Recent Extensions of the Discontinuous Enrichment Method (DEM) to Advection-Dominated Fluid Mechanics Problems

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Motivation

Scalar Advection-Diffusion Equation

$$\mathcal{L}u = \underbrace{-\kappa\Delta u}_{\text{diffusion}} + \underbrace{\mathbf{a}\cdot\nabla u}_{\text{advection}} = f$$

- Advection velocity: $\mathbf{a} = (a_1, a_2)^T = |\mathbf{a}|(\cos\phi, \sin\phi)^T$.
- ϕ = advection direction.
- $\kappa \equiv 1$ = diffusivity.

- Describes many transport phenomena in fluid mechanics.
- Usual scalar model for the more challenging Navier-Stokes equations.
- Global Péclet number (L = length scale associated with Ω):

$$Pe = \frac{\text{rate of advection}}{\text{rate of diffusion}} = \frac{L|\mathbf{a}|}{\kappa} = Re \cdot \begin{cases} Pr & \text{(thermal diffusion)} \\ Sc & \text{(mass diffusion)} \end{cases}$$

Advection-Dominated Regime
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Sharp gradients in exact solution
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Galerkin FEM inadequate: spurious oscillations (Fig. 1)

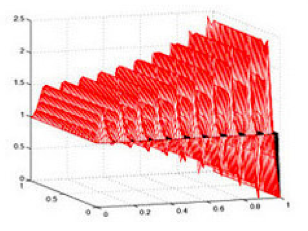
Figure 1: Spurious oscillations in the standard Galerkin Q_1 solution at high Pe number

Some Classical Remedies	
Stabilized FEMs (SUPG, GLS, USFEM)	RFB, VMS, PUM
Add a weighted residual (numerical diffusion) to variational equation to damp out oscillations.	Construct conforming spaces that incorporate knowledge of local behavior of the solution.

Discontinuous Enrichment Method

- First proposed and developed by Farhat *et. al.* in [1] for the solution of the Helmholtz equation.

Idea of DEM

"Enrich" the usual Galerkin polynomial field $\mathcal{V}^P$ by the free-space solutions to the governing constant-coefficient homogeneous PDE.

$$\begin{array}{ccccc} \mathcal{V}^h & = & \mathcal{V}^P & \oplus & \mathcal{V}^E \\ \Psi & & \Psi & & \Psi \\ u^h & = & u^P & + & u^E. \end{array}$$

- Relation to multi-scale methods: splitting of solution into coarse (polynomial) and fine (enrichment) scales.
- Unlike PUM, VMS & RFB: enrichment field in DEM is not required to vanish at element boundaries.
- Continuity across element boundaries is enforced weakly using Lagrange multipliers $\lambda^h \in \mathcal{W}^h$.

Two Variants of DEM: True DEM vs. Pure DGM

- Primal unknown $u^h \in \mathcal{V}^h$ has one of the two forms:

	DGM	DEM
$\mathcal{V}^h$	$\mathcal{V}^E$	$\mathcal{V}^P \oplus (\mathcal{V}^E \backslash \mathcal{V}^P)$
u^h	u^E	$u^P + u^E$

Enrichment-only "pure DGM": Contribution of the standard polynomial field is dropped from the approximation entirely

Genuine or "full" DEM: Splitting of the approximation into coarse (polynomial) and fine (enrichment) scales

Implementation

- Element matrix problem (uncondensed):

$$\begin{pmatrix} \mathbf{k}^{PP} & \mathbf{k}^{PE} & \mathbf{k}^{PC} \\ \mathbf{k}^{EP} & \mathbf{k}^{EE} & \mathbf{k}^{EC} \\ \mathbf{k}^{CP} & \mathbf{k}^{CE} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{u}^P \\ \mathbf{u}^E \\ \lambda \end{pmatrix} = \begin{pmatrix} \mathbf{r}^P \\ \mathbf{r}^E \\ \mathbf{r}^C \end{pmatrix}$$

Due to the discontinuous nature of $\mathcal{V}^E$, u^E can be eliminated at the element level by a static condensation.

