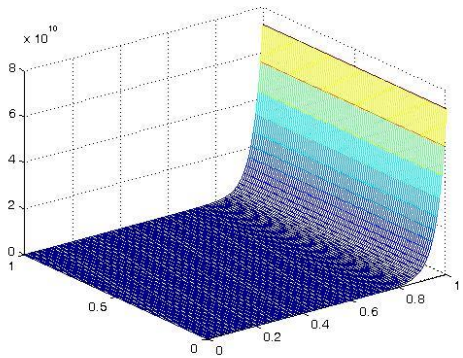
$$\lambda_i^{lr} = e^{\left(\frac{a_1}{2}+\|\boldsymbol{a}\|\cos\theta_i\right)(x-x_r)}, \quad \lambda_i^{tb} = e^{\left(\frac{a_2}{2}+\|\boldsymbol{a}\|\sin\theta_i\right)(y-y_r)}$$

(x_r, y_r) = arbitrary reference point added to alleviate ill-conditioning

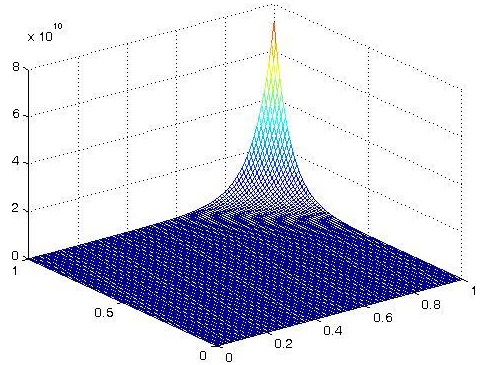
$\{\theta_i\}$ = set of angles related to flow direction that specify the enrichment basis

<u>Remark</u>: the nature of the enrichment functions makes it possible to evaluate all integrals analytically ⇒ bypass accuracy and cost issues associated with numerical quadrature

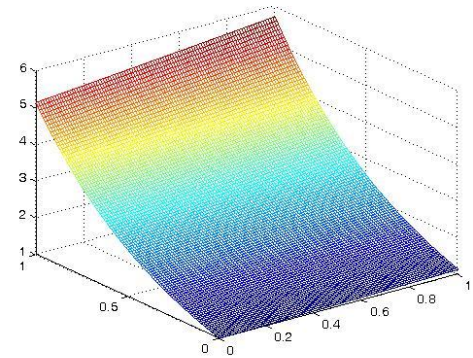
Plots of enrichment basis functions[2]



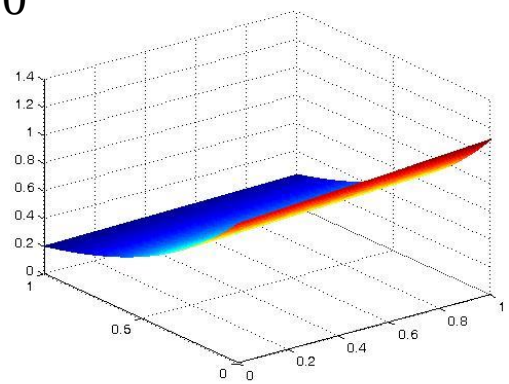
$\theta = 0$



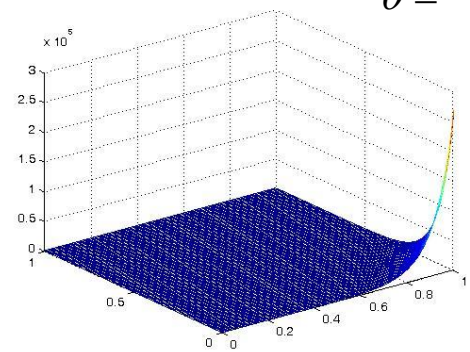
$\theta = \pi/2$



$\theta = 23\pi/24$



$\theta = 25\pi/24$



$\theta = 3\pi/2$

[2] The given plots are for Pé = 25, $a_2 = 0$.

