# LOCAL BEHAVIOR OF HARMONIC FUNCTIONS ON THE SIERPINSKI GASKET 

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## A THESIS

in

Mathematics

Presented to the Faculties of the University of Pennsylvania in Partial Fulfillment of the Requirements for the Degree of Master of Arts

2006

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Dedicated to my parents.
Without your sacrifices and support, I would not be the person I am today.

## Acknowledgements

I would like to acknowledge my advisor, Dr. Alexandre A. Kirillov, for his guidance and assistance. Thank you for providing me with the opportunity to explore a very interesting and still subdeveloped domain in the field of mathematics. I would also like to thank the faculty of the University of Pennsylvania Mathematics Department for their encouragement and support.


#### Abstract

In this thesis, we summarize what is known about the local behavior of the four basic functions $\chi, \phi, \psi$ and $\xi$ on the Sierpinski Gasket and present some new results concerning their differentiability. We prove that the basic functions are differentiable at all rational nondyadic points and that their derivatives at these points are jointly 0 or jointly $\infty$. We then identify the rational nondyadic points at which all basic functions necessarily have the same derivative and group the operators $W$ which fix these points into conjugacy classes according to their traces. We end with a discussion of several open problems that come out of the said analysis.


Key words and phrases: Sierpinski Gasket, harmonic, Laplacian, basic function, $\chi, \phi, \psi, \xi$, derivative, fixed point, operator, word, dyadic, rational nondyadic.

## 1 Introduction

In very simple terms, a fractal is a geometric object or set that displays selfsimilarity in a non-trivial manner and on all scales. The modern theory of fractals was pioneered in the 1970s by Benoit Mandelbrot, who coined the term from the Latin fractus, meaning broken or irregular. Mandelbrot defined a fractal as a "rough or fragmented geometric shape that can be subdivided in parts, each of which is, at least approximately, a reduced copy of the whole". Fractals are generated by recursion and can be grouped into three broad categories:

1. Iterated Function Systems: fractals defined according to a fixed geometric replacement rule (e.g., the Cantor set, the Sierpinski Gasket, the Koch snowflake)
2. Escape-time Fractals: fractals defined by a recurrence relation at each point in a space such as the complex plane (e.g., the Mandelbrot set, the Lyapunov fractal)
3. Random Fractals: fractals generated by stochastic rather than deterministic processes (e.g., fractal landscapes, objects in nature)

While it can be argued that all Euclidean objects, e.g., the real line, are exactly self-similar and therefore fractals, this argument is a distinct minority position. Unlike traditional geometric forms, fractals are not comprised of elements having integer dimensions, such as lines, planes, arcs, spheres and so on ${ }^{1}$. After sufficient magnification, it would be impossible to tell the difference between an ordinary Euclidean circle and a straight line. Yet no fractal exhibits this property because fractals possess infinite detail and self-similarity on all scales, even at infinite magnification. In this sense, fractals are fundamentally different from ordinary Euclidean objects and are therefore not definable by traditional geometry. Figure ${ }^{2} 1$ illustrates the self-similarity and infinite detail of the Mandelbrot set.

While fractals have been observed in various branches of science for hundreds of years, the rigorous mathematical study of this phenomenon began only recently. One of the key features of fractal analysis has been the invention of the Laplacian on fractals, which originated in the physics literature. The theory of differential equations on fractals was created by J. Kigami. The Laplacian on the Sierpinski Gasket

[^1]

Figure 1: Mandelbrot Set
was first constructed as the generator of a diffusion process by S. Kusuoka and S. Goldstein. Much of the current research in the domain involves spectral analysis of the Laplacian and the theory of Dirchlet forms ${ }^{3}$. The spectrum of the eigenvalues and eigenfunctions for the Laplacian was studied in detail by M. Fukushima and T. Shima. More recent advancements can be attributed to R. Strichartz, L. Malozemov and A. Teplyaev ${ }^{4}$, who examined the spectral properties of the Laplacian on infinite Sierpinski Gaskets.

The main objective of this thesis is to classify the derivatives of the four so-called basic functions $\chi, \phi, \psi$ and $\xi$ on the Sierpinski Gasket $S$. The first third of the thesis provides the necessary foundation for our analysis. We begin with a rigorous definition of self-similarity, the Sierpinski Gasket and the Laplacian (Sections 2 and 3) and then use the properties of the basic functions to prove a number of important theorems about their local behavior. In particular, we prove that any two of the basic functions, together with the identity, serve as a basis for the 3-dimensional vector space of all harmonic functions on $S$, denoted $\mathcal{H}(S)$; that the basic functions have a unique continuous extension to $S$; that they are rational at all rational points; and

[^2]that they are strictly increasing on $[0,1]$. We also show that the derivative of any basic function at any dyadic point exists and is either 0 or $\infty$. These facts motivate our discussion of the basic functions' derivatives at rational nondyadic points, which begins in Section 4.

Instead of determining the derivative of each basic function individually, the procedure ${ }^{5}$ outlined in Section 4.2.1 involves computing the derivative of a linear combination of the basis functions $\chi, \psi$, and 1 . Using the fact that every nondyadic rational has a periodic binary expansion and is the fixed point of some linear transformation $w(t)$, we construct a difference equation which makes it possible to determine the rate of change of the eigenfunctions $f_{W}^{ \pm}(t)$. The derivatives of the basic functions depend on the stability of the said difference equation and hence the eigenvalues of the operator $W$ which fixes $t$. In order to understand which rational nondyadic points have operators with equal eigenvalues, we group the operators which fix such points into conjugacy classes of matrices having the same trace (Section 4.2.3). Using the cyclic property of the trace and the symmetries of the basic functions, we relate the derivatives of $\phi$ and $\xi$ to those of $\chi$ and $\psi$, concluding that each of the basic functions is an example of a continuous, monotone function with vanishing derivative on one dense set and infinite derivative on another.

It is worth noting that the results of this thesis lead to other considerations and conjectures, which are summarized in Section 5. In particular, we discuss the combinatorics of operator classification, in particular the possibility of using 2-ary necklaces to identify a pattern in the derivatives of the basic functions, if such a pattern exists. We also conjecture about the derivatives of the basic functions at irrational points, a current open problem of unknown difficulty.

[^3]
## 2 Harmonic Functions on the Sierpinski Gasket

### 2.1 Self-Similar Fractals and the Sierpinski Gasket

As mentioned in the introduction, one of the defining properties of fractals is that they are self-similar, meaning when any part of a fractal is magnified, the resulting picture bears an exact resemblance to the whole and this likeness continues to repeat through further magnifications, to infinity. We will use the Fixed Point Theorem for Contracting Mappings ${ }^{6}$ to give a rigorous definition of this idea.

Suppose $(M, d)$ is a metric space and let $\mathbb{K}(M)$ denote the collection of all nonempty compact subsets of $M$. Recall that a subset of a metric space is compact if every sequence $\left\{x_{n}\right\}$ in the subset has a convergent subsequence. We would like to define a distance $d_{H}$ between two compact sets such that $\left(\mathbb{K}(M), d_{H}\right)$ is itself a metric space. We first define the distance between a point $x$ and a compact set $Y$ :

$$
\begin{equation*}
d_{H}(x, Y)=\min _{y \in Y} d(x, y) \tag{2.1}
\end{equation*}
$$

Now, if $X$ is also a compact set, the Hausdorff distance between $X$ and $Y$ is

$$
\begin{align*}
d_{H}(X, Y) & =\max _{x \in X} d_{H}(x, Y)+\min _{y \in Y} d_{H}(y, X)  \tag{2.2}\\
& =\max _{x \in X} \min _{y \in Y} d(x, y)+\max _{y \in Y} \min _{x \in X} d(x, y) . \tag{2.3}
\end{align*}
$$

In other words, two sets are within Hausdorff distance $d$ from each other if and only if any point of one set is within distance $d$ from some point of the other set. One can check that $d_{H}: \mathbb{K}(M) \times \mathbb{K}(M) \rightarrow \mathbb{R}$ defined in (2.2) satisfies the metric

[^4]axioms.

Theorem 2.1. If the metric space $M$ is complete, then the space $\mathbb{K}(M)$ is also complete ${ }^{7}$.

Proof. Let $\left\{X_{n}\right\}$ be a Cauchy sequence in $\mathbb{K}(M)$ Let $X$ denote the set of all points $x \in M$ such that, for any neighborhood $U$ of $x$ one has that $U \cap X_{n} \neq \varnothing$ for infinitely many $n$. One can check that $X \in \mathbb{K}(M)$. We show that $X_{n} \rightarrow X$.

Fix $\epsilon>0$ and let $N$ be such that $d_{H}\left(X, X_{n}\right)<\epsilon$ for all $m, n \geq N$. It suffices to show that $d_{H}\left(X, X_{n}\right)<2 \epsilon$ for any $n \geq N$.

Let $D_{\epsilon}(x)$ denote the $\epsilon$-ball around $x \in X$ for any $x$. There exists an $m \geq N$ such that $D_{\epsilon}(x) \cap X_{m} \neq \varnothing$. In other words, there is a point $y \in X_{m}$ such that $d(x, y)<\epsilon$. Since $X_{n}$ is Cauchy, $d_{H}\left(X_{n}, X_{m}\right)<\epsilon$, so $d_{H}\left(y, X_{n}\right)<\epsilon$ and therefore $d_{H}\left(x, X_{n}\right)<2 \epsilon$ by the triangle inequality.

We claim that $d_{H}(x, X)<2 \epsilon$ for any $x \in X_{n}$ : let $n_{1}=n$ and for every integer $k>1$ choose an index $n_{k}$ such that $n_{k}>n_{k+1}$ and $d_{H}\left(X_{a}, X_{b}\right)<\epsilon / 2^{k}$ for all $a, b \geq n_{k}$.

Now define a sequence of points $\left\{x_{k}\right\}$, where $x_{k} \in X_{n_{k}}$ by letting $x_{1}=x$ and $x_{k+1}$ be the point of $X_{n_{k+1}}$ such that $d\left(x_{k}, x_{k+1}\right)<\epsilon / 2^{k}$ for all $k$. We know such a point can be found is because $d_{H}\left(X_{n_{k}}, X_{n_{k+1}}\right)<\epsilon / 2^{k}$. Since $\sum_{k=1}^{\infty} d\left(x_{k}, x_{k+1}\right)<2 \epsilon<\infty$, the sequence $\left\{x_{k}\right\}$ is a Cauchy sequence and hence converges to a point $y \in M$. Then $d(x, y)=\lim _{n \rightarrow \infty} d\left(x, x_{n}\right) \leq \sum_{k=1}^{\infty} d\left(x_{k}, x_{k+1}\right)<2 \epsilon$. Because $y \in X$ by construction, it follows that $d_{H}(x, X)<2 \epsilon$ and therefore $d_{H}\left(X, X_{n}\right)<2 \epsilon$.

Recall that map $f$ from a metric space $(M, d)$ to itself is called contracting if there is a real number $\lambda \in(0,1)$ such that $d(f(x), f(y)) \leq \lambda \cdot d(x, y)$ for all $x, y \in M$. Let $\left\{f_{1}, f_{2}, \ldots, f_{k}\right\}$ in $M$ be a finite sequence of functions where each $f_{i}: M \rightarrow M$

[^5]is a contracting mapping. Define the transformation $F: \mathbb{K}(M) \rightarrow \mathbb{K}(M)$ by
\[

$$
\begin{equation*}
F(X)=f_{1}(X) \cup f_{2}(X) \cup \cdots \cup f_{k}(X) \tag{2.4}
\end{equation*}
$$

\]

Theorem 2.2. If $(M, d)$ is a nonempty complete metric space, the map $F$ given by (2.4) is contracting, meaning there exists a unique non-empty compact subset $X \subset M$ satisfying $F(X)=X$. In particular, if $X_{0} \in M$ and we define $X_{1}=F\left(X_{0}\right), X_{2}=$ $F\left(X_{1}\right), \ldots, X_{n+1}=F\left(X_{n}\right), \ldots$, then

$$
\lim _{n \rightarrow \infty} X_{n}=X
$$

We call $X$ the invariant set for the functions $\left\{f_{1}(X) \cup f_{2}(X) \cup \cdots \cup f_{k}(X)\right\}$.

Proof. Since $M$ is complete, by Theorem 2.1, the space $\left(\mathbb{K}(M), d_{H}\right)$ is also complete. The desired conclusion will follow from the Contraction Mapping Principle ${ }^{8}$ if we can show that $F: \mathbb{K}(M) \rightarrow \mathbb{K}(M)$ defined by $F(X)=f_{1}(X) \cup f_{2}(X) \cup \cdots \cup f_{k}(X)$ for every nonempty compact set $X$ is contracting.

Let $A$ and $B$ be any two nonempty compact subsets of $M$ and consider any $\delta \geq d_{H}(A, B)$. Since the $f_{i}: M \rightarrow M$ are contracting mappings, there is some $0<\lambda_{i}<1$ such that $d\left(f_{i}(a), f_{i}(b)\right) \leq \lambda_{i} \cdot d(a, b)$ for all $a, b \in M$. Let $\lambda=$ $\max \left\{\lambda_{1}, \ldots, \lambda_{n}\right\}$. We claim that $d_{H}(F(A), F(B)) \leq \lambda \cdot d_{H}(A, B)$. First, for any $x \in F(A)=f_{1}(A) \cup f_{2}(A) \cup \cdots \cup f_{k}(A)$, there is some $a_{i} \in A$ such that $x=f_{i}\left(a_{i}\right)$. Since $\delta \geq d_{H}(A, B)$, there is some $b_{i} \in B$ such that $d\left(a_{i}, b_{i}\right) \leq \delta$. Hence,

$$
d\left(x, f_{i}\left(b_{i}\right)\right)=d\left(f_{i}\left(a_{i}\right), f_{i}\left(b_{i}\right)\right) \leq \lambda_{i} \cdot d\left(a_{i}, b_{i}\right) \leq \lambda \delta
$$

[^6]This shows that $F(A) \subseteq D_{\lambda \delta}(F(B)$ ) (the "ball" of radius $\lambda \delta$ around $F(B)$ ). One can similarly show that $F(B) \subseteq D_{\lambda \delta}(F(A))$. Since this holds for all $\delta \geq d_{H}(A, B)$, we have that $d_{H}(F(A), F(B)) \leq \lambda \cdot d_{H}(A, B)$ where $\lambda=\max \left\{\lambda_{1}, \ldots, \lambda_{n}\right\}$. Since $0<\lambda_{i}<1$ for all $i$, we have $0<\lambda<1$, so $F$ is indeed a contracting mapping with invariant set $X$.

We call the invariant set X a homogeneous self-similar fractal set and the system of functions $\left\{f_{1}, \ldots, f_{k}\right\}$, the Iterated Function System (or IFS for short) defining $X$.


Figure 2: Sierpinski Gasket

Let us now use Theorem 2.2 to provide a rigorous definition of the Sierpinski Gasket. Begin by fixing three contractions $F_{1}, F_{2}, F_{3}: \mathbb{R}^{2} \rightarrow \mathbb{R}^{2}$ given by

$$
\begin{equation*}
F_{1}(x)=\frac{x}{2}, \quad F_{2}(x)=\frac{x}{2}+\left(\frac{1}{2}, 0\right), F_{3}(x)=\frac{x}{2}+\left(\frac{1}{4}, \frac{\sqrt{3}}{4}\right) \quad \forall x \in \mathbb{R}^{2} \tag{2.5}
\end{equation*}
$$

The Sierpinski Gasket is the unique compact set $S$ such that $S_{n}=F_{1}\left(S_{n-1}\right) \cup$ $F_{2}\left(S_{n-1}\right) \cup F_{3}\left(S_{n-1}\right)$, where $S_{0}=\left\{(0,0),(1,0),\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)\right\}$,

$$
\begin{equation*}
S_{\infty}=\cup_{n=1}^{\infty} S_{n} \quad \text { and } \quad S=\bar{S}_{\infty} \tag{2.6}
\end{equation*}
$$

Iterating accordingly generates Figure 2 above. In our notation, $S_{n}$ is the $n^{\text {th }}$ approximation of the (in this case) 0-dimensional Sierpinski Gasket.

One should note that a similar iterative scheme can be used to define a higher dimensional gasket. For example, let $M=\mathbb{C}$ and the initial set be the solid triangle with vertices $S_{0}=\left\{1, \omega=e^{\frac{2 \Pi i}{3}}, \bar{\omega}=e^{-\frac{2 \Pi i}{3}}\right\}$. Then, if

$$
F_{1}(z)=\frac{z+1}{2}, F_{2}(z)=\frac{z+\omega}{2}, F_{3}(z)=\frac{z+\bar{\omega}}{2}
$$

the sequence $S_{n}=F^{n}\left(X_{0}\right)$ is decreasing and $S_{\infty}=\cup_{n=1}^{\infty} S_{n}$. Since the initial set was a solid triangle rather than a set of points, $\bar{S}_{\infty}$ is the 2-dimensional Sierpinski Gasket. Figure 3 shows the first two iterations.


Figure 3: 2-dimensional Sierpinski Gasket

### 2.2 Harmonic Functions and the Laplace Operator

In physics, the Laplacian, or Laplace Operator, denoted by $\Delta$, represents the net flux into an infinitesimal volume at each point in space. Mathematically, $\Delta$ on $\mathbb{R}^{n}$ is given by

$$
\Delta f=\sum_{k=1}^{n}\left(\frac{\partial}{\partial x^{k}}\right)^{2} f .
$$

It turns out that this operator can be defined on any Riemannian manifold $M$. A function is said to be harmonic if it satisfies Laplace's Equation, $\Delta f=0$. Two important properties of harmonic functions can be deduced from Laplace's equation.

Theorem 2.3 (Mean Value Property). If $D(x, r)$ is a ball with center $x$ and radius $r$ which is completely contained in $U$, then the value $f(x)$ of the harmonic function $f$ at the center of the ball is given by the average value of $f$ on the surface (and interior) of the ball. In other words

$$
u(x)=\frac{1}{\omega_{n} r^{n-1}} \oint_{\partial D(x, r)} u d S=\frac{n}{\omega_{n} r^{n}} \int_{D(x, r)} u d V
$$

where $\omega_{n}$ is the surface area of the $n$-dimensional sphere.

Theorem 2.4 (Maximum Principle). If $M$ is a connected manifold with a boundary, then any non-constant real harmonic function on $M$ attains its maximal value only on the boundary $\partial M$.

The proofs of Theorems 2.3 and 2.4 can be found in any elementary real analysis text and are left out here. The following corollary is relevant to our study of harmonic fractal functions on $S$.

Corollary 2.5. The boundary values of a harmonic function on a manifold $M$ determine its values at all interior points of $M$.

Corollary 2.5 is proven for $M=S_{n}$, the $n$-dimensional Sierpinski Gasket, in the following section.

### 2.3 Harmonic Functions on $S$

A powerful mathematical method for studying complicated fractal sets is to instead study the space of functions on these sets. We begin by giving a precise definition of a harmonic function on $S$. Let $f_{A B}^{C}(t)$ denote a function that takes on the values $A, B$ and $C$ at each boundary point on the triangle defined by the set $S_{0}=\left\{(0,0),(1,0),\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)\right\}$ (see Figure 4 on the next page). Let $x \in S_{n}$ be a boundary value on the $n^{\text {th }}$ approximation of the Sierpinski Gasket, where $n \neq 0$. We call the four points $y_{i}$ such that $\left|x-y_{i}\right|=\frac{1}{2^{n}}$ the neighbors of $x$. For example, in Figure 4, the neighbors of $b$ on $S_{3}$, the $3^{r d}$ approximation of the gasket, are $a_{3}$, $c_{3}, a_{1}$ and $c_{1}$.

