APPLICATION OF A NONLOCAL VECTOR CALCULUS TO THE ANALYSIS OF LINEAR PERIDYNAMIC MATERIALS

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Abstract. A nonlocal vector calculus is applied to the peridynamic model of mechanics. We use nonlocal field operators to express the constitutive relation for an ordinary peridynamic elastic material. The linear peridynamic models and associated nonlocal volume-constrainted problems are defined and analyzed within the nonlocal vector calculus framework. As an example, the well-posedness of the peridynamic model for a linear homogeneous and anisotropic material is demonstrated and a relation between the nonlocal linear isotropic peridynamic model and the local linear Navier model of classical elasticity is also established.

Key words. peridynamics, nonlocal operators, vector calculus, volume-constrained problems

AMS subject classifications. 35B40, 35J20, 35J25, 35Q99, 45A05, 45K05

1. Introduction.

The peridynamic continuum theory was introduced by Silling in [11] to allow for discontinuous deformation. The internal force density denoted by the divergence of the stress tensor was replaced by an integral operator and leads to a nonlocal model of force interaction. An important generalization of the peridynamic theory was given by Silling et. al. in [13]. In particular, a sophisticated deformation operator—the deformation state—was introduced generalizing the approach discussed in [11, § 15] to allow for deformation dependent upon collective motion. The significance is that for homogeneous deformations of a linear isotropic material, Poisson's ratio is no longer limited to be equal to one-fourth. The bond- and state-based peridynamic theory refer to the case when Poisson's ratio is one-fourth or not, respectively. The state-based linear peridynamic theory introduced in [13] was generalized by Silling in [12].

The goal of our paper is to determine the well-posedeness of the linear peridynamic equilibrium equation by exploiting the nonlocal vector calculus developed in [6] given volume constraints. These constraints represent the nonlocal analogue [11, § 13] of boundary conditions necessary for the stability of the linear peridynamic equilibrium operator. Our major contribution is the well-posedeness of the state-based linear peridynamic equilibrium equation, a first of a kind result. Previous work has focused on the well-posedness of the bond-based model; see [3, 8, 9] when the deformation is vector-valued and [2, 4, 5, 10] when the deformation is scalar-valued or the related case

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of nonlocal diffusion. Thus, the present study fills a significant gap in the analysis of peridynamic models and also demonstrates the efficacy of the nonlocal vector calculus framework introduced by [6] for the analysis of nonlocal problems.

We represent the peridynamic deformation fields such as the extension scalar, dilatation, etc., in terms of operators from the nonlocal vector calculus. Based upon the representation of the deformation state, we then formulate variational principles and rewrite both the linear bond-based and the linear state-based peridynamic elastic model via the nonlocal divergence operators and their corresponding adjoints. Our variational formulation coincides with the linearization of the original peridynamic equation derived in [11, 13]. Moreover, we prove the well-posedness of peridynamic "boundary"-value problem, or more appropriately, volume-constrained problem, that is, the linear peridynamic elastic model subject to suitable constraints that are either explicitly enforced on volumes of non-zero measure or naturally implied by the variational principle. We also show in what sense, for linear isotropic materials, that the state-based peridynamic operator, in the local limit, tends to the Navier operator of classical elasticity with arbitrary Poisson ratio.

The paper is organized as follows. In Section 2, we briefly review the nonlocal vector calculus operators that are used for nonlocal peridynamic modeling. In Sections 3 and 4, we rewrite the peridynamic constitutive relation and the linearized peridynamic models for elastic materials as variational problems involving the nonlocal calculus operators by minimizing the total potential energy. Finally, in Sections 5 and 6, we establish the well-posedness of the peridynamic models and prove the convergence of the isotropic peridynamic models to their classical elasticity counterparts. As a result, we show that whereas the bond-based peridynamic model leads, in the local limit, to materials with Poisson ratio one-fourth, the state-based peridynamic model has more general representations as it can correspond to an isotropic elastic material with any Poisson ratio.

2. A review of a nonlocal vector calculus. Our formulation of the linear peridynamic constitutive relation employs the nonlocal divergence operator for tensor functions $\mathcal{D}_t$, the weighted nonlocal divergence operators for tensor functions $\mathcal{D}_{t,\omega}$ and their adjoint operators $\mathcal{D}_t^*$ and $\mathcal{D}_{t,\omega}^*$, respectively. These operators were introduced within the nonlocal vector calculus [6].

Let Ω denote an open subset of $\mathbb{R}^d$ and let $\boldsymbol{\alpha} = \boldsymbol{\alpha}(\mathbf{x},\mathbf{y})$ denote an anti-symmetric mapping from $\Omega\times\Omega$ to $\mathbb{R}^d$, i.e., $\boldsymbol{\alpha}(\mathbf{x},\mathbf{y}) = -\boldsymbol{\alpha}(\mathbf{y},\mathbf{x})$. Given the tensor two-point function $\boldsymbol{\Psi}\colon \Omega\times\Omega\to\mathbb{R}^{n\times k}$ and the point function $\mathbf{v}:\Omega\to\mathbb{R}^n$, the *nonlocal point divergence operator* $\mathcal{D}_t(\boldsymbol{\Psi}):\Omega\to\mathbb{R}^n$ *for tensors* and adjoint operator $\mathcal{D}_t^*(\mathbf{v}):\Omega\times\Omega\to\mathbb{R}^{n\times k}$ are defined as

$$\mathcal{D}_t(\boldsymbol{\Psi})(\mathbf{x}) = \int_\Omega \big(\boldsymbol{\Psi}(\mathbf{x},\mathbf{y}) + \boldsymbol{\Psi}(\mathbf{y},\mathbf{x})\big)\cdot\boldsymbol{\alpha}(\mathbf{x},\mathbf{y})\,d\mathbf{y} \quad \text{for } \mathbf{x}\in\Omega, \tag{2.1a}$$

$$\mathcal{D}_t^*(\mathbf{v})(\mathbf{x},\mathbf{y}) = -\big(\mathbf{v}(\mathbf{y}) - \mathbf{v}(\mathbf{x})\big)\otimes\boldsymbol{\alpha}(\mathbf{x},\mathbf{y}) \quad \text{for } \mathbf{x},\mathbf{y}\in\Omega. \tag{2.1b}$$

