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Finite range jump processes and volume–constrained diffusion problems.

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Abstract

A nonlocal convection–diffusion model is introduced for the master equation of Markov jump processes in bounded domains. With minimal assumptions on the model parameters, the nonlocal steady and unsteady state master equations are shown to be well–posed in a weak sense. Then, the nonlocal operator is shown to be the generator of finite range nonsymmetric jump processes and, when certain conditions on the model parameters hold, the generators of finite and infinite activity Lévy and Lévy–type jump processes are shown to be special instances of the nonlocal operator.

Keywords: Nonlocal diffusion, nonlocal operators, nonlocal vector calculus, variational forms, master equation, Markov processes, Lévy processes.

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1 Introduction

This work is motivated by nonsymmetric jump processes of finite range; a general model for their description is a nonlocal convection–diffusion equation introduced in [18] and further analyzed in [17]. This paper extends the results of [17] to infinite activity processes and to a broader class of jump processes; moreover, it studies both the steady and unsteady state equations in bounded domains.

The main contribution is the analysis of a general class of nonlocal diffusion problems; with minimal assumptions on the parameters, we prove that the nonlocal equations are well–posed, generalizing the results in [17] to singular and not necessarily positive kernels[1]. We also provide the basis for the analysis of a large class of stochastic processes confined to bounded domains; in fact, we show that for non–negative kernel functions the convection–diffusion operator is the generator of a Markov jump process and that the corresponding master equation has a unique solution.

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[1] The case of negative kernel functions has no link with stochastic processes. However, it is of interest in many other applications; see [30] where a class of sign changing kernels is analyzed.

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Standard probabilistic methods analyze the strong form of the equations governing the process; instead, we treat an associated variational problem. To the best of our knowledge, the use of variational methods is a non–conventional probabilistic approach and we are not aware of other works that prove that the master equation of a general Markov jump process is well–posed in a weak sense. Our approach allows us to prove the well–posedness of the problem using classical arguments of the variational theory and makes Galerkin–type numerical methods (e.g., the finite element methods) naturally suitable for numerical approximation. On the other hand, the weak formulation does not allow for point–wise estimates of the solution; however, it provides (optimal) energy estimates. Improving such estimates requires a regularity result for the nonlocal convection-diffusion equations, a current topic of research.

As opposed to local classical models, nonlocal models allow for discontinuities in the solution. In a nonlocal model, the interactions between points can occur at a finite distance, whereas in the local case they occur only due to contact. The need for nonlocality in modeling stochastic processes comes from the possibility of having a jump in the sample path; this happens, e.g., in Lévy jump processes whereas it does not happen in a Brownian process that features a continuous sample path.

Nonlocal symmetric diffusion models have been widely used and studied not only in the field of stochastic processes but, more generally, in image analyses [8, 24, 25, 27], machine learning [32], kinetic equations [5, 26], phase transitions [6, 23], nonlocal heat conduction [7], and a linearized peridynamic model for mechanics [35]. Their analysis has been improved by a recently developed nonlocal vector calculus that provides tools that allow one to study nonlocal equations in a similar manner as one studies the associated local partial differential equations. The nonlocal vector calculus, which is a nonlocal counterpart of the classical vector calculus, was introduced in [16] and applied to volume–constrained nonlocal diffusion problems in [15]. Moreover, several numerical methods for nonlocal diffusion equations with volume constraints have been introduced; see, e.g., [1, 9, 10, 12, 14, 13, 17, 19, 31, 34].

Nonlocal convection–diffusion can be interpreted as nonlocal nonsymmetric diffusion in the sense that nonlocal convection does not have a drift effect; instead, it is a non–uniform diffusion, i.e., it occurs in some random direction. Nonsymmetric diffusion is used to describe nonsymmetric jump processes; we mention the works by Meerschaert and collaborators [4, 28, 29] where the equations are set either in free space or in bounded domains with boundary conditions. Ervin and Roop [20] consider the variational form of the equations introduced by Meerschaert in a bounded domain; there, they prescribe boundary conditions. In their work they do not provide a stochastic interpretation of the process underlying the equation and limit their analysis to operators associated to infinite activity processes with infinite variation (see [11] for the classification of jump processes). Felsinger et al. [22] also analyze the variational formulation of the diffusion equations; they consider integrable and non–integrable, symmetric and nonsymmetric kernels. Their work is similar to the one presented in this paper; however, there are differences that make their models less suitable for stochastic applications. In fact, they do not provide any stochastic interpretation and, in general, the operators treated in [22] are not generators of stochastic processes unless further conditions (that guarantee the conservation of probability) on the kernels are prescribed. Different assumptions on the model parameters still yield the well–posedness of the steady state equation for a large class of kernels (several examples are provided in [22, §6]); in particular, they do not allow the kernels to take on negative values and allow the nonlocal interactions to be infinite. Andreu and collaborators [2] consider the strong form of nonsymmetric diffusion equations to which they prescribe volume constraints; their analysis is limited to integrable, positive, and translation invariant kernels. In this work we consider a more general class of operators and we augment the nonlocal equations with volume constraints; this choice is motivated by the fact that the sample path is not continuous, instead, it might jump outside of the domain without passing through the boundary. Furthermore, prescribing volume constraints is a key assumption to prove that the problem is well–posed.

In the remainder of this section we introduce the notation used throughout the paper. In Section 2, we introduce the nonlocal operator and the steady state convection–diffusion equation. Using standard variational arguments and the nonlocal vector calculus, we prove that the weak solution of the problem exists, is unique and depends continuously on the data. Then, using a Fredholm alternative argument, we show that the equation is well–posed, relaxing the assumptions on the parameters. In Section 3, we treat the time–dependent

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problem and show that it is well–posed in a weak sense. In Section 4 we provide an interpretation of the problem in terms of stochastic processes and show that the nonlocal equation evolves the probability density of a general Markov jump process.

1.1 Notation

We introduce the notation of the nonlocal vector calculus used throughout the paper. For an extensive introduction to that calculus, see [16].

We define the action of the nonlocal divergence operator $\mathcal{D}(\boldsymbol{\nu})\colon \mathbb{R}^n \to \mathbb{R}$ as

$$\mathcal{D}(\boldsymbol{\nu})(\mathbf{x}) := \int_{\mathbb{R}^n} \big(\boldsymbol{\nu}(\mathbf{x},\mathbf{y}) + \boldsymbol{\nu}(\mathbf{y},\mathbf{x})\big) \cdot \boldsymbol{\alpha}(\mathbf{x},\mathbf{y})\, d\mathbf{y} \qquad \text{for } \mathbf{x} \in \mathbb{R}^n, \tag{1a}$$

where $\boldsymbol{\nu}(\mathbf{x},\mathbf{y}), \boldsymbol{\alpha}(\mathbf{x},\mathbf{y})\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ with $\boldsymbol{\alpha}$ anti–symmetric. The action of the operator $\mathcal{D}^*(u)\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, that is the formal adjoint to $\mathcal{D}$, is given by

$$\mathcal{D}^*\big(u\big)(\mathbf{x},\mathbf{y}) := -\big(u(\mathbf{y}) - u(\mathbf{x})\big)\boldsymbol{\alpha}(\mathbf{x},\mathbf{y}) \qquad \text{for } \mathbf{x},\mathbf{y} \in \mathbb{R}^n, \tag{1b}$$

where $u\colon \mathbb{R}^n \to \mathbb{R}$ is a given mapping. In [16], this operator is shown to be the nonlocal analog of the negative of the classical gradient operator. Let $\mathcal{L}u : \mathbb{R}^n \to \mathbb{R}$ be a nonlocal operator defined as

$$\mathcal{L}u(\mathbf{x}) = -\mathcal{D}(\boldsymbol{\Theta}\mathcal{D}^* u)(\mathbf{x}) + \mathcal{D}(\boldsymbol{\mu} u)(\mathbf{x}), \tag{2a}$$

where, without loss of generality, $\boldsymbol{\Theta}(\mathbf{x},\mathbf{y}) = \boldsymbol{\Theta}(\mathbf{y},\mathbf{x}) = \boldsymbol{\Theta}^T(\mathbf{x},\mathbf{y})$ and $\boldsymbol{\mu}(\mathbf{x},\mathbf{y}) = \boldsymbol{\mu}(\mathbf{y},\mathbf{x})$. We refer to the second order tensor $\boldsymbol{\Theta}$ as the nonlocal diffusion parameter and to the vector $\boldsymbol{\mu}$ as the nonlocal convection parameter. For

$$\gamma(\mathbf{x},\mathbf{y}) = \boldsymbol{\alpha}(\mathbf{x},\mathbf{y}) \cdot \big(\boldsymbol{\Theta}(\mathbf{x},\mathbf{y})\boldsymbol{\alpha}(\mathbf{x},\mathbf{y})\big) - \boldsymbol{\mu}(\mathbf{x},\mathbf{y}) \cdot \boldsymbol{\alpha}(\mathbf{x},\mathbf{y}), \tag{2b}$$

the operator $\mathcal{L}$ has the explicit form

$$\mathcal{L}u(\mathbf{x}) = \int_{\mathbb{R}^n} \big(u(\mathbf{y})\gamma(\mathbf{y},\mathbf{x}) - u(\mathbf{x})\gamma(\mathbf{x},\mathbf{y})\big)\, d\mathbf{y} \qquad \text{for } \mathbf{x} \in \mathbb{R}^n. \tag{2c}$$

The same nonlocal operator (and the associated time–dependent problem, treated in Section 3) has been analyzed in [17] for integrable kernel functions.