Computational complexity

Static condensation	?	cost of solving matrix problem is independent of n^E (dimension of enrichment field)

Notation:

DGM: $R - X - N$

DEM: $R - X - N^{+} = [R - X - N] \cup Q_1$

where
$$\begin{cases} R = \text{"rectangular"} \\ X = n^E \\ N = n^{\lambda} \end{cases}$$

Q_1 : 4-noded bilinear quadrilateral Galerkin element

Q_2 : 8-noded biquadratic quadrilateral Galerkin element

Table 1:

Element	Asymptotic # of dofs
Q_1	n^2
Q_2	$4n^2$
R-X-1	$2n^2$
R-X-1$^+$	$3n^2$
R-X-2	$4n^2$
R-X-2$^+$	$5n^2$

Test problem 1: smooth unforced boundary layer problem[3]

- Constant advection coefficients, no distributed source (f = 0).

- Inhomogeneous Dirichlet data are specified such that the exact solution on Ω=(0,1) x (0,1) is (for a chosen advection angle ϕ and global Péclet number Pé):

$$u_{ex}(x,y) = \frac{e^{\text{Pé}\cos f\,(x-1)+\text{Pé}\sin f\,(y-1)} - 1}{e^{-\text{Pé}\cos f - \text{Pé}\sin f} - 1}$$

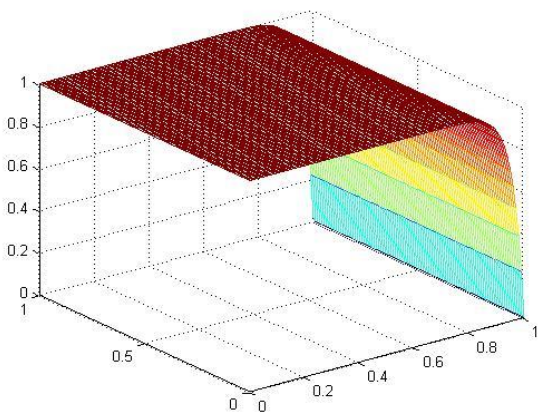


Figure 2(a): u_{ex} for ϕ = 0 (Pé=25)

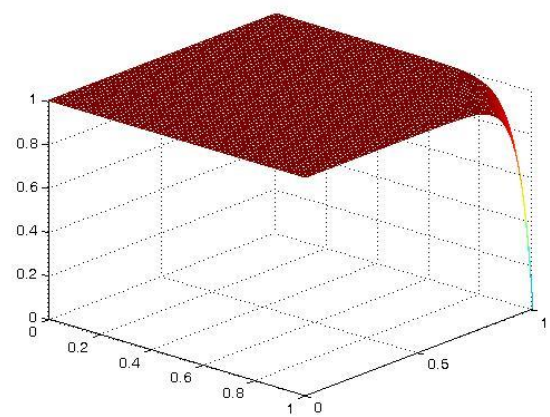


Figure 2(b): u_{ex} for ϕ = π/4 (Pé=25)

[3] Borrowed from §5.1 of [4]: I. Harari, L.P. Franca, S.P. Oliveira. "Streamline design of stability parameters for advection-diffusion problems". J. Comput. Phys. 171 (2001) 115-131.

Test problem 1: numerical results

Table 2: L^2 relative errors (%) at 400 dofs[4]

Pé	ϕ/π	Q_1	STR	EST	FFH	R-8-2
1e2	0	8.97	7.62	7.62	8.59	5.61e-13
	1/6	1.31	1.14	1.15	1.25	2.51e-2
	1/5	1.31	?	?	?	1.13e-2
	1/4	1.31	1.14	1.15	1.26	1.98e-13
	1/3	1.31	?	?	?	2.31e-2
1e3	0	57.7	12.8	12.8	12.9	3.17e-12
	1/6	2.53	1.67	1.67	1.75	1.82e-2
	1/5	2.57	?	?	?	8.72e-2
	1/4	2.62	1.67	1.67	1.77	2.80e-12
	1/3	2.53	?	?	?	1.31e-2

Q_1: 4-node bilinear quadrilateral Galerkin element

STR: stabilized FE with streamline parameter

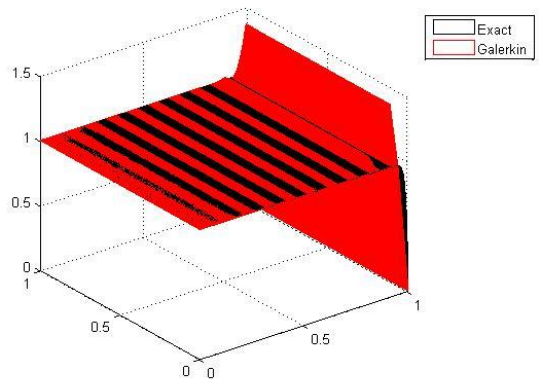
EST: stabilized FE with (another) estimated streamline parameter