Let $\omega\colon\Omega\times\Omega\to\mathbb{R}$ denote a non-negative scalar function. Given the tensor point function $\mathbf{U}\colon\Omega\to\mathbb{R}^{n\times k}$ and vector point functions $\mathbf{u}\colon\Omega\to\mathbb{R}^n$, the *weighted nonlocal divergence operator* $\mathcal{D}_{t,\omega}(\mathbf{U}):\Omega\to\mathbb{R}^n$ *for tensors* and its adjoint operator are defined by

$$\mathcal{D}_{t,\omega}\big(\mathbf{U}\big)(\mathbf{x}) = \mathcal{D}_t\big(\omega(\mathbf{x},\mathbf{y})\mathbf{U}(\mathbf{x})\big)(\mathbf{x}) \quad \text{for } \mathbf{x}\in\Omega, \tag{2.2a}$$

$$\mathcal{D}^*_{t,\omega}\big(\mathbf{u}\big)(\mathbf{x}) = \int_\Omega \mathcal{D}^*_t(\mathbf{u})(\mathbf{x},\mathbf{y})\,\omega(\mathbf{x},\mathbf{y})\,d\mathbf{y} \quad \text{for } \mathbf{x}\in\Omega. \tag{2.2b}$$

One verifies that $\big(\mathcal{D}_t(\mathbf{\Psi}),\mathbf{v}\big)_\Omega = \big(\mathbf{\Psi},\mathcal{D}^*_t(\mathbf{v})\big)_{\Omega\times\Omega}$ and $\big(\mathcal{D}_{t,\omega}(\mathbf{U}),\mathbf{u}\big)_\Omega = \big(\mathbf{U},\mathcal{D}^*_{t,\omega}(\mathbf{u})\big)_\Omega$ by direct calculation, where $\big(\mathbf{\Psi},\mathcal{D}^*_t(\mathbf{v})\big)_{\Omega\times\Omega}$ is the Frobenius inner product on $\Omega\times\Omega$.

From (2.2b), one can also see that

$$\mathrm{Tr}\big(\mathcal{D}^*_{t,\omega}(\mathbf{u})\big)(\mathbf{x}) = \int_\Omega \mathrm{Tr}\big(\mathcal{D}^*_t(\mathbf{u})\big)(\mathbf{x},\mathbf{y})\,\omega(\mathbf{x},\mathbf{y})\,d\mathbf{y} \quad \text{for } \mathbf{x}\in\Omega. \tag{2.3a}$$

By the relations $(\mathbf{a}\otimes\mathbf{b})\,\mathbf{c} = (\mathbf{a}\otimes\mathbf{c})\,\mathbf{b} = (\mathbf{b}\cdot\mathbf{c})\,\mathbf{a}$ that holds for any d-dimensional vectors $\mathbf{a},\mathbf{b},\mathbf{c}$ and that $\mathrm{Tr}\big(\mathcal{D}^*_t(\mathbf{u})\big) = \mathcal{G}^*(\mathbf{u})$ and $\mathrm{Tr}\big(\mathcal{D}^*_{t,\omega}(\mathbf{u})\big) = \mathcal{G}^*_\omega(\mathbf{u})$ in which $\mathcal{G}^*$ and $\mathcal{G}^*_\omega$ are defined in [6], we obtain the nonlocal Green's identities for the *trace* of tensor divergences:

$$\int_\Omega\int_\Omega \mathrm{Tr}(\mathcal{D}^*_t\mathbf{u})\mathrm{Tr}(\mathcal{D}^*_t\mathbf{v})\;d\mathbf{y}\,d\mathbf{x} = \int_\Omega \mathcal{D}_t\Big(\big(\mathcal{D}^*_t(\mathbf{u})\big)^T\Big)\cdot\mathbf{v}\;d\mathbf{x}, \tag{2.3b}$$

$$\int_\Omega \mathrm{Tr}(\mathcal{D}^*_{t,\omega}\mathbf{u})\mathrm{Tr}(\mathcal{D}^*_{t,\omega}\mathbf{v})\;d\mathbf{y}\,d\mathbf{x} = \int_\Omega \mathcal{D}_{t,\omega}\big(\mathrm{Tr}(\mathcal{D}^*_{t,\omega}\mathbf{u})\mathrm{I}\big)\cdot\mathbf{v}\;d\mathbf{x}. \tag{2.3c}$$

3. Constitutive relations in peridynamic modeling. In this section, we rewrite the constitutive relation for a peridynamic ordinary elastic material in terms of the nonlocal vector calculus operators. In [13], a set of deformation states and functions of deformation states are introduced in order to represent the nonlocal analogue of the "stress-strain relation".