We are now ready to define the Laplacian on $S$. Suppose we assign values to all the points on $S_{n}$, the nth approximation of $S$, and call this function $f$. Let $x \in S_{n}$ and let $y_{1}, y_{2}, y_{3}, y_{4} \in S_{n}$ be the four neighbors of $x$. Then the Laplacian on $S_{n}$ is given by

$$
\begin{equation*}
\Delta_{n} f(x)=\frac{1}{4} \sum_{i=1}^{4} y_{i}-f(x) \tag{2.7}
\end{equation*}
$$

Theorem 2.6. There exists a harmonic function on $S_{n}$ given by (2.7) that is uniquely determined by its boundary values.

Proof. Let $f$ be the desired harmonic function on $S_{n}$, assuming it exists. By (2.7), the harmonicity of $f$ is equivalent to a system of $\frac{3^{n}-3}{2}$ linear non-homogeneous equations whose right hand side depends on the boundary values of $f$. Let $A$ be the coefficient matrix of the system, $x$ the vector of unknowns and $b$ the vector corresponding to the right hand side. Recall that if $A x=0$ has a solution other than
$x=0, A$ is not invertible and the system has infinitely many solutions. By the Maximum Principle, if $b=0$, meaning $f$ has zero boundary values, $f$ is identically zero. So $x=0$ is the only solution to $A x=0$. The non-homogeneous system describing the harmonicity of $f$ has a unique solution.


We now give a precise definition of a harmonic function on $S_{\infty}$, where $S_{\infty}=$ $\cup_{n=1}^{\infty} S_{n}$, as defined in (2.6). A function on $S_{\infty}$ is called harmonic if its restriction to every $S_{n}$ is harmonic. In other words, $S_{\infty}$ is harmonic if and only if

$$
\begin{equation*}
\Delta_{n} f(x)=0, \quad \forall n \tag{2.8}
\end{equation*}
$$

Figure 4: $f_{A B}^{C}$ on $S_{3}$
Let us consider the set of all harmonic functions on $S_{\infty}$. It turns out that this set is closed under addition and scalar multiplication and hence forms a vector space, namely the space of all harmonic functions on $S_{\infty}$. Let us denote this vector space by $\mathcal{H}\left(S_{\infty}\right)$. Since a harmonic function is uniquely determined by its boundary values (Theorem 2.6), $\mathcal{H}\left(S_{\infty}\right)$ has dimension 3. Indeed, the natural coordinates for a function $f_{A B}^{C} \in \mathcal{H}\left(S_{\infty}\right)$ are its boundary values $A, B$ and $C$. Note that we can restrict $f$ to $S_{k} \in S_{n}$ (i.e., to any $k$-dimensional approximation of $S$, where $k<n$ ). For example, in Figure 4 on the previous page,

$$
\begin{align*}
& 4 b_{1}=A+c+a_{1}+c_{1} \\
& 4 c=b_{1}+a_{1}+b_{2}+a_{2} \\
& \Delta_{3} f=\quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots  \tag{2.9}\\
& 4 a_{3}=C+b+c_{3}+b_{3} \\
& 4 c_{3}=a_{3}+b_{3}+b+a \\
& 4 c=A+b+a+B \\
& \left.\Delta_{3} f\right|_{S_{2}}=4 b=A+c+a+C  \tag{2.10}\\
& 4 a=C+b+c+B
\end{align*}
$$

The system in (2.9) has 12 equations and 12 unknowns whereas the system in (2.10) has 3 equations and 3 unknowns. One can see that (2.10) is a restricted system. We use this notation for the restriction of $f$ to some $S_{k}, k<n$, in the proof of the following theorem.

Theorem 2.7. The restriction of any harmonic function on $S_{n}$ to $S_{n-1}$ is also harmonic.

Proof. Let $f$ be a harmonic function on $S_{n}$. By (2.7), $\Delta_{n} f=0$. Suppose $f$ has a nonzero boundary. We can view $f$ as an eigenfunction of the Laplace operator

$$
\begin{equation*}
\Delta_{n} f=\lambda f \tag{2.11}
\end{equation*}
$$

Here, $\lambda$ is the eigenvalue of the unrestricted system on $S_{n}$. Since, by the Maximum Principle, $f \neq 0,(2.7)$ implies $\lambda=0$. Now, let $\mu$ be the eigenvalue of the Laplacian determining the restriction of $f$ to $S_{n-1}$.

$$
\begin{equation*}
\left.\Delta_{n} f\right|_{S_{n-1}}=\mu f \tag{2.12}
\end{equation*}
$$

Our goal is to show that $\left.\Delta_{n} f\right|_{S_{n-1}}=0$, meaning $f$ restricted to $S_{n-1}$ is also harmonic. Refer to Figure 4. By (2.8), $\Delta_{n} f=\lambda f$ gives the following system of equations ${ }^{9}$

$$
\begin{aligned}
& 4(\lambda+1) c=b_{1}+a_{1}+b_{2}+a_{2}, \\
& 4(\lambda+1) a_{1}=b+c_{1}+b_{1}+c, \quad 4(\lambda+1) b_{1}=A+c+c_{1}+a_{1}, \\
& 4(\lambda+1) c_{1}=A+b_{1}+a_{1}+b, \quad 4(\lambda+1) a_{2}=c+B+c_{2}+b_{2}, \\
& 4(\lambda+1) b_{2}=c+a_{2}+c_{2}+a, \quad 4(\lambda+1) c_{2}=a+B+a_{2}+b_{2}
\end{aligned}
$$

Adding the last four equations, we have that $4(\lambda+1)\left(b_{1}+a_{1}+b_{2}+a_{2}\right)=\left(b_{1}+a_{1}+\right.$ $\left.b_{2}+a_{2}\right)+(A+B+b+a)+2\left(c_{1}+c_{2}\right)+4 c$. Adding the first two equations gives $4(\lambda+1)\left(c_{1}+c_{2}\right)=\left(b_{1}+a_{1}+b_{2}+a_{2}\right)+(A+B+b+a)$. Combining these expressions, we can write $\left(c_{1}+c_{2}\right)$ in terms of $(A+B+b+a)$ and $c$. The the first equation of the original system becomes

$$
\begin{equation*}
(2 \lambda+3)(A+B+b+a)=4(\lambda+1)(2 \lambda+3)(4 \lambda+1) c \tag{2.13}
\end{equation*}
$$

But (2.12) implies

$$
\begin{equation*}
A+B+b+a=4(\mu+1) c \tag{2.14}
\end{equation*}
$$

Since $\lambda \neq-\frac{2}{3}$ (because $\lambda=0$ ),

$$
\begin{equation*}
4(\mu+1)=4(\lambda+1)(4 \lambda+1) \quad \text { or } \quad \mu=\lambda(4 \lambda+5) \tag{2.15}
\end{equation*}
$$

Therefore, $\mu=0$, meaning (2.12) reduces to $\left.\Delta_{n} f\right|_{S_{n-1}}=0$. The restriction of $f$ to $S_{n-1}$ is also harmonic.

[^7]Theorem 2.7 has an important corollary. Recall from real analysis that a continuous function on a compact set is uniformly continuous on that set ${ }^{10}$. Another well-known theorem states that if $M$ and $N$ are metric spaces and $f: U \in M \rightarrow N$ is a uniform map, where $N$ is complete, $f$ can be extended to a unique continuous function on $\bar{U}$. We use these theorems to prove the following corollary.

Corollary 2.8. Any harmonic function on $S_{\infty}$ is uniformly continuous and hence has a unique continuous extension to $S$.

Proof. First, let $f$ be a harmonic function on $S_{n}$ and define the variation of $f$ by

$$
\operatorname{var} f_{A B}^{C}=\sup _{x, y \in S_{\infty}}|f(x)-f(y)|=\max \{|A-B|,|B-C|,|C-A|\}
$$

Assuming $A \leq B \leq C$, we have by Theorem 2.4 that $A \leq f(x) \leq C$ and $A \leq$ $f(y) \leq C$ for $x, y \in S_{\infty}$, which implies the second equality. Let $x, y \in S_{n}$ be any two neighbors. By (2.16) (below),

$$
\left|f_{A B}^{C}(x)-f_{A B}^{C}(y)\right| \leq \operatorname{var} f \cdot\left(\frac{3}{5}\right)^{n}=\operatorname{var} f \cdot 2^{-n \beta} \leq \text { const } \cdot d(x, y)^{\beta}, \quad \beta=\log _{2} \frac{5}{3} \approx 0.73697 \ldots
$$

The above inequality shows that $f$ is continuous. Since $S_{n}$ is a compact set, it follows from the Uniform Continuity Theorem that $f$ is uniformly continuous on $S_{n}$. Therefore $f$ has a unique continuous extension to $S=\bar{S}_{\infty}$, as desired.

We now derive explicit formulas for the values of a harmonic function $f$ given the boundary values $A, B$ and $C$. Let $x_{1}, x_{2}, x_{3} \in \mathbb{R}^{2}$ be neighbor points of $S_{n}$ which form an equilateral triangle (see Figure 4). Let $y_{1}=\frac{x_{2}+x_{3}}{2}, y_{2}=\frac{x_{1}+x_{3}}{2}$, and $y_{3}=\frac{x_{1}+x_{2}}{2} \in \mathbb{R}^{2}$. Then, $y_{1}, y_{2}, y_{3}$ are neighbor points in $S_{n+1}$ and for any harmonic

[^8]function $f$ on $S_{n+1}$ we have
\[

$$
\begin{align*}
& f\left(y_{1}\right)=\frac{f\left(x_{1}\right)+2 f\left(x_{2}\right)+2 f\left(x_{3}\right)}{5}=\frac{A+2 B+2 C}{5}  \tag{2.16}\\
& f\left(y_{2}\right)=\frac{2 f\left(x_{1}\right)+f\left(x_{2}\right)+2 f\left(x_{3}\right)}{5}=\frac{2 A+B+2 C}{5}  \tag{2.17}\\
& f\left(y_{3}\right)=\frac{2 f\left(x_{1}\right)+2 f\left(x_{2}\right)+f\left(x_{3}\right)}{5}=\frac{2 A+2 B+C}{5} \tag{2.18}
\end{align*}
$$
\]

These relations follow directly from the system of equations in (2.10).
Having provided an overview of harmonic functions on the Sierpinski Gasket, we are now ready to define the four basic functions.

## 3 The Four Basic Functions $\chi, \phi, \psi, \xi$

### 3.1 Definition and Identities

We mentioned in the previous section that $\mathcal{H}\left(S_{\infty}\right)$, the vector space of all harmonic functions on $S$ has dimension 3. Let us now choose a basis for this space. Since $\mathcal{H}\left(S_{\infty}\right)$ must contain the constant function $f \equiv 1$, the basis must contain two non-trivial harmonic functions. The functions $\left\{f_{11}^{1}, f_{01}^{0}, f_{01}^{1}\right\}$ form one such basis. Although these functions span $\mathcal{H}\left(S_{\infty}\right)$, it is convenient to introduce two additional functions $f_{01}^{-1}$ and $f_{01}^{2}$ and study all four functions simultaneously.

As an aside, note that $S_{3}$, the group of all permutations of three symbols (also the group of symmetries of a regular triangle, or $D_{3}$, the dihedral group) has a natural action on $S$. This action corresponds to a permutation of the boundary values of a basic function $f_{A B}^{C} \in S$ : for any function $f_{A B}^{C} \in S$ and any permutation $s \in S_{3}$ we associate the linear operator $T_{s}$ such that

$$
\begin{equation*}
T_{s} f_{A B}^{C}=f_{A^{\prime} B^{\prime}}^{C^{\prime}} \tag{3.1}
\end{equation*}
$$

Using this action, one can show that

$$
\begin{equation*}
f_{A B}^{C}(t)=f_{B A}^{C}(1-t) \quad \text { and } \quad f_{A B}^{C}+f_{B C}^{A}+f_{C A}^{B} \equiv a+b+c \tag{3.2}
\end{equation*}
$$

Due to the geometry of $S$, coming up with a systematic approach to the study of harmonic functions on the set can be complicated. To make things simpler, rather


Figure 5: Geometric Interpretation of $\chi, \phi, \psi, \xi$ on $\left.S\right|_{[0,1]}$
than viewing a function $f_{A B}^{C}$ on all of $S$, we consider its restriction to the bottom side of the gasket, and identify with it the unit interval $I=[0,1]$ (see Figures 5 and 6). We call the restrictions of the four functions $f_{01}^{-1}, f_{01}^{0}, f_{01}^{1}, f_{01}^{2}$ to $I \chi, \psi, \phi, \xi$ respectively. Figure 6 gives a plot of these functions. Note that, since the boundary values of $f_{01}^{-1}, f_{01}^{0}, f_{01}^{1}, f_{01}^{2}$ form an arithmetic progression so do the restrictions $\chi, \psi, \phi, \xi$.

The following corollary gives the basic function identities and some key relations.

Corollary 3.1. The basic functions satisfy the following identities and relations:


Figure 6: Graphs of $\chi, \phi, \psi, \xi$ on $[0,1]$

$$
\begin{array}{ccc}
\text { (i) } & \text { (ii) } & \text { (iii) } \\
\chi\left(\frac{t}{2}\right)=\frac{1}{5} \cdot \chi(t) & \chi(t)=1-\xi(1-t) & \phi(t)=\frac{2 \chi(t)-\chi(1-t)+1}{3} \\
\phi\left(\frac{1+t}{2}\right)=\frac{2}{5}+\frac{3}{5} \cdot \phi(t) & \psi(t)=1-\phi(1-t) & \psi(t)=\frac{\chi(t)-2 \chi(1-t)+2}{3} \\
\psi\left(\frac{t}{2}\right)=\frac{3}{5} \cdot \psi(t) & & \\
\xi\left(\frac{1+t}{2}\right)=\frac{4}{5}+\frac{1}{5} \cdot \xi(t) & &
\end{array}
$$

Proof. By writing in the missing values in Figure 5, one sees that the boundary values of the restriction of $f_{01}^{-1}$ to the boundary of the small triangle is proportional to the boundary values of the initial function on the big triangle with coefficient $\frac{1}{5}$. This proves the first identity. The same argument can be used to prove the identity for $\psi$, except now the coefficient of proportionality is $\frac{3}{5}$. The remaining relations
come from writing the basic functions as linear combinations of $\chi$ and $\psi$.
Now that we have introduced the basic functions and proven some theorems about the space they span, several questions come to mind. First, it is clear that one can use the identities in Corollary 3.1 to evaluate the basic functions at any dyadic point. Since doing this directly can be somewhat tedious, one wonders if there is any shortcut. Moreover, we currently have no method for evaluating the basic functions at nondyadic points. Since any harmonic function on $S_{\infty}$ has a unique continuous extension to $S$, one should be able to compute the exact value of a basic function at any point in $[0,1]$. We address these questions in the coming pages.

### 3.2 Symmetries

It is clear from Corollary 3.1 that the basic functions exhibit a number of symmetries which are so to speak encoded in the relations of the previous section. These symmetries greatly facilitate computations involving the said functions. There are two kinds of symmetries that arise: multiplicative symmetry and reflexive symmetry. A function $f$ is called multiplicatively symmetric if $f(t)=c f(k \cdot t), \forall t$, where $c, k \in \mathbb{R} ; f$ is reflexively symmetric about a point $s \in \mathbb{R}$ if $f(s-t)=f(s+t), \forall t$.

Theorem 3.2. The basic functions exhibit the following symmetries:
(i) The graph in Figure 7 is central symmetric about the point $\left(\frac{1}{2}, \frac{1}{2}\right) \in \mathbb{R}^{2}$.
(ii) All four basic functions are multiplicatively reflexive: $\chi$ and $\xi$ and consist of the self-similar $I^{0}$ and its 180 rotation $I^{180} ; \phi$ and $\psi$ and consist of the self-similar $I I^{0}$ and its $180^{\circ}$ rotation, $I I^{180}$.


Figure 7: Symmetries of $\chi, \phi, \psi, \xi$
(iii) The functions $\phi$ and $\psi$ exhibit reflexive symmetry. Each segment $I I^{0} \in\left[\frac{1}{2^{n+1}}, \frac{1}{2^{n}}\right]$ is reflexively symmetric about the point $\frac{3}{2^{n+2}}$.

Proof. Parts (i) and (ii) follow directly from Corollary 3.1. Since $\phi$ and $\psi$ are mirror images of each other, it is enough to prove (iii) for $\psi$. Because $\psi$ is multiplicatively symmetric, we can consider any interval $\left[\frac{1}{2^{n+1}}, \frac{1}{2^{n}}\right] \in[0,1]$. Let $n=0$, so our interval is $\left[\frac{1}{2}, 1\right]$. Let

$$
\Delta \psi(t)=\psi(t)-\psi\left(t-\frac{1}{2^{n}}\right), \quad t \in\left[\frac{1}{2}, 1\right]
$$

be the first difference of $\psi$. To prove that $\psi(t)$ is reflexive, we show that $\Delta \psi(t)$ is symmetric about $t=\frac{3}{4}$. Computing $\Delta \psi(t)$ numerically ${ }^{11}$ gives Figure 8 below. The

[^9]

Figure 8: $\Delta \psi(t)=\psi(t)-\psi\left(t-\frac{1}{2^{n}}\right)$
symmetry and reversed orientation of $\Delta \psi(t)$ indicates the segment $I I^{0}$ is reflexively symmetric about the point $t=\frac{3}{4}$.

### 3.3 Linear Operators on $\mathcal{H}$

Since the set of harmonic functions on $S$ is a vector space, it is natural to define some linear transformations on $\mathcal{H}(S)$. Let $\mathcal{H}$ denote the space of real-values functions on $[0,1]$ spanned by the restrictions of harmonic functions on $S . \mathcal{H}$ is spanned by any two of the basic functions and the constant function, so we can choose from a number of bases. Since we are interested in all four basic functions, let us introduce the following vector-valued functions representing two possible bases, namely $\{1, \chi, \psi\}$ of $\psi$ to the positive real line: $\psi(k)=\frac{\chi(k)+2 \cdot 5^{n}-2 \chi\left(2^{n}-k\right)}{3^{n+1}}$. Extensions of the basic functions go beyond the scope of this thesis.
and $\{1,1-\xi, 1-\phi\}$.

$$
\vec{h}(t)=\left(\begin{array}{c}
\chi(t)  \tag{3.3}\\
\psi(t) \\
1
\end{array}\right) \quad \text { and } \quad \vec{g}(t)=\left(\begin{array}{c}
1-\xi(t) \\
1-\phi(t) \\
1
\end{array}\right)
$$

Note that $\vec{h}(1-t)=\vec{g}(t)$. Let us also introduce the following three transformations

$$
\begin{equation*}
\alpha: t \rightarrow \frac{t}{2}, \quad \beta: t \rightarrow \frac{1+t}{2}, \quad \gamma: t \rightarrow 1-t \tag{3.4}
\end{equation*}
$$

$\alpha$ and $\beta$ are contractions to the left hand side and the right hand side of $I$, respectively, and $\gamma$ is the reflection of $t$ to the other side of the interval. Each of these transformations induces a linear operator with corresponding matrix $A, B$ or $C^{12}$. In other words, if $\vec{h} \in \mathcal{H}, A \vec{h}(t)=\vec{h}(\alpha(t)), B \vec{h}(t)=\vec{h}(\beta(t))$ and $C \vec{h}(t)=\vec{h}(\gamma(t))$. All three operators preserve the 3 -dimensional subspace $\mathcal{H}$ and are found to be

$$
A=\left(\begin{array}{ccc}
\frac{1}{5} & 0 & 0  \tag{3.5}\\
0 & \frac{3}{5} & 0 \\
0 & 0 & 1
\end{array}\right), \quad B=\left(\begin{array}{ccc}
\frac{1}{2} & \frac{3}{10} & \frac{1}{5} \\
\frac{1}{10} & \frac{3}{10} & \frac{3}{5} \\
0 & 0 & 1
\end{array}\right), \quad C=\left(\begin{array}{ccc}
\frac{1}{2} & -\frac{3}{2} & 1 \\
-\frac{1}{2} & -\frac{1}{2} & 1 \\
0 & 0 & 1
\end{array}\right)
$$

The eigenvalues of both $A$ and $B$ are $\frac{1}{5}, \frac{3}{5}$ and 1 , a fact we will make use of in the next section.