FFH: stabilized FE with the FFH parameter

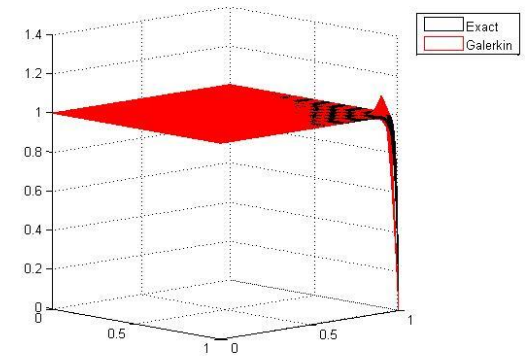
R-8-2: DGM element specified by $q_i \in \{0, p/2, 23p/24, 25p/24, 3p/2, p/4, 15p/8, f - p\}$ with 2 LMs/edge

[4] In [3], results are given for a 20 x 20 mesh. From Table 1, if n=20 for a Q_1 element, one must limit n to n=10 for the R-8-2 element to stay at 400 dofs.

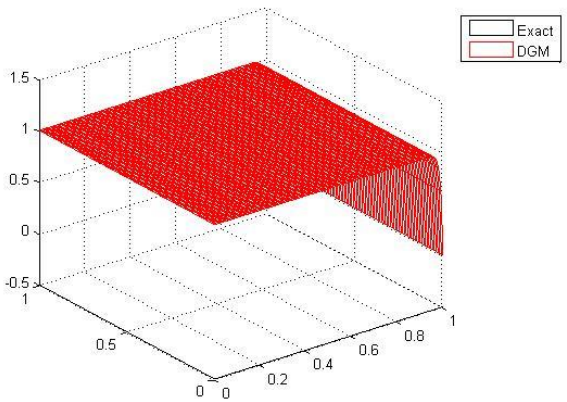
Test problem 1: solution plots



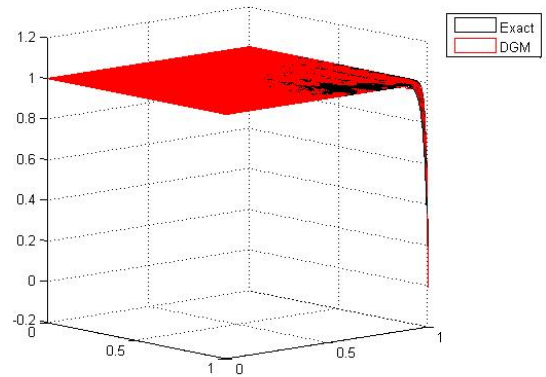
Q_1 solution (ϕ=0, Pé=1e2)



Q_1 solution ($\phi=\pi/6$, Pé=1e2)



R-8-2 solution (ϕ=0, Pé=1e2)



R-8-2 solution ($\phi=\pi/6$, Pé=1e2)