To simplify the notation, we let $\underline{y} = |\mathbf{u}(\mathbf{y})+\mathbf{y}-\mathbf{u}(\mathbf{x})-\mathbf{x}|$ and $\underline{x} = |\mathbf{y}-\mathbf{x}|$ for any given pair $\mathbf{x}$ and $\mathbf{y}$, where $\mathbf{u}$ denotes the displacement field. We choose $\underline{\omega}(\mathbf{x},\mathbf{y})$ as an influence function which provides the contribution of a bond to the determination of the force state; for a detailed definition, see Definition 3.2 in [13, page 156]. The fundamental deformation quantities used in peridynamic modeling can be expressed as follows:

$$\text{extension scalar state} \qquad \underline{e} = \underline{y} - \underline{x}, \tag{3.1a}$$

$$\text{weighted volume} \qquad m = (\underline{\omega x})\bullet\underline{x}, \tag{3.1b}$$

$$\text{dilatation} \qquad \widehat{\theta} = \frac{d}{m}\underline{\omega x}\bullet\underline{e}, \tag{3.1c}$$

$$\text{isotropic part of } \underline{e} \qquad \underline{e}^i = \frac{\widehat{\theta}\underline{x}}{d}, \tag{3.1d}$$

$$\text{deviatoric part of } \underline{e} \qquad \underline{e}^d = \underline{e} - \underline{e}^i, \tag{3.1e}$$

where d denotes the space dimension. The operation $\bullet$ between different states may be viewed as an inner product in a Hilbert space whose precise form is not required in the current work; interested readers can refer to sections 2 and 3 in [13] for detailed explanations.

We now specialize the definitions of the nonlocal calculus to the peridynamic theory. Let

$$
\begin{aligned}
\boldsymbol{\alpha}(\mathbf{x},\mathbf{y}) &= (\mathbf{x}-\mathbf{y})/|\mathbf{y}-\mathbf{x}|, \\
n(\mathbf{x}) &= \frac{1}{d}\int_{\Omega} |\mathbf{y}-\mathbf{x}|^2 \underline{\omega}(\mathbf{x},\mathbf{y})\; d\mathbf{y}, \qquad (3.2)\\
\omega(\mathbf{x},\mathbf{y}) &= |\mathbf{y}-\mathbf{x}|\underline{\omega}(\mathbf{x},\mathbf{y})/n(\mathbf{x}).
\end{aligned}
$$

From this definition, we have the relation

$$n(\mathbf{x}) = m/d.$$

where m is the *weighted volume* defined in (3.1b). Then, after linearizing the extension scalar state with respect to $\mathbf{u}(\mathbf{y})-\mathbf{u}(\mathbf{x})$, we obtain

$$
\begin{aligned}
\underline{e} &= |\mathbf{u}(\mathbf{y})+\mathbf{y}-\mathbf{u}(\mathbf{x})-\mathbf{x}| - |\mathbf{y}-\mathbf{x}| \\
&= \big(\mathbf{y}-\mathbf{x}\big)\cdot\big(\mathbf{u}(\mathbf{y})-\mathbf{u}(\mathbf{x})\big)/|\mathbf{y}-\mathbf{x}| \qquad (3.3)\\
&= \mathrm{Tr}\big(\mathcal{D}_t^*\mathbf{u}\big)\,.
\end{aligned}
$$

By the definition of the scalar product of two states, we have the following description of a *linear peridynamic elastic material*:

$$\widehat{\theta} = \mathrm{Tr}\big(\mathcal{D}_{t,\omega}^*\mathbf{u}\big)\,, \qquad (3.4a)$$

$$\underline{e}^i = \mathrm{Tr}\big(\mathcal{D}_{t,\omega}^*\mathbf{u}\big)\,|\mathbf{y}-\mathbf{x}|/d, \qquad (3.4b)$$

$$\underline{e}^d = \mathrm{Tr}\big(\mathcal{D}_t^*\mathbf{u}\big) - \mathrm{Tr}\big(\mathcal{D}_{t,\omega}^*\mathbf{u}\big)\,|\mathbf{y}-\mathbf{x}|/d, \qquad (3.4c)$$

where $\mathcal{D}_t^*\mathbf{u} = \mathcal{D}_t^*(\mathbf{u})$, $\mathcal{D}_{t,\omega}^*\mathbf{u} = \mathcal{D}_{t,\omega}^*(\mathbf{u})$ and "Tr" denotes the *trace* operator. We see that, when small deformations are considered, see Definition 4.1 in [12] for more explanations, the peridynamic deformation quantities can be represented in terms of the nonlocal adjoint tensor divergence operator $\mathcal{D}_t^*$ and the weighted adjoint tensor divergence operator $\mathcal{D}_{t,\omega}^*$.

Let $\delta > 0$ denote the horizon, which is treated as a material property. For any two points $\mathbf{x}\,,\mathbf{y}\in\Omega$, if $|\mathbf{x}-\mathbf{y}|<\delta$, the vector $\boldsymbol{\xi}=\mathbf{y}-\mathbf{x}$ is called a bond. Then for the bond-based peridynamic model, the influence function $\underline{\omega}(\mathbf{x},\mathbf{y})$ is a singular measure, that is, a delta-function $\Delta(\boldsymbol{\xi}\,,\boldsymbol{\zeta})\chi_{|\mathbf{x}-\mathbf{y}|<\delta}$, where $\boldsymbol{\xi}$ is the bond formed by $\mathbf{x}$ and $\mathbf{y}$, $\boldsymbol{\zeta}$ is the bond formed by $\mathbf{x}$ and any other point in Ω, $\chi_{|\mathbf{x}-\mathbf{y}|<\delta}$ is the characteristic function which equals to 1 if $|\mathbf{x}-\mathbf{y}|<\delta$, 0 otherwise. Then, the deformation quantities for linear bond-based peridynamic materials are given as

$$\widehat{\theta} = d\,\mathrm{Tr}\big(\mathcal{D}_t^*\mathbf{u}\big)/|\mathbf{y}-\mathbf{x}| \qquad (3.5a)$$