- Statically-condensed true DEM element system:

$$\begin{pmatrix} \tilde{\mathbf{k}}^{PP} & \tilde{\mathbf{k}}^{PC} \\ \tilde{\mathbf{k}}^{CP} & \tilde{\mathbf{k}}^{CC} \end{pmatrix} \begin{pmatrix} \mathbf{u}^P \\ \lambda \end{pmatrix} = \begin{pmatrix} \tilde{\mathbf{r}}^P \\ \tilde{\mathbf{r}}^C \end{pmatrix},$$

- Statically-condensed pure DGM element system:

$$-\mathbf{k}^{CE}(\mathbf{k}^{EE})^{-1}\mathbf{k}^{EC}\lambda = \mathbf{r}^C - \mathbf{k}^{CE}(\mathbf{k}^{EE})^{-1}\mathbf{r}^E.$$

where

$$\begin{aligned} \tilde{\mathbf{k}}^{PP} &= \mathbf{k}^{PP} - \mathbf{k}^{PE}(\mathbf{k}^{EE})^{-1}\mathbf{k}^{EP}, & \tilde{\mathbf{k}}^{PC} &= \mathbf{k}^{PC} - \mathbf{k}^{PE}(\mathbf{k}^{EE})^{-1}\mathbf{k}^{EC} \\ \tilde{\mathbf{k}}^{CP} &= \mathbf{k}^{CP} - \mathbf{k}^{CE}(\mathbf{k}^{EE})^{-1}\mathbf{k}^{EP}, & \tilde{\mathbf{k}}^{CC} &= -\mathbf{k}^{CE}(\mathbf{k}^{EE})^{-1}\mathbf{k}^{EC} \\ \tilde{\mathbf{r}}^{P} &= \mathbf{r}^{P} - \mathbf{k}^{PE}(\mathbf{k}^{EE})^{-1}\mathbf{r}^{E}, & \tilde{\mathbf{r}}^{C} &= \mathbf{r}^{C} - \mathbf{k}^{CE}(\mathbf{k}^{EE})^{-1}\mathbf{r}^{E}. \end{aligned}$$

*Joint work with Dr. Charbel Farhat and Dr. Radek Tezaur.

Hybrid Variational Formulation of DEM for Advection-Diffusion

- Strong form:

$$(S): \begin{cases} \text{Find } u \in H^1(\Omega) \text{ such that} \\ -\Delta u + \mathbf{a}\cdot\nabla u = f, & \text{in } \Omega \\ u = g, & \text{on } \Gamma = \partial\Omega \\ [\![u]\!] = 0 & \text{on } \Gamma_{int} \end{cases}$$

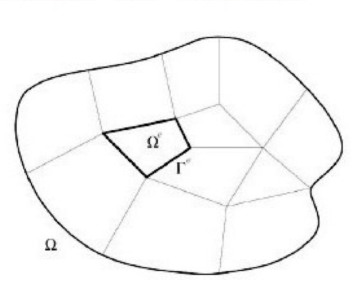
Figure 2: Decomposition of domain Ω into elements Ω^e

- Weak hybrid variational form:

$$(W): \begin{cases} \text{Find } (u,\lambda) \in \mathcal{V}\times\mathcal{W} \text{ such that} \\ a(v,u) + b(\lambda, v) = r(v), & \forall v \in \mathcal{V} \\ b(\mu, u) = -r_d(\mu) & \forall \mu \in \mathcal{W} \end{cases}$$

Notation:
$\tilde{\Omega} = \cup_{e=1}^{n_{el}}\Omega^e$
$\tilde{\Gamma} = \cup_{e=1}^{n_{el}}\Gamma^e$
$\Gamma^{e,e'} = \Gamma^e \cap \Gamma^{e'}$

where

$$\mathcal{V} \equiv \left\{ v \in L^2(\tilde{\Omega}) : v|_{\Omega^e} \in H^1(\Omega^e)\right\}, \qquad \mathcal{W} = \Pi_e \Pi_{e'<e} H^{-1/2}(\Gamma^{e,e'}) \times H^{-1/2}(\Gamma)$$

$$a(v,u) = (\nabla v + v\mathbf{a}, \nabla u)_{\tilde{\Omega}}, \qquad r(v) = (f, v)$$

$$b(\lambda, v) = \sum_e \sum_{e'<e} \int_{\Gamma^{e,e'}} \lambda(v_{e'} - v_e)d\Gamma + \int_\Gamma \lambda v\, d\Gamma, \qquad r_d(\mu) = \int_\Gamma \mu g d\Gamma$$