Test problem 1: convergence rates

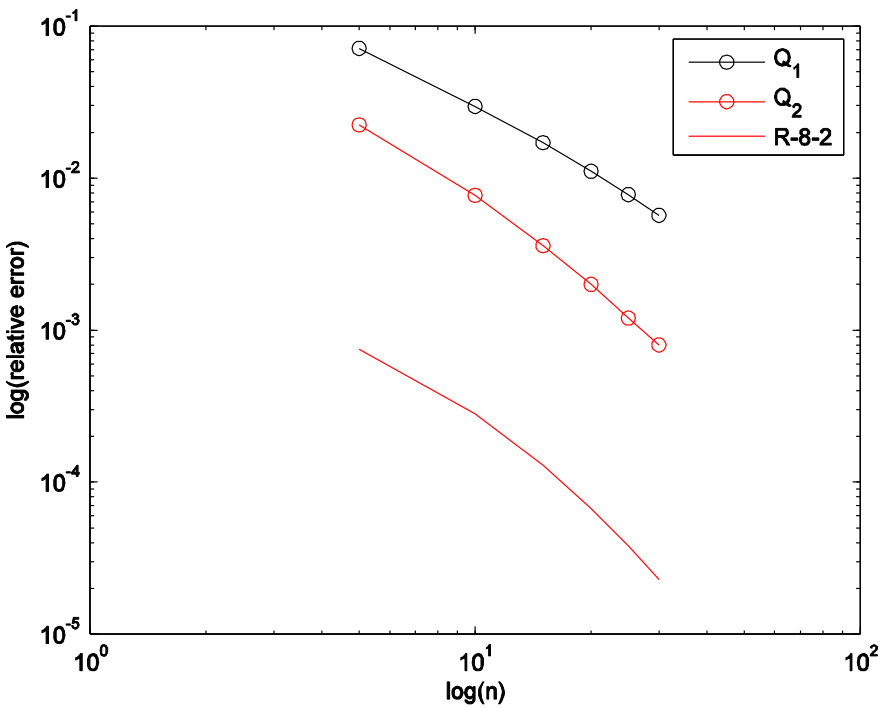


Figure 3: log-log plot showing convergence rates (Pé = 1e2, $\phi = \pi/6$)

Table 3: convergence rates (Pé = 1e2, $\phi = \pi/6$)

Element	Convergence rate[5]
Q_1	1.72
Q_2	2.68
R-8-2	2.76

- R-8-2 element comparable to Q_2 in terms of convergence rate
- Error is at least 1 order of magnitude smaller than Galerkin Q_2

[5] These numbers are below the theoretical rates because the convergence rates decline slightly from the theoretical rates as Pé↑.

Test problem 2: inhomogeneous BVP

<u>To highlight the role of the polynomial field:</u> consider the previous BVP but with a source term $f = 2\{(\text{Pé}\cos f)x + (\text{Pé}\sin f)y - 2\}$

$$u_{ex}(x,y) = \underbrace{x^2 + y^2}_{\text{slowly varying}} + \underbrace{\frac{e^{\text{Pé}\cos f\,(x-1)+\text{Pé}\sin f\,(y-1)} - e^{-\text{Pé}\cos f - \text{Pé}\sin f}}{e^{-\text{Pé}\cos f - \text{Pé}\sin f} - 1}}_{\text{rapidly varying}}$$

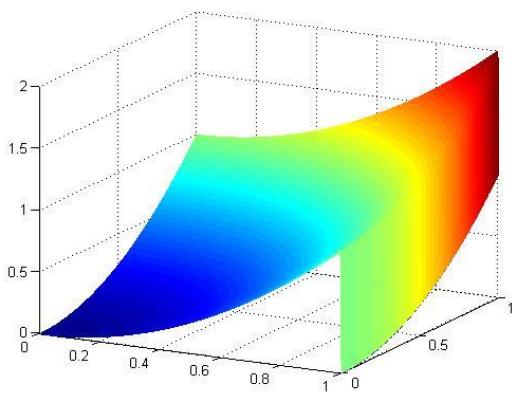


Figure 4(a): u_{ex} for $\phi = 0$ (Pé = 1e3)

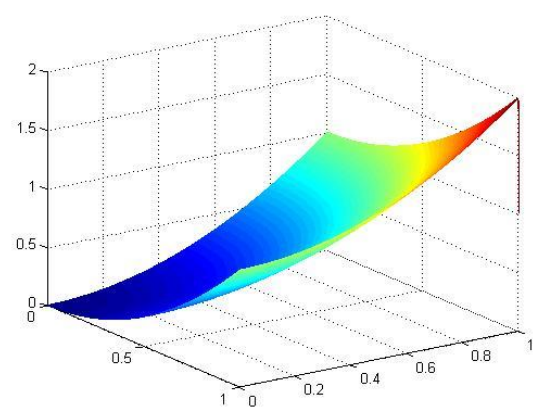


Figure 4(b): u_{ex} for $\phi = \pi/4$ (Pé = 1e3)

Test problem 2: numerical results

Table 3: L^2 relative errors (%) at 400 dofs[6]

Pé	ϕ/π	Q_1	R-9-2⁺
1e2	0	11.5	2.54e-1
	1/6	1.66	5.02e-1
	1/5	1.67	3.03e-1
	1/4	1.67	1.98e-1
1e3	0	73.3	1.69
	1/6	3.21	9.24e-1
	1/5	3.26	9.20e-1
	1/4	3.33	3.02e-1