$$\underline{e}^i = \mathrm{Tr}\big(\mathcal{D}_t^*\mathbf{u}\big), \qquad (3.5b)$$

$$\underline{e}^d = 0. \qquad (3.5c)$$

In the sequel, we focus on the linearized theory of a peridynamic solid whose microelastic energy is given by [13, page 23],

$$W(\theta,\underline{e}^d) = \frac{k}{2}\widehat{\theta}^2 + \frac{\eta}{2}(\underline{\omega e^d})\bullet\underline{e}^d\,, \qquad (3.6)$$

where k and η are material parameters. Taking state-based peridynamic materials as an example, we can rewrite the microelastic energy in terms of nonlocal vector calculus as

$$W(\theta, \underline{e}^d) = \frac{k(\mathbf{x})\big(\mathrm{Tr}(\mathcal{D}^*_{t,\omega}\mathbf{u})\big)^2}{2} + \frac{\eta(\mathbf{x})}{2}\int_\Omega \underline{\omega}(\mathbf{x},\mathbf{y})\big(\mathrm{Tr}(\mathcal{D}^*_t\mathbf{u}) - \mathrm{Tr}(\mathcal{D}^*_{t,\omega}\mathbf{u})\,|\mathbf{y}-\mathbf{x}|/d\big)^2\, d\mathbf{y}.$$

4. Variational principles for linear peridynamic models. In this section, we use an energy minimization principle to derive equations for linear peridynamic materials.

By (3.6), the total potential energy of a state-based peridynamic material under an external force $\mathbf{b}(\mathbf{x})$ is given by

$$\begin{aligned} E_s(\mathbf{u}) &= \int_\Omega \frac{k(\mathbf{x})\,\widehat{\theta}^2}{2}\, d\mathbf{x} + \int_\Omega\int_\Omega \frac{\eta(\mathbf{x})}{2}\underline{\omega}(\mathbf{x},\mathbf{y})(\underline{e}^d)^2\, d\mathbf{y}\, d\mathbf{x} - \int_\Omega \mathbf{u}\cdot\mathbf{b}\, d\mathbf{x} \\ &= \int_\Omega \frac{k(\mathbf{x})\big(\mathrm{Tr}(\mathcal{D}^*_{t,\omega}\mathbf{u})\big)^2}{2}\, d\mathbf{x} - \int_\Omega \mathbf{u}\cdot\mathbf{b}\, d\mathbf{x} \\ &\quad + \int_\Omega\int_\Omega \frac{\eta(\mathbf{x})}{2}\underline{\omega}(\mathbf{x},\mathbf{y})\big(\mathrm{Tr}(\mathcal{D}^*_t\mathbf{u}) - \mathrm{Tr}(\mathcal{D}^*_{t,\omega}\mathbf{u})\,|\mathbf{y}-\mathbf{x}|/d\big)^2\, d\mathbf{y}\, d\mathbf{x}. \end{aligned} \tag{4.1}$$

According to the constitutive relation for bond-based peridynamic materials in Section 3 and [11], we similarly have the energy of a bond-based material given by

$$\begin{aligned} E_b(\mathbf{u}) &= \int_\Omega\int_\Omega \frac{\tilde{k}(\mathbf{x},\mathbf{y})\,\widehat{\theta}^2}{2}\, d\mathbf{y}\, d\mathbf{x} - \int_\Omega \mathbf{u}\cdot\mathbf{b}\, d\mathbf{x} \\ &= \int_\Omega\int_\Omega \frac{d^2\tilde{k}(\mathbf{x},\mathbf{y})\big(\mathrm{Tr}(\mathcal{D}^*_t\mathbf{u})\big)^2}{2|\mathbf{y}-\mathbf{x}|^2}\, d\mathbf{y}\, d\mathbf{x} - \int_\Omega \mathbf{u}\cdot\mathbf{b}\, d\mathbf{x} \\ &= \int_\Omega\int_\Omega \frac{\gamma(\mathbf{x},\mathbf{y})\big(\mathrm{Tr}(\mathcal{D}^*_t\mathbf{u})\big)^2}{2}\, d\mathbf{y}\, d\mathbf{x} - \int_\Omega \mathbf{u}\cdot\mathbf{b}\, d\mathbf{x}, \end{aligned} \tag{4.2}$$

where $\gamma(\mathbf{x},\mathbf{y}) = d^2\tilde{k}(\mathbf{x},\mathbf{y})/|\mathbf{y}-\mathbf{x}|^2$.

4.1. Variation of the potential energy. The model problems for peridynamic materials we consider in this work are described by the minimization of the potential energy so as to find the equilibrium state of the material. Thus, to determine a minimizer of the potential energy, we seek a displacement field $\mathbf{u}$ such that $\delta E(\mathbf{u})/\delta\mathbf{u} = 0$ with certain constraints. To define the contraints, we first introduce the following notation:

Ω_s: the solution domain of the model equation

Ω_c: the constraint domain where the displacement field $\mathbf{u}$ is specified.

$\Omega = (\Omega_s \cup \Omega_c) \cup (\overline{\Omega}_s \cap \overline{\Omega}_c)^0$

$U(\Omega)$: a function space defined on Ω such that the total potential energy is finite.

Note that the constraint domain Ω_c is either empty or has positive volume, that is, it cannot be a lower-dimensional manifold.

A variety of choices for the region Ω_c having positive volume can be allowed as part of the the material domain. See Figure 4.1 for illustrations of some of the possibilities. We note that most of our results can also be applied to cases where Ω is a union of the domains shown in the figure.