- Space of Lagrange Multiplier Approximations $\mathcal{W}^h$:

$$\begin{aligned} a(u,v) &= \textstyle\int_{\tilde{\Omega}}(\mathbf{a}\cdot\nabla u - \Delta u)vd\Omega + \int_\Gamma \nabla u \cdot \mathbf{n} v d\Gamma \\ &\quad + \textstyle\sum_e \sum_{e'} \int_{\Gamma^{e,e'}} (\nabla u_e \cdot \mathbf{n}_e v_e + \nabla u_{e'} \cdot \mathbf{n}_{e'} v_{e'})d\Gamma \end{aligned}$$

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Suggests approximating:

$$\boxed{\lambda^h \approx \nabla u_e^E \cdot \mathbf{n}^e = -\nabla u_{e'}^E \cdot \mathbf{n}^{e'} \quad \text{on } \Gamma^{e,e'}}$$

and

$$\boxed{\lambda^h \approx -\nabla u^E \cdot \mathbf{n} \quad \text{on } \Gamma}$$

if a Dirichlet boundary condition is to be enforced on Γ.

Approximation Spaces for 2D Advection-Diffusion

Exponential Enrichment Functions

- Derived by solving $\mathcal{L}u^E = \mathbf{a}\cdot\nabla u^E - \Delta u^E = 0$ analytically.

Enrichment functions for 2D advection-diffusion:

$$u^E(\mathbf{x};\theta_i) = e^{\frac{Pe}{2}(\cos\phi + \cos\theta_i)(x - x_{r,i})} e^{\frac{Pe}{2}(\sin\phi + \sin\theta_i)(y - y_{r,i})}$$

$\Theta^u \equiv \{\theta_i\}_{i=1}^{n^E} \in [0, 2\pi) =$ set of angles specifying $\mathcal{V}^E$

$(x_{r,i}, y_{r,i}) =$ reference point for u_i^E

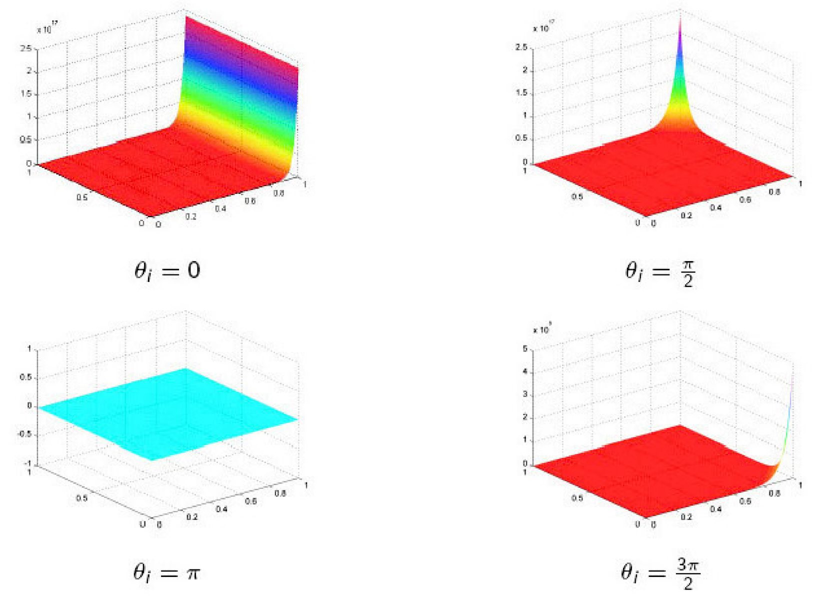

Figure 3: Plots of enrichment function $u^E(\mathbf{x};\theta_i)$ for several values of θ_i ($\phi = 0$)