One can easily compile a table of values of $\chi, \phi, \psi$ and $\xi$ at any dyadic point using these operators. To see how, consider the point $\frac{3}{4}$. Beginning at $t=0$, one must contract to the right twice in order to arrive at the specified point. We must therefore apply first $\beta$ twice and use this composition of functions to computing $\vec{h}(\beta(\beta(0)))$.

[^10]This translates to left multiplication by $B^{2}$. So $\vec{h}\left(\frac{3}{4}\right)=B^{2} \vec{h}(0)$. Remark that there is a connection between the binary representation of a point $t \in[0,1]$ and $W$. Suppose $t=\left(0 . t_{1} t_{2} \ldots\right)_{2}, t_{i} \in\{0,1\}$ in binary and observe that $\alpha(t)=\left(0.0 t_{1} t_{2} \ldots\right)_{2}$ and $\beta(t)=\left(0.1 t_{1} t_{2} \ldots\right)_{2}$. This suggests that a 0 in the binary representation of $t$ corresponds to multiplication by $A$ and a 1 to multiplication by $B$. Indeed, $\frac{3}{4}=$ $(0.11)_{2}$.

The values in Table 3.3 were computed using these transformations and operators.

Table 3.3. Some Values of the Basic Functions at Dyadic Points

| $f \backslash t$ | 0 | $\frac{1}{8}$ | $\frac{1}{4}$ | $\frac{3}{8}$ | $\frac{1}{2}$ | $\frac{5}{8}$ | $\frac{3}{4}$ | $\frac{7}{8}$ | 1 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\chi(t)$ | 0 | $\frac{1}{125}$ | $\frac{1}{25}$ | $\frac{12}{125}$ | $\frac{1}{5}$ | $\frac{41}{125}$ | $\frac{12}{25}$ | $\frac{85}{125}$ | 1 |
| $\phi(t)$ | 0 | $\frac{14}{125}$ | $\frac{5}{25}$ | $\frac{36}{125}$ | $\frac{2}{5}$ | $\frac{65}{125}$ | $\frac{16}{25}$ | $\frac{98}{125}$ | 1 |
| $\psi(t)$ | 0 | $\frac{27}{125}$ | $\frac{9}{25}$ | $\frac{60}{125}$ | $\frac{3}{5}$ | $\frac{89}{125}$ | $\frac{20}{25}$ | $\frac{111}{125}$ | 1 |
| $\xi(t)$ | 0 | $\frac{40}{125}$ | $\frac{13}{25}$ | $\frac{84}{125}$ | $\frac{4}{5}$ | $\frac{113}{125}$ | $\frac{24}{25}$ | $\frac{124}{125}$ | 1 |

The linear operators on $\mathcal{H}$ enable us to prove the following two theorems.

Theorem 3.4. All basic functions are strictly increasing on $[0,1]$.

Proof. By Corollary 3.1, it is enough to show that any one of the basic functions is strictly increasing. We show that $\chi(t) \leq \chi(s), \forall t \leq s$. Consider the following binary representations of $t$ and $s: t=\left(0 . t_{1} t_{2} \ldots t_{k} \ldots\right)_{2}, s=\left(0 . s_{1} s_{2} \ldots s_{k} \ldots\right)_{2}$, where $t_{i}, s_{i} \in\{0,1\}$. Assume $t_{i}=s_{i}$ for $i<m$ and $t_{m}=1, s_{m}=0$. Any binary representation has an associated operator $W$ which is a product of the matrices $A$ and $B$. As noted on the previous page, since $A$ is a contraction to the left and $B$ a contraction to the right, a 0 in the binary expansion of some number
corresponds to multiplication by $A$ and a 1 in the binary expansion of some number corresponds to multiplication by $B$ (e.g., if $\frac{3}{8}=.375=(.011)_{2}, W=A B B$, meaning $\left.\vec{h}\left(\frac{3}{8}\right)=A B B \vec{h}(0)\right)$.

Let $W_{m-1}$ be the operator corresponding to $\left(0 . t_{1} t_{2} \ldots t_{m-1}\right)_{2}=\left(0 . s_{1} s_{2} \ldots s_{m-1}\right)_{2}$. Then

$$
\vec{h}(t)=W_{m-1} B \vec{h}(x), \quad \vec{h}(s)=W_{m-1} A \vec{h}(y)
$$

where $x=\left(0 . t_{m+1} t_{m+2} \ldots\right)_{2} \in[0,1]$ and $y=\left(0 . s_{m+1} s_{m+2} \ldots\right)_{2} \in[0,1]$. Since the $W_{i}$ have nonnegative entries, it is enough to show that $B \vec{h}(x)>A \vec{h}(y)$ Computing these directly, we find that

$$
B \vec{h}(x)=\left(\begin{array}{ccc}
\frac{1}{2} & \frac{3}{10} & \frac{1}{5} \\
\frac{1}{10} & \frac{3}{10} & \frac{3}{5} \\
0 & 0 & 1
\end{array}\right)\left(\begin{array}{c}
\chi(x) \\
\psi(x) \\
1
\end{array}\right) \geq\left(\begin{array}{c}
\frac{1}{5} \\
\frac{3}{5} \\
1
\end{array}\right), \quad \text { since }\left(\begin{array}{c}
\chi(x) \\
\psi(x) \\
1
\end{array}\right) \geq\left(\begin{array}{c}
0 \\
0 \\
1
\end{array}\right)
$$

while

$$
A \vec{h}(y)=\left(\begin{array}{ccc}
\frac{1}{5} & 0 & 0 \\
0 & \frac{3}{5} & 0 \\
0 & 0 & 1
\end{array}\right)\left(\begin{array}{c}
\chi(y) \\
\psi(y) \\
1
\end{array}\right) \leq\left(\begin{array}{c}
\frac{1}{5} \\
\frac{3}{5} \\
1
\end{array}\right), \quad \text { since }\left(\begin{array}{c}
\chi(y) \\
\psi(y) \\
1
\end{array}\right) \leq\left(\begin{array}{c}
1 \\
1 \\
1
\end{array}\right)
$$

Equality holds if and only if $x=0$ and $y=1$. But then $t=s$.

We now turn our attention to the rational nondyadic points.

Example 3.5. Suppose $t=\frac{1}{3}$ and we desire to evaluate the basic functions at this point. In binary, $t=\frac{1}{3}=(0 . \overline{01})_{2}$. As expected, the associated operator is
$W=(A B)^{\infty}$, where

$$
A B=\left(\begin{array}{cc|c}
\frac{1}{10} & \frac{3}{50} & \frac{1}{25} \\
\frac{3}{50} & \frac{9}{50} & \frac{9}{25} \\
\hline 0 & 0 & 1
\end{array}\right)=\left(\begin{array}{c|c}
A_{w} & \vec{v} \\
\hline \overrightarrow{0}^{T} & 1
\end{array}\right)
$$

We are therefore interested in the following limit ${ }^{13}$

$$
\begin{equation*}
\vec{h}\left(\frac{1}{3}\right)=\lim _{n \rightarrow \infty}(A B)^{n} \vec{h}(1) \tag{3.6}
\end{equation*}
$$

Applying the formula for the sum of a geometric series gives

$$
\begin{align*}
\lim _{n \rightarrow \infty}\left(\begin{array}{cc}
A_{w} & \vec{v} \\
\overrightarrow{0}^{T} & 1
\end{array}\right)^{n}\binom{1}{\overrightarrow{1}} & =\lim _{n \rightarrow \infty}\left(\begin{array}{cc}
A_{w}^{n} & \sum_{k=0}^{n-1} A_{w}^{k} \vec{v} \\
\overrightarrow{0}^{T} & 1
\end{array}\right)\binom{1}{\overrightarrow{1}}  \tag{3.7}\\
& =\lim _{n \rightarrow \infty}\left(\begin{array}{cc}
A_{w}^{n} & \left(1-A_{w}^{n}\right)\left(1-A_{w}\right)^{-1} \vec{v} \\
\overrightarrow{0}^{T} & 1
\end{array}\right)\binom{1}{\overrightarrow{1}}(3  \tag{3.8}\\
& =\left(\begin{array}{cc}
0 & \left(1-A_{w}\right)^{-1} \vec{v} \\
\overrightarrow{0}^{T} & 1
\end{array}\right)\binom{1}{\overrightarrow{1}} \tag{3.9}
\end{align*}
$$

Since $\left(1-A_{w}\right)$ is invertible, the desired limit is ${ }^{14}$

$$
\left(1-A_{w}\right)^{-1} \vec{v}=\left(\begin{array}{cc}
\frac{1025}{918} & \frac{25}{306} \\
\frac{25}{306} & \frac{125}{102}
\end{array}\right)\binom{\frac{1}{25}}{\frac{9}{25}}=\binom{\frac{2}{27}}{\frac{4}{9}}=\binom{\chi\left(\frac{1}{3}\right)}{\psi\left(\frac{1}{3}\right)}
$$

[^11]Example 3.5 motivates the following theorem.
Theorem 3.6. The values of all basic functions at rational points are rational.
Proof. If $t \in \mathbb{Z}_{\langle 2\rangle}$, the theorem follows directly from Corollary 3.1. Suppose $t \in$ $[0,1] \cap\left(\mathbb{Q}-\mathbb{Z}_{\langle 2\rangle}\right)$. Then $t$ has a periodic binary expansion, say $t=0 . \overline{t_{1} t_{2} \ldots t_{k}}$, $t_{i} \in\{0,1\}$. As in the proof of Theorem 3.4, any binary representation has an associated operator $W$ which is some product of the matrices $A$ and $B^{15}$. So

$$
\vec{h}(t)=\lim _{n \rightarrow \infty} W^{n} \vec{h}(1)
$$

where $W$ has the form $W=\left(\begin{array}{cc}A_{w} & \vec{v} \\ \overrightarrow{0}^{T} & 1\end{array}\right)$, as in Example 3.5, where $A_{w}$ is a $2 \times 2$ matrix and $\vec{v}$ is a vector in $\mathbb{R}^{2}$. Applying the formula for the sum of a geometric series,

$$
W^{n}=\left(\begin{array}{cc}
A_{w} & \vec{v} \\
\overrightarrow{0}^{T} & 1
\end{array}\right)^{n}=\left(\begin{array}{cc}
A_{w}^{n} & \sum_{k=0}^{n-1} A_{w}^{k} \vec{v} \\
\overrightarrow{0}^{T} & 1
\end{array}\right)=\left(\begin{array}{cc}
A_{w}^{n} & \left(1-A_{w}^{n}\right)\left(1-A_{w}\right)^{-1} \vec{v} \\
\overrightarrow{0}^{T} & 1
\end{array}\right)
$$

For any word $W$, the entries of $A_{w}$ are too small for the matrix to have an eigenvalue of 1 . Therefore $1-A_{w}$ is invertible and the right-most expression makes sense. Since $A_{w}$ has entries $<1, A_{w}^{n} \rightarrow 0$. Thus,

$$
\vec{h}(t)=\left(\begin{array}{cc}
0 & \left(1-A_{w}\right)^{-1} \vec{v} \\
\overrightarrow{0}^{T} & 1
\end{array}\right)\binom{1}{\overrightarrow{1}}
$$

Both $A_{w}$ and $\vec{v}$ had rational entries. Therefore $\vec{h}(t)$ must also be rational.

Theorems 3.4 and 3.6 rely on the fact that each point $t \in[0,1]$ has associated to

[^12]it a corresponding operator $W$ which is some "word" in $A$ and $B$. If $t$ is dyadic $W$ is finite; if $t$ is rational but nondyadic, $W$ is infinite and periodic; if $t$ is irrational, $W$ is infinite and aperiodic. Moreover, we saw in Example 3.5 that one can determine $W$ for any point $t$ by finding its binary expansion and converting each 0 to an $A$ and each 1 to a $B$. We restate these facts here because they are what make possible the classification of the derivatives of the basic functions, the subject of the next section and the primary objective of this thesis.

## 4 Derivatives

First and foremost, when we talk about the derivative of a basic function, we mean its rate of change with respect to $t^{16}$. In Theorem 3.4, we showed that all basic functions are strictly increasing. According to a well-known theorem in real analysis, if $f:(a, b) \rightarrow \mathbb{R}$ is a monotone function and $(a, b)$ is a bounded interval, then $f$ has a finite derivative almost everywhere. Unfortunately, this theorem tells us nothing about the points at which the derivative fails to exist or if there are indeed any such points. In order to address this issue, let us first give a precise definition of the derivative of a function on $\mathcal{H}(S)$. Without loss of generality, consider the function $\chi$ and some point $t_{0} \in[0,1]$. If $\chi^{\prime}(t)$ exists at $t_{0}$, then

$$
\begin{equation*}
\chi^{\prime}\left(t_{0}\right)=\lim _{\epsilon \rightarrow 0} \frac{\chi\left(t_{0}+\epsilon\right)-\chi\left(t_{0}\right)}{\epsilon}=\lim _{\epsilon \rightarrow 0} \frac{\chi\left(t_{0}-\epsilon\right)-\chi\left(t_{0}\right)}{\epsilon} \tag{4.1}
\end{equation*}
$$

In other words, for $\chi^{\prime}\left(t_{0}\right)$ to exist, the left and right derivatives exist and agree at $t_{0}$. Given the fractal nature of our functions, $\epsilon$ can be viewed as the rate of contraction toward the point $t_{0}$. This interpretation is discussed at length in Section 4.2.

[^13]To classify the derivatives of the basic functions, recall the transformations $\alpha(t)=$ $\frac{t}{2}$ and $\beta(t)=\frac{1+t}{2}$ and the corresponding linear operators, $A$ and $B$. As discussed in the previous section, each point $t \in[0,1]$ can be reached through a series of contractions and we call $w(t)$, the corresponding composition of functions $\alpha$ and $\beta$, the word which fixes $t$. The matrix $W$ is referred to as the corresponding operator of the associated word. There are three types of words to consider:

## 1. Finite Words $\leftrightarrow$ Dyadic Points

2. Infinite Periodic Words $\leftrightarrow$ Rational Points
3. Infinite Aperiodic Words $\leftrightarrow$ Irrational Points

Each type has a different procedure for derivative classification at the corresponding point. It turns out the derivatives of the basic functions exist and are either 0 or $\infty$ at all rational points. Keep in mind that both the left and right derivatives of a given function need to be computed to determine if the derivative exists at that point. The procedures discussed in this thesis can be used to find either of the necessary limits. Although we have no examples of a point at which the limits do not agree, this possibility cannot be ruled out, say for some irrational point.

We begin with the simplest case, namely points in $\mathbb{Z}_{\langle 2\rangle}$.

### 4.1 Derivatives at Dyadic Points

As noted in Section 3, one arrives at a dyadic point through a finite number of contractions toward that point. We begin by considering $\chi$ and $\psi$ only and then use the symmetry of the basic functions to extend our analysis to $\phi$ and $\xi$. The following lemma shows that the former two functions roughly resemble $y=x^{2}$ and $y=\sqrt{x}$ respectively.

Lemma 4.1. For $\frac{1}{2^{n+1}} \leq t \leq \frac{1}{2^{n}}$, the following inequalities hold
(i) $\frac{1}{5} t^{\beta} \leq \chi(t) \leq 5 t^{\beta}, \quad$ where $\quad \beta=\log _{2}(5) \approx 2.322 \ldots$
(ii) $\frac{3}{5} t^{\alpha} \leq \psi(t) \leq \frac{5}{3} t^{\alpha}$, where $\quad \alpha=\log _{2}\left(\frac{5}{3}\right) \approx 0.737 \ldots$

Proof. To prove (i), suppose $\frac{1}{2^{n+1}} \leq t \leq \frac{1}{2^{n}}$ and let $\beta=\log _{2}(5)$. Then $\left(\frac{1}{2^{n}}\right)^{\beta}=\frac{1}{5^{n}}$ and $\left(\frac{1}{2^{n+1}}\right)^{\beta}=\frac{1}{5^{n+1}}$ So $5^{n+1} \geq \frac{1}{t^{\beta}} \geq 5^{n}$. Also, since $\chi$ is strictly increasing and by Corollary 3.1, $\frac{1}{5^{n+1}} \leq \chi(t) \leq \frac{1}{5^{n}}$. Combining these results gives

$$
\begin{align*}
5^{n} \cdot \frac{1}{5^{n+1}} & \leq \frac{\chi(t)}{t^{\beta}} \leq \frac{1}{5^{n}} \cdot 5^{n+1}  \tag{4.2}\\
\frac{1}{5} t^{\beta} & \leq \chi(t) \leq 5 t^{\beta} \tag{4.3}
\end{align*}
$$

The proof of the (ii) is analogous.
We are now ready to classify the derivatives of the basic functions at all dyadic points.

Theorem 4.2 (Classification of Derivatives at Dyadic Points).
(i) $\chi^{\prime}(0)=\xi^{\prime}(1)=0$.
(ii) Except for the two cases in (i), every basic function has infinite derivative at all dyadic points.

Proof. We begin by proving (i). Let $\epsilon>0$. By Lemma 4.1,

$$
\frac{\epsilon^{\beta}}{5 \epsilon} \leq \frac{\chi(\epsilon)}{\epsilon} \leq \frac{5 \epsilon^{\beta}}{\epsilon}
$$

Now, since $\beta=\log _{2}(5) \approx 2.322 \ldots$,

$$
\lim _{\epsilon \rightarrow 0} \frac{\epsilon^{\beta-1}}{5}=0 \quad \text { and } \quad \lim _{\epsilon \rightarrow 0} 5 \epsilon^{\beta-1}=0
$$

By the sandwich lemma,

$$
\chi^{\prime}(0)=\lim _{\epsilon \rightarrow 0} \frac{\chi(\epsilon)-\chi(0)}{\epsilon}=\lim _{\epsilon \rightarrow 0} \frac{\chi(\epsilon)}{\epsilon}=0
$$

Since $\chi(t)=1-\xi(1-t), \forall t$, it follows that $\xi^{\prime}(1)=0$.