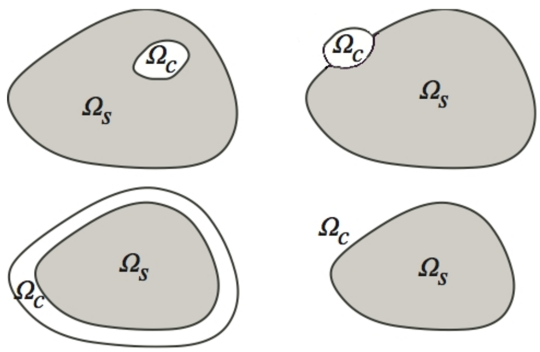

FIG. 4.1. *Four of the possible configurations for* $\Omega = (\Omega_s \cup \Omega_c) \cup (\overline{\Omega}_s \cap \overline{\Omega}_c)^0$

We consider the following constrained energy minimization problem that corresponds to conventional boundary-value problems with essential boundary conditions:

$$\min_{\mathbf{u}\in U(\Omega)} E_s(\mathbf{u}) \quad \text{subject to} \quad \mathbf{u} = \mathbf{h}_d \text{ for } \mathbf{x} \in \Omega_c. \tag{4.3}$$

We define the test-function space U_0 as

$$U_0 = \{\mathbf{v} \in U(\Omega) \colon \mathbf{v} = 0 \text{ on } \Omega_c\}.$$

Let $\mathbf{v}$ denote an arbitrary function in U_0 and $f(\epsilon) = E_s(\mathbf{u} + \epsilon\mathbf{v})$. We compute $f'(\epsilon)|_{\epsilon=0}$ to obtain

$$\begin{aligned}
f'(0) &= \int_\Omega k(\mathbf{x})\mathrm{Tr}(\mathcal{D}^*_{t,\omega}\mathbf{u})\mathrm{Tr}(\mathcal{D}^*_{t,\omega}\mathbf{v})\;d\mathbf{x} - \int_\Omega \mathbf{b}\cdot\mathbf{v}\;d\mathbf{x}\\
&\quad + \int_\Omega\int_\Omega \eta(\mathbf{x})\underline{\omega}(\mathbf{x},\mathbf{y})\big(\mathrm{Tr}(\mathcal{D}^*_t\mathbf{u}) - \mathrm{Tr}(\mathcal{D}^*_{t,\omega}\mathbf{u})\frac{|\mathbf{y}-\mathbf{x}|}{d}\big)\big(\mathrm{Tr}(\mathcal{D}^*_t\mathbf{v}) - \mathrm{Tr}(\mathcal{D}^*_{t,\omega}\mathbf{v})\frac{|\mathbf{y}-\mathbf{x}|}{d}\big)\;d\mathbf{y}\,d\mathbf{x}\\
&= \int_\Omega k(\mathbf{x})\mathrm{Tr}(\mathcal{D}^*_{t,\omega}\mathbf{u})\mathrm{Tr}(\mathcal{D}^*_{t,\omega}\mathbf{v})\;d\mathbf{x} - \int_\Omega \mathbf{b}\cdot\mathbf{v}\;d\mathbf{x} + \int_\Omega\int_\Omega \eta(\mathbf{x})\underline{\omega}(\mathbf{x},\mathbf{y})\mathrm{Tr}(\mathcal{D}^*_t\mathbf{u})\mathrm{Tr}(\mathcal{D}^*_t\mathbf{v})\;d\mathbf{x}\\
&\quad + \int_\Omega \eta(\mathbf{x})\mathrm{Tr}(\mathcal{D}^*_{t,\omega}\mathbf{u})\mathrm{Tr}(\mathcal{D}^*_{t,\omega}\mathbf{v})\frac{1}{d^2}\int_\Omega \underline{\omega}(\mathbf{x},\mathbf{y})|\mathbf{y}-\mathbf{x}|^2\;d\mathbf{y}\,d\mathbf{x}\\
&\quad - \int_\Omega \eta(\mathbf{x})\mathrm{Tr}(\mathcal{D}^*_{t,\omega}\mathbf{v})\frac{1}{d}\int_\Omega \underline{\omega}(\mathbf{x},\mathbf{y})\mathrm{Tr}(\mathcal{D}^*_t\mathbf{u})|\mathbf{y}-\mathbf{x}|\;d\mathbf{y}\,d\mathbf{x}\\
&\quad - \int_\Omega \eta(\mathbf{x})\mathrm{Tr}(\mathcal{D}^*_{t,\omega}\mathbf{u})\frac{1}{d}\int_\Omega \underline{\omega}(\mathbf{x},\mathbf{y})\mathrm{Tr}(\mathcal{D}^*_t\mathbf{v})|\mathbf{y}-\mathbf{x}|\;d\mathbf{y}\,d\mathbf{x}\,.
\end{aligned}$$