Exponential Lagrange Multipliers on a Straight Edge

Lagrange Multiplier Approximations: $\lambda^h \approx \nabla u^E \cdot \mathbf{n}|_{\Gamma_{e,e'}}$

$$\lambda^h(s)|_{\Gamma^{ij}} = \sum_{k=1}^{n^E} \lambda_k \underbrace{\exp\left\{\frac{|\mathbf{a}|}{2}\left[\cos(\phi - \alpha^{ij}) + \cos(\theta_k - \alpha^{ij})\right](s - s_r^{ij})\right\}}_{\equiv \lambda^h(s;\theta_k)},$$

- Discrete Babuška-Brezzi *inf-sup* condition: a.e. in the mesh

$$\boxed{\#\text{ Lagrange multipliers per edge} = n^\lambda \leq \frac{n^E}{2}}$$

- Bound is a necessary, but in general *not* a sufficient condition for ensuring a non-singular global interface problem.

Numerical tests [5] suggest to limit:

$$n^\lambda = \left\lfloor \frac{n^E}{4} \right\rfloor \equiv \max\left\{ n \in \mathbb{Z} | n \leq \frac{n^E}{4} \right\}$$

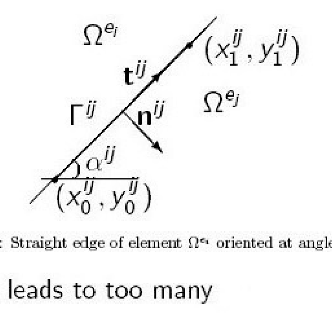

Figure 4: Straight edge of element Ω^{e_i} oriented at angle α^{ij}

- The set Θ^u that defines the enrichment field typically leads to too many Lagrange multiplier dofs.

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$$\boxed{\Theta^\lambda = \{\theta_k^\lambda\}_{k=1}^{n^\lambda} = \text{set of angles that specifies the Lagrange multipliers} \not\subset \Theta^u}$$

DEM/DGM Element Design for 2D Advection-Diffusion

Lagrange Multiplier Selection & Truncation

Lemma 1

Two Lagrange multipliers $\lambda^h(s;\theta_1^\lambda)$ and $\lambda^h(s;\theta_2^\lambda)$ given on a straight edge Γ^{ij} are redundant (that is, $\lambda^h(s;\theta_1^\lambda) = C\lambda^h(s;\theta_2^\lambda)$ for some real constant C) if

$$\frac{\theta_1^\lambda - \theta_2^\lambda}{2} = n\pi, \quad \text{or} \quad \frac{\theta_1^\lambda + \theta_2^\lambda}{2} = \alpha^{ij} + n\pi,$$

for any $n \in \mathbb{Z}$, where α^{ij} is the angle at which Γ^{ij} is oriented.

- Key Observation: If Θ^λ as a set of angles that are clustered around α^{ij}:

$$\boxed{\Theta^\lambda = \alpha^{ij} + \{\beta_k^\lambda\}_{k=1}^{n^\lambda}, \qquad \{\beta_k^\lambda\}_{k=1}^{n^\lambda} \in [0, 2\pi)}$$

the necessary redundancy condition (Lemma 1) becomes mesh-independent:

$$\frac{\theta_k^\lambda + \theta_l^\lambda}{2} = \alpha^{ij} + n\pi \quad \Leftrightarrow \quad \frac{\beta_k^\lambda + \beta_l^\lambda}{2} = n\pi.$$

Mesh Independent Element Design Procedure

Enrichment Functions: $u^E(\mathbf{x};\theta_i) = e^{\frac{Pe}{2}(\cos\phi + \cos\theta_i)(x - x_{r,i})} e^{\frac{Pe}{2}(\sin\phi + \sin\theta_i)(y - y_{r,i})}$

Lagrange multipliers: $\lambda^h(s)|_{\Gamma^{ij}} = e^{\frac{|\mathbf{a}|}{2}[\cos(\phi - \alpha^{ij}) + \cos\beta_k^\lambda](s - s_{r,k})}$