The reason why we can assume $\boldsymbol{\Theta} = \boldsymbol{\Theta}^T$ without loss of generality is because in γ the anti–symmetric part of $\boldsymbol{\Theta}$ (as a tensor) has no contribution. In fact, if we decompose $\boldsymbol{\Theta}$ in its symmetric and anti–symmetric parts (in a tensor sense) we have

$$\boldsymbol{\alpha} \cdot (\boldsymbol{\Theta}\boldsymbol{\alpha}) = \boldsymbol{\alpha} \cdot (\boldsymbol{\Theta}_S\boldsymbol{\alpha}) + \boldsymbol{\alpha} \cdot (\boldsymbol{\Theta}_A\boldsymbol{\alpha}) = \boldsymbol{\alpha} \cdot (\boldsymbol{\Theta}_S\boldsymbol{\alpha}).$$

On the other hand, we assume that $\boldsymbol{\Theta}(\mathbf{x},\mathbf{y}) = \boldsymbol{\Theta}(\mathbf{y},\mathbf{x})$ because the anti–symmetric part of $\boldsymbol{\Theta}$ (as a function of $\mathbf{x}$ and $\mathbf{y}$) can be included in the convection term. In fact, consider the decomposition of $\boldsymbol{\Theta}$ in its symmetric and anti–symmetric parts (in a function sense), we have

$$\gamma = \boldsymbol{\alpha} \cdot (\boldsymbol{\Theta}_s\boldsymbol{\alpha}) + \boldsymbol{\alpha} \cdot (\boldsymbol{\Theta}_a\boldsymbol{\alpha}) + \boldsymbol{\mu} \cdot \boldsymbol{\alpha}.$$

Let $\boldsymbol{\sigma}(\mathbf{x},\mathbf{y}) = \boldsymbol{\sigma}(\mathbf{y},\mathbf{x})$ be defined as $\boldsymbol{\sigma} = \boldsymbol{\Theta}_a\boldsymbol{\alpha}$, we can rewrite the kernel as

$$\gamma = \boldsymbol{\alpha} \cdot (\boldsymbol{\Theta}_s\boldsymbol{\alpha}) + (\boldsymbol{\mu} + \boldsymbol{\sigma}) \cdot \boldsymbol{\alpha}.$$

With a similar argument we show that the assumption of a symmetric nonlocal convection parameter is not restrictive. In fact, assume that $\boldsymbol{\mu}$ is nonsymmetric and consider the decomposition of $\boldsymbol{\mu}$ into its symmetric and anti–symmetric parts (in a function sense), we have

$$\gamma = \boldsymbol{\alpha} \cdot (\boldsymbol{\Theta}\boldsymbol{\alpha}) - \boldsymbol{\mu}_a \cdot \boldsymbol{\alpha} - \boldsymbol{\mu}_s \cdot \boldsymbol{\alpha};$$

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where, the first and the second term are symmetric. We write the second term as

$$\boldsymbol{\mu}_a \cdot \boldsymbol{\alpha} = \boldsymbol{\alpha} \cdot (\boldsymbol{\Psi}\boldsymbol{\alpha}),$$

with $\boldsymbol{\Psi}(\mathbf{x},\mathbf{y}) = \boldsymbol{\Psi}(\mathbf{y},\mathbf{x})$ and, e.g., $\boldsymbol{\Psi} = (\boldsymbol{\mu}_a \otimes \boldsymbol{\alpha})/(\boldsymbol{\alpha}^T\boldsymbol{\alpha})$ so that when $\boldsymbol{\alpha} = 0$ on a measure–zero set, $\boldsymbol{\Psi}$ is well–defined almost everywhere. Then, we can write $\gamma(\mathbf{x},\mathbf{y})$ as

$$\gamma = \boldsymbol{\alpha} \cdot \big((\boldsymbol{\Theta} - \boldsymbol{\Psi})\boldsymbol{\alpha}\big) - \boldsymbol{\mu}_s \cdot \boldsymbol{\alpha}.$$

Thus, the nonsymmetric part of the nonlocal diffusion operator can be interpreted as nonlocal convection and $\mathcal{L}$ becomes the sum of a diffusion and a convection term with symmetric diffusion and convection parameters.

Letting $\Omega \subset \mathbb{R}^n$ denote a bounded open domain, we define the interaction domain $\Omega_{\mathcal{I}} \subset \mathbb{R}^n$ as

$$\Omega_{\mathcal{I}} := \{\mathbf{y} \in \mathbb{R}^n \backslash \Omega \quad \text{such that} \quad \boldsymbol{\alpha}(\mathbf{x},\mathbf{y}) \neq \mathbf{0} \quad \text{for } \mathbf{x} \in \Omega\},$$

i.e., $\Omega_{\mathcal{I}}$ consists of those points outside of Ω that interact with points inside of Ω. We suppose that γ is a localized kernel, i.e., for all $\mathbf{x} \in \Omega \cup \Omega_{\mathcal{I}}$,

$$\gamma(\mathbf{x},\mathbf{y}) = 0 \quad \forall\, \mathbf{y} \in (\Omega \cup \Omega_{\mathcal{I}}) \backslash B_\lambda(\mathbf{x}), \tag{3}$$

where λ is a positive constant and $B_\lambda(\mathbf{x}) := \{\mathbf{y} \in \Omega \cup \Omega_{\mathcal{I}} \colon |\mathbf{y} - \mathbf{x}| \leqslant \lambda\}$. Note that, in general, γ is not radial nor translation invariant.

We define the nonlocal energy semi–norm, nonlocal energy space, and nonlocal volume–constrained energy space by

$$|||v|||^2 := \frac{1}{2} \int_{\Omega \cup \Omega_{\mathcal{I}}} \int_{\Omega \cup \Omega_{\mathcal{I}}} \left|\mathcal{D}^*(v)(\mathbf{x},\mathbf{y})\right|^2 d\mathbf{y}\, d\mathbf{x} \tag{4a}$$

$$V(\Omega \cup \Omega_{\mathcal{I}}) := \left\{v \in L^2(\Omega \cup \Omega_{\mathcal{I}}) \ : \ |||v||| < \infty\right\} \tag{4b}$$

$$V_c(\Omega \cup \Omega_{\mathcal{I}}) := \left\{v \in V(\Omega \cup \Omega_{\mathcal{I}}) \ : \ v = 0 \text{ on } \Omega_{\mathcal{I}}\right\}, \tag{4c}$$

respectively, and we assume that the energy norm satisfies a Poincaré–like inequality: $\|v\|_{L^2(\Omega\cup\Omega_{\mathcal{I}})} \leqslant C_p |||v|||$ for all $v \in V_c$. We refer to C_p as the Poincaré constant. This property holds for a large class of kernel functions, see, e.g., cases 1 and 2 in [15, §4.2]. We denote by $V_c'(\Omega)$ the dual space of $V_c(\Omega \cup \Omega_{\mathcal{I}})$ with respect to the standard duality pairing; a norm on $V_c'(\Omega)$ can be defined as

$$\|g\|_{V_c'} := \sup_{v \in V_c(\Omega\cup\Omega_{\mathcal{I}}), v \neq 0} \frac{\int_\Omega g\, v\, d\mathbf{x}}{|||v|||} \quad \forall\ g \in V_c'(\Omega). \tag{5}$$

In [16] one can find results such as the nonlocal integration by parts and the nonlocal first and second Green's identities.

2 The steady state problem

In this section we show that the weak form of the nonlocal convection–diffusion problem is well–posed. Using standard arguments of the classical variational theory, we prove two well–posedness results; their combination provides a weighted sufficient condition on the model parameters granting the existence and uniqueness of a solution. Then, we rely on a Fredholm alternative argument [3] to prove a more general result.