From (3.2) and (2.3a) and the Green's identities (2.3b) and (2.3c), we obtain

$$
\begin{aligned}
f'(0) &= \int_\Omega k(\mathbf{x})\mathrm{Tr}(\mathcal{D}^*_{t,\omega}\mathbf{u})\mathrm{Tr}(\mathcal{D}^*_{t,\omega}\mathbf{v})\,d\mathbf{x} - \int_\Omega \mathbf{b}\cdot\mathbf{v}\,d\mathbf{x} + \int_\Omega\int_\Omega \eta(\mathbf{x})\underline{\omega}(\mathbf{x},\mathbf{y})\mathrm{Tr}(\mathcal{D}^*_t\mathbf{u})\mathrm{Tr}(\mathcal{D}^*_t\mathbf{v})\,d\mathbf{x}\\
&\quad + \int_\Omega \frac{\eta(\mathbf{x})n(\mathbf{x})}{d}\mathrm{Tr}(\mathcal{D}^*_{t,\omega}\mathbf{u})\mathrm{Tr}(\mathcal{D}^*_{t,\omega}\mathbf{v})\,d\mathbf{x} - \int_\Omega \frac{\eta(\mathbf{x})n(\mathbf{x})}{d}\mathrm{Tr}(\mathcal{D}^*_{t,\omega}\mathbf{u})\mathrm{Tr}(\mathcal{D}^*_{t,\omega}\mathbf{v})\,d\mathbf{x}\\
&\quad - \int_\Omega \frac{\eta(\mathbf{x})n(\mathbf{x})}{d}\mathrm{Tr}(\mathcal{D}^*_{t,\omega}\mathbf{u})\mathrm{Tr}(\mathcal{D}^*_{t,\omega}\mathbf{v})\,d\mathbf{x}\\
&= \int_\Omega \mathcal{D}_{t,\omega}\big(\big(k(\mathbf{x}) - \frac{1}{d}\eta(\mathbf{x})n(\mathbf{x})\big)\mathrm{Tr}(\mathcal{D}^*_{t,\omega}\mathbf{u})\mathrm{I}\big)\cdot\mathbf{v}\,d\mathbf{x}\\
&\quad + \mathcal{D}_t\big(\eta(\mathbf{x})\underline{\omega}(\mathbf{x},\mathbf{y})(\mathcal{D}^*_t\mathbf{u})^T\big)\cdot\mathbf{v}\,d\mathbf{x} - \int_\Omega \mathbf{b}\cdot\mathbf{v}\,d\mathbf{x}\,.
\end{aligned}
$$

Thus, the constrained energy minimization problem (4.3) results in the equivalent nonlocal system

$$
\begin{cases} -\mathcal{L}(\mathbf{u})(\mathbf{x}) = \mathbf{b}(\mathbf{x}), & \mathbf{x}\in\Omega_s,\\ \mathbf{u}(\mathbf{x}) = \mathbf{h}_d(\mathbf{x}), & \mathbf{x}\in\Omega_c, \end{cases} \tag{4.4}
$$

where the peridynamic operator $\mathcal{L}(\cdot)$ is given by

$$
-\mathcal{L}(\cdot) = \mathcal{D}_t\big(\eta(\mathbf{x})\underline{\omega}(\mathbf{x},\mathbf{y})\big(\mathcal{D}^*_t(\cdot)\big)^T\big) + \mathcal{D}_{t,w}\big(\big(k(\mathbf{x}) - n(\mathbf{x})\eta(\mathbf{x})/d\big)\mathrm{Tr}\big(\mathcal{D}^*_{t,\omega}(\cdot)\big)\mathrm{I}\big). \tag{4.5}
$$

By direct calculation, one sees that the nonlocal system (4.4) derived using a variational principle is exactly the same as the one obtained in [13] based upon the balance of linear momentum.

Setting

$$
c(\mathbf{x}) = (k(\mathbf{x}) - \eta(\mathbf{x})n(\mathbf{x})/d)/n^2(\mathbf{x}), \tag{4.6}
$$

we have the following equivalent form of the peridynamic operator.

Lemma 4.1. *The peridynamic operator can be alternatively expressed in the following form. For a given function* $\mathbf{u}\in U(\Omega)$,

$$
\mathcal{L}(\mathbf{u})(\mathbf{x}) = \int_\Omega \mathbb{C}(\mathbf{x},\mathbf{y})\big(\mathbf{u}(\mathbf{y}) - \mathbf{u}(\mathbf{x})\big)\,d\mathbf{y} \tag{4.7}
$$

where $\mathbb{C}(\mathbf{x},\mathbf{y}) = K_1(\mathbf{x},\mathbf{y}) + C_0(\mathbf{x},\mathbf{y})$ *with*

$$
K_1(\mathbf{x},\mathbf{y}) = 2\eta(\mathbf{x})\frac{\underline{\omega}(\mathbf{x},\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^2}(\mathbf{y}-\mathbf{x})\otimes(\mathbf{y}-\mathbf{x}), \tag{4.8a}
$$

$$
\begin{aligned}
C_0(\mathbf{x},\mathbf{y}) = \int_\Omega \Big(& c(\mathbf{z})\underline{\omega}(\mathbf{x},\mathbf{z})\underline{\omega}(\mathbf{z},\mathbf{y})(\mathbf{x}-\mathbf{z})\otimes(\mathbf{z}-\mathbf{y})\\
& - c(\mathbf{y})\underline{\omega}(\mathbf{x},\mathbf{y})\underline{\omega}(\mathbf{y},\mathbf{z})(\mathbf{x}-\mathbf{y})\otimes(\mathbf{y}-\mathbf{z})\\
& + c(\mathbf{x})\underline{\omega}(\mathbf{x},\mathbf{z})\underline{\omega}(\mathbf{x},\mathbf{y})(\mathbf{x}-\mathbf{z})\otimes(\mathbf{x}-\mathbf{y})\Big)\,d\mathbf{z}.
\end{aligned} \tag{4.8b}
$$