Algorithm 1. DGM/DEM element design

Fix $n^E \in \mathbb{N}$ (the desired number of angles defining $\mathcal{V}^E$).
Select a set of n^E distinct angles $\{\theta_k\}_{k=1}^{n^E}$ between $[0, 2\pi)$.
Let $\Theta^u = \phi + \{\theta_i\}_{i=1}^{n^E}$.
Let $n^\lambda = \lfloor \frac{n^E}{4} \rfloor$.
Choose a set of n^λ distinct angles $\{\beta_k\}_{k=1}^{n^\lambda}$ between $[0, \pi)$.
for each edge $\Gamma^{ij} \in \Gamma^{\text{int}}$ having slope α^{ij}
  Let $\Theta^\lambda = \alpha^{ij} + \{\beta_k\}_{k=1}^{n^\lambda}$ (the set of angles defining the Lagrange multiplier approximations on $\Gamma^{e,e'}$).
end for

Some DGM/DEM Elements

Notation

DGM Element: $Q - n^E - n^\lambda$
DEM Element: $Q - n^E - n^{\lambda +} \equiv [Q - n^E - n^\lambda] \cup [Q_1]$

'Q': Quadrilateral
n^E: Number of Enrichment Functions
n^λ: Number of Lagrange Multipliers per Edge
Q_1: Galerkin Bilinear Quadrilateral Element

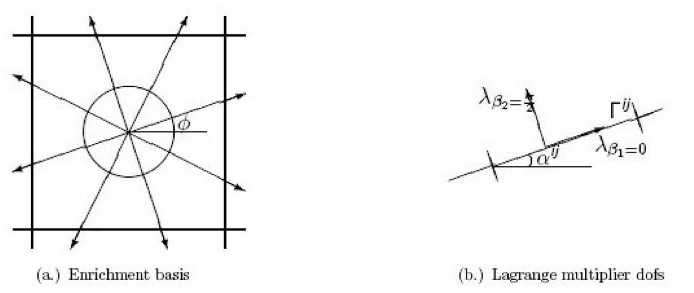

Figure 5: Illustration of the sets Θ^u and Θ^λ that define the $Q-8-2$ element

Table 1: Some DGM and DEM elements

Name	n^E	Θ^u	n^λ	Θ^λ
$Q-4-1$	4	$\phi + \{\frac{m\pi}{2} : m = 0,...,3\}$	1	ϕ
$Q-8-2$	8	$\phi + \{\frac{m\pi}{4} : m = 0,...,7\}$	2	$\alpha^{ij} + \{0, \frac{\pi}{2}\}$
$Q-12-3$	12	$\phi + \{\frac{m\pi}{6} : m = 0,...,11\}$	3	$\alpha^{ij} + \{\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}\}$
$Q-16-4$	16	$\phi + \{\frac{m\pi}{8} : m = 0,...,15\}$	4	$\alpha^{ij} + \{0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}\}$
$Q-5-1^+$	5	$\phi + \{\frac{2m\pi}{5} : m = 0,...,4\}$	1	$\phi - \pi$
$Q-9-2^+$	9	$\phi + \{\frac{2m\pi}{9} : m = 0,...,8\}$	2	$\alpha^{ij} + \{0, \frac{\pi}{2}\}$
$Q-13-3^+$	13	$\phi + \{\frac{2m\pi}{13} : m = 0,...,12\}$	3	$\alpha^{ij} + \{\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}\}$
$Q-17-4^+$	17	$\phi + \{\frac{2m\pi}{17} : m = 0,...,16\}$	4	$\alpha^{ij} + \{0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}\}$

Computational Complexity & Properties

Table 2: Computational complexity of some DGM, DEM, and standard Galerkin elements

Element	Asymptotic # of dofs	Stencil width for uniform $n \times n$ mesh
Q_1	n_{el}	9
Q_2	$3n_{el}$	21
Q_3	$5n_{el}$	33
Q_4	$7n_{el}$	45
$Q-4-1$	$2n_{el}$	7
$Q-8-2$	$4n_{el}$	14
$Q-12-3$	$6n_{el}$	21
$Q-16-4$	$8n_{el}$	28
$Q-5-1^+$	$3n_{el}$	21
$Q-9-2^+$	$5n_{el}$	33
$Q-13-3^+$	$7n_{el}$	45
$Q-17-4^+$	$9n_{el}$	57