We formulate the steady state nonlocal convection–diffusion problem as

$$\begin{cases} -\mathcal{L}u = & g \quad \text{in } \Omega, \\ \quad\ \ u = & 0 \quad \text{in } \Omega_{\mathcal{I}}, \end{cases} \tag{6}$$

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where $g \in V_c'$. The inner product of (6) over Ω with a test function $v \in V_c$ gives

$$\int_\Omega \mathcal{D}(\boldsymbol{\Theta}\mathcal{D}^* u)(\mathbf{x}) v(\mathbf{x})\, d\mathbf{x} - \int_\Omega \mathcal{D}(\boldsymbol{\mu} u)(\mathbf{x})\, v(\mathbf{x})\, d\mathbf{x} = \int_\Omega g(\mathbf{x}) v(\mathbf{x})\, d\mathbf{x}.$$

The nonlocal Green's identity (see [16, §4.3]) grants

$$\int_{\Omega\cup\Omega_\mathcal{I}} \int_{\Omega\cup\Omega_\mathcal{I}} \mathcal{D}^*(u)(\mathbf{x},\mathbf{y}) \cdot (\boldsymbol{\Theta}\mathcal{D}^* v)(\mathbf{x},\mathbf{y})\, d\mathbf{y}\, d\mathbf{x} - \int_\Omega \mathcal{D}(\boldsymbol{\mu} u)(\mathbf{x})\, v(\mathbf{x})\, d\mathbf{x} = \int_\Omega g(\mathbf{x}) v(\mathbf{x})\, d\mathbf{x}. \tag{7}$$

The weak form of problem (6) can be formulated as: *given $g \in V_c'$, find $u \in V_c$ that satisfies* (7) *for all $v \in V_c$.*

Now define the bilinear form

$$a(u,v) = \int_{\Omega\cup\Omega_\mathcal{I}} \int_{\Omega\cup\Omega_\mathcal{I}} \mathcal{D}^*(u)(\mathbf{x},\mathbf{y}) \cdot (\boldsymbol{\Theta}\mathcal{D}^* v)(\mathbf{x},\mathbf{y})\, d\mathbf{y}\, d\mathbf{x} - \int_\Omega \mathcal{D}(\boldsymbol{\mu} u)(\mathbf{x})\, v(\mathbf{x})\, d\mathbf{x} \tag{8}$$

and the linear functional

$$G(v) = \int_\Omega g(\mathbf{x}) v(\mathbf{x})\, d\mathbf{x} \tag{9}$$

for all $v \in V_c$. Then, the weak solution solves $a(u,v) = G(v)$ for all $v \in V_c$.

Theorem 1. *For $g \in V_c'$, $a(\cdot,\cdot)$ and $G(\cdot)$ given by* (8) *and* (9)*, $\boldsymbol{\Theta}$ such that there exist $\vartheta_*, \vartheta^* > 0$ satisfying*

$$0 < \vartheta_* \leqslant \inf_{\mathbf{x}\in\mathbb{R}^n} (\min_i \theta_i), \qquad \sup_{\mathbf{x}\in\mathbb{R}^n} (\max_i \theta_i) \leqslant \vartheta^* < \infty, \tag{10}$$

where θ_i are the singular values of $\boldsymbol{\Theta}$, and $\boldsymbol{\mu}$ such that $C_p^2 \|\mathcal{D}\boldsymbol{\mu}\|_\infty \leqslant 2\vartheta_*$ and $\| \, |\boldsymbol{\mu}| \, \|_\infty \leqslant \mu^*$, the problem

$$a(u,v) = G(v) \qquad \forall\, v \in V_c \tag{11}$$

has a unique solution $u^* \in V_c$. Furthermore, that solution satisfies the a priori estimate

$$|||u^*||| \leqslant C \|g\|_{V_c'}, \tag{12}$$

where $C = \frac{1}{C_{\text{coer}}}$ and $C_{\text{coer}} = \vartheta_* - \frac{1}{2} C_p^2 \|\mathcal{D}\boldsymbol{\mu}\|_\infty$.

Proof. By using the nonlocal integration by parts formula (see [16, §4.3]) we rewrite $a(\cdot,\cdot)$ as

$$\begin{aligned} a(u,v) &= \int_{\Omega\cup\Omega_\mathcal{I}} \int_{\Omega\cup\Omega_\mathcal{I}} \mathcal{D}^*(u)(\mathbf{x},\mathbf{y}) \cdot (\boldsymbol{\Theta}\mathcal{D}^* v)(\mathbf{x},\mathbf{y})\, d\mathbf{y}\, d\mathbf{x} \\ &\quad - \int_{\Omega\cup\Omega_\mathcal{I}} \int_{\Omega\cup\Omega_\mathcal{I}} \mathcal{D}^*(v)(\mathbf{x},\mathbf{y}) \cdot \boldsymbol{\mu}(\mathbf{x},\mathbf{y})\, u(\mathbf{x})\, d\mathbf{y} d\mathbf{x} - \int_{\Omega_\mathcal{I}} v(\mathbf{x}) \mathcal{N}(\boldsymbol{\mu} u)(\mathbf{x})\, d\mathbf{x} \\ &= \int_{\Omega\cup\Omega_\mathcal{I}} \int_{\Omega\cup\Omega_\mathcal{I}} \mathcal{D}^*(u)(\mathbf{x},\mathbf{y}) \cdot (\boldsymbol{\Theta}\mathcal{D}^* v)(\mathbf{x},\mathbf{y})\, d\mathbf{y}\, d\mathbf{x} - \int_{\Omega\cup\Omega_\mathcal{I}} \int_{\Omega\cup\Omega_\mathcal{I}} \mathcal{D}^*(v)(\mathbf{x},\mathbf{y}) \cdot \boldsymbol{\mu}(\mathbf{x},\mathbf{y})\, u(\mathbf{x})\, d\mathbf{y} d\mathbf{x}, \end{aligned}$$

where the last equality follows from the homogeneous volume constraint. By the Lax–Milgram theorem, sufficient conditions so that the problem (11) is well–posed are the coercivity and the continuity of $a(\cdot,\cdot)$ and the continuity of $G(\cdot)$. The assumptions on $\boldsymbol{\Theta}$ imply that

$$\begin{aligned} a(u,u) &\geqslant \vartheta_* \int_{\Omega\cup\Omega_\mathcal{I}} \int_{\Omega\cup\Omega_\mathcal{I}} \mathcal{D}^*(u)(\mathbf{x},\mathbf{y}) \cdot \mathcal{D}^*(u)(\mathbf{x},\mathbf{y})\, d\mathbf{x} \\ &\quad - \int_{\Omega\cup\Omega_\mathcal{I}} \int_{\Omega\cup\Omega_\mathcal{I}} \mathcal{D}^*(u)(\mathbf{x},\mathbf{y}) \cdot \boldsymbol{\mu}(\mathbf{x},\mathbf{y})\, u(\mathbf{x})\, d\mathbf{y}\, d\mathbf{x} \\ &= \vartheta_* |||u|||^2 - \int_{\Omega\cup\Omega_\mathcal{I}} \int_{\Omega\cup\Omega_\mathcal{I}} \mathcal{D}^*(u)(\mathbf{x},\mathbf{y}) \cdot \boldsymbol{\mu}(\mathbf{x},\mathbf{y})\, u(\mathbf{x})\, d\mathbf{y}\, d\mathbf{x}. \end{aligned} \tag{13}$$

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We analyze the second term in (13):

$$
\begin{aligned}
&\int_{\Omega\cup\Omega_{\mathcal{I}}}\int_{\Omega\cup\Omega_{\mathcal{I}}} \mathcal{D}^*(u)(\mathbf{x},\mathbf{y})\cdot\boldsymbol{\mu}(\mathbf{x},\mathbf{y})\,u(\mathbf{x})\,d\mathbf{y}\,d\mathbf{x}\\
&=\int_{\Omega\cup\Omega_{\mathcal{I}}}\int_{\Omega\cup\Omega_{\mathcal{I}}} u(\mathbf{x})\boldsymbol{\mu}(\mathbf{x},\mathbf{y})\cdot\boldsymbol{\alpha}(\mathbf{x},\mathbf{y})\big(u(\mathbf{x})-u(\mathbf{y})\big)\,d\mathbf{y}\,d\mathbf{x}\\
&=\int_{\Omega\cup\Omega_{\mathcal{I}}} u^2(\mathbf{x})\int_{\Omega\cup\Omega_{\mathcal{I}}}(\boldsymbol{\mu}\cdot\boldsymbol{\alpha})(\mathbf{x},\mathbf{y})\,d\mathbf{y}\,d\mathbf{x}\\
&-\int_{\Omega\cup\Omega_{\mathcal{I}}}\int_{\Omega\cup\Omega_{\mathcal{I}}} u(\mathbf{x})u(\mathbf{y})(\boldsymbol{\mu}\cdot\boldsymbol{\alpha})(\mathbf{x},\mathbf{y})\,d\mathbf{y}\,d\mathbf{x}\\
&=\frac{1}{2}\int_{\Omega\cup\Omega_{\mathcal{I}}} u^2(\mathbf{x})\mathcal{D}\boldsymbol{\mu}\,d\mathbf{x},
\end{aligned}
\tag{14}
$$

where we exploited the fact that the second integrand is anti–symmetric and that

$$
\mathcal{D}\boldsymbol{\mu}(\mathbf{x})=\int_{\Omega\cup\Omega_{\mathcal{I}}}(\boldsymbol{\mu}(\mathbf{x},\mathbf{y})+\boldsymbol{\mu}(\mathbf{y},\mathbf{x}))\cdot\boldsymbol{\alpha}(\mathbf{x},\mathbf{y})\,d\mathbf{y}=2\int_{\Omega\cup\Omega_{\mathcal{I}}}(\boldsymbol{\mu}\cdot\boldsymbol{\alpha})(\mathbf{x},\mathbf{y})\,d\mathbf{y}. \tag{15}
$$

Note that when $(\boldsymbol{\mu}\cdot\boldsymbol{\alpha})(\mathbf{x},\mathbf{y})$ is singular, (15) should be intended in a principal value sense, i.e.

$$
\mathcal{D}\boldsymbol{\mu}(\mathbf{x})=2\lim_{\varepsilon\to 0}\int_{(\Omega\cup\Omega_{\mathcal{I}})\setminus B_\varepsilon(\overline{\mathbf{x}})}(\boldsymbol{\mu}\cdot\boldsymbol{\alpha})(\mathbf{x},\mathbf{y})\,d\mathbf{y},
$$

being $\overline{\mathbf{x}}$ the point where the singularity occurs.