To prove (ii), we first show that $\psi^{\prime}(0)=\infty$. By Lemma 4.1, $\psi(t) \geq \frac{3}{5} t^{\alpha}$, where $\alpha=\log _{2}\left(\frac{5}{3}\right)<1$. This means

$$
\psi^{\prime}(0)=\lim _{\epsilon \rightarrow 0} \frac{\psi(\epsilon)-\psi(0)}{\epsilon}=\lim _{\epsilon \rightarrow 0} \frac{\psi(\epsilon)}{\epsilon} \geq \lim _{\epsilon \rightarrow 0} \frac{3 \epsilon^{\alpha}}{5 \epsilon}=\lim _{\epsilon \rightarrow 0} \frac{3}{5 \epsilon^{1-\alpha}}=\infty .
$$

Recall the vector-valued functions

$$
\vec{h}(t)=\left(\begin{array}{c}
\chi(t) \\
\psi(t) \\
1
\end{array}\right) \quad \text { and } \quad \vec{g}(t)=\left(\begin{array}{c}
1-\xi(t) \\
1-\phi(t) \\
1
\end{array}\right)
$$

Applying the contraction $\gamma$ in (3.4),

$$
\vec{g}(t)=\left(\begin{array}{ccc}
\frac{1}{2} & -\frac{3}{2} & 1  \tag{4.4}\\
-\frac{1}{2} & -\frac{1}{2} & 1 \\
0 & 0 & 1
\end{array}\right) \vec{h}(t)
$$

In other words, the $\phi$ and $\xi$ can each be expressed as a linear combination of $\chi, \psi$ and 1 (which of course makes sense since these are the basis vectors). Differentiating
both sides ${ }^{17}$ gives

$$
\binom{-\xi^{\prime}(t)}{-\phi^{\prime}(t)}=\left(\begin{array}{cc}
\frac{1}{2} & -\frac{3}{2} \\
-\frac{1}{2} & -\frac{1}{2}
\end{array}\right)\binom{\chi^{\prime}(t)}{\psi^{\prime}(t)}
$$

Taking $t=0$, it follows that $\xi^{\prime}(0)=\phi^{\prime}(0)=\infty$. By Corollary 3.1, $\chi^{\prime}(1)=\psi^{\prime}(1)=$ $\infty$.

Due to the symmetry of the basic functions, one need only consider one particular dyadic point and show that the derivatives of all the basic functions are infinite at that point. Let $t=\frac{1}{2}$ and recall the contraction $\alpha(t)=\frac{t}{2}$, with corresponding matrix $A$ defined in (3.5). Since $\frac{1}{2} \vec{h}^{\prime}\left(\frac{1}{2}\right)=A \vec{h}^{\prime}(1), \chi^{\prime}\left(\frac{1}{2}\right)=\psi^{\prime}\left(\frac{1}{2}\right)=\infty$. Applying (4.4) once more gives that $\phi^{\prime}\left(\frac{1}{2}\right)=\xi^{\prime}\left(\frac{1}{2}\right)=\infty$. The same argument can be extended inductively to all dyadic points.

It should be noted that Lemma 4.1 implies that $\chi(t)$ also has a vanishing second derivative at 0 , since $2<\beta<3$. It turns out that we can compute $\chi^{\prime \prime}(t)$ at dyadic points $t$ even though $\chi^{\prime}(t)=\infty$. This makes it possible to determine the concavity change at these points ${ }^{18}$.

We now arrive at the main objective of this work: to classify the derivatives of the basic functions at the rational nondyadic points.

[^14]
### 4.2 Derivatives at Rational Nondyadic Points

### 4.2.1 Motivation

To motivate our procedure for computing the derivatives of the basic functions at rational nondyadic points, let us begin with the following example.

Example 4.3. Suppose we desire to compute $\chi^{\prime}\left(\frac{1}{3}\right)$. Since $\frac{1}{3}$ is nondyadic, no triangle of the gasket will ever have a vertex at this point. It was shown in Example 3.5 that the word $W=A B$ fixes $\frac{1}{3}$. If one considers the geometry of the gasket, what this is saying is that, to compute $\chi\left(\frac{1}{3}\right)$, we must "build" smaller and smaller triangles around $\frac{1}{3}$, first on the right, then on the left, then on the right, and so on according to the pattern $\alpha \beta \alpha \beta \ldots$. Assuming it exists, we can apply the Mean Value Theorem to numerically approximate the derivative of $\chi$ at $\frac{1}{3}$ by "zooming in" on a neighborhood around the point and comparing the rate of change of $\chi$ in this neighborhood to the rate at which the triangles converge to $\frac{1}{3}$, marked in red in the following two figures.

## Approximation 1:



## Approximation 2:

$$
\begin{aligned}
& \chi^{\prime}\left(\frac{1}{3}\right) \approx \frac{\chi\left(\frac{3}{8}\right)-\chi\left(\frac{5}{16}\right)}{\frac{1}{16}} \\
& \quad=\frac{\chi\left(\frac{1}{3}+\left(\frac{3}{8}-\frac{1}{3}\right)\right)-\chi\left(\frac{1}{3}-\left(\frac{1}{3}-\frac{5}{16}\right)\right.}{\frac{1}{16}}
\end{aligned}
$$



Constructing smaller and smaller triangles within the gasket leads to the following difference quotient

$$
\begin{equation*}
\chi^{\prime}\left(\frac{1}{3}\right)=\lim _{\substack{\epsilon_{n}, \epsilon_{n}^{\prime} \rightarrow 0 \\ n \rightarrow \infty}} \frac{\chi\left(\frac{1}{3}+\epsilon_{n}\right)-\chi\left(\frac{1}{3}-\epsilon_{k}^{\prime}\right)}{\left(\frac{1}{2}\right)^{2 n}} \tag{4.5}
\end{equation*}
$$

where $\epsilon_{n}$ and $\epsilon_{n}^{\prime}$ both depend on $n$. Computing the limit in (4.5) numerically gives $\chi^{\prime}\left(\frac{1}{3}\right)=0$. We verify this result in Section 4.2.2.

The setup in Example 4.3 involved finding the derivative of $\chi$ by approaching $\frac{1}{3}$ from both sides. As noted earlier, we really need to consider the left and right hand limits separately and make sure they agree. Since computing $\chi^{\prime}\left(t_{0}\right)$ at any rational nondyadic point $t_{0}$ involves beginning at some dyadic point $t$ either to the left or to the right of $t_{0}$ and approaching $t_{0}$, Example 4.3 motivates the following difference quotient

$$
\begin{equation*}
\chi^{\prime}\left(t_{0}\right)=\lim _{\substack{t \rightarrow t_{0} \\ n \rightarrow \infty}} \frac{\chi\left(t_{0}+\frac{t-t_{0}}{2^{k}}\right)-\chi(t)}{\left(\frac{1}{2}\right)^{n k}} \tag{4.6}
\end{equation*}
$$

where $t_{0}$ is rational nondyadic and $t \in \mathbb{Z}_{\langle 2\rangle}$. Letting $\epsilon=t-t_{0}$, we have

$$
\begin{equation*}
\chi^{\prime}\left(t_{0}\right)=\lim _{\substack{\epsilon \rightarrow 0 \\ n \rightarrow \infty}} \frac{\chi\left(t_{0}+\frac{\epsilon}{2^{k}}\right)-\chi\left(t_{0}+\epsilon\right)}{\left(\frac{1}{2}\right)^{n k}} \tag{4.7}
\end{equation*}
$$



Figure 9: Construction of the Right Derivative

Figure 9 illustrates this construction in terms of the geometry of the gasket. Notice that if $t>t_{0}$, (4.7) is the right derivative; if $t<t_{0}$, (4.7) is the left derivative. It is our claim that the derivatives of all basic functions at all rational nondyadic points are either 0 or $\infty$. The next few pages are devoted to developing the machinery ${ }^{19}$ needed to arrive at this conclusion.

[^15]
### 4.2.2 Procedure for Derivative Computation

Begin by recalling the transformations $\alpha: t \rightarrow \frac{t}{2}$ and $\beta: t \rightarrow \frac{1+t}{2}$, which have the corresponding matrices

$$
A=\left(\begin{array}{ccc}
\frac{1}{5} & 0 & 0 \\
0 & \frac{3}{5} & 0 \\
0 & 0 & 1
\end{array}\right) \quad B=\left(\begin{array}{ccc}
\frac{1}{2} & \frac{3}{10} & \frac{1}{5} \\
\frac{1}{10} & \frac{3}{10} & \frac{3}{5} \\
0 & 0 & 1
\end{array}\right)
$$

As previously noted, $\alpha$ and $\beta$ are contraction maps with fixed points 0 and 1 respectively. It turns out that

$$
\begin{equation*}
w(t)=\frac{t+l}{2^{k}}, \quad l, k \in \mathbb{Z} \tag{4.8}
\end{equation*}
$$

fixes some rational nondyadic point $t$ and is some word in $\alpha$ and $\beta$. It is a known fact that all rational nondyadic fractions are fixed points of some linear transformation $f\left(t_{0}\right)=t_{0}$. We claim that $w(t)$ is the desired transformation. We first prove the following lemma.

Lemma 4.4. $\alpha$ and $\beta$ generate a semigroup $G$ which is free on finite words.

Proof. To show that $G$ is free, we must show that no product of $\alpha$ and $\beta$ can ever be expressed more simply in terms of other elements. Consider the action of words in $G$ on the point $t=\frac{1}{2}$. We proceed by induction on $k$, the length of the word. Since $\alpha\left(\frac{1}{2}\right)=\frac{1}{4} \neq \frac{3}{4}=\beta\left(\frac{1}{2}\right)$, words of length 1 have distinct images. Suppose words of length $k-1$ give $2^{k-1}$ distinct images $\frac{m}{2^{k}}$ where $\operatorname{gcd}\left(m, 2^{k}\right)=1$ and $0<m<2^{k}$. Now $\alpha\left(\frac{m_{1}}{2^{k}}\right)=\alpha\left(\frac{m_{2}}{2^{k}}\right)$ implies that $\frac{m_{1}}{2^{k}}=\frac{m_{2}}{2^{k}}$, meaning $m_{1}=m_{2}$. The same is true if $\beta\left(\frac{m_{1}}{2^{k}}\right)=\beta\left(\frac{m_{2}}{2^{k}}\right)$. Suppose $\alpha\left(\frac{m_{1}}{2^{k}}\right)=\beta\left(\frac{m_{2}}{2^{k}}\right)$. Then $\frac{m_{1}}{2^{k+1}}=\frac{2^{k}+m_{2}}{2^{k+1}}$, so $m_{1}=2^{k}+m_{2}$. But this is impossible since $0<m_{1}, m_{2}<2^{k}$. Hence, words of length $k$ give distinct images of $\frac{1}{2}$, meaning $G$ is free on finite words in $\alpha$ and $\beta$.

Lemma 4.5. Let $t$ be any rational of the form $\frac{p}{q}$ such that $t \in[0,1] \cap\left(\mathbb{Q}-\mathbb{Z}_{\langle 2\rangle}\right)$, $q$ is $\operatorname{odd}^{20}$, and $\operatorname{gcd}(p, q)=1$. Then
(i) $t$ can be written as $\frac{l}{2^{k}-1}$, some $k, l \in \mathbb{Z}$.
(ii) $t$ is the fixed point of some finite word in $G$ of length $k$ and this word is given by $w(t)=\frac{t+l}{2^{k}}$.

Proof.
(i) If $q=2^{k}-1$, the rational is already in the desired form. Suppose $q \neq 2^{k}-1$. Since $q$ is odd by hypothesis, $q$ divides $2^{k}-1$ for some $k \in \mathbb{Z}$.
(ii) Suppose $\operatorname{gcd}(p, q)=1$. Since $q \mid\left(2^{k}-1\right)$ for some $k \in \mathbb{Z},\left(\frac{p}{q}\right)_{2}$ repeats after $k$ digits. Hence, $\left(\frac{p}{q}\right)_{2}=\left(0 . \overline{t_{1} t_{1} \ldots t_{k}}\right)_{2}$, where $t_{i} \in\{0,1\}$. Now $\alpha\left(\frac{p}{q}\right)=$ $\left(0.0 \overline{t_{1} t_{2} \ldots t_{k}}\right)_{2}$ and $\beta\left(\frac{p}{q}\right)=\left(0.1 \overline{t_{1} t_{2} \ldots t_{k}}\right)_{2}$. Let $w$ be a word in $G$ such that $w=w\left(t_{1}\right) w\left(t_{2}\right) \ldots w\left(t_{k}\right)$ where $w(0)=\alpha$ and $w(1)=\beta$. Then

$$
\begin{align*}
w\left(t_{1}\right) w\left(t_{2}\right) \ldots w\left(t_{k}\right)\left(0 . \overline{t_{1} t_{2} \ldots t_{k}}\right)_{2} & =w\left(t_{1}\right) w\left(t_{2}\right) \ldots w\left(t_{k-1}\right)\left(0 . t_{k} \overline{t_{1} t_{2} \ldots t_{k}} \cdot \frac{2}{2}\right) \\
& =\vdots  \tag{4.10}\\
& =w\left(t_{1}\right)\left(0 . t_{2} \ldots t_{k} \overline{t_{1} t_{2} \ldots t_{k}}\right)_{2}  \tag{4.11}\\
& =\left(0 . \overline{t_{1} t_{2} \ldots t_{k}}\right)_{2}  \tag{4.12}\\
& =\frac{p}{q} \tag{4.13}
\end{align*}
$$

So there exists a word $w \in G$ of length $k$ which fixes $t$. To show $w(t)=\frac{t+l}{2^{k}}$ is the desired word, substitute $t=\frac{l}{2^{k}-1}$ :

$$
w(t)=\frac{\frac{l}{2^{k}-1}+l}{2^{k}}=\frac{l+\left(2^{k}-1\right) l}{2^{k}\left(2^{k}-1\right)}=\frac{l}{2^{k}-1}=t
$$

[^16]Let $t=\frac{l}{2^{k}-1}$ be given. By Lemma 4.5, $t$ is the fixed point of the word $w(t)=\frac{t+l}{2^{k}}$. Since the contractions $\alpha$ and $\beta$ are in one-to-one correspondence with the matrices $A$ and $B$, we have that

$$
\vec{h}\left(\frac{t+l}{2^{k}}\right)=W \vec{h}(t) \quad \text { where } \quad \vec{h}(t)=\left(\begin{array}{c}
\chi(t)  \tag{4.14}\\
\psi(t) \\
1
\end{array}\right)
$$

$W$ is the product of the matrices $A$ and $B$; therefore $W$ has the form

$$
W=\left(\begin{array}{cc}
A_{w} & \vec{v}  \tag{4.15}\\
\overrightarrow{0}^{T} & 1
\end{array}\right) \quad \text { where } \quad A_{w}=\left(\begin{array}{cc}
a_{1} & a_{2} \\
a_{3} & a_{4}
\end{array}\right), \vec{v}=\binom{v_{1}}{v_{2}}
$$

as we saw in Example 3.5. We are interested in the difference quotient

$$
\begin{equation*}
\lim _{\substack{\epsilon \rightarrow 0 \\ n \rightarrow \infty}} \frac{f\left(t+\frac{\epsilon}{2^{k}}\right)-f(t+\epsilon)}{\left(\frac{1}{2}\right)^{n k}} \tag{4.16}
\end{equation*}
$$

for $f \in \mathcal{H}(S)$. Since $\chi$ and $\psi$ are our chosen basis functions, we know that any harmonic function can be expressed as a linear combination of them, plus the identity. In particular, given $t \in \mathbb{R}$,

$$
\begin{align*}
& \chi\left(t+\frac{\epsilon}{2^{k}}\right)=a_{1}+a_{2} \chi(\epsilon)+a_{3} \psi(\epsilon)  \tag{4.17}\\
& \psi\left(t+\frac{\epsilon}{2^{k}}\right)=b_{1}+b_{2} \chi(\epsilon)+b_{3} \psi(\epsilon) \tag{4.18}
\end{align*}
$$

for some $a_{i}, b_{i} \in \mathbb{R}$. Since determining the coefficients $a_{i}$ and $b_{i}$ individually is
a rather arduous task, it makes sense to consider a function $f$ which is a linear combination of $\chi$ and $\psi$ and try to compute its derivative. The logical choices for $f$ are the eigenfunctions of $W$, where $W$ is the word which fixes a given rational nondyadic point $t$. Ignoring the eigenvalue 1 which has the trivial eigenfunction $1(t)$, let $\lambda_{W}^{+}$and $\lambda_{W}^{-}$be the other two eigenvalues of $W$ and $f_{W}^{ \pm}(t)$ the corresponding eigenfunctions. From (4.15), these are computed to be

$$
\begin{gather*}
\lambda_{W}^{ \pm}=\frac{\left(a_{1}+a_{4}\right) \pm \sqrt{\left(a_{1}-a_{4}\right)^{2}+4 a_{2} a_{3}}}{2}  \tag{4.19}\\
f_{W}^{ \pm}(t)=\chi(t)+\frac{\lambda_{W}^{ \pm}-a_{1}}{a_{3}} \psi(t)+\frac{a_{3} v_{1}+v_{2}\left(\lambda_{W}^{ \pm}-a_{1}\right)}{a_{3}\left(\lambda_{W}^{ \pm}-1\right)} 1(t) \tag{4.20}
\end{gather*}
$$

Note that $\lambda_{W}^{ \pm}$are also the eigenvalues of the matrix $A_{w}$ in (4.15).

Remark 4.6. Any operator $W$ has real eigenvalues and hence real eigenfunctions, meaning $f_{W}^{ \pm}(t) \in \mathcal{H}(\mathcal{S}, \mathbb{R})$ for any operator $W$.

Proof. Let $W_{i j}$ denote the entry in the $i^{\text {th }}$ row and $j^{\text {th }}$ column of $W$. Since the components of $A$ and $B$ are greater than or equal to zero, $W_{i j} \geq 0$ for all $W$. Hence, $\left(W_{11}-W_{22}\right)^{2} \geq 0$ and $4 W_{12} W_{21} \geq 0$. Since $W_{11}=a_{1}, W_{12}=a_{2}, W_{21}=a_{3}$ and $W_{22}=a_{4}$, we have that $\left(a_{1}-a_{4}\right)^{2}+4 a_{2} a_{3} \geq 0$ so $W$ has real eigenvalues. It follows that the eigenfunctions $f_{W}^{ \pm}(t) \in \mathcal{H}(\mathcal{S}, \mathbb{R})$.

Let $t \in[0,1] \cap\left(\mathbb{Q}-\mathbb{Z}_{\langle 2\rangle}\right)$ be given and notice that

$$
\begin{equation*}
w(t \pm \epsilon)=\frac{(t \pm \epsilon)+l}{2^{k}}=\frac{t+l}{2^{k}} \pm \frac{\epsilon}{2^{k}}=w(t) \pm \frac{\epsilon}{2^{k}} \tag{4.21}
\end{equation*}
$$

so $w(t \pm \epsilon)-w(t)= \pm \frac{\epsilon}{2^{k}}$ and the rate of contraction toward $t$ is $\frac{1}{2^{k}}$, as expected. If $t$ is the fixed point of $w(t)$, then (4.21) becomes $w(t+\epsilon)=t+\frac{\epsilon}{2^{k}}$ and we arrive at
the following difference equation

$$
\begin{equation*}
\binom{\chi\left(t+\frac{\epsilon}{2^{k}}\right)}{\psi\left(t+\frac{\epsilon}{2^{k}}\right)}=A_{w}\binom{\chi(t+\epsilon)}{\psi(t+\epsilon)} \tag{4.22}
\end{equation*}
$$

Iterating (4.22) gives

$$
\begin{equation*}
\binom{\chi\left(t+\frac{\epsilon}{2^{n k}}\right)}{\psi\left(t+\frac{\epsilon}{2^{n k}}\right)}=A_{w}^{n}\binom{\chi(t+\epsilon)}{\psi(t+\epsilon)} \tag{4.23}
\end{equation*}
$$

As $n \rightarrow \infty$, the left hand side approaches $\left(\begin{array}{ll}\chi(t) & \psi(t)\end{array}\right)^{T}$. We are moving $\frac{1}{2^{k}}$ closer to $t$ at each step of the iteration so $2^{-k}$ is the rate of contraction toward $t$, as shown in (4.21). This result is consistent with the earlier analysis involving the geometry of $S$.