- Exponential enrichments ⇒ all integrations can be computed analytically.
- $\mathcal{L}u^E = 0$ ⇒ convert volume integrals to boundary integrals:

$$a(v^E, u^E) = \int_{\tilde{\Omega}} (\nabla v^E \cdot \nabla u^E + \mathbf{a}\cdot\nabla u^E v^E)\, d\Omega = \int_{\tilde{\Gamma}} \nabla u^E \cdot \mathbf{n} v^E d\Gamma.$$

Numerical Results

Double Ramp Problem on an L–shaped Domain

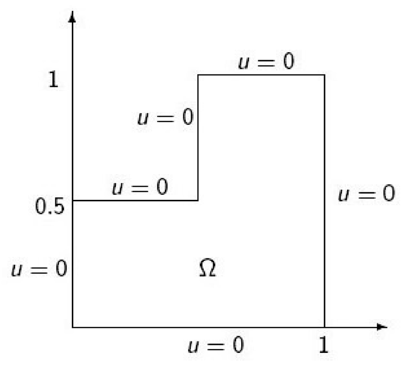

Figure 6: L-shaped domain

- Homogeneous Dirichlet boundary conditions are prescribed on all six sides of L–shaped domain Ω
- Advection direction: $\phi = 0$
- Source: $f = 1$
- Strong outflow boundary layer along the line $x = 1$
- Two crosswind boundary layers along $y = 0$ and $y = 1$
- A crosswind internal layer along $y = 0.5$

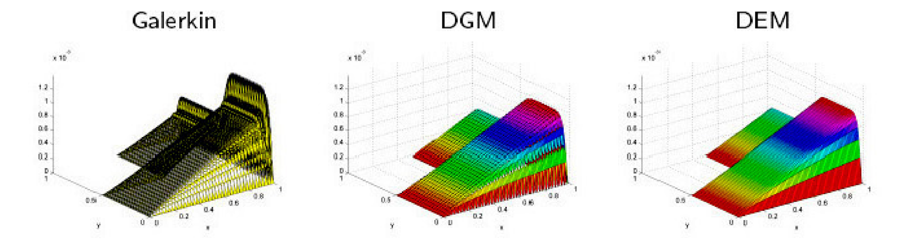

Figure 7: L-shaped double ramp problem solutions: $Pe = 10^3$, 1200 elts.

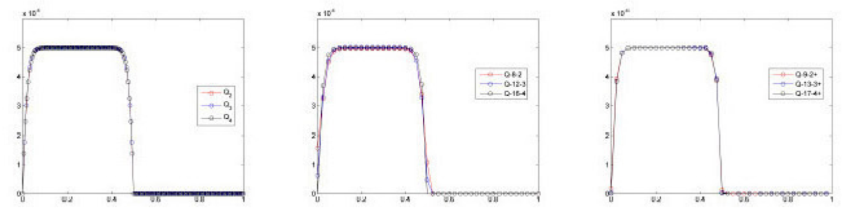
Figure 8: Solution to L-shaped double ramp problem along the line $x = 0.5$: $Pe = 10^3$, 1200 elts.

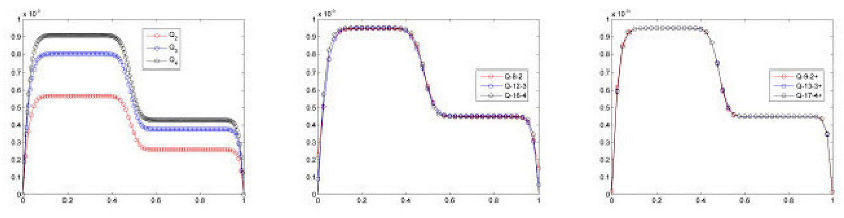
Figure 9: Solution to L-shaped double ramp problem along the line $x = 0.9$: $Pe = 10^3$, 1200 elts.

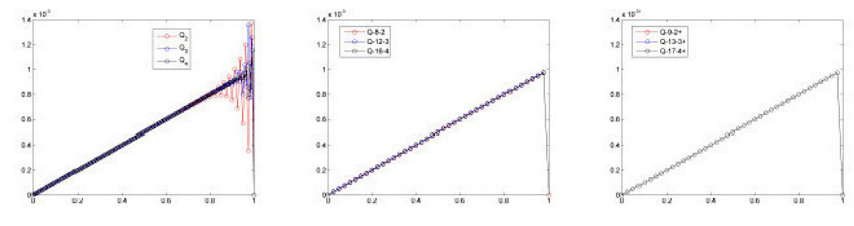
Figure 10: Solution to L-shaped double ramp problem along the line $y = 0.25$: $Pe = 10^3$, 1200 elts.

