

## Approaches for building stable projection-based reduced order models for compressible flow

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Projection-based reduced order modeling is a promising tool for bridging the gap between high-fidelity and real-time/multi-query applications such as uncertainty quantification (UQ), optimization and control design. A popular approach to building projection-based reduced order models (ROMs) for fluid problems is the proper orthogonal decomposition (POD)/Galerkin projection method. This method consists of two steps: (1) the computation of the POD basis from a set of snapshots of the solution field, followed by (2) the Galerkin projection of the governing equations onto this reduced basis in some inner product. POD is a mathematical procedure that constructs a reduced basis for an ensemble of snapshots collected from a high-fidelity simulation. This basis is optimal in the sense that it describes more energy on average of the ensemble than any other linear basis of the same reduced dimension.

Unfortunately, ROMs constructed via the POD/Galerkin method using the  $L^2$  inner product lack, in general, an *a priori* stability guarantee when applied to compressible flow problems. This leads to practical limitations of ROMs obtained using the POD/Galerkin method: a ROM aimed to capture a flow in a physically stable (i.e., bounded as  $t \rightarrow \infty$ ) regime might be stable for a given number of modes, but unstable (i.e., unbounded as  $t \rightarrow \infty$ ), and therefore inaccurate, for other choices of basis size [1, 2]. The situation can be exacerbated by basis truncation: the removal of POD modes having low energy. Although necessary for model reduction, truncation can destroy the balance between energy production and dissipation in a fluid ROM, thereby leading to ROMs that are inaccurate and/or unstable.

This talk will describe several approaches for building stable projection-based reduced order models (ROMs) for compressible flows developed by the speaker during the past 10 years.

The first part of the talk will focus on approaches to building projection-based ROMs using the POD method and continuous Galerkin projection for compressible flows that have an *a priori* stability guarantee. The inner product used to define the Galerkin projection step of the model reduction procedure is closely tied to the stability of the resulting ROM. It will be demonstrated that if the Galerkin projection step of the model reduction is performed in this inner product, whose associated norm represents the total energy of the fluid system, the ROM numerical solution will be “energy-stable”, that is, bounded in a way that is consistent with the behavior of the energy of the exact solution to the governing fluid equations. Energy inner products for the linearized as well as nonlinear compressible flow equations will be derived [1, 2]. POD/Galerkin ROMs constructed using these inner products will be evaluated on test cases involving an inviscid pressure pulse in a box, and a viscous laminar cavity problem.

The second part of the talk will focus on approaches to stabilizing *a posteriori* unstable ROMs. These approaches are less intrusive than those presented in the first part of the talk, and can be applied in a black-box fashion to a variety of ROMs constructed using continuous or discrete projection. First, attention will be focused on linear

time-invariant (LTI) systems. A method for stabilizing unstable ROMs for LTI systems termed optimization-based eigenvalue reassignment will be described and evaluated [3]. In this approach, the unstable eigenvalues of the ROM system these eigenvalues are moved into the stable half of the complex plane. To ensure accuracy of the resulting ROM, a constrained nonlinear least-squares optimization problem for the stabilized ROM eigenvalues in which the error in the ROM output is minimized is formulated. Next, attention is turned to the nonlinear compressible flow case. Here, we develop an approach, termed minimal subspace rotation, that is similar in flavor to the eigenvalue reassignment approach described above, but has a number of advantages, namely it is consistent, applicable to nonlinear problems, and enables extreme model reduction [4]. Rather than modifying the ROM dynamical system, we modify the reduced basis. Specifically, we derive a transformation of the projection subspace that accounts for modes truncated by a typical reduced basis approach, e.g., POD. The proposed approach can be formulated mathematically as a trace minimization problem on the Stiefel manifold. Because only the projection subspace is modified, consistency between the ROM and the Navier-Stokes equations is retained. The reproductive as well as predictive capabilities of the method are evaluated on several nonlinear compressible flow problems.

To conclude the talk, perspectives on the various approaches presented, including relative pros and cons, will be discussed, in addition to ongoing/future research directions.

#### References:

- [1] M.F. Barone, I. Kalashnikova, D.J. Segalman, H. Thornquist. Stable Galerkin Reduced Order Models for Linearized Compressible Flow. *J. Comput. Phys.* 288 1932-1946, 2009.
- [2] I. Kalashnikova, S. Arunajatesan. A Stable Galerkin Reduced Order Model (ROM) for Compressible Flow, WCCM-2012-18407, *10th World Congress on Computational Mechanics (WCCM)*, Sao Paulo, Brazil, 2012.
- [3] I. Kalashnikova, B.G. van Bloemen Waanders, S. Arunajatesan, M.F. Barone. "Stabilization of Projection-Based Reduced Order Models for Linear Time-Invariant Systems via Optimization-Based Eigenvalue Reassignment". *Comput. Meth. Appl. Mech. Engng.* 272 (2014) 251-270.
- [4] M. Balajewicz, I. Tezaur, E. Dowell. "Minimal subspace rotation on the Stiefel manifold for stabilization and enhancement of projection-based reduced order models for the compressible Navier-Stokes equations", *J. Comput. Phys.* 321 (2016) 224-241.