We now write the expression for the difference quotient of the eigenfunctions $f_{W}^{ \pm}(t)$.

$$
\begin{align*}
f_{W}^{\prime}(t) & =\lim _{\substack{\epsilon \rightarrow 0 \\
n \rightarrow \infty}} \frac{f_{W}^{ \pm}\left(t+\frac{\epsilon}{2^{k}}\right)-f_{W}^{ \pm}(t+\epsilon)}{\left(\frac{1}{2}\right)^{n k}}  \tag{4.24}\\
& =\lim _{\substack{\epsilon \rightarrow 0 \\
n \rightarrow \infty}} \frac{\chi\left(t+\frac{\epsilon}{2^{k}}\right)-\chi(t+\epsilon)}{\left(\frac{1}{2}\right)^{n k}}+\frac{\lambda_{W}^{ \pm}-a_{1}}{a_{3}} \cdot \frac{\psi\left(t+\frac{\epsilon}{2^{k}}\right)-\psi(t+\epsilon)}{\left(\frac{1}{2}\right)^{n k}}  \tag{4.25}\\
& =\chi^{\prime}(t)+\frac{\lambda_{W}^{ \pm}-a_{1}}{a_{3}} \psi^{\prime}(t) . \tag{4.26}
\end{align*}
$$

Our goal is to determine the rate of contraction of $f_{W}^{ \pm}(t+\epsilon)$ towards $f_{W}^{ \pm}(t)$ and compare it to $2^{-k}$. Note that we could let $\epsilon$ approach 0 from either the left or the right, depending on which derivative we want to compute. We use the following theorem to perform the necessary analysis.

Theorem 4.7 (Stability of Difference Equations ${ }^{21}$ ).
The difference equation $u_{k+1}=A u_{k}$ is
(i) stable if all eigenvalues satisfy $\left|\lambda_{i}\right|<1$
(ii) neutrally stable if some $\left|\lambda_{i}\right|=1$ and all other $\left|\lambda_{i}\right|<1$
(iii) unstable if at least one eigenvalue has $\left|\lambda_{i}\right|>1$.

In the stable case, the powers $A^{k}$ approach zero and so does $u_{k}=A^{k} u_{0}$. In the unstable case, at least one of the components of $u_{k}$ grows without bound.

Note that, given a rational nondyadic point $t_{0}$ with corresponding word $W, W^{n}$ also fixes $t_{0}$, so $\lambda_{W^{n}}^{ \pm}=\left(\lambda_{W}^{ \pm}\right)^{n}$. We would like to identify the word $W$ for which the eigenvectors $f_{W}^{ \pm}(t)$ satisfy $f_{W}^{ \pm}\left(t_{0}+\frac{t-t_{0}}{2^{k}}\right)=\lambda_{W}^{ \pm} f_{W}^{ \pm}(t)$. But by Lemma $4.5, w\left(t_{0}\right)=t_{0}$ is fixed by the eigenfunctions $f_{W}^{ \pm}(t)$. These eigenfunctions are therefore contractions toward $t_{0}$ as long as $t_{0}$ is fixed by $w \in G$. It follows that

$$
\begin{align*}
f_{W}^{\prime \pm}\left(t_{0}\right) & =\lim _{\substack{\epsilon \rightarrow 0 \\
n \rightarrow \infty}} \frac{f_{W}^{ \pm}\left(t_{0}+\frac{\epsilon}{2^{k}}\right)-f_{W}^{ \pm}\left(t_{0}+\epsilon\right)}{\left(\frac{1}{2}\right)^{n k}}  \tag{4.27}\\
& =\lim _{n \rightarrow \infty} \frac{\lambda_{W_{n}}^{ \pm}}{\left(2^{-k}\right)^{n}}  \tag{4.28}\\
& =\lim _{n \rightarrow \infty}\left(\frac{\lambda_{W}^{ \pm}}{2^{-k}}\right)^{n}  \tag{4.29}\\
& =\lim _{n \rightarrow \infty}\left(2^{k} \lambda_{W}^{ \pm}\right)^{n} \tag{4.30}
\end{align*}
$$

where $W$ is the operator which fixes $t_{0} \in[0,1] \cap\left(\mathbb{Q}-\mathbb{Z}_{\langle 2\rangle}\right)$. The task of classifying the derivatives of the basic functions at the rational points therefore amounts to computing the eigenvalues corresponding to each eigenfunction and comparing them to $2^{-k}$.

[^17]Lemma 4.8. $\lambda_{W}^{ \pm} \neq 2^{-k}$ for any operator $W$.

Proof. First, we prove that for any word $W$ of length $k, \operatorname{tr}\left(A_{w}\right)=\frac{c}{2 \cdot 5^{k}}$ for some $c \in \mathbb{Z}$. Let $W_{k-1}$ be a word of length $k-1$ and suppose $A_{w_{k-1}}=\left(\begin{array}{ll}a_{1} & a_{2} \\ a_{3} & a_{4}\end{array}\right)$. Since the denominators of all the entries of $A$ and $B$ are divisible by 5,5 divides each of the denominators of $a_{1}, a_{2}, a_{3}$ and $a_{4}$ for any $A_{w}$. So we can write $a_{1}=\frac{c_{1}}{5^{k-1}}, a_{2}=\frac{c_{2}}{5^{k-1}}$, $a_{3}=\frac{c_{3}}{5^{k-1}}$ and $a_{4}=\frac{c_{4}}{5^{k-1}}$ for some $c_{i} \in \mathbb{Z}$. Let $A_{w_{k}}$ denote the matrix corresponding to a word of length $k$. Now, $A_{w_{k}}=\left(\begin{array}{cc}a_{1} & a_{2} \\ a_{3} & a_{4}\end{array}\right)\left(\begin{array}{cc}\frac{1}{5} & 0 \\ 0 & \frac{3}{5}\end{array}\right)=\left(\begin{array}{cc}\frac{1}{5} a_{1} & \frac{3}{5} a_{2} \\ \frac{1}{5} a_{3} & \frac{3}{5} a_{4}\end{array}\right)$ or $A_{w_{k}}=\left(\begin{array}{cc}a_{1} & a_{2} \\ a_{3} & a_{4}\end{array}\right)\left(\begin{array}{cc}\frac{1}{2} & \frac{3}{10} \\ \frac{1}{10} & \frac{3}{10}\end{array}\right)=\frac{1}{2}\left(\begin{array}{ll}a_{1}+\frac{1}{5} a_{2} & \frac{3}{5} a_{1}+\frac{3}{5} a_{2} \\ a_{3}+\frac{1}{5} a_{4} & \frac{3}{5} a_{3}+\frac{3}{5} a_{4}\end{array}\right)$. If $A_{w_{k}}$ is of the first form, $\operatorname{tr}\left(A_{w_{k}}\right)=\frac{c_{1}}{5^{k}}+\frac{3 c_{4}}{5^{k}}=\frac{2\left(c_{1}+3 c_{4}\right)}{2.5^{k}}$, which satisfies the claim. If $A_{w_{k}}$ has the second form, $\operatorname{tr}\left(A_{w_{k}}\right)=\left(\frac{1}{2}\right)\left(\frac{c_{1}}{5^{k-1}}+\frac{c_{2}}{5^{k}}+\frac{3 c_{3}}{5^{k}}+\frac{3 c_{4}}{5^{k}}\right)=\frac{5 c_{1}+c_{2}+3 c_{3}+3 c_{4}}{2 \cdot 5^{k}}$, which again satisfies the claim. Hence, $\operatorname{tr}\left(A_{w}\right)=\frac{c}{2 \cdot 5^{k}}$, some $c \in \mathbb{Z}$, for any operator $W$.

Now, suppose $\lambda_{W}^{+}=2^{-k}$. Then $\lambda_{W}^{+} \cdot \lambda_{W}^{-}=\operatorname{det}\left(A_{w}\right)=\operatorname{det}(W)=\left(\frac{3}{25}\right)^{k}$, so $\lambda_{W}^{-}=\left(\frac{6}{25}\right)^{k}$, which means $\lambda_{W}^{+}+\lambda_{W}^{-}=\operatorname{tr}\left(A_{w}\right)=\frac{25^{k}+12^{k}}{2^{k} \cdot 5^{k}}$. By the claim proven above, $\operatorname{tr}\left(A_{w}\right)=\frac{c}{2 \cdot 5^{k}}$ for some $c \in \mathbb{Z}$. Setting these two equal to each other, we find that $c=\frac{2\left(25^{k}+12^{k}\right)}{10^{k}}$. We want to see if $2\left(25^{k}+12^{k}\right)$ is divisible by $10^{k}$ for any nonnegative integer $k$. Remark that since 25 is odd, $25^{k}$ is odd for all $k$; since 12 is even, $12^{k}$ is even for all $k$. Therefore $\left(25^{k}+12^{k}\right)$ is odd for all $k$, as it is it is the sum of an even and an odd integer. But $\frac{10^{k}}{2}$ is even for all $k>1$, so $\frac{10^{k}}{2} \nmid\left(25^{k}+12^{k}\right)$ for any $k$. Therefore $\lambda_{W}^{+} \neq 2^{-k}$. Assuming $\lambda_{W}^{-}=2^{-k}$ would imply that $\lambda_{W}^{+}=\left(\frac{6}{25}\right)^{k}<2^{-k}$, which is also a contradiction.

Lemma 4.8 has the following corollary.

Corollary 4.9. $\lambda_{W}^{-}<2^{-k}$ for any $W$.

Proof. Suppose $\lambda_{W}^{-}>2^{-k}$ for some operator $W$. Since $\lambda_{W}^{+} \cdot \lambda_{W}^{-}=\operatorname{det}\left(A_{w}\right)=$ $\operatorname{det}(W)=\left(\frac{3}{25}\right)^{k}, \lambda_{W}^{+}<\left(\frac{6}{25}\right)^{k}<\left(\frac{1}{2}\right)^{k}=\lambda_{W}^{-}$, which is clearly a contradiction.

Corollary 4.9 implies that either $\lambda_{W}^{ \pm}<2^{-k}$ or $\lambda_{W}^{-}<2^{-k}$ and $\lambda_{W}^{+}>2^{-k}$. In other words, it is impossible for both eigenvalues to be $>2^{-k}$. This fact has some very important consequences for the classification of the derivatives of the basic functions. Before presenting our main classification theorem, we summarize some of the above results in the following criterion and consider two examples.

Criterion for Classifying Derivatives at Rational Nondyadic Points: Let $\mu_{W}^{ \pm}=2^{k} \lambda_{W}^{ \pm}$. Then
(i) if $\mu_{W}^{ \pm}<1, f_{W}^{\prime \pm}(t)=0$.
(ii) if $\mu_{W}^{ \pm}>1, f_{W}^{\prime \pm}(t)=\infty$.

By Lemma 4.8, there are no rational points at which $f_{W}^{ \pm}(t)$ has nonzero finite derivative.

Example 4.10. Suppose once more that $t=\frac{1}{3}$ which we know by now is fixed by $w=\alpha \beta \in G$. Recalling the matrix $A B$ from Example 3.5 and applying (4.19) and (4.20), we have that

$$
\begin{gathered}
\lambda_{A B}^{ \pm}=\frac{7 \pm \sqrt{13}}{50} \\
f_{A B}^{ \pm}(t)=\chi(t)+\frac{2 \pm \sqrt{13}}{3} \psi(t)-\frac{10 \pm 4 \sqrt{13}}{27} 1(t)
\end{gathered}
$$

The values of $\chi$ and $\psi$ at $t$ can be found in Appendix A: $\chi\left(\frac{1}{3}\right)=\frac{2}{27}$ and $\psi\left(\frac{1}{3}\right)=\frac{4}{9}$. Plugging these into $f_{A B}^{ \pm}(t)$ gives $f_{A B}^{ \pm}\left(\frac{1}{3}\right)=0$. As expected, the eigenfunctions $f_{A B}^{ \pm}\left(\frac{1}{3}\right)$ vanish at $\frac{1}{3}$.

Let us now compute the derivative $f_{A B}^{\prime \pm}\left(\frac{1}{3}\right)$. Since $\mu_{A B}^{+}=2^{2} \lambda_{A B}^{+}<1, f_{W}^{\prime \pm}\left(\frac{1}{3}\right)=$ 0 by the above criterion. In other words, $\chi^{\prime}\left(\frac{1}{3}\right)+\frac{2+\sqrt{13}}{3} \psi^{\prime}\left(\frac{1}{3}\right)=0$ and $\chi^{\prime}\left(\frac{1}{3}\right)+$ $\frac{2-\sqrt{13}}{3} \psi^{\prime}\left(\frac{1}{3}\right)=0$. We apply some logic to deduce what $\chi^{\prime}$ and $\psi^{\prime}$ must be. Since $\chi$ and $\psi$ are strictly increasing, $\chi^{\prime}\left(\frac{1}{3}\right)=\psi^{\prime}\left(\frac{1}{3}\right)=0$. As noted in Example 4.3, a precise enough numeric approximation gives the same result.

Example 4.11. Let us now give an example of a point at which $f_{W}^{\prime \pm}(t)=\infty$. It turns out $\frac{1}{15}$ is such a point. Since $\frac{1}{15}=(0 . \overline{0001})_{2}, w=\alpha \alpha \alpha \alpha \beta \in G$ fixes $\frac{1}{15}$. We compute the eigenvalues and eigenfunctions to be

$$
\begin{gathered}
\lambda_{A^{3} B}^{ \pm}=\frac{43 \pm \sqrt{61}}{1250} \\
f_{A^{3} B}^{ \pm}(t)=\chi(t)+\frac{38 \pm 5 \sqrt{61}}{27} \psi(t)-\left(\frac{9254417448}{47373828125} \pm \frac{6225 \sqrt{61}}{242554}\right) 1(t)
\end{gathered}
$$

The values of $\chi$ and $\psi$ at $\frac{1}{15}$ can be found in Appendix A. Once again, one can check that $f_{A^{3} B}^{ \pm}\left(\frac{1}{15}\right)=0$ as desired, meaning our eigenfunctions do indeed vanish at $\frac{1}{15}$. To determine $f_{A^{3} B}^{\prime}\left(\frac{1}{15}\right)$, Remark that $\mu_{A^{3} B}^{-}=2^{4} \lambda_{A^{3} B}^{-}<1$ and $\mu_{A^{3} B}^{+}=2^{4} \lambda_{A^{3} B}^{+}>1$. By our criterion, $f_{A^{3} B}^{\prime}\left(\frac{1}{15}\right)=0$ and $f_{A^{3} B}^{\prime}\left(\frac{1}{15}\right)=\infty$. Since $\chi^{\prime}\left(\frac{1}{15}\right)+\frac{38+5 \sqrt{61}}{27} \psi^{\prime}\left(\frac{1}{15}\right)=$ $\infty$, at least one of $\chi^{\prime}\left(\frac{1}{15}\right)$ and $\psi^{\prime}\left(\frac{1}{15}\right)$ is $\infty$. But we must also have that $\chi^{\prime}\left(\frac{1}{15}\right)+$ $\frac{38-5 \sqrt{61}}{27} \psi^{\prime}\left(\frac{1}{15}\right)=0$. Note that $\frac{38-5 \sqrt{61}}{27}<0$. The only way for $f_{A^{3} B}^{-}\left(\frac{1}{15}\right)$ to be 0 is if both $\chi^{\prime}\left(\frac{1}{15}\right)$ and $\psi^{\prime}\left(\frac{1}{15}\right)$ are $\infty$.

It is worth noting that Example 4.11 motivates an alternate definition of the derivative. Since $f_{A^{3} B}^{\prime-}\left(\frac{1}{15}\right)=0$, we have that $\chi^{\prime}\left(\frac{1}{15}\right)+\frac{38-5 \sqrt{61}}{27} \psi^{\prime}\left(\frac{1}{15}\right)=0$. Rearranging, we find that $\chi^{\prime}\left(\frac{1}{15}\right)=\frac{-38+5 \sqrt{61}}{27} \psi^{\prime}\left(\frac{1}{15}\right)$, or $\frac{d \chi(1 / 15)}{d \psi(1 / 15)}=\frac{-38+5 \sqrt{61}}{27}$. We know that both $\chi^{\prime}\left(\frac{1}{15}\right)$ and $\psi^{\prime}\left(\frac{1}{15}\right)$ are $\infty$; this rate of change gives us an idea of just how
big these $\infty$ 's are relative to one another. Derivatives of the basic functions with respect to each other go beyond the scope of this thesis but it should be noted that such derivatives can be computed at points other than $\frac{1}{15}$ using the eigenfunctions in Table 6.6 of Appendix C.

The following theorem summarizes the results of this section.

Theorem 4.12 (Classification of Derivatives at Rational Nondyadic Points: Part I). The derivative of any basic function at a rational nondyadic point is either 0 or $\infty$. Moreover if $t$ is rational nondyadic and $W$ fixes $t$,
(i) $f_{W}^{\prime \pm}(t)=0 \Longrightarrow \chi^{\prime}(t)=\psi^{\prime}(t)=0$.
(ii) $f_{W}^{\prime+}(t)=\infty$ and $f_{W}^{\prime}-(t)=0 \Longrightarrow \chi^{\prime}(t)=\psi^{\prime}(t)=\infty$

It is impossible for both $f_{W}^{\prime} \pm(t)=\infty$; therefore there are no rational nondyadic points at which $\chi^{\prime}(t)=0$ and $\psi^{\prime}(t)=\infty$ or vice versa.

Proof. We first prove (i) and (ii).
(i) Suppose $f_{W}^{\prime \pm}(t)=0$, so $\chi^{\prime}(t)+\frac{\lambda_{W}^{ \pm}-a_{1}}{a_{3}} \psi^{\prime}(t)=0$. Since $\chi$ and $\psi$ are strictly increasing, we must have that $\chi^{\prime}(t)=\psi^{\prime}(t)=0$.
(ii) Suppose $f_{W}^{\prime+}(t)=\infty$ and $f_{W}^{\prime-}(t)=0$. Then $\chi^{\prime}(t)+\frac{\lambda_{W}^{+}-a_{1}}{a_{3}} \psi^{\prime}(t)=\infty$ and $\chi^{\prime}(t)+\frac{\lambda_{W}^{-}-a_{1}}{a_{3}} \psi^{\prime}(t)=0$. The first relation implies that either $\chi^{\prime}(t)$ or $\psi^{\prime}(t)$ is $\infty$. In order for the second statement to hold, we must have that $\frac{\lambda_{W}^{+}-a_{1}}{a_{3}}<0$ and that the remaining function also has infinite derivative.