Thus, we have

$$
\begin{aligned}
a(u,u)&\geqslant \vartheta_*|||u|||^2-\frac{1}{2}\int_{\Omega\cup\Omega_{\mathcal{I}}} u^2(\mathbf{x})\mathcal{D}(\boldsymbol{\mu})\,d\mathbf{x}\\
&\geqslant \vartheta_*|||u|||^2-\frac{1}{2}\|\mathcal{D}\boldsymbol{\mu}\|_\infty\|u\|^2_{L^2(\Omega)}\\
&\geqslant \vartheta_*|||u|||^2-\frac{C_p^2}{2}\|\mathcal{D}\boldsymbol{\mu}\|_\infty|||u|||^2\\
&=(\vartheta_*-\frac{C_p^2}{2}\|\mathcal{D}\boldsymbol{\mu}\|_\infty)|||u|||^2=C_{\mathrm{coer}}|||u|||^2.
\end{aligned}
\tag{16}
$$

Then, the coercivity follows from the assumptions on $\boldsymbol{\mu}$. Next, we show the continuity of $a(\cdot,\cdot)$. We have

$$
\begin{aligned}
|a(u,v)|&=\left|\int_{\Omega\cup\Omega_{\mathcal{I}}}\int_{\Omega\cup\Omega_{\mathcal{I}}}\mathcal{D}^*(u)(\mathbf{x},\mathbf{y})\cdot(\boldsymbol{\Theta}\mathcal{D}^*v)(\mathbf{x},\mathbf{y})\,d\mathbf{y}d\mathbf{x}-\int_{\Omega\cup\Omega_{\mathcal{I}}}\int_{\Omega\cup\Omega_{\mathcal{I}}}\mathcal{D}^*(v)(\mathbf{x},\mathbf{y})\cdot\boldsymbol{\mu}(\mathbf{x},\mathbf{y})\,u(\mathbf{x})\,d\mathbf{y}\,d\mathbf{x}\right|\\
&\leqslant \vartheta^*\,|||u|||\,|||v|||+\mu^*\left|\int_{\Omega\cup\Omega_{\mathcal{I}}}u(\mathbf{x})\int_{\Omega\cup\Omega_{\mathcal{I}}}|\mathcal{D}^*(v)(\mathbf{x},\mathbf{y})|\,d\mathbf{y}\,d\mathbf{x}\right|\\
&\leqslant \vartheta^*|||u|||\,|||v|||+\mu^*\|u\|_{L^2(\Omega\cup\Omega_{\mathcal{I}})}\left\|\int_{\Omega\cup\Omega_{\mathcal{I}}}|\mathcal{D}^*(v)(\mathbf{x},\mathbf{y})|\,d\mathbf{y}\right\|_{L^2(\Omega\cup\Omega_{\mathcal{I}})}\\
&\leqslant \vartheta^*|||u|||\,|||v|||+\mu^*C_p|||u|||C_\lambda|||v|||\\
&=(\vartheta^*+\mu^*C_pC_\lambda)|||u|||\,|||v|||=C_{\mathrm{cont}}|||u|||\,|||v|||,
\end{aligned}
$$

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where the last inequality follows from the Poincaré inequality and from

$$\left\| \int_{\Omega\cup\Omega_{\mathcal{I}}} |\mathcal{D}^*(v)(\mathbf{x},\mathbf{y})|\, d\mathbf{y} \right\|_{L^2(\Omega\cup\Omega_{\mathcal{I}})} = \left(\int_{\Omega\cup\Omega_{\mathcal{I}}} \left(\int_{\Omega\cup\Omega_{\mathcal{I}}} |\mathcal{D}^*(v)(\mathbf{x},\mathbf{y})|\, d\mathbf{y} \right)^2 d\mathbf{x} \right)^{1/2}$$
$$\leqslant \left(\int_{\Omega\cup\Omega_{\mathcal{I}}} \int_{\Omega\cup\Omega_{\mathcal{I}}} |\mathcal{D}^*(v)(\mathbf{x},\mathbf{y})|^2\, d\mathbf{y}\, d\mathbf{x} \right)^{1/2} \left(\int_{\Omega\cup\Omega_{\mathcal{I}}} 1\, d\mathbf{y} \right)^{1/2} = C_\lambda |||v|||.$$

The last condition for the well–posedness is the continuity of G; we have

$$|G(v)| = \left| \int_\Omega g\, v\, d\mathbf{x} \right| \leqslant \|g\|_{V_c'} |||v|||.$$

Finally, because

$$C_{\text{coer}} |||u^*|||^2 \leqslant a(u^*, u^*) = G(u^*) \leqslant \|g\|_{V_c'} |||u^*|||,$$

we have the following a priori estimate

$$|||u^*||| \leqslant \frac{1}{C_{\text{coer}}} \|g\|_{V_c'},$$

i.e., u^* depends continuously upon the data. □

This theorem covers the case $\mathcal{D}\boldsymbol{\mu} = 0$ that is the nonlocal counterpart of $\nabla \cdot \boldsymbol{\beta} = 0$, where $\boldsymbol{\beta}$ is the convection field of the drift term $\boldsymbol{\beta} \cdot \nabla u$ in a partial differential equation. This is a very common assumption in local convection–diffusion problems. In Theorem 1 we use the assumption that $\|\mathcal{D}\boldsymbol{\mu}\|_\infty$ is bounded by a constant depending on C_p and $\boldsymbol{\Theta}$; this is a condition on a weighted average of $\boldsymbol{\mu}$. A different approach in showing the coercivity of $a(\cdot,\cdot)$ leads to different assumptions that involve the spectral properties of the model parameters.

Theorem 2. *Let $m^* = \sup_{\mathbf{x}\in\mathbb{R}^n} (\max_i m_i)$, where m_i are the eigenvalues of $\boldsymbol{\mu}\boldsymbol{\mu}^T$; for $g \in V_c'$, $a(\cdot,\cdot)$ and $G(\cdot)$ given by* (8) *and* (9)*, $\boldsymbol{\Theta}$ such that* (10) *holds, $\| \, |\boldsymbol{\mu}| \, \|_\infty \leqslant \mu^*$, and $\vartheta_*/m^* \leqslant C_p\, C_\lambda$, the problem* (11) *has a unique solution $u^* \in V_c$. Furthermore, that solution satisfies the a priori estimate*

$$|||u^*||| \leqslant \widetilde{C} \|g\|_{V_c'}, \tag{17}$$

where, $\widetilde{C} = \frac{1}{\widetilde{C}_{\text{coer}}}$, $\widetilde{C}_{\text{coer}} = \vartheta_* - C_p\, C_\lambda m^*$, and $C_\lambda = \|1\|_{L^2(\Omega\cup\Omega_{\mathcal{I}})}$.

Proof. The only thing that we need to show is the coercivity of $a(\cdot,\cdot)$ using the assumptions on $\boldsymbol{\Theta}$ and $\boldsymbol{\mu}$. The continuity of $a(\cdot,\cdot)$ and $G(\cdot)$ follow from the same arguments as in Theorem 1. From (13) and (14), we have that

$$a(u,u) \geqslant \vartheta_* |||u|||^2 - \int_{\Omega\cup\Omega_{\mathcal{I}}} \int_{\Omega\cup\Omega_{\mathcal{I}}} u(\mathbf{x}) \boldsymbol{\mu}(\mathbf{x},\mathbf{y}) \cdot \boldsymbol{\alpha}(\mathbf{x},\mathbf{y}) \big(u(\mathbf{x}) - u(\mathbf{y})\big)\, d\mathbf{y}\, d\mathbf{x} = \vartheta_* |||u|||^2 - I_1.$$

We find a bound for $|I_1|$:

$$|I_1| = \left| \int_{\Omega\cup\Omega_{\mathcal{I}}} u(\mathbf{x}) \int_{\Omega\cup\Omega_{\mathcal{I}}} \boldsymbol{\mu}(\mathbf{x},\mathbf{y}) \cdot \boldsymbol{\alpha}(\mathbf{x},\mathbf{y}) \big(u(\mathbf{x}) - u(\mathbf{y})\big)\, d\mathbf{y}\, d\mathbf{x} \right|$$
$$\leqslant \|u\|_{L^2(\Omega\cup\Omega_{\mathcal{I}})} \left\| \int_{\Omega\cup\Omega_{\mathcal{I}}} \boldsymbol{\mu}(\mathbf{x},\mathbf{y}) \cdot \boldsymbol{\alpha}(\mathbf{x},\mathbf{y}) \big(u(\mathbf{x}) - u(\mathbf{y})\big)\, d\mathbf{y} \right\|_{L^2(\Omega\cup\Omega_{\mathcal{I}})} = \|u\|_{L^2(\Omega\cup\Omega_{\mathcal{I}})}\, I_2.$$