R-9-2⁺:defined by the set of angles

$$q_i \hat{\mathrm{I}} \{0, p/2, 23p/24, 25p/24, 3p/2, p/4, 5p/4, p/8, 15p/8\}$$

+ the polynomial field of the Q_1 element with 2 LMs/edge

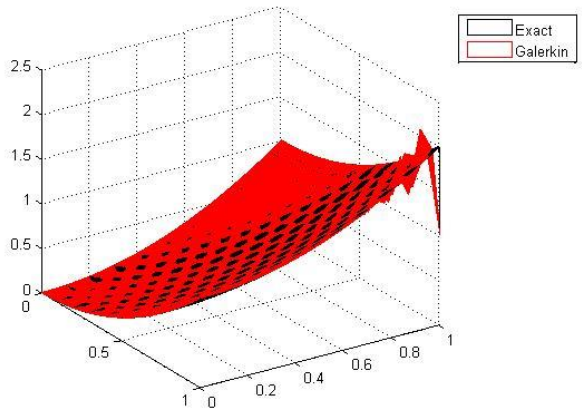


Figure 5(a): Q_1 solution (ϕ = π/6, Pé = 1e3)

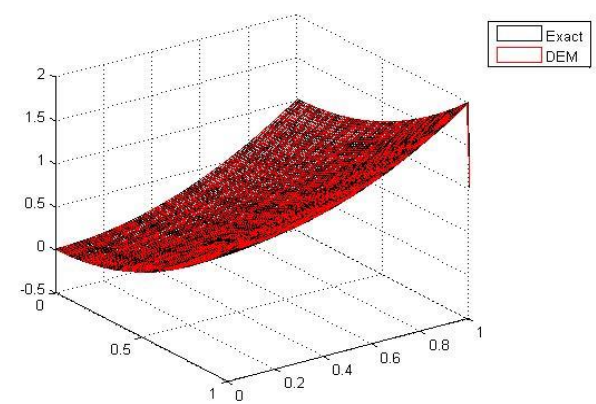


Figure 5(b): R-9-2⁺ solution (ϕ = π/6, Pé = 1e3)

[6] If a 20 x 20 mesh is used with the Q_1 element, from Table 1, one must limit n to n = 9 for the cost of the R-9-2⁺ element to stay at 400 dofs.

Technical advances/contributions

Technical advances:

- The DEM alleviates the difficulties in applying FEs to high Pé flows:
  - No more oscillations!
  - Errors reduced by several orders of magnitude compared to stabilized methods.
  - Efficient & easy to implement.

- Impressive results for advection-diffusion suggest that the DEM has a significant potential for improving FE computations in the field of fluid mechanics.

- Potential depends on success for non-linear problems (next task).

Personal contributions:

- Extended the DEM to the 2D advection-diffusion equation:
  - Derived the enrichment basis and Lagrange multiplier approximations.
  - Implemented/tested the method.
  - Formulated a convenient way to systematically design DEM elements of arbitrary orders based on flow directions (the $\{\theta\}$-form of the DEM basis).
  - Improved conditioning of matrices through use of reference point.

- Future work will involve extending the DEM to 3D advection-diffusion and to the Navier-Stokes equations.

References

[1] C. Farhat, I. Harari, L.P. Franca. "The discontinuous enrichment method". Comput. Methods Appl. Mech. Engrg. 190 (2001) 6455-6479.

[2] C. Farhat, I. Harari, U. Hetmaniuk. "A discontinuous Galerkin method with Lagrange multipliers for the solution of Helmholtz problems in the mid-frequency regime". Comput. Methods Appl. Mech. Engrg. 192 (2003) 1389-1419.

[3] C. Farhat, R. Tezaur. "Three-dimensional discontinuous Galerkin elements with plane waves and Lagrange multipliers for the solution of mid-frequency Helmholtz problem". Int. J. Numer. Meth. Engng 66 (2006) 796-815.

[4] I. Harari, L.P. Franca, S.P. Oliveira. "Streamline design of stability parameters for advection-diffusion problems". J. Comput. Phys. 171 (2001) 115-131.