To prove the last part of the theorem, assume without loss of generality $\chi^{\prime}(t)=0$ and $\psi^{\prime}(t)=\infty$. Then, if $\frac{\lambda_{W}^{-}-a_{1}}{a_{3}}>0, \chi^{\prime}(t)+\frac{\lambda_{W}^{+}-a_{1}}{a_{3}} \psi^{\prime}(t)=\infty$ and $\chi^{\prime}(t)+\frac{\lambda_{W}^{-}-a_{1}}{a_{3}} \psi^{\prime}(t)=$ $\infty$. By the criterion above, both eigenvalues $\mu_{W}^{ \pm}>1$. But this is impossible by

Corollary 4.9. Note that if $\frac{\lambda_{W}^{-}-a_{1}}{a_{3}}<0$, we have that $\chi^{\prime}(t)+\frac{\lambda_{W}^{+}-a_{1}}{a_{3}} \psi^{\prime}(t)=-\infty$ which contradicts the fact that all basic functions are strictly increasing. The same problem arises if we assume $\chi^{\prime}(t)=\infty$ and $\psi^{\prime}(t)=0$

Since $\phi(t)$ and $\xi(t)$ consist of segments which are rotations of $\chi(t)$ and $\psi(t)$ on smaller scales, we must have that the derivatives of these functions are also either both 0 or both $\infty$.

Some values of the derivatives of the basic functions, as well as a step-by-step procedure for derivative computation, can be found in Appendix B.

Theorem 4.12 has some important consequences. First, it shows that, after considering all rational points in $[0,1]$, we still have no example of a point at which a basic function has nonzero finite derivative. Moreover, the last part of the theorem is potentially counterintuitive. In Theorem 4.2, it was shown that $\chi^{\prime}(0)=0$ whereas $\phi^{\prime}(0)=\psi^{\prime}(0)=\xi^{\prime}(0)=\infty$ and $\xi^{\prime}(1)=0$ whereas $\chi^{\prime}(1)=\phi^{\prime}(1)=\psi^{\prime}(1)=\infty$. One might therefore expect for there to be other points at which the derivatives of three of the basic functions vanish and the other is $\infty$. Our construction shows that this does not happen at any rational point: since both $\chi^{\prime}(t)$ and $\psi^{\prime}(t)$ either vanish or blow up at every rational, so must all four basic functions. This fact becomes more apparent in the next subsection, where we discuss the relationship between the derivatives of all four basic functions.

### 4.2.3 Conjugacy Classes of Operators $W$

Since the basic functions are multiplicatively and reflexively symmetric, one might wonder if knowing the derivative of $\chi$ and $\psi$ at a point $t$ tells us anything about these functions' derivatives at other points, say those with the same denomi-
nator. It turns out the answer to this question is yes. The goal of this subsection is to enumerate all rational nondyadic points at which the basic functions necessarily have the same derivative. In other words, we are interested in determining which operators $W$ have the same eigenvalues.

Recall the difference equation (4.23):

$$
\binom{\chi\left(t+\frac{\epsilon}{2^{n k}}\right)}{\psi\left(t+\frac{\epsilon}{2^{n k}}\right)}=A_{w}^{n}\binom{\chi(t+\epsilon)}{\psi(t+\epsilon)}
$$

Since $\mu_{W}^{ \pm}=2^{k} \lambda_{W}^{ \pm}$are the eigenvalues of the matrix $2^{k} A_{w}$, we introduce the following notation.

Notation 4.13. Let $\tau \equiv \operatorname{trace}\left(2^{k} A_{w}\right)$ and $\Delta \equiv \operatorname{det}\left(2^{k} A_{w}\right)$. We say two words $W$ are in the same conjugacy class if they have the same $\tau, \Delta$ and length $k$ (and therefore the same eigenvalues).

Definition 4.13 is motivated by the following relations

$$
\begin{align*}
\tau & =\mu_{W}^{+}+\mu_{W}^{-}  \tag{4.31}\\
\Delta & =\mu_{W}^{+} \cdot \mu_{W}^{-} \tag{4.32}
\end{align*}
$$

which imply that

$$
\begin{equation*}
\mu_{W}^{ \pm}=\frac{\tau \pm \sqrt{\tau^{2}-4 \Delta}}{2} \tag{4.33}
\end{equation*}
$$

By the Criterion for Classifying Derivatives at Rational Points, $f_{W}^{\prime \pm}(t)=0$ if $\mu_{W}^{ \pm}<1$
and $f_{W}^{\prime \pm}(t)=\infty$ if $\mu_{W}^{ \pm}>1$. Remark that

$$
\begin{align*}
\Delta & \equiv \mu_{W}^{+} \cdot \mu_{W}^{-}=\operatorname{det}\left(2^{k} A_{w}\right)=2^{2 k} \operatorname{det}\left(A_{w}\right)=2^{2 k} \operatorname{det}(W)  \tag{4.34}\\
& =2^{2 k}\left(\frac{3}{25}\right)^{k}=\left(\frac{12}{25}\right)^{k}  \tag{4.35}\\
\tau & \equiv \mu_{W}^{+}+\mu_{W}^{-}=\operatorname{tr}\left(2^{k} A_{w}\right)=2^{k} \operatorname{tr}\left(A_{w}\right)=2^{k}(\operatorname{tr}(W)-1) \tag{4.36}
\end{align*}
$$

The reason it is natural to group the words $W$ into conjugacy classes of operators having equal eigenvalues because each of $\chi$ and $\psi$ must have the same derivative at points fixed by operators in each conjugacy class. Because it is so important, we state this fact as a lemma.

Lemma 4.14. Suppose $W$ fixes $t_{1}$ and $W^{\prime}$ fixes $t_{2}$. If $W$ and $W^{\prime}$ are in the same conjugacy class, then $\chi^{\prime}\left(t_{1}\right)=\chi^{\prime}\left(t_{2}\right)$ and $\psi^{\prime}\left(t_{1}\right)=\psi^{\prime}\left(t_{2}\right)$.

Proof. Suppose $W$ fixes $t_{1}$ and $W^{\prime}$ fixes $t_{2}$. Since the operators are in the same conjugacy class, they have the same $\tau, \Delta$ and length $k$. By (4.33), they have the same eigenvalues, so $f_{W}^{\prime \pm}\left(t_{1}\right)=f_{W}^{\prime \pm}\left(t_{2}\right)$. Now by Theorem 4.12, $\chi^{\prime}\left(t_{1}\right)=\chi^{\prime}\left(t_{2}\right)$ and $\psi^{\prime}\left(t_{1}\right)=\psi^{\prime}\left(t_{2}\right)$.

We now determine which words fall into the same conjugacy class. Let $W=A^{r_{1}} B^{m_{1}} A^{r_{2}} B^{m_{2}} \ldots A^{r_{i}} B^{m_{j}}$ be a word of length $k$ and define $\sigma$ to be the cyclic permutation

$$
\begin{equation*}
\sigma: A^{r_{1}} B^{m_{1}} A^{r_{2}} B^{m_{2}} \ldots A^{r_{i}} B^{m_{j}} \rightarrow B A^{r_{1}} B^{m_{1}} A^{r_{2}} B^{m_{2}} \ldots A^{r_{i}} B^{m_{j-1}} \tag{4.37}
\end{equation*}
$$

$\sigma$ shifts the matrices $A$ and $B$ in the product $W$ with the elements shifted off the end inserted back at the beginning. The mapping $\sigma^{n}$ is therefore $W_{i} \rightarrow W_{(i+n) \bmod k}$, where $W_{i} \in\{A, B\}$. This permutation is important because of the cyclic property of the trace.

Theorem 4.15 (Cyclic Property of the Trace). The trace of a product of $n$ square matrices is the trace of any cyclic permutation of this product

$$
\begin{equation*}
\operatorname{tr}(W)=\operatorname{tr}(\sigma(W))=\operatorname{tr}\left(\sigma^{n}(W)\right), \quad n \in \mathbb{Z} \tag{4.38}
\end{equation*}
$$

We arrive at the following lemma.
Lemma 4.16. Let $t_{0}=\frac{l}{2^{k}-1}$ be a given rational with period $k$. Then the points $t_{i}=\frac{2^{i} l \bmod \left(2^{k}-1\right)}{2^{k}-1}$ correspond to words $\sigma^{n}(W)$ (that is, cyclic permutations of $W$ ) for all $i$. There are exactly $k$ such points.

Proof. We prove Lemma 4.16 by recalling some properties of binary expansions. Suppose $t_{0}=\frac{l}{2^{k}-1}$ has the binary expansion $\left(0 . \overline{s_{1} s_{2} \ldots s_{k}}\right)_{2}$, where $s_{i} \in\{0,1\}$ and $k$ is the period. Then $t_{0}=\frac{s_{1}}{2}+\frac{s_{2}}{2^{2}}+\cdots+\frac{s_{k}}{2^{k}}+\frac{s_{1}}{2^{k+1}}+\frac{s_{2}}{2^{k+2}}+\ldots$ Now, multiplying $t_{0}$ by $2^{i}$ and modding out by $2^{k}-1$ gives $2^{i} t_{0}=\frac{s_{1}}{2^{1-i}}+\frac{s_{2}}{2^{2-i}}+\cdots+\frac{s_{k}}{2^{k-i}} \frac{s_{1}}{2^{k+1-i}}+\frac{s_{2}}{2^{k+2-i}}+\ldots$. This corresponds to a left hand shift of the decimal point $i$ places and does not affect the period or the order of the letters: $2^{i} t_{0}=\left(0 . \overline{s_{i} s_{i+1} \ldots s_{k} s_{1} \ldots s_{i-1}}\right)_{2}$. In other words, multiplication by $a^{i}$ cyclically permutes the letters of the word $W$ as long as $a^{i} l \equiv 2^{i} l \bmod \left(2^{k}-1\right)$. Since there are exactly $k$ cyclic permutations of $W$, there are exactly $k$ such points.

Note that if $t_{0}=\frac{l}{2^{k}-1}$ is a point fixed by a word of length $k$, then $k=$ the number of distinct integers in the set $\left\{2^{i} l \bmod \left(2^{k}-1\right) \mid i=1,2, \ldots, k\right\}$ : since multiplication of $t_{0}$ by an integer congruent to $2^{i} \bmod \left(2^{k}-1\right)$ results in a cyclic permutation of the letters in the word corresponding to $t_{0}$ and there are exactly $k$ such permutations, $k$ is the number of distinct integers that satisfy the congruence. Lemma 4.16 has the following corollary.

Corollary 4.17. $\chi^{\prime}\left(t_{i}\right)=\chi^{\prime}\left(t_{0}\right)$ and $\psi^{\prime}\left(t_{i}\right)=\psi^{\prime}\left(t_{0}\right)$ for all $\frac{2^{i l} \bmod \left(2^{k}-1\right)}{2^{k}-1}$ and $t_{0}=$ $\frac{l}{2^{k}-1}$.

Proof. Let $t_{0}=\frac{l}{2^{k}-1}$ and let $W$ be the corresponding operator. By Lemma 4.16, multiplying $t_{0}$ by $2^{i} \bmod \left(2^{k}-1\right)$ cyclically permutes the "letters" of $W$. If we call the new word $W_{i}$, then $W_{i}=\sigma^{i}(W)$. Since the length of $W_{i}$ is the same as the length of $W, \operatorname{det}(W)=\operatorname{det}\left(W_{i}\right)=\left(\frac{12}{25}\right)^{k}$. Now, by the cyclic property of the trace (Theorem 4.15), $\operatorname{tr}(W)=\operatorname{tr}\left(W_{i}\right)$. Since $W$ and $W_{i}$ have the same determinant and trace, they must have the same eigenvalues and are therefore in the same conjugacy class. The conclusion follows from Lemma 4.16.

Before proceeding further, let us identify the conjugacy classes of operators for a particular point, say $\frac{1}{31}$.

## Example 4.18.

Let $t_{0}=\frac{1}{31}=\frac{1}{2^{5}-1}$. Since the binary expansion of $t_{0}$ is $(0 . \overline{00001})_{2}, w=\alpha \alpha \alpha \alpha \beta \in$ $G$ fixes $\frac{1}{31}$. Clearly, $k=5$. Let us check that this is consistent with the claim that $k=$ the number of distinct integers in the set $\left\{2^{i} \bmod \left(2^{k}-1\right) \mid i=1,2, \ldots, k\right\}$.

$$
\begin{equation*}
2^{i} \bmod 31=\{1,2,4,8,16\} \tag{4.39}
\end{equation*}
$$

As one would expect, there are five distinct integers in this set.
Before computing the derivatives of $\chi$ and $\psi$ all points of the form $\frac{a}{31}$, let us identify some of the different conjugacy classes. In doing so, we are looking for operators $W$ which have the same determinant and trace. The preceding analysis showed that points with numerators congruent to $2^{i} \bmod 31$ are in the same conjugacy class as
$W=A A A A B$ and that their operators are cyclic permutations of the letters in $W$ :

$$
\begin{align*}
& \frac{1}{31}=A A A A B  \tag{4.40}\\
& \frac{2}{31}=A A A B A  \tag{4.41}\\
& \frac{4}{31}=A A B A A  \tag{4.42}\\
& \frac{8}{31}=A B A A A  \tag{4.43}\\
& \frac{16}{31}=B A A A A \tag{4.44}
\end{align*}
$$

We have not exhausted all possible numerators $a$ such that $\frac{a}{31}$ remains in lowest terms. Since $\varphi(31)=30$, there are 30 such points ${ }^{22}$ and the length of each of the words which fix each of these points is also 5 . By (4.31), all operators of length $k=5$ have the same determinant, so it is enough to group these operators by trace. Such words could consist of $1 A$ and $4 B s, 2 A s$ and $3 B s, 3 A s$ and $2 B s$ or $4 A s$ and 1 $B$. Table ${ }^{23} 4.19$ below gives all possibilities of words of length 5 in two letters $A$ and $B$. We know a priori that words which are cyclic permutations of each other will be in the same conjugacy class, so we do not need to enumerate them separately. The eigenvalues and derivatives found in the table were computed using the procedure discussed in the previous section.

[^18]Table 4.19. Conjugacy Classes of Words of Length 5

| Word | Numerator | $\mu_{W}^{+}$ | $\mu_{W}^{-}$ | $\chi^{\prime}$ and $\psi^{\prime}$ |
| :---: | :---: | :---: | :---: | :---: |
| $\sigma^{n}(A A A A B)$ | $2^{i} \bmod 31$ <br> $=1,2,4,8,16$ | 1.2494 | 0.0204 | $\infty$ |
| $\sigma^{n}(A A A B B)$ | $3 \cdot 2^{i} \bmod 31$ <br> $=3,6,12,24,17$ | 0.8719 | 0.0292 | 0 |
| $\sigma^{n}(A A B A B)$ | $5 \cdot 2^{i} \bmod 31$ <br> $=5,10,20,9,18$ | 0.7440 | 0.0342 | 0 |
| $\sigma^{n}(A A B B B)$ | $7 \cdot 2^{i} \bmod 31$ <br> $=7,14,28,25,19$ | 0.8719 | 0.0292 | 0 |
| $\sigma^{n}(B B A B A)$ | $11 \cdot 2^{i} \bmod 31$ <br> $=11,13,21,22,26$ | 0.7440 | 0.0342 | 0 |
| $\sigma^{n}(B B B B A)$ | $15 \cdot 2^{i} \bmod 31$ <br> $=$ <br> $15,30,29,27,23$ | 1.2494 | 0.0204 | $\infty$ |

Notice that the derivatives at all points having denominator 31 are not equal. This is because all possible operators of length 5 do not have the same trace. Also note that one can construct more precise conjugacy classes than just those consisting of matrix products which are cyclic permutations of each other. Three of the conjugacy classes in Table 4.19 can be combined because their corresponding operators have equal eigenvalues. This observation motivates the following lemma.

Lemma 4.20. The words $W=A^{r_{1}} B^{m_{1}} A^{r_{2}} B^{m_{2}} \ldots A^{r_{i}} B^{m_{j}}$ and $W^{c}=B^{r_{1}} A^{m_{1}} B^{r_{2}} A^{m_{2}} \ldots B^{r_{i}} A^{m_{j}}$ are in the same conjugacy class. Moreover, if $W$ fixes the rational nondyadic point $t \in[0,1]$ then $W^{c}$ fixes the point $1-t$. We say that $W^{c}$ is the binary two's complement of $W$.

Proof. Since $W$ and $W^{c}$ have the same length and therefore determinant, all that must be shown is that they have the same trace. Begin by writing $A=\left(\begin{array}{cc}L & \vec{v}_{A} \\ \overrightarrow{0}^{T} & 1\end{array}\right)$ and $B=\left(\begin{array}{cc}R & \vec{v}_{B} \\ \overrightarrow{0}^{T} & 1\end{array}\right)$, where $L=\left(\begin{array}{cc}\frac{1}{5} & 0 \\ 0 & \frac{3}{5}\end{array}\right)$ and $R=\left(\begin{array}{cc}\frac{1}{2} & \frac{3}{10} \\ \frac{1}{10} & \frac{3}{10}\end{array}\right)$. Then

$$
A^{n}=\left(\begin{array}{cc}
L & \vec{v}_{A} \\
\overrightarrow{0}^{T} & 1
\end{array}\right)^{n}=\left(\begin{array}{cc}
L^{n} & \sum_{k=0}^{n-1} L^{k} \vec{v}_{A} \\
\overrightarrow{0}^{T} & 1
\end{array}\right)
$$

and

$$
B^{m}=\left(\begin{array}{cc}
R & \vec{v}_{B} \\
\overrightarrow{0}^{T} & 1
\end{array}\right)^{m}=\left(\begin{array}{cc}
R^{m} & \sum_{k=0}^{m-1} B^{k} \vec{v}_{B} \\
\overrightarrow{0}^{T} & 1
\end{array}\right)
$$

Showing that $\operatorname{tr}\left(A^{n} B^{m}\right)=\operatorname{tr}\left(B^{n} A^{m}\right)$ therefore amounts to showing that $\operatorname{tr}\left(L^{n} R^{m}\right)=$ $\operatorname{tr}\left(R^{n} L^{m}\right)$. Note that $L$ and $R$ have the same trace of $\frac{4}{5}$ and the same eigenvalues, namely $\frac{1}{5}$ and $\frac{3}{5}$. $L$ is diagonal; since $R$ has distinct eigenvalues, it is diagonalizable. Let $S$ be the matrix whose columns are the eigenvectors of $R$. Computing these eigenvectors, one finds that $S=\left(\begin{array}{cc}-1 & 3 \\ 1 & 1\end{array}\right)$ and $S^{-1}=\frac{1}{4} S$. Diagonalizing,
$R=S L S^{-1}=S^{-1} L S$. Now,

$$
\begin{align*}
\operatorname{tr}(W)-1 & =\operatorname{tr}\left(L^{r_{1}} R^{m_{1}} L^{r_{2}} R^{m_{2}} \ldots L^{r_{i}} R^{m_{j}}\right)  \tag{4.45}\\
& =\operatorname{tr}\left(L^{r_{1}}\left(S L^{m_{1}} S^{-1}\right) L^{r_{2}}\left(S L^{m_{2}} S^{-1}\right) \ldots L^{r_{i}}\left(S L^{m_{j}} S^{-1}\right)\right)  \tag{4.46}\\
& =\operatorname{tr}\left(\left(S^{-1} L^{r_{1}} S\right) L^{m_{1}}\left(S^{-1} L^{r_{2}} S\right) L^{m_{2}} \ldots\left(S^{-1} L^{r_{i}} S\right) L^{m_{j}}\right)  \tag{4.47}\\
& \left.=\operatorname{tr}\left(\left(S L^{r_{1}} S^{-1}\right) L^{m_{1}}\left(S L^{r_{2}} S^{-1}\right) L^{m_{2}} \ldots\left(S L^{r_{i}} S^{-1}\right) L^{m_{j}}\right)\right)  \tag{4.48}\\
& =\operatorname{tr}\left(R^{r_{1}} L^{m_{1}} R^{r_{2}} \ldots R^{m_{i}} L^{r_{j}}\right)  \tag{4.49}\\
& =\operatorname{tr}\left(W^{c}\right)-1 \tag{4.50}
\end{align*}
$$

The third step is justified by the cyclic property of the trace.
To prove the second part of the claim, we use the fact that $W=A^{r_{1}} B^{m_{1}} A^{r_{2}} B^{m_{2}} \ldots A^{r_{i}} B^{m_{j}}$ implies that the binary period of $t \in[0,1]$ fixed by $w \in G$ is
$(0 . \underbrace{00 \ldots 0}_{r_{1} \text { times }} \underbrace{11 \ldots 1}_{m_{1} \text { times }} \underbrace{00 \ldots 0}_{r_{2} \text { times }} \ldots \underbrace{11 \ldots 1}_{m_{j} \text { times }})_{2}$. By the properties of binary numbers, the two's complement of $t$, or $-t$, is $(0 . \underbrace{11 \ldots 1}_{r_{1} \text { times }} \underbrace{00 \ldots 0}_{m_{1} \text { times }} \underbrace{11 \ldots 1}_{r_{2} \text { times }} \ldots \underbrace{00 \ldots 0}_{m_{j} \text { times }})_{2}$. Now, translating binary subtraction to binary addition, we have that $(1-t)_{2}=(1+(-t))_{2}=$ ( $0 . \underbrace{11 \ldots 1}_{r_{1} \text { times }} \underbrace{00 \ldots 0}_{m_{1} \text { times }} \underbrace{11 \ldots 1}_{r_{2} \text { times }} \ldots \underbrace{00 \ldots 0}_{m_{j} \text { times }})_{2}$. The corresponding word is $W^{c}=B^{r_{1}} A^{m_{1}} B^{r_{2}} A^{m_{2}} \ldots B^{r_{i}} A^{m_{j}}$.