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Next, we find a bound for I_2:

$$
\begin{aligned}
I_2 &= \left(\int_{\Omega\cup\Omega_{\mathcal{I}}}\left(\int_{\Omega\cup\Omega_{\mathcal{I}}}\boldsymbol{\mu}(\mathbf{x},\mathbf{y})\cdot\boldsymbol{\alpha}(\mathbf{x},\mathbf{y})\big(u(\mathbf{x})-u(\mathbf{y})\big)\,d\mathbf{y}\right)^2 d\mathbf{x}\right)^{1/2}\\
&\leqslant \left(\int_{\Omega\cup\Omega_{\mathcal{I}}}\|\boldsymbol{\mu}(\mathbf{x},\mathbf{y})\cdot\boldsymbol{\alpha}(\mathbf{x},\mathbf{y})\big(u(\mathbf{x})-u(\mathbf{y})\big)\|^2_{L^2(\Omega\cup\Omega_{\mathcal{I}})}\,\|1\|^2_{L^2(\Omega\cup\Omega_{\mathcal{I}})}\,d\mathbf{x}\right)^{1/2}\\
&\leqslant C_\lambda\left(\int_{\Omega\cup\Omega_{\mathcal{I}}}\int_{\Omega\cup\Omega_{\mathcal{I}}}(\boldsymbol{\mu}(\mathbf{x},\mathbf{y})\cdot\boldsymbol{\alpha}(\mathbf{x},\mathbf{y}))^2\big(u(\mathbf{x})-u(\mathbf{y})\big)^2\,d\mathbf{y}\,d\mathbf{x}\right)^{1/2}\\
&\leqslant C_\lambda\left(\int_{\Omega\cup\Omega_{\mathcal{I}}}\int_{\Omega\cup\Omega_{\mathcal{I}}}\boldsymbol{\alpha}(\mathbf{x},\mathbf{y})\cdot((\boldsymbol{\mu}(\mathbf{x},\mathbf{y})\boldsymbol{\mu}(\mathbf{x},\mathbf{y})^T)\boldsymbol{\alpha}(\mathbf{x},\mathbf{y}))\big(u(\mathbf{x})-u(\mathbf{y})\big)^2\,d\mathbf{y}\,d\mathbf{x}\right)^{1/2}\\
&\leqslant C_\lambda\left((m^*)^2\int_{\Omega\cup\Omega_{\mathcal{I}}}\int_{\Omega\cup\Omega_{\mathcal{I}}}(\mathcal{D}^*u)^2\,d\mathbf{y}\,d\mathbf{x}\right)^{1/2} = C_\lambda\,m^*\,|||u|||.
\end{aligned}
$$

Then, we have that

$$|I_1| \leqslant C_\lambda\,m^*\|u\|_{L^2(\Omega\cup\Omega_{\mathcal{I}})}|||u||| \leqslant C_p\,C_\lambda\,m^*\,|||u|||^2$$

and

$$a(u,u) \geqslant \vartheta_*|||u|||^2 - C_p\,C_\lambda\,m^*\,|||u|||^2 = \widetilde{C}_{\text{coer}}\,|||u|||^2,$$

where $\widetilde{C}_{\text{coer}}$ is positive by assumption. Then, (17) is obtained using the same argument as in Theorem 1. □

If we rewrite (14) as

$$\frac{\omega}{2}\int_{\Omega\cup\Omega_{\mathcal{I}}}u^2(\mathbf{x})\mathcal{D}\boldsymbol{\mu}(\mathbf{x})\,d\mathbf{x} + (1-\omega)\int_{\Omega\cup\Omega_{\mathcal{I}}}\int_{\Omega\cup\Omega_{\mathcal{I}}}u(\mathbf{x})\boldsymbol{\mu}(\mathbf{x},\mathbf{y})\cdot\boldsymbol{\alpha}(\mathbf{x},\mathbf{y})\big(u(\mathbf{x})-u(\mathbf{y})\big)\,d\mathbf{y}\,d\mathbf{x},$$

where the weight $\omega\in[0,1]$, we obtain a weighted condition on $\boldsymbol{\mu}$ that coincides with $C_p^2\|\mathcal{D}\boldsymbol{\mu}\|_\infty \leqslant 2\vartheta_*$ when $\omega=1$ and with $\vartheta_*/m^* < C_p\,C_\lambda$ when $\omega=0$.

2.1 Fredholm alternative

Using standard variational arguments, i.e., the Lax–Milgram theorem, Theorems 1 and 2 provide sufficient conditions on the parameters so that problem (11) is well–posed. However, using an argument based on the Fredholm alternative theorem [3], a more general result can be achieved; the steps in our proof are based on the approach utilized in [30] for symmetric, translation invariant, sign changing kernels. We assume that the energy space is a closed subspace of $L^2(\Omega\cup\Omega_{\mathcal{I}})$, compactly embedded in $L^2(\Omega\cup\Omega_{\mathcal{I}})$[2]. First, we rewrite the nonlocal convection parameter as $\boldsymbol{\mu}=\check{\boldsymbol{\mu}}+\widehat{C}\widehat{\boldsymbol{\mu}}$, for $\check{\boldsymbol{\mu}},\widehat{\boldsymbol{\mu}}:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ and $\widehat{C}\in\mathbb{R}$, so that the operator $\mathcal{L}:V_c\to V_c'(\Omega)$ can be written as

$$\mathcal{L}u = -\mathcal{D}(\boldsymbol{\Theta}\mathcal{D}^*u) + \mathcal{D}(\check{\boldsymbol{\mu}}\,u) + \widehat{C}\mathcal{D}(\widehat{\boldsymbol{\mu}}\,u) = \check{\mathcal{L}}\,u + \widehat{C}\widehat{\mathcal{L}}u.$$

The vector $\check{\boldsymbol{\mu}}$ is such that the bilinear form associated with $\check{\mathcal{L}}:V_c\to V_c'(\Omega)$ is coercive; thus, $\widehat{C}\widehat{\mathcal{L}}:V_c\to V_c'(\Omega)$ is a perturbation of $\check{\mathcal{L}}$ such that the bilinear form associated with $\mathcal{L}$ is not necessarily coercive.

If u is a solution of (11), we have that

$$(\check{\mathcal{L}}u,v)+\widehat{C}(\widehat{\mathcal{L}}u,v) = (g,v)\quad\forall\,v\in V_c,\qquad\text{or equivalently}\qquad(\check{\mathcal{L}}u,v) = (g-\widehat{C}\widehat{\mathcal{L}}u,v)\quad\forall\,v\in V_c.$$

[2] As an example we might consider the Sobolev space $H^s(\Omega\cup\Omega_{\mathcal{I}})$. In [15] it is shown that the energy space is equivalent to $H^s(\Omega\cup\Omega_{\mathcal{I}})$ for a class of kernel functions (see case 1 in §4.2); among these we mention the kernel associated with the fractional Laplacian operator.

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Thus, in operator form, we can write

$$u = \check{\mathcal{L}}^{-1}(g - \widehat{C}\widehat{\mathcal{L}}u) \qquad \text{that implies} \qquad (I + \widehat{C}\check{\mathcal{L}}^{-1}\widehat{\mathcal{L}})u = \check{\mathcal{L}}^{-1}g,$$

where $I : V_c \to V_c$ is the identity operator. Now define $K = \check{\mathcal{L}}^{-1}\widehat{\mathcal{L}} : V_c \to V_c$; in order to apply the Fredholm alternative theorem we have to show that K is a compact perturbation of the identity. First, we show that $\widehat{\mathcal{L}} : V_c \to V_c'(\Omega)$ is a compact operator, i.e., for any sequence $\{u_j\} \subset V_c$ such that $u_j \xrightarrow{w} 0$ in V_c, $\widehat{\mathcal{L}}u_j \to 0$ in V_c'. Here $\xrightarrow{w}$ stands for weak convergence.