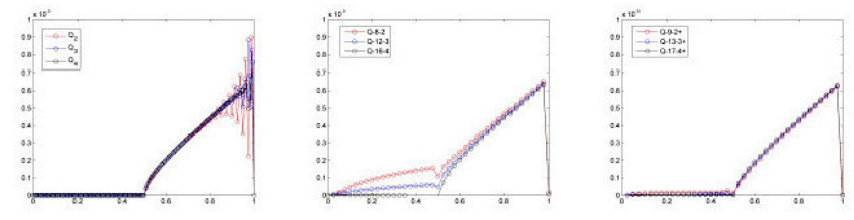
Figure 11: Solution to L-shaped double ramp problem along the line $y = 0.5$: $Pe = 10^3$, 1200 elts.

- No oscillations can be seen in the computed DGM and DEM solutions.
- DEM elements outperform DGM elements in general for this problem.
- Pure DGM elements experience some difficulty along the $y = 0.5$ line, the location of the crosswind internal layer.

Table 3: $L^2(\Omega)$ errors relative to a reference solution*: L–shaped double ramp problem, $Pe = 10^3$

# elements	Q_2	$Q-8-2$	$Q-9-2^+$
300	2.72×10^{-1}	1.19×10^{-1}	4.11×10^{-2}
1200	1.23×10^{-1}	6.07×10^{-2}	8.47×10^{-3}
4800	5.26×10^{-2}	2.81×10^{-2}	1.65×10^{-3}
10,800	2.92×10^{-2}	1.54×10^{-2}	7.43×10^{-4}
# elements	**Q_3**	**$Q-12-3$**	**$Q-13-3^+$**
300	1.49×10^{-1}	1.11×10^{-1}	2.80×10^{-2}
1200	6.57×10^{-2}	5.00×10^{-2}	4.71×10^{-3}
4800	2.36×10^{-2}	1.02×10^{-2}	8.24×10^{-4}
10,800	1.08×10^{-2}	4.54×10^{-3}	9.75×10^{-5}
# elements	**Q_4**	**$Q-16-4$**	**$Q-17-4^+$**
300	9.58×10^{-2}	8.32×10^{-2}	2.16×10^{-2}
1200	3.78×10^{-2}	1.33×10^{-2}	2.94×10^{-3}
4800	1.03×10^{-2}	9.17×10^{-3}	1.26×10^{-4}
10,800	3.70×10^{-3}	4.92×10^{-4}	2.12×10^{-5}

* Since an analytical solution to this problem is not available, in computing the relative error, we use in place of the exact solution a reference solution, computed using a Galerkin Q_4 polynomial element on a $43,200 = 3\cdot(120\times120)$ element mesh.

Conclusions & Ongoing Work

- DGM/DEM Elements outperform their Galerkin and stabilized Galerkin counterparts of comparable complexity by at least one (and sometimes many) orders of magnitude difference.
- For $Pe = 10^3$, to achieve a 0.1% level of relative error:
  - $Q-8-2$ and $Q-9-2^+$ elements: reduce the dof requirement of the Q_2 element by a factor of ≈ 5.
  - $Q-12-3$ and $Q-13-3^+$ elements: reduce the dof requirement of the Q_3 element by a factor of ≈ 15.
  - $Q-16-4$ and $Q-17-4^+$ elements: reduce the dof requirement of the Q_4 element by a factor of ≈ 15.
- In a high Péclet regime, DGM and DEM solutions are almost completely oscillation-free, in contrast with the Galerkin solutions.
- Results presented herein demonstrate the potential of DEM for realistic advection-dominated transport problems in fluid mechanics.
- Ongoing/future work:
  - DEM for variable-coefficient problems.
  - Projection-method based DEM for incompressible, time dependent Navier-Stokes.

References

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