By Lemma 4.20, given a word $W$ of length $k$, the conjugacy class of $W$ consists of all cyclic permutations of $W$ and all cyclic permutations of its two's complement. This is a very important conclusion because it enables us to finally link the derivatives of all four basic functions.

Theorem 4.21. For all rational nondyadic $t \in[0,1], \phi^{\prime}(t)=\psi^{\prime}(t)$ and $\xi^{\prime}(t)=\chi^{\prime}(t)$.

Proof. Recall that $\phi(t)=1-\psi(1-t)$ and $\xi(t)=1-\chi(1-t)$. By symmetry, the behavior of $\chi$ at $1-t$ is the same as the behavior of $\xi$ at $t$. The same is true for $\psi$
and $\phi$. Thus, $\xi^{\prime}(t)=\chi^{\prime}(1-t)$ and $\phi^{\prime}(t)=\psi^{\prime}(1-t)$. Since the words which fix $t$ and $1-t$ are in the same conjugacy class, $\chi^{\prime}(1-t)=\chi^{\prime}(t)$ and $\psi^{\prime}(1-t)=\psi^{\prime}(t)$. The conclusion follows immediately.

In Example 4.18, we had that the points $\frac{1}{31}, \frac{2}{31}, \frac{4}{31}, \frac{8}{31}$ and $\frac{16}{31}$ are in the same conjugacy class as $\frac{15}{31}=1-\frac{16}{31}, \frac{23}{31}=1-\frac{8}{31}, \frac{27}{31}=1-\frac{4}{31}, \frac{29}{31}=1-\frac{2}{31}$ and $\frac{30}{31}=1-\frac{1}{31}$ and that $\chi^{\prime}=\psi^{\prime}=\infty$ at all these points. The symmetry in the conjugacy classes combined with the symmetry of the basic functions enables us to conclude that $\psi^{\prime}$ and $\xi^{\prime}$ are also $\infty$ at every single one of these points.

We summarize the results of this subsection in the following theorem.

Theorem 4.22 (Classification of Derivatives at Dyadic Points: Part II). Suppose $W$ and $W_{i}$ are in the same conjugacy class and $W$ fixes $t=\frac{p}{q}$, where $q$ is odd and $\operatorname{gcd}(p, q)=1$. Suppose $W^{c}$ fixes $t_{i} \in[0,1]$. Then
(i) $W_{i}=\sigma^{n}(W)$ and $t_{i} \equiv 2^{i} \bmod q$ or $W_{i}$ is the binary two's complement of $W$ (i.e., if $W=A^{r_{1}} B^{m_{1}} A^{r_{2}} B^{m_{2}} \ldots A^{r_{i}} B^{m_{j}}$, then $W^{c}=B^{r_{1}} A^{m_{1}} B^{r_{2}} A^{m_{2}} \ldots B^{r_{i}} A^{m_{j}}$ ) and $t_{i}=1-t$.
(ii) $\chi^{\prime}(t)=\chi^{\prime}\left(t_{i}\right)=\xi^{\prime}(t)=\xi^{\prime}\left(t_{i}\right)$ and $\psi^{\prime}(t)=\psi^{\prime}\left(t_{i}\right)=\phi^{\prime}(t)=\phi^{\prime}\left(t_{i}\right)$.

Proof. (i) follows from Lemmas 4.16 and 4.20. (ii) is a restatement of Theorem 4.21 coupled with Theorem 4.12, which states that $\chi^{\prime}(t)=\psi^{\prime}(t)$ for all rational nondyadic points $t$.

We knew from the results of the previous section that, by symmetry, the derivatives of $\psi$ and $\xi$ are either 0 or $\infty$ since these are the only possibilities for $\chi^{\prime}$ and $\psi^{\prime}$. We also showed that $\chi^{\prime}$ and $\psi^{\prime}$ are jointly 0 or jointly $\infty$ at any rational nondyadic
point. Since any nondyadic rational $t$ and its binary two's complement $1-t$ have operators which are in the same conjugacy class, it follows that all four basic functions have the same derivative at any rational nondyadic point.

Several tables summarizing these results for operators of length $\leq 6$ can be found in Appendix C. Computing the traces of the operators $W$ would enable one to fully classify the derivatives of the basic functions at all rational nondyadic points.

## 5 Conclusions and Conjectures

The analysis of the preceding two sections prompts some interesting open questions, which we formulate here.

The first of these is whether it is possible to identify some pattern in the derivatives of the basic functions. While our procedure makes possible the complete classification of the derivatives of the basic functions at all rational points, it involves multiplying together arbitrarily many matrices, a task we would like to avoid if at all possible. Examining the tables in Appendix C more closely, one sees that words containing more than twice as many $A s$ than $B s$ or vice versa seem to correspond to points with $\infty$ derivative and those having more or less the same number of $A s$ and Bs seem to correspond to points with 0 derivative. The greater the trace, the more likely the derivatives are $\infty$, so it may be possible to identify which combinations of $A$ and $B$ give a small enough trace for the derivatives at the corresponding point to be 0 and vice versa. Since the trace of an operator $W$ depends not only on the number of $A s$ and $B s$ but also their arrangement, one can use 2-ary necklaces ${ }^{24}$ to

[^19]associate a combinatorical picture to each conjugacy class (see Figure 10 and Appendix C). The location of black and white beads on each necklace may serve as a guideline for determining whether an operator corresponds to a point having 0 or $\infty$ derivative.




Figure 10: 2-ary Necklaces of Lengths 1-4

Another possible approach is to compute the average traces and eigenvalues of the operators in all conjugacy classes of length $k$ and use this information to try to guess which words have 0 and which have $\infty$ derivative. Notice that $(A+B)^{k}$ is the sum of all words in $A$ and $B$ of length $k$, e.g., $(A+B)^{2}=A^{2}+A B+B A+B^{2}=$ $2 \cdot C C\left(A^{2}\right)+2 \cdot C C(A B),(A+B)^{3}=A^{3}+A B A+B A A+B^{2} A+A^{2} B+A B^{2}+$ $B A B+B^{3}=2 \cdot C C\left(A^{3}\right)+6 \cdot C C(A A B)$ and so on. Recalling that $\tau=2^{k}(\operatorname{tr}(W)-1)$, we have the following expressions

$$
\begin{equation*}
\sum_{\substack{\text { Conjugacy Classes } \\ \text { of length } k}} \operatorname{tr}(W)=\operatorname{tr}\left((A+B)^{k}\right) \tag{5.1}
\end{equation*}
$$

$$
\begin{align*}
\sum_{\substack{\text { Conjugacy Classes } \\
\text { of length } k}} \tau & =2^{k}\left(\sum_{\substack{\text { Conjugacy Classes } \\
\text { of length } k}} \operatorname{tr}(W)-2^{k}\right)  \tag{5.2}\\
& =2^{k}\left(\operatorname{tr}\left((A+B)^{k}\right)-2^{k}\right) \tag{5.3}
\end{align*}
$$

Since there are $2^{k}$ possible arrangements of $k$ letters, the average trace of a class of operators having length $k$ is $\bar{\tau}_{k}=\operatorname{tr}\left((A+B)^{k}\right)-2^{k}$. It was shown in (4.31) that all operators of length $k$ have the same determinant $\Delta=\left(\frac{12}{25}\right)^{k}$, so the average eigenvalues are given by $\bar{\mu}_{W}^{ \pm}=\frac{\bar{\tau} \pm \sqrt{\bar{\tau}^{2}-4\left(\frac{12}{25}\right)^{k}}}{2}$. One should note that an expansion of $(A+B)^{k}$ contains conjugacy classes that would have been considered redundant in our prior analysis. Obviously, if $k$ is the period of a word, so is $n \cdot k$ for any positive integer $n$. Although $(A+B)^{2}=A^{2}+A B+B A+B^{2}$, the trace of $A^{2}$ is not needed to compute the derivatives of the basic functions at the corresponding point since $A$ and $A^{2}$, not to mention $A^{3}, A^{4}$ and so on, fix the same points.

The average class traces enable one to compute the average eigenvalues for all operators of length $k$, so they tell us whether the average value of $f_{W}^{\prime \pm}(t)$ per conjugacy class is 0 or $\infty$. This gives us an idea of where the breakdown between words corresponding to 0 derivatives and words corresponding to $\infty$ derivative occurs. For example, take $k=4$. Then $(A+B)^{4}=2\binom{4}{0} \cdot C C\left(A^{4}\right)+2\binom{4}{1} \cdot C C\left(A^{3} B\right)+4$. $C C\left(A^{2} B^{2}\right)+2 \cdot C C\left((A B)^{2}\right)$ after the operators are grouped into their respective conjugacy classes. The coefficients 4 and 2 are the number of unique words which are cyclic permutations of $A^{2} B^{2}$ and $(A B)^{2}$ respectively and the sum of these coefficients is $\binom{4}{2}$. One knows the first two of these are likely to have correspond to points having $\infty$ derivative because one letter "overpowers" the other (i.e., there are more As than $B s$ or vice versa). Using $\bar{\tau}_{4}$, given in Appendix C, one finds that the aver-
age value of $f_{W}^{\prime+}(t)$ is $\infty$. There are 10 conjugacy classes having more of one letter than another and only 6 having the same number of letters, so one might guess that the breakdown occurs after this second conjugacy class, which is indeed the case. Of course, to test this guideline, one would need to compute the derivatives and average class traces of many more operators as well as look at their corresponding necklaces. The average class traces and the 2-ary necklaces for the conjugacy classes of operators of short length $(k \leq 4)$ can be found in Table 6.7 of Appendix C.

While the operators $W$ have been classified, one has yet to find a shortcut for determining if an operator of arbitrary length $k$ corresponds to a point having 0 or $\infty$ derivative without actually computing the necessary matrix.

Another open problem worth noting is the task of computing the derivatives of the basic functions at irrational points. The procedure discussed in this thesis relied on the fact that rational nondyadic points have periodic binary expansions, which made possible the construction of the difference equation (4.23). But irrationals have infinite aperiodic binary expansions so some alternate procedure needs to be devised. At first glance, one may conjecture that these derivatives are also 0 or $\infty$, as that seems to be the trend. However, we know from real analysis that all basic functions, being monotone, have nonzero finite derivative almost everywhere, so there must be some irrational points at which the derivatives of the basic functions exist and are neither 0 nor $\infty$. Moreover, this set of points cannot be one having measure zero. Our earlier analysis showed that $\chi, \phi, \psi$ and $\xi$ are examples of continuous monotone functions with vanishing derivative on a dense set and infinite derivative on another dense set. It may be that they take on finite nonzero values on some dense set in $\mathbb{R}-\mathbb{Q}$. Since little more is known about the local behavior of the basic functions at the irrationals, the stated question remains an unsolved problem of unknown
difficulty.

## 6 Appendix

### 6.1 Appendix A: Values of the Basic Functions

Table 6.1. Some Values of the Basic Functions at Rational Points

|  | 0 | $\frac{1}{15}$ | $\frac{2}{15}$ | $\frac{1}{5}$ | $\frac{4}{15}$ | $\frac{1}{3}$ | $\frac{2}{5}$ | $\frac{7}{15}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\chi(t)$ | 0 | $\frac{706}{363831}$ | $\frac{3530}{363831}$ | $\frac{4}{179}$ | $\frac{17650}{363831}$ | $\frac{2}{27}$ | $\frac{20}{179}$ | $\frac{57706}{363831}$ |
| $\phi(t)$ | 0 | $\frac{1967}{27987}$ | $\frac{2305}{19149}$ | $\frac{29}{179}$ | $\frac{78875}{363831}$ | $\frac{7}{27}$ | $\frac{55}{179}$ | $\frac{130331}{363831}$ |
| $\psi(t)$ | 0 | $\frac{16812}{121277}$ | $\frac{28020}{121277}$ | $\frac{54}{179}$ | $\frac{46700}{121277}$ | $\frac{4}{9}$ | $\frac{90}{179}$ | $\frac{5204}{9329}$ |
| $\xi(t)$ | 0 | $\frac{75301}{363831}$ | $\frac{124325}{363831}$ | $\frac{79}{179}$ | $\frac{201325}{363831}$ | $\frac{17}{27}$ | $\frac{125}{179}$ | $\frac{275581}{363831}$ |

### 6.2 Appendix B: Derivatives of the Basic Functions

Procedure for Computing the Derivatives of the Basic Functions at Rational Nondyadic Points:

Given any $t=\frac{p}{q} \in[0,1] \cap\left(\mathbb{Q}-\mathbb{Z}_{\langle 2\rangle}\right)$, where $q$ is odd and $\operatorname{gcd}(p, q)=1$, we can apply the following procedure to compute the derivatives of the four basic functions at that point.

1. Find the binary expansion of $t$ and determine its period $k$.
2. Write the word $w$ which fixes $t$ by converting each 0 in the binary expansion to an $\alpha$ and each 1 to a $\beta$.
3. Compute the corresponding operator $W$ and find its trace and determinant. Then use (4.33) to solve for $\mu_{W}^{ \pm}$.
4. Compare $\mu_{W}^{ \pm}$to 1 as follows to determine the derivatives:

- If $\mu_{W}^{ \pm}<1$, then $\chi^{\prime}(t)=\phi^{\prime}(t)=\psi^{\prime}(t)=\xi^{\prime}(t)=0$.
- If $\mu_{W}^{-}<1$ but $\mu_{W}^{+}>1$, then $\chi^{\prime}(t)=\phi^{\prime}(t)=\psi^{\prime}(t)=\xi^{\prime}(t)=\infty$

Table 6.2. Derivatives at the Points $t=\frac{m}{63}, m \in \mathbb{Z}$

|  | $\frac{1}{63}$ | $\frac{2}{63}$ | $\frac{3}{63}$ | $\frac{4}{63}$ | $\frac{5}{63}$ | $\frac{6}{63}$ | $\frac{7}{63}$ | $\frac{8}{63}$ | $\frac{9}{63}$ | $\frac{10}{63}$ | $\frac{11}{63}$ | $\frac{12}{63}$ | $\frac{13}{63}$ | $\frac{14}{63}$ | $\frac{15}{63}$ | $\frac{16}{63}$ |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Derivatives | $\infty$ | $\infty$ | $\infty$ | $\infty$ | 0 | $\infty$ | 0 | $\infty$ | 0 | 0 | 0 | $\infty$ | 0 | 0 | $\infty$ | $\infty$ |


| $\frac{17}{63}$ | $\frac{18}{63}$ | $\frac{19}{63}$ | $\frac{20}{63}$ | $\frac{21}{63}$ | $\frac{22}{63}$ | $\frac{23}{63}$ | $\frac{24}{63}$ | $\frac{25}{63}$ | $\frac{26}{63}$ | $\frac{27}{63}$ | $\frac{28}{63}$ | $\frac{29}{63}$ | $\frac{30}{63}$ | $\frac{31}{63}$ | $\frac{32}{63}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\infty$ | 0 | 0 | 0 | 0 | 0 | $\infty$ | $\infty$ | $\infty$ |

### 6.3 Appendix C: Conjugacy Classes of Operators $W$

Table 6.3. Conjugacy Classes of Operators ${ }^{25} W, k \leq 4$

| Class | $t$ | $k$ | $\left[W_{q}\right]$ | $\tau$ | $\Delta$ | $\mu_{W}^{ \pm}$ | Derivatives |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| A | 0,1 | 1 | 2 | $\frac{8}{5}$ | $\frac{12}{25}$ | $\begin{aligned} & 0.4000 \\ & 1.2000 \end{aligned}$ | $\begin{gathered} \chi^{\prime}(0)=\xi^{\prime}(1)=0 \\ \phi^{\prime}(0)=\psi^{\prime}(0)=\xi^{\prime}(0)=\infty \\ \chi^{\prime}(1)=\phi^{\prime}(1)=\psi^{\prime}(1)=\infty \end{gathered}$ |
| $A^{2}$ | 0,1 | 2 | 2 | $\frac{8}{5}$ | $\left(\frac{12}{25}\right)^{2}$ | $\begin{aligned} & 0.1600 \\ & 1.4400 \end{aligned}$ | 0 or $\infty$ |
| $A B$ | $\frac{1}{3}, \frac{2}{3}$ | 2 | 2 | $\frac{28}{25}$ | $\left(\frac{12}{25}\right)^{2}$ | $\begin{aligned} & 0.2716 \\ & 0.8484 \end{aligned}$ | 0 |
| $A^{3}$ | 0,1 | 3 | 2 | $\frac{224}{125}$ | $\left(\frac{12}{25}\right)^{3}$ | $\begin{aligned} & 0.0640 \\ & 1.4400 \end{aligned}$ | 0 or $\infty$ |
| $A^{2} B$ | $\begin{aligned} & \frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \\ & \frac{4}{7}, \frac{5}{7}, \frac{6}{7} \end{aligned}$ | 3 | 6 | $\frac{128}{125}$ | $\left(\frac{12}{25}\right)^{3}$ | $\begin{aligned} & 0.1227 \\ & 0.9013 \end{aligned}$ | 0 |
| $A^{4}$ | 0,1 | 4 | 2 | $\frac{1312}{625}$ | $\left(\frac{12}{25}\right)^{4}$ | $\begin{gathered} 0.0256 \\ 2.074 \end{gathered}$ | 0 or $\infty$ |
| $A^{3} B$ | $\begin{aligned} & \frac{1}{15}, \frac{2}{15}, \frac{4}{15}, \frac{7}{15}, \\ & \frac{8}{15}, \frac{11}{15}, \frac{13}{15}, \frac{14}{15} \end{aligned}$ | 4 | 8 | $\frac{688}{625}$ | $\left(\frac{12}{25}\right)^{4}$ | $\begin{aligned} & 0.0505 \\ & 1.0503 \end{aligned}$ | $\infty$ |
| $A^{2} B^{2}$ | $\frac{3}{15}, \frac{6}{15}, \frac{12}{15}, \frac{9}{15}$ | 4 | 4 | $\frac{544}{625}$ | $\left(\frac{12}{25}\right)^{4}$ | $\begin{aligned} & 0.0660 \\ & 0.8044 \end{aligned}$ | 0 |
| $(A B)^{2}$ | $\frac{1}{3}, \frac{2}{3}$ | 4 | 2 | $\frac{496}{625}$ | $\left(\frac{12}{25}\right)^{4}$ | $\begin{gathered} 0.0737 \\ 2.074 \end{gathered}$ | 0 or $\infty$ |