Because of the compact embedding of V_c in L^2, $u_j \to 0$ in $L^2(\Omega \cup \Omega_{\mathcal{I}})$; also for all $v \in V_c$ there exists a positive constant $\overline{C}$ such that

$$\|\widehat{\mathcal{L}}v\|_{V_c'} \leqslant \overline{C}\|v\|_{L^2(\Omega\cup\Omega_{\mathcal{I}})}. \tag{18}$$

To see this, consider the following statements. According to definition (5), for all $v \in V_c$ we have

$$\|\widehat{\mathcal{L}}v\|_{V_c'} := \sup_{w\in V_c(\Omega\cup\Omega_{\mathcal{I}}),\, w\neq 0} \frac{\int_\Omega \widehat{\mathcal{L}}vw\, d\mathbf{x}}{|||w|||}.$$

We analyze the numerator,

$$\begin{aligned}
\left|\int_\Omega \widehat{\mathcal{L}}vw\, d\mathbf{x}\right| &= \left|\int_\Omega \int_{\Omega\cup\Omega_{\mathcal{I}}} \big(\widehat{\boldsymbol{\mu}}(\mathbf{x},\mathbf{y})v(\mathbf{x}) + \widehat{\boldsymbol{\mu}}(\mathbf{y},\mathbf{x})v(\mathbf{y})\big)\cdot\boldsymbol{\alpha}(\mathbf{x},\mathbf{y})w(\mathbf{x})\, d\mathbf{y}\, d\mathbf{x}\right| \\
&= 2\left|\int_\Omega \int_{\Omega\cup\Omega_{\mathcal{I}}} \big(v(\mathbf{x}) + v(\mathbf{y})\big)\widehat{\boldsymbol{\mu}}(\mathbf{y},\mathbf{x})\cdot\boldsymbol{\alpha}(\mathbf{x},\mathbf{y})w(\mathbf{x})\, d\mathbf{y}\, d\mathbf{x}\right| \\
&= 2\left|\int_\Omega v(\mathbf{x})w(\mathbf{x}) \int_{\Omega\cup\Omega_{\mathcal{I}}} \widehat{\boldsymbol{\mu}}(\mathbf{y},\mathbf{x})\cdot\boldsymbol{\alpha}(\mathbf{x},\mathbf{y})\, d\mathbf{y}\, d\mathbf{x}\right| \\
&+ 2\left|\int_\Omega w(\mathbf{x}) \int_{\Omega\cup\Omega_{\mathcal{I}}} v(\mathbf{y})\widehat{\boldsymbol{\mu}}(\mathbf{y},\mathbf{x})\cdot\boldsymbol{\alpha}(\mathbf{x},\mathbf{y})\, d\mathbf{y}\, d\mathbf{x}\right|.
\end{aligned}$$

Let

$$\alpha_1(\mathbf{x}) = \int_{\Omega\cup\Omega_{\mathcal{I}}} \widehat{\boldsymbol{\mu}}(\mathbf{x},\mathbf{y})\cdot\boldsymbol{\alpha}(\mathbf{x},\mathbf{y})\, d\mathbf{y} \quad \text{and} \quad \alpha_2(\mathbf{x}) = \int_{\Omega\cup\Omega_{\mathcal{I}}} \big(\widehat{\boldsymbol{\mu}}(\mathbf{x},\mathbf{y})\cdot\boldsymbol{\alpha}(\mathbf{x},\mathbf{y})\big)^2\, d\mathbf{y} \tag{19}$$

and assume that[3]

$$\overline{\alpha_1} = \|\alpha_1\|_\infty < \infty \quad \text{and} \quad \overline{\alpha_2}^2 = \|\alpha_2\|_\infty < \infty; \tag{20}$$

we have

$$\begin{aligned}
\left|\int_\Omega \widehat{\mathcal{L}}vw\, d\mathbf{x}\right| &\leqslant 2\overline{\alpha_1}\|v\|_{L^2(\Omega)}\|w\|_{L^2(\Omega)} + 2\|w\|_{L^2(\Omega)}\left\|\int_{\Omega\cup\Omega_{\mathcal{I}}} v(\mathbf{y})\widehat{\boldsymbol{\mu}}(\mathbf{x},\mathbf{y})\cdot\boldsymbol{\alpha}(\mathbf{x},\mathbf{y})\, d\mathbf{y}\right\|_{L^2(\Omega)} \\
&\leqslant 2\overline{\alpha_1}\|v\|_{L^2(\Omega)}\|w\|_{L^2(\Omega)} + 2\|w\|_{L^2(\Omega)}\left[\int_\Omega\left(\int_{\Omega\cup\Omega_{\mathcal{I}}} v(\mathbf{y})\widehat{\boldsymbol{\mu}}(\mathbf{x},\mathbf{y})\cdot\boldsymbol{\alpha}(\mathbf{x},\mathbf{y})\, d\mathbf{y}\right)^2 d\mathbf{x}\right]^{\frac{1}{2}} \\
&\leqslant 2\overline{\alpha_1}\|v\|_{L^2(\Omega)}\|w\|_{L^2(\Omega)} + 2\|w\|_{L^2(\Omega)}\left[\int_\Omega \|v(\mathbf{y})\widehat{\boldsymbol{\mu}}(\mathbf{x},\mathbf{y})\cdot\boldsymbol{\alpha}(\mathbf{x},\mathbf{y})\|^2_{L^2(\Omega\cup\Omega_{\mathcal{I}})}\|1\|^2_{L^2(\Omega\cup\Omega_{\mathcal{I}})}\, d\mathbf{x}\right]^{\frac{1}{2}} \\
&\leqslant 2\overline{\alpha_1}\|v\|_{L^2(\Omega)}\|w\|_{L^2(\Omega)} + 2C_\lambda\|w\|_{L^2(\Omega)}\left[\int_{\Omega\cup\Omega_{\mathcal{I}}}\int_{\Omega\cup\Omega_{\mathcal{I}}} v^2(\mathbf{y})\big(\widehat{\boldsymbol{\mu}}(\mathbf{x},\mathbf{y})\cdot\boldsymbol{\alpha}(\mathbf{x},\mathbf{y})\big)^2\, d\mathbf{y}\, d\mathbf{x}\right]^{\frac{1}{2}} \\
&\leqslant 2\overline{\alpha_1}\|v\|_{L^2(\Omega)}\|w\|_{L^2(\Omega)} + 2\overline{\alpha_2}C_\lambda\|v\|_{L^2(\Omega\cup\Omega_{\mathcal{I}})}\|w\|_{L^2(\Omega\cup\Omega_{\mathcal{I}})} \\
&\leqslant 2(\overline{\alpha_1} + C_\lambda\overline{\alpha_2})\|v\|_{L^2(\Omega\cup\Omega_{\mathcal{I}})}\|w\|_{L^2(\Omega\cup\Omega_{\mathcal{I}})} \leqslant 2C_p(\overline{\alpha_1} + C_\lambda\overline{\alpha_2})\|v\|_{L^2(\Omega\cup\Omega_{\mathcal{I}})}|||w|||.
\end{aligned}$$

[3] When the integrands are singular the same considerations as in Theorem 1 apply.

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Thus,

$$\sup_{w\in V_c(\Omega\cup\Omega_{\mathcal{I}}),w\neq 0}\frac{\int_\Omega \widehat{\mathcal{L}}vw\,d\mathbf{x}}{|||w|||}\leqslant \sup_{w\in V_c(\Omega\cup\Omega_{\mathcal{I}}),w\neq 0}\frac{2C_p(\overline{\alpha_1}+C_\lambda\overline{\alpha_2})\|v\|_{L^2(\Omega\cup\Omega_{\mathcal{I}})}|||w|||}{|||w|||}=2C_p(\overline{\alpha_1}+C_\lambda\overline{\alpha_2})\|v\|_{L^2(\Omega\cup\Omega_{\mathcal{I}})}.$$

From (18) it follows that for the weakly convergent sequence $\{u_j\}\subset V_c$, the sequence $\{\widehat{\mathcal{L}}u_j\}\subset V_c'$ converges strongly to 0 in V_c', thus, $\widehat{\mathcal{L}}:V_c\to V_c'$ is a compact operator. Because $\check{\mathcal{L}}^{-1}:V_c'\to V_c$ is continuous, then K is a compact operator whose eigenvalues, denoted by k_j, form a countable set. We can now apply the Fredholm alternative theorem and state the following result.

Theorem 3. *Assume that* (20) *holds. Then, there exists a countable set* $\mathcal{S}=\{1/k_j\}$*, with* $k_j\neq 0$ *such that* (11) *is well–posed for all* $g\in V_c'$ *if and only if* $\widehat{C}\notin\mathcal{S}$*.*

3 The unsteady state problem

We consider the following time–dependent functional spaces: $L^2(0,T;V_c)=\{v(\cdot,t)\in V_c:\ |||v(\cdot,t)|||\in L^2(0,T)\}$ and $L^2(0,T;V_c')=\{g(\cdot,t)\in V_c':\ \|g(\cdot,t)\|_{V_c'}\in L^2(0,T)\}$, for $T>0$. We formulate the time–dependent nonlocal convection–diffusion problem as follows

$$\begin{cases} u_t-\mathcal{L}u &= g & \mathbf{x}\in\Omega,\ t\in(0,T]\\ u(\mathbf{x},t) &= 0 & \mathbf{x}\in\Omega_{\mathcal{I}},\ t\in(0,T]\\ u(\mathbf{x},0) &= u_0(\mathbf{x}) & \mathbf{x}\in\Omega,\end{cases}\tag{21}$$

where $g\in L^2(0,T;V_c')$ and $u_0\in V_c$. A weak form of problem (21) is: *given* $g\in L^2(0,T;V_c')$ *and* $u_0\in V_c$*, find* $u\in L^2(0,T;V_c)$ *that satisfies, for all* $v\in V_c$*,*

$$\int_\Omega u_t\,v\,d\mathbf{x}+\int_{\Omega\cup\Omega_{\mathcal{I}}}\int_{\Omega\cup\Omega_{\mathcal{I}}}\mathcal{D}^*(u)\cdot(\boldsymbol{\Theta}\mathcal{D}^*v)\,d\mathbf{y}\,d\mathbf{x}-\int_\Omega\mathcal{D}(\boldsymbol{\mu}u)\,v\,d\mathbf{x}=\int_\Omega g\,v\,d\mathbf{x},\tag{22}$$

such that $u(\mathbf{x},0)=u_0(\mathbf{x})$*.*

According to the notation introduced in the previous section, (22) is equivalent to

$$(u_t,v)_\Omega+a(u,v)=G(v),\tag{23}$$

where $(\cdot,\cdot)_{\tilde{\Omega}}$ is the L^2 inner product over $\widetilde{\Omega}$. When $a(\cdot,\cdot)$ is coercive and continuous and $G(\cdot)$ is continuous, the weak formulation is well–posed; however, the weak coercivity of $a(\cdot,\cdot)$, that requires weaker assumptions on $\boldsymbol{\mu}$ (see the following Theorem 4), is also a sufficient condition for the well–posedness of (23) [21].