Proof. We first calculate $\mathcal{D}_t\big(\eta(\mathbf{x})\underline{\omega}(\mathbf{x},\mathbf{y})\big(\mathcal{D}_t^*(\mathbf{u})\big)^T\big)$:

$$\begin{aligned}
\mathcal{D}_t\big(\eta(\mathbf{x})\underline{\omega}(\mathbf{x},\mathbf{y})\big(\mathcal{D}_t^*(\mathbf{u})\big)^T\big) &= -2\eta(\mathbf{x})\int_\Omega \underline{\omega}(\mathbf{x},\mathbf{y})\boldsymbol{\alpha}(\mathbf{x},\mathbf{y})\otimes\boldsymbol{\alpha}(\mathbf{x},\mathbf{y})\big(\mathbf{u}(\mathbf{y})-\mathbf{u}(\mathbf{x})\big)\ d\mathbf{y}\\
&= -\int_\Omega 2\eta(\mathbf{x})\frac{\underline{\omega}(\mathbf{x},\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^2}(\mathbf{y}-\mathbf{x})\otimes(\mathbf{y}-\mathbf{x})\big(\mathbf{u}(\mathbf{y})-\mathbf{u}(\mathbf{x})\big)\ d\mathbf{y}\\
&= -\int_\Omega K_1(\mathbf{x},\mathbf{y})\big(\mathbf{u}(\mathbf{y})-\mathbf{u}(\mathbf{x})\big)\ d\mathbf{y}.
\end{aligned}$$

Next, we consider $\mathcal{D}_{t,w}\big(\big(k(\mathbf{x})-n(\mathbf{x})\eta(\mathbf{x})/d\big)\mathrm{Tr}\big(\mathcal{D}_{t,\omega}^*(\mathbf{u})\big)\mathrm{I}\big)$:

$$\begin{aligned}
&\mathcal{D}_{t,w}\big(\big(k(\mathbf{x})-n(\mathbf{x})\eta(\mathbf{x})/d\big)\mathrm{Tr}\big(\mathcal{D}_{t,\omega}^*(\mathbf{u})\big)\mathrm{I}\big)\\
&\quad= \int_\Omega\bigg(c(\mathbf{y})\int_\Omega\big(\mathbf{u}(\mathbf{z})-\mathbf{u}(\mathbf{y})\big)\cdot\boldsymbol{\alpha}(\mathbf{y},\mathbf{z})\underline{\omega}(\mathbf{y},\mathbf{z})|\mathbf{y}-\mathbf{z}|\ d\mathbf{z}\\
&\qquad + c(\mathbf{x})\int_\Omega\big(\mathbf{u}(\mathbf{z})-\mathbf{u}(\mathbf{x})\big)\cdot\boldsymbol{\alpha}(\mathbf{x},\mathbf{z})\underline{\omega}(\mathbf{x},\mathbf{z})|\mathbf{x}-\mathbf{z}|\ d\mathbf{z}\bigg)\cdot\boldsymbol{\alpha}(\mathbf{x},\mathbf{y})\underline{\omega}(\mathbf{x},\mathbf{y})|\mathbf{y}-\mathbf{x}|\ d\mathbf{y}\\
&\quad= \int_\Omega\int_\Omega c(\mathbf{y})\underline{\omega}(\mathbf{x},\mathbf{y})\underline{\omega}(\mathbf{y},\mathbf{z})\big((\mathbf{y}-\mathbf{x})\otimes(\mathbf{z}-\mathbf{y})\big)\big(\mathbf{u}(\mathbf{z})-\mathbf{u}(\mathbf{y})\big)\ d\mathbf{z}\,d\mathbf{y}\\
&\qquad + \int_\Omega\int_\Omega c(\mathbf{x})\underline{\omega}(\mathbf{x},\mathbf{y})\underline{\omega}(\mathbf{x},\mathbf{z})\big((\mathbf{y}-\mathbf{x})\otimes(\mathbf{z}-\mathbf{x})\big)\big(\mathbf{u}(\mathbf{z})-\mathbf{u}(\mathbf{x})\big)\ d\mathbf{z}\,d\mathbf{y}\\
&\quad= \int_\Omega\int_\Omega c(\mathbf{y})\underline{\omega}(\mathbf{x},\mathbf{y})\underline{\omega}(\mathbf{y},\mathbf{z})\big((\mathbf{y}-\mathbf{x})\otimes(\mathbf{z}-\mathbf{y})\big)\mathbf{u}(\mathbf{z})\ d\mathbf{z}\,d\mathbf{y}\\
&\qquad - \int_\Omega\int_\Omega c(\mathbf{y})\underline{\omega}(\mathbf{x},\mathbf{y})\underline{\omega}(\mathbf{y},\mathbf{z})\big((\mathbf{y}-\mathbf{x})\otimes(\mathbf{z}-\mathbf{y})\big)\mathbf{u}(\mathbf{y})\ d\mathbf{z}\,d\mathbf{y}\\
&\qquad + \int_\Omega\int_\Omega c(\mathbf{x})\underline{\omega}(\mathbf{x},\mathbf{y})\underline{\omega}(\mathbf{x},\mathbf{z})\big((\mathbf{y}-\mathbf{x})\otimes(\mathbf{z}-\mathbf{x})\big)\mathbf{u}(\mathbf{z})\ d\mathbf{z}\,d\mathbf{y}\\
&\qquad - \int_\Omega\int_\Omega c(\mathbf{x})\underline{\omega}(\mathbf{x},\mathbf{y})\underline{\omega}(\mathbf{x},\mathbf{z})\big((\mathbf{y}-\mathbf{x})\otimes(\mathbf{z}-\mathbf{x})\big)\mathbf{u}(\mathbf{x})\ d\mathbf{z}\,d\mathbf{y}.
\end{aligned}$$