[^20]Table 6.4. Conjugacy Classes of Operators $W, k=5$

| Class | $t$ | $k$ | $\left[W_{q}\right]$ | $\tau$ | $\Delta$ | $\mu_{W}^{ \pm}$ | Derivatives |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $A^{5}$ | 0,1 | 5 | 2 | $\frac{7808}{3125}$ | $\left(\frac{12}{25}\right)^{5}$ | 0.0102 | 0 or $\infty$ |
|  |  |  |  |  |  | 2.4883 |  |
| $A^{4} B$ | $\frac{1}{31}, \frac{2}{31}, \frac{4}{31}, \frac{8}{31}, \frac{15}{31}$, | 5 | 10 | $\frac{3968}{3125}$ | $\left(\frac{12}{25}\right)^{5}$ | 0.0204 | $\infty$ |
|  | $\frac{16}{31}, \frac{23}{31}, \frac{27}{31}, \frac{29}{31}, \frac{30}{31}$ |  |  |  |  | 1.2494 |  |
| $A^{3} B^{2}$ | $\frac{3}{31}, \frac{6}{31}, \frac{7}{31}, \frac{12}{31}, \frac{14}{31}$, | 5 | 10 | $\frac{2816}{3125}$ | $\left(\frac{12}{25}\right)^{5}$ | 0.0292 | 0 |
|  | $\frac{17}{31}, \frac{19}{31}, \frac{24}{31}, \frac{25}{31}, \frac{28}{31}$ |  |  |  |  |  |  |
| $A^{2} B A B$ | $\frac{5}{31}, \frac{9}{31}, \frac{10}{31}, \frac{11}{31}, \frac{13}{31},$ | 5 | 10 | $\frac{2432}{3125}$ | $\left(\frac{12}{25}\right)^{5}$ | 0.0342 | 0 |
|  | $\frac{18}{31}, \frac{20}{31}, \frac{21}{31}, \frac{22}{31}, \frac{26}{31}$ |  |  |  |  | 0.7440 |  |

Table 6.5. Conjugacy Classes of Operators $W, k=6$

| Class | $t$ | $k$ | $\left[W_{q}\right]$ | $\tau$ | $\Delta$ | $\mu_{W}^{ \pm}$ | Derivatives |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $A^{6}$ | 0,1 | 6 | 2 | $\frac{9344}{3125}$ | $\left(\frac{12}{25}\right)^{6}$ | 0.0041 | 0 or $\infty$ |
|  |  |  |  |  |  | 2.9860 |  |
| $A^{5} B$ | $\frac{1}{63}, \frac{2}{63}, \frac{4}{63}, \frac{8}{63}, \frac{16}{63}, \frac{31}{63}$ | 6 | 12 | $\frac{23488}{15625}$ | $\left(\frac{12}{25}\right)^{6}$ | 0.0082 | $\infty$ |
|  | $\frac{32}{63}, \frac{47}{63}, \frac{55}{63}, \frac{59}{63}, \frac{61}{63}, \frac{62}{63}$ |  |  |  |  | 1.4951 |  |
| $A^{4} B^{2}$ | $\frac{3}{63}, \frac{6}{63}, \frac{12}{63}, \frac{15}{63}, \frac{24}{63}, \frac{30}{63}$ | 6 | 12 | $\frac{128}{125}$ | $\left(\frac{12}{25}\right)^{6}$ | 0.0121 | $\infty$ |
|  | $\frac{33}{63}, \frac{39}{63}, \frac{48}{63}, \frac{51}{63}, \frac{57}{63}, \frac{60}{63}$ |  |  |  |  | 1.0119 |  |
| $A^{3} B^{3}$ | $\frac{7}{63}, \frac{14}{63}, \frac{28}{63}, \frac{35}{63}, \frac{49}{63}, \frac{56}{63},$ | 6 | 6 | $\frac{14272}{15625}$ | $\left(\frac{12}{25}\right)^{6}$ | 0.0136 | 0 |
|  |  |  |  |  |  | 0.8998 |  |
| $A^{3} B A B$ | $\frac{5}{63}, \frac{10}{63}, \frac{17}{63}, \frac{20}{63}, \frac{23}{63}, \frac{29}{63}$, | 6 | 12 | $\frac{13504}{15625}$ | $\left(\frac{12}{25}\right)^{6}$ | 0.0143 | 0 |
|  | $\frac{34}{63}, \frac{40}{63}, \frac{43}{63}, \frac{46}{63}, \frac{53}{63}, \frac{58}{63}$ |  |  |  |  |  |  |
| $A^{2} B A B^{2}$ | $\frac{11}{63}, \frac{13}{63}, \frac{19}{63}, \frac{22}{63}, \frac{25}{63}, \frac{26}{63}$, | 6 | 12 | $\frac{448}{625}$ | $\left(\frac{12}{25}\right)^{6}$ | 0.0175 | 0 |
|  | $\frac{37}{63}, \frac{38}{63}, \frac{41}{63}, \frac{44}{63}, \frac{50}{63}, \frac{52}{63}$ |  |  |  |  | 0.6993 |  |
| $\left(A^{2} B\right)^{2}$ | $\frac{9}{63}, \frac{18}{63}, \frac{27}{63}, \frac{36}{63}, \frac{45}{63}, \frac{54}{63},$ | 6 | 6 | $\frac{12928}{15625}$ | $\left(\frac{12}{25}\right)^{6}$ | 0.0151 | 0 |
|  |  |  |  |  |  | 0.8123 |  |
| $(A B)^{2}$ | $\frac{21}{63}, \frac{42}{63}$ | 6 | 2 | $\frac{9856}{15625}$ | $\left(\frac{12}{25}\right)^{6}$ | 0.0200 | 0 |
|  |  |  |  |  |  | 0.6108 |  |

Table 6.6. Derivatives of $f_{W}^{ \pm}(t)$ by Conjugacy Class

| Word | $t$ | $k$ | $\mu_{W}^{ \pm}$ | $f_{W}^{\prime}(t)$ |
| :---: | :---: | :---: | :---: | :---: |
| $A B$ | $\frac{1}{3}, \frac{2}{3}$ | 2 | $\frac{14 \pm 2 \sqrt{13}}{25}$ | $\chi^{\prime}(t)+\frac{2 \pm \sqrt{13}}{3} \psi^{\prime}(t)$ |
| $A^{2} B$ | $\begin{aligned} & \frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \\ & \frac{4}{7}, \frac{5}{7}, \frac{6}{7} \end{aligned}$ | 3 | $\frac{64 \pm 8 \sqrt{37}}{125}$ | $\chi^{\prime}(t)+\frac{11 \pm 2 \sqrt{37}}{9} \psi^{\prime}(t)$ |
| $A^{3} B$ | $\begin{aligned} & \frac{1}{15}, \frac{2}{15}, \frac{4}{15}, \frac{7}{15}, \\ & \frac{8}{15}, \frac{11}{15}, \frac{13}{15}, \frac{14}{15} \end{aligned}$ | 4 | $\frac{344 \pm 40 \sqrt{61}}{625}$ | $\chi^{\prime}(t)+\frac{38 \pm 5 \sqrt{61}}{27} \psi^{\prime}(t)$ |
| $A^{2} B^{2}$ | $\frac{3}{15}, \frac{6}{15}, \frac{12}{15}, \frac{9}{15}$ | 4 | $\frac{272 \pm 64 \sqrt{13}}{625}$ | $\chi^{\prime}(t)+\frac{5 \pm 2 \sqrt{13}}{9} \psi^{\prime}(t)$ |
| $A^{4} B$ | $\begin{aligned} & \frac{1}{31}, \frac{2}{31}, \frac{4}{31}, \frac{8}{31}, \frac{15}{31}, \\ & \frac{16}{31}, \frac{23}{31}, \frac{27}{31}, \frac{29}{31}, \frac{30}{31} \end{aligned}$ | 5 | $\frac{1984 \pm 32 \sqrt{3601}}{3125}$ | $\chi^{\prime}(t)+\frac{119 \pm 2 \sqrt{3601}}{81} \psi^{\prime}(t)$ |
| $A^{3} B^{2}$ | $\begin{aligned} & \frac{3}{31}, \frac{6}{31}, \frac{7}{31}, \frac{12}{31}, \frac{14}{31}, \\ & \frac{17}{31}, \frac{19}{31}, \frac{24}{31}, \frac{25}{31}, \frac{28}{31} \end{aligned}$ | 5 | $\frac{1408 \pm 32 \sqrt{1693}}{3125}$ | $\chi^{\prime}(t)+\frac{37 \pm \sqrt{1693}}{54} \psi^{\prime}(t)$ |
| $A^{2} B A B$ | $\begin{aligned} & \frac{5}{31}, \frac{9}{31}, \frac{10}{31}, \frac{11}{31}, \frac{13}{31}, \\ & \frac{18}{31}, \frac{20}{31}, \frac{21}{31}, \frac{22}{31}, \frac{26}{31} \end{aligned}$ | 5 | $\frac{1216 \pm 32 \sqrt{1201}}{3125}$ | $\chi^{\prime}(t)+\frac{59 \pm 2 \sqrt{1201}}{63} \psi^{\prime}(t)$ |
| $A^{5} B$ | $\frac{1}{63}, \frac{2}{63}, \frac{4}{63}, \frac{8}{63}, \frac{16}{63}, \frac{31}{63}$, $\frac{32}{63}, \frac{47}{63}, \frac{55}{63}, \frac{59}{63}, \frac{61}{63}, \frac{62}{63}$ | 6 | $\frac{11744 \pm 32 \sqrt{131773}}{15625}$ | $\chi^{\prime}(t)+\frac{362 \pm \sqrt{131773}}{243} \psi^{\prime}(t)$ |
| $A^{4} B^{2}$ | $\frac{3}{63}, \frac{6}{63}, \frac{12}{63}, \frac{15}{63}, \frac{24}{63}, \frac{30}{63}$, $\frac{33}{63}, \frac{39}{63}, \frac{48}{63}, \frac{51}{63}, \frac{57}{63}, \frac{60}{63}$ | 6 | $\frac{64}{125} \pm \frac{1792 \sqrt{19}}{15625}$ | $\chi^{\prime}(t)+\frac{59 \pm 14 \sqrt{19}}{81} \psi^{\prime}(t)$ |
| $A^{3} B^{3}$ | $\frac{7}{63}, \frac{14}{63}, \frac{28}{63}, \frac{35}{63}, \frac{49}{63}, \frac{56}{63},$ | 6 | $\frac{\frac{7136 \pm 416 \sqrt{277}}{15625}}{}$ | $\chi^{\prime}(t)+\frac{14 \pm \sqrt{277}}{27} \psi^{\prime}(t)$ |
| $A^{3} B A B$ | $\frac{5}{63}, \frac{10}{63}, \frac{17}{63}, \frac{20}{63}, \frac{23}{63}, \frac{29}{63}$, <br> $\frac{34}{63}, \frac{40}{63}, \frac{43}{63}, \frac{46}{63}, \frac{53}{63}, \frac{58}{63}$ | 6 | $\frac{6752 \pm 32 \sqrt{41605}}{15625}$ | $\chi^{\prime}(t)+\frac{194 \pm \sqrt{41605}}{189} \psi^{\prime}(t)$ |
| $A^{2} B A B^{2}$ | $\frac{11}{63}, \frac{13}{63}, \frac{19}{63}, \frac{22}{63}, \frac{25}{63}, \frac{26}{63}$, <br> $\frac{37}{63}, \frac{38}{63}, \frac{41}{63}, \frac{44}{63}, \frac{50}{63}, \frac{52}{63}$ | 6 | $\frac{224}{625} \pm \frac{352 \sqrt{229}}{15625}$ | $\chi^{\prime}(t)+\frac{122 \pm 11 \sqrt{229}}{225} \psi^{\prime}(t)$ |

Table 6.7. Average Class Traces $\bar{\tau}_{k}$ and 2-ary Necklaces

| Necklace | Class | $t$ | $\tau$ | Derivative |
| :---: | :---: | :---: | :---: | :---: |
| Class Size $=1$ |  |  | $\bar{\tau}_{1}=\frac{8}{5}$ |  |
|  |  |  |  |  |
| Class Size $=2$ |  |  |  |  |


| Necklace | Class | $t$ | $\tau$ | Derivative |
| :---: | :---: | :---: | :---: | :---: |
| Class Size $=4$ |  |  | $\tau_{4}=\frac{706}{625}$ |  |
|  |  |  |  |  |

## References

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[2] A. Kirillov. A Tale of Two Fractals, work in progress.
[3] J. Marsden, M. Hoffman. Elementary Classical Analysis. 2nd Ed., W.H. Freeman and Company, New York, NY (1993).
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[^21]
[^0]:    Supervisor of Thesis

[^1]:    ${ }^{1} \mathrm{~A}$ unique property of fractals is that they have fractional Hausdorff dimension and this dimension strictly exceeds their topological dimension. If $\Theta$ is a bounded subset of $\mathbb{R}^{n}$ and $N_{\Theta}(\epsilon)$ is the minimum number of balls of radius $\epsilon$ needed to cover $\Theta$ the Hausdorff dimension of $\Theta$ is defined by $d_{H}(\Theta):=-\lim _{\epsilon \rightarrow 0} \frac{\log N_{\Theta}(\epsilon)}{\log \epsilon}$. The Sierpinski Gasket has dimension $\frac{\log (3)}{\log (2)}$. See Section 2.1 and [2] for more on Hausdorff measure and dimension.
    ${ }^{2}$ The majority of the figures in this thesis were generated using iterative programs written in Maple 10. Maple 10 was also used to compute the eigenvalues and eigenfunctions and to test our claims about the derivatives of the basic functions numerically.

[^2]:    ${ }^{3}$ These topics go beyond the scope of this thesis.
    ${ }^{4}$ See [6].

[^3]:    ${ }^{5}$ A similar procedure is described by Rupinski in [4].

[^4]:    ${ }^{6}$ See footnote 8 .

[^5]:    ${ }^{7}$ It is also true that if $M$ is compact, $\mathbb{K}(M)$ is compact.

[^6]:    ${ }^{8}$ The Contraction Mapping Principle states that if $M$ is a complete metric space and $f$ is a contracting map from $M$ to $M$, then there is a unique fixed point for $f$ in $M$ satisfying $f(x)=x$. The proof of this theorem can be found in any real analysis, in particular [3], p. 301.

[^7]:    ${ }^{9}$ The remaining 5 equations of the system are not relevant to this proof.

[^8]:    ${ }^{10}$ Also known as the Uniform Continuity Theorem. See [3], p. 215.

[^9]:    ${ }^{11}$ The values of $\Delta \psi(t)$ needed to generate Figure 8 were computed using the following extension

[^10]:    ${ }^{12}$ From this point forward, the letters $A, B$ and $C$ refer to these operators and NOT the boundary values of a function $f_{A B}^{C} \in \mathcal{H}(S)$. Despite the potential confusion, we keep this notation to be consistent with the literature, in particular [2].

[^11]:    ${ }^{13}$ See Example 4.3 for a corresponding figure.
    ${ }^{14}$ The values of $\chi\left(\frac{1}{3}\right)$ and $\psi\left(\frac{1}{3}\right)$ can be found in Appendix A.

[^12]:    ${ }^{15}$ See Lemma 4.5 for a detailed proof of this fact.

[^13]:    ${ }^{16}$ One can also consider the derivatives of the basic functions with respect to one another. This idea is briefly discussed in Example 4.11.

[^14]:    ${ }^{17}$ One should keep in mind that when we write $\chi^{\prime}(t)$ and $\psi^{\prime}(t)$, we mean really mean the difference quotient. The shorthand notation only makes sense if the functions are differentiable at the points in question. Since we have shown that limits in (4.1) exist using Lemma 4.1, differentiability is not a problem at any dyadic point.
    ${ }^{18}$ For more on the second derivatives of the basic functions, see [2] and [4].

[^15]:    ${ }^{19} \mathrm{~A}$ similar procedure is described by Rupinski in [4].

[^16]:    ${ }^{20}$ We consider nondyadic rational points with odd denominator only because any basic function evaluated at a point with an even denominator can be simplified using Corollary 3.1.

[^17]:    ${ }^{21}$ Quoted from [5], pp. 254-262.

[^18]:    ${ }^{22} \varphi(n)$, also known as Euler's Totient Function is defined as the number of positive integers relatively prime to an integer $n . \varphi(n)$ can be computed using the formula $\varphi(n)=n \prod_{p \mid n}\left(1-\frac{1}{p}\right)$ with the product ranging over the distinct primes $p_{r}$ which divide $n$.
    ${ }^{23} \mathrm{~A}$ more detailed table can be found in Appendix C.

[^19]:    ${ }^{24}$ In the technical combinatorial sense, an $a$-ary necklace of length $n$ is a string of $n$ characters, each of $a$ possible types. Rotation is ignored, in the sense that $b_{1} b_{2} \ldots b_{n}$ is equivalent to $b_{k} b_{k+1} \ldots b_{n} b_{1} b_{2} \ldots b_{k-1}$ for any $k$.

[^20]:    ${ }^{25}\left[W_{q}\right]$ denotes the class size of a word $w \in G$ which fixes $t=\frac{p}{q}$ where $q$ is odd and $\operatorname{gcd}(p, q)=1$. In other words, $\left[W_{q}\right]$ is the number of matrices having the same $\tau, \Delta$ and length $k$. " 0 or $\infty$ " is our shorthand notation for $\chi^{\prime}(0)=\xi^{\prime}(1)=0, \phi^{\prime}(0)=\psi^{\prime}(0)=\xi^{\prime}(0)=\chi^{\prime}(1)=\phi^{\prime}(1)=\psi^{\prime}(1)=\infty$.

[^21]:    ${ }^{26}$ This paper was written for Dr. A.A. Kirillov's Math 550 class at the University of Pennsylvania during the Fall 2005 semester.