Lemma 1. *If* $\|\mathcal{D}\boldsymbol{\mu}\|_\infty<\infty$*, then, the bilinear form* $a(\cdot,\cdot)$ *is weakly coercive.*

Proof. Equation (16) implies that

$$a(u,u)+C_{\rm wc}\|u\|^2_{L^2(\Omega\cup\Omega_{\mathcal{I}})}\geqslant\vartheta_*|||u|||^2,\tag{24}$$

where $C_{\rm wc}=\frac{1}{2}C_p\|\mathcal{D}\boldsymbol{\mu}\|_\infty$. A bilinear form satisfying (24) is, by definition, weakly coercive. □

Theorem 4. *For* $g\in L^2(0,T;V_c')$ *and* $u_0\in V_c$*,* $a(\cdot,\cdot)$ *continuous and weakly coercive, and* $G(\cdot)$ *continuous, problem* (23) *has a unique solution* $u^*\in L^2(0,T;V_c)$*. Furthermore, if* $a(\cdot,\cdot)$ *is coercive, that solution satisfies the following a priori estimate*

$$\|u^*(\cdot,t)\|^2_{L^2(\Omega)}+K_{\rm coer}\int_0^t|||u^*(\cdot,s)|||^2\,ds\leqslant\|u_0\|^2_{L^2(\Omega)}+\frac{C_p^2}{2K_{\rm coer}}\int_0^t\|g(\cdot,s)\|^2_{V_c'}\,ds,\quad\forall\,t>0,\tag{25}$$

where $K_{\rm coer}\in\{C_{\rm coer},\,\widetilde{C}_{\rm coer}\}$*.*

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Proof. The weak coercivity and the continuity of $a(\cdot,\cdot)$ and the continuity of $G(\cdot)$ imply the existence and uniqueness of a solution $u^* \in L^2(0,T;V_c)$ [21]. Then, (25) follows from arguments of the classical theory of partial differential equations [33]. □

4 The relation to Markov processes

In the previous sections the kernel function γ is allowed to take on negative values. However, when the nonlocal equation is associated with a jump process, γ denotes the *jump rate*; thus, we make the assumption that $\gamma : \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$.

Let X_t be a jump process conditioned on $X_0 \in \Omega$ that is absorbed when $X_t \in \Omega_{\mathcal{I}}$. For a non–negative initial condition $u_0(\mathbf{x})$ such that

$$\int_\Omega u_0(\mathbf{x})\, d\mathbf{x} = 1 \tag{26}$$

and $g = 0$, the nonlocal system (21) over $\Omega \times (0,\infty)$ describes the evolution of the probability density for the process X_t with jump rate $\gamma \geqslant 0$, i.e.,

$$\mathbb{P}(X_{\tilde{t}} \in \tilde{\Omega}) = \int_{\tilde{\Omega}} u(\mathbf{x},\tilde{t})\, d\mathbf{x} \qquad \text{for } \tilde{\Omega} \subset \Omega.$$

We refer to $(21)_1$ as the *master equation* for the jump process. The condition on u_0 ensures that $X_0 \in \Omega$ and the homogeneous volume constraint grants that the process does not re–enter the domain, i.e., is absorbed if it exits the domain. The first integrand of $\mathcal{L}$ in (2c) represents the rate $\gamma(\mathbf{y},\mathbf{x})\, d\mathbf{x}$ to $d\mathbf{x}$ from $\mathbf{y}$ given the probability $u(\mathbf{y},t)d\mathbf{y}$ whereas the second integrand represents the rate $\gamma(\mathbf{x},\mathbf{y})\, d\mathbf{y}$ to $d\mathbf{y}$ from $\mathbf{x}$ given the probability $u(\mathbf{x},t)d\mathbf{x}$. The difference in these two rates gives the rate of change of the probability $u(\mathbf{x},t)d\mathbf{x}$; the assumption on g implies that at steady state the rates are equal.

Because $X_{t+t'}$ for $t' > 0$ only depends upon X_t we see that the process X_t is Markov; thus, the nonlocal convection–diffusion operator $\mathcal{L}$ is the generator of a Markov process. Also, because γ is a nonsymmetric localized kernel, the nonlocal jumps are, in general, not symmetrically distributed and have finite length. We refer to such a process as a finite range nonsymmetric Markov jump process, a generalization of a continuous–time Markov chain over the state space Ω. This observation leads to a particle tracking method for realizing the process; see [17, §5.1].

The paper [17] demonstrates that for processes governed by the master equation (21) with integrable kernels, the probability is conserved over Ω; this statement holds regardless of whether γ is integrable or not.

4.1 Exit–time problem for the jump Markov process

The solution of the evolution equation (21) for the probability density in bounded domains allows us to solve the *exit–time problem* for jump processes. We introduce the random variable

$$\tau := \inf\{t > 0,\, X_t \in \Omega_{\mathcal{I}} \,|\, X_0 \in \Omega\}$$

that denotes the first exit time of X_t from Ω. Its probability distribution is given by

$$F_\tau(t) = 1 - \int_\Omega u(\mathbf{x},t)\, d\mathbf{x}.$$

The expected exit time from Ω is given by the expected value of the random variable τ:

$$\mathbb{E}(\tau) = \int_0^\infty \int_\Omega u(\mathbf{x},t)\, d\mathbf{x}\, dt.$$

The paper [11] establishes that for symmetric infinite and finite activity Lévy jump processes the expected exit time is finite as long as the initial condition is square integrable. Following the same argument we show

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that such a statement holds also for the expected exit time of the Markov jump process associated with the master equation (21), provided that the bilinear form in (23) is coercive and that the initial condition is such that $u_0 \in L^2(\Omega)$ [4].

Lemma 2. *If $u_0(\mathbf{x}) : \Omega \to [0, \infty)$ is such that $u_0 \in L^2(\Omega)$ and (26) holds, then the expected exit time $\mathbb{E}(\tau)$ for the master equation (21) is finite. Furthermore,*

$$\mathbb{E}(\tau) \leqslant C_\tau \|u_0\|_{L^2(\Omega)}, \tag{27}$$

where $C_\tau = \frac{C_p^2|\Omega|}{K_{\text{coer}}}$, $K_{\text{coer}} \in \{C_{\text{coer}}, \widetilde{C}_{\text{coer}}\}$.

Proof. Consider the weak formulation (23) for $g = 0$

$$(u_t, v) + a(u, v) = 0 \qquad \forall\, v \in V_c.$$

With $v = u$ we have

$$(u_t, u) + a(u, u) = 0 \qquad \text{or equivalently} \qquad \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega u^2(\mathbf{x}, t)\, d\mathbf{x} = -2a(u, u),$$

For $K_{\text{coer}} \in \{C_{\text{coer}}, \widetilde{C}_{\text{coer}}\}$, the coercivity of $a(\cdot, \cdot)$ implies that

$$-a(u, u) \leqslant -K_{\text{coer}} |||u(\cdot, t)|||^2 \leqslant -\frac{K_{\text{coer}}}{C_p^2} \|u(\cdot, t)\|^2_{L^2(\Omega)}.$$

Thus,

$$\frac{\mathrm{d}}{\mathrm{d}t} \|u(\cdot, t)\|^2_{L^2(\Omega)} \leqslant -\frac{2K_{\text{coer}}}{C_p^2} \|u(\cdot, t)\|^2_{L^2(\Omega)}.$$

Letting $c_\tau = \frac{2K_{\text{coer}}}{C_p^2}$, we have

$$\|u(\cdot, t)\|^2_{L^2(\Omega)} \leqslant e^{-c_\tau t} \|u(\cdot, 0)\|^2_{L^2(\Omega)} \qquad \forall\, t > 0.$$

By the Cauchy–Schwarz inequality we have

$$\int_\Omega u(\mathbf{x}, t)\, d\mathbf{x} \leqslant e^{-\frac{c_\tau}{2} t}\, |\Omega|\, \|u_0\|_{L^2(\Omega)} \qquad \forall\, t > 0, \tag{28}$$

i.e., the probability of remaining in Ω decreases exponentially in time. Then, for $C_\tau = \frac{2\,|\Omega|}{c_\tau}$, (27) is obtained by integrating both sides of (28) in time. □

4.2 Special cases of the nonlocal operator

When certain conditions on the nonlocal diffusion and convection parameters hold, the nonlocal convection–diffusion operator is the generator of Lévy or Lévy–type processes. For a Lévy measure ϕ, the generator of a Lévy jump process in $\mathbb{R}^n$ is defined as [29]

$$\mathcal{G}f(\mathbf{x}) = \int_{\mathbb{R}^n} \big(f(\mathbf{x} - \mathbf{y}) - f(\mathbf{x}) + \mathbf{y} \cdot \nabla f(\mathbf{x})\, \mathbf{1}(\|\mathbf{y}\| \leqslant R)\big)\phi(d\mathbf{y}), \quad R < \infty,\ \mathbf{x} \in \mathbb{R}^n.$$

[4] When the initial condition of the master equation $(21)_1$ is the Dirac measure, the theory of L^p spaces, $p \geqslant 1$, allows us to apply the result in Lemma 2.

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For an integrable Lévy measure ϕ, $\mathcal{G}$ is the generator of a finite activity jump process; for a singular ϕ, $\mathcal{G}$ is the generator of an infinite activity jump process. In the latter case, $\mathcal{G}$ has to be interpreted in a principal value sense, i.e.,

$$\mathcal{G}f(\mathbf{x}) = \lim_{\varepsilon\to 0}\int_{\mathbb{R}^n\setminus B_\varepsilon(\mathbf{0})} \big(f(\mathbf{x}-\mathbf{y}) - f(\mathbf{x}) + \mathbf{y}\cdot\nabla f(\mathbf{x})\,\mathbf{1}(\|\mathbf{y}\|\leqslant R)\big)\phi(d\mathbf{y}) = \lim_{\varepsilon\to 0}\mathcal{P}_\varepsilon f(\mathbf{x}).$$

In words, for a singular Lévy measure ϕ, the generator of the Lévy process is the limit as $\varepsilon\to 0$ of the generator of a compound Poisson process $\mathcal{P}_\varepsilon$. Assuming that $\phi(d\mathbf{y}) = \phi(\mathbf{y})d\mathbf{y}$ and splitting the integral, we have

$$\mathcal{G}f(\mathbf{x}) = \int_{\mathbb{R}^n}\big(f(\mathbf{x}-\mathbf{y}) - f(\mathbf{x})\big)\,\phi(\mathbf{y})\,d\mathbf{y} + \mathbf{d}\cdot\nabla f(\mathbf{x}), \tag{29}$$

where the advection term is such that $\mathbf{d}\cdot\nabla f(\mathbf{x}) = \int_{\|\mathbf{y}\|\leqslant R}\mathbf{y}\cdot\nabla f(\mathbf{x})\,\phi(\mathbf{y})\,d\mathbf{y}$.

Consider now the operator $\mathcal{L}$; if we assume that $\boldsymbol{\alpha},\boldsymbol{\Theta},\boldsymbol{\mu}$ are translation invariant over $\mathbb{R}^n$ and not necessarily of compact support, we may then rewrite (2c) as

$$\mathcal{L}u(\mathbf{x}) = \int_{\mathbb{R}^n}\big(u(\mathbf{y})\gamma(\mathbf{y}-\mathbf{x}) - u(\mathbf{x})\gamma(\mathbf{x}-\mathbf{y})\big)\,d\mathbf{y}, \qquad \mathbf{x}\in\mathbb{R}^n, \tag{30}$$

and because

$$\int_{\mathbb{R}^n}\gamma(\mathbf{x}-\mathbf{y})\,d\mathbf{y} = \int_{\mathbb{R}^n}\gamma(\mathbf{y}-\mathbf{x})\,d\mathbf{y} \qquad \forall\,\mathbf{x}\in\mathbb{R}^n$$

regardless of whether or not $\gamma(\mathbf{y}-\mathbf{x}) = \gamma(\mathbf{x}-\mathbf{y})$ holds, we have

$$\mathcal{L}u(\mathbf{x}) = \int_{\mathbb{R}^n}\big(u(\mathbf{y}) - u(\mathbf{x})\big)\gamma(\mathbf{y}-\mathbf{x})\,d\mathbf{y} = \int_{\mathbb{R}^n}\big(u(\mathbf{x}-\mathbf{z}) - u(\mathbf{x})\big)\gamma(\mathbf{z})\,d\mathbf{z}. \tag{31}$$

When γ is a Lévy measure, comparing (29) and (31) we see that $\mathcal{G}$ is an instance of $\mathcal{L}$ and advection. As a matter of fact, $\mathcal{G}$ can generate only a small class of jump processes; for example, as soon as we confine the process to a bounded domain, the jump rate is not translation invariant and therefore $\mathcal{G}$ cannot be the generator. As an example, consider the Lévy jump rate $\gamma_l(\mathbf{x}-\mathbf{y})$; when the process is confined to Ω the jump rate becomes $\gamma_c(\mathbf{x},\mathbf{y}) = \gamma_l(\mathbf{x}-\mathbf{y})\mathbf{1}(\mathbf{x}\in\Omega\cup\Omega_{\mathcal{I}})\mathbf{1}(\mathbf{y}\in\Omega\cup\Omega_{\mathcal{I}})$; in this case, the more general form (2c) is required.

Another class of processes of interest consists in those whose jump rate (not necessarily symmetric nor translation invariant) satisfies

$$\int_{\Omega\cup\Omega_{\mathcal{I}}}\gamma(\mathbf{x},\mathbf{y})\,d\mathbf{y} = \int_{\Omega\cup\Omega_{\mathcal{I}}}\gamma(\mathbf{y},\mathbf{x})\,d\mathbf{y} \qquad \forall\,\mathbf{x}\in\Omega\cup\Omega_{\mathcal{I}}. \tag{32}$$

In this case (2c) can be rewritten as

$$\mathcal{L}u(\mathbf{x}) = \int_{\Omega\cup\Omega_{\mathcal{I}}}\big(u(\mathbf{y}) - u(\mathbf{x})\big)\gamma(\mathbf{x},\mathbf{y})\,d\mathbf{y}.$$

Moreover, the condition (32) can be interpreted as an intrinsic property of the nonlocal convection parameter. In fact, from (15) we see that the following relations are equivalent:

$$\mathcal{D}\boldsymbol{\mu} = 0 \qquad \mathbf{x}\in\Omega \tag{33a}$$

$$\int_{\Omega\cup\Omega_{\mathcal{I}}}(\boldsymbol{\mu}\cdot\boldsymbol{\alpha})(\mathbf{x},\mathbf{y})\,d\mathbf{y} = 0 \qquad \mathbf{x}\in\Omega. \tag{33b}$$

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Integrating (33b) over Ω we have

$$\int_\Omega \int_\Omega (\boldsymbol{\mu} \cdot \boldsymbol{\alpha})(\mathbf{x}, \mathbf{y})\, d\mathbf{y} + \int_\Omega \int_{\Omega_\mathcal{I}} (\boldsymbol{\mu} \cdot \boldsymbol{\alpha})(\mathbf{x}, \mathbf{y})\, d\mathbf{y} = \int_\Omega \int_{\Omega_\mathcal{I}} (\boldsymbol{\mu} \cdot \boldsymbol{\alpha})(\mathbf{x}, \mathbf{y})\, d\mathbf{y} = 0.$$

Thus, that $\boldsymbol{\mu}$ is nonlocally divergence free is a statement on the flux density, i.e., the probability flux from Ω into $\Omega_\mathcal{I}$ must be zero. Because $\boldsymbol{\alpha} \cdot (\boldsymbol{\Theta}\boldsymbol{\alpha})$ is symmetric, (33) also allows us to conclude that

$$\int_{\Omega \cup \Omega_\mathcal{I}} \gamma(\mathbf{x}, \mathbf{y})\, d\mathbf{y} = \int_{\Omega \cup \Omega_\mathcal{I}} \gamma(\mathbf{y}, \mathbf{x})\, d\mathbf{y} \Longleftrightarrow \mathcal{D}\boldsymbol{\mu} = 0 \qquad \forall\, \mathbf{x} \in \Omega.$$

We also mention that, for certain kernel functions, the operator $\mathcal{L}$ in (2c) is equivalent to a class of fractional differential operators; see [14] where the equivalence between the nonlocal operator $\mathcal{L}$ and the fractional Laplacian $(-\Delta)^s$ is analyzed for all $s \in (0, 1)$, and see [15] where, for symmetric and translation invariant kernel functions, the authors show the equivalence of $\mathcal{L}$ and the fractional operators introduced in [28]. Using the nonsymmetric kernel in (2b), even more general fractional operators associated with nonsymmetric diffusion can be represented as special instances of $\mathcal{L}$; this topic is the subject of our current research.

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