We also note that, by changing the order of integration,

$$\begin{aligned}
&\int_\Omega\int_\Omega c(\mathbf{y})\underline{\omega}(\mathbf{x},\mathbf{y})\underline{\omega}(\mathbf{y},\mathbf{z})\big((\mathbf{y}-\mathbf{x})\otimes(\mathbf{z}-\mathbf{y})\big)\mathbf{u}(\mathbf{z})\ d\mathbf{z}\,d\mathbf{y}\\
&\quad= \int_\Omega\int_\Omega c(\mathbf{z})\underline{\omega}(\mathbf{x},\mathbf{z})\underline{\omega}(\mathbf{z},\mathbf{y})\big((\mathbf{z}-\mathbf{x})\otimes(\mathbf{y}-\mathbf{z})\big)\mathbf{u}(\mathbf{y})\ d\mathbf{z}\,d\mathbf{y}
\end{aligned}$$

and

$$\begin{aligned}
&\int_\Omega\int_\Omega c(\mathbf{x})\underline{\omega}(\mathbf{x},\mathbf{y})\underline{\omega}(\mathbf{x},\mathbf{z})\big((\mathbf{y}-\mathbf{x})\otimes(\mathbf{z}-\mathbf{x})\big)\mathbf{u}(\mathbf{z})\ d\mathbf{z}\,d\mathbf{y}\\
&\quad= \int_\Omega\int_\Omega c(\mathbf{x})\underline{\omega}(\mathbf{x},\mathbf{z})\underline{\omega}(\mathbf{x},\mathbf{y})\big((\mathbf{z}-\mathbf{x})\otimes(\mathbf{y}-\mathbf{x})\big)\mathbf{u}(\mathbf{y})\ d\mathbf{z}\,d\mathbf{y}.
\end{aligned}$$

Thus, we have

$$\mathcal{D}_{t,w}\big(\big(k(\mathbf{x})-n(\mathbf{x})\eta(\mathbf{x})/d\big)\mathrm{Tr}\big(\mathcal{D}_{t,\omega}^*(\mathbf{u})\big)\mathrm{I}\big) = \int_\Omega C_0(\mathbf{x},\mathbf{y})\big(\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y})\big)\ d\mathbf{y}\,.$$

The conclusion of the lemma now follows. $\square$

By (4.8a) and (4.8b), the double state $\underline{\mathbb{K}}[\mathbf{x}]\langle \mathbf{z}-\mathbf{x}, \mathbf{y}-\mathbf{x}\rangle$ defined in [12, eqns 23, 24] for the peridynamic solid is obtained as

$$\begin{aligned}\underline{\mathbb{K}}[\mathbf{x}]\langle \mathbf{z}-\mathbf{x}, \mathbf{y}-\mathbf{x}\rangle &= K_1(\mathbf{x},\mathbf{y})\Delta(\mathbf{z}-\mathbf{x}, \mathbf{y}-\mathbf{x}) \\ &\quad + c(\mathbf{x})\underline{\omega}(\mathbf{x},\mathbf{z})\underline{\omega}(\mathbf{x},\mathbf{y})(\mathbf{z}-\mathbf{x})\otimes(\mathbf{y}-\mathbf{x}),\end{aligned}$$

in which $\Delta(\cdot,\cdot)$ denotes the delta-function, and the force state $\underline{\mathbf{T}}(\mathbf{x},\mathbf{y})$ can be found by [12, eqn 23]. Both of these states coincide with the ones derive from the balance law. We can also refer to Table 1 in [12, page 27] for the mechanical explanations for the nonlocal quantities.

Through a similar but much simpler derivation, the nonlocal system corresponding to the constrained energy minimization problem for the bond-based peridynamic solid is given by

$$\begin{cases} -\mathcal{L}_b(\mathbf{u})(\mathbf{x}) = \mathbf{b}(\mathbf{x}), & \mathbf{x}\in\Omega_s, \\ \mathbf{u}(\mathbf{x}) = \mathbf{h}_d(\mathbf{x}), & \mathbf{x}\in\Omega_c, \end{cases} \tag{4.9}$$

where the bond-based peridynamic operator is given by

$$-\mathcal{L}_b(\cdot) = \mathcal{D}_t\big(\gamma(\mathbf{x},\mathbf{y})\big(\mathcal{D}_t^*(\cdot)\big)^T\big). \tag{4.10}$$

5. Well-posedness of the peridynamic solid models. In the previous section, the peridynamic models are derived from energy minimization principles. The derivations are implicitly based on corresponding weak formulations of the nonlocal models, that is,

$$\begin{cases} \displaystyle\int_\Omega\int_\Omega \eta(\mathbf{x})\underline{\omega}(\mathbf{x},\mathbf{y})\mathrm{Tr}(\mathcal{D}_t^*\mathbf{u})\mathrm{Tr}(\mathcal{D}_t^*\mathbf{v}))\; d\mathbf{y}\, d\mathbf{x} \\ \displaystyle + \int_\Omega \big(k(\mathbf{x}) - n(\mathbf{x})\eta(\mathbf{x})/d\big)\mathrm{Tr}(\mathcal{D}_{t,\omega}^*\mathbf{u})\mathrm{Tr}(\mathcal{D}_{t,\omega}^*\mathbf{v})\; d\mathbf{x} = \int_\Omega \mathbf{b}\cdot\mathbf{v}\; d\mathbf{x} \quad \forall\,\mathbf{v}\in U_0, \\ \mathbf{u}(\mathbf{x}) = \mathbf{h}_d(\mathbf{x}), \quad \mathbf{x}\in\Omega_c \end{cases} \tag{5.1}$$

for the linearized state-based peridynamic solid and

$$\begin{cases} \displaystyle\int_\Omega \gamma(\mathbf{x},\mathbf{y})\mathrm{Tr}(\mathcal{D}_t^*\mathbf{u})\mathrm{Tr}(\mathcal{D}_t^*\mathbf{v})\; d\mathbf{x} = \int_\Omega \mathbf{b}\cdot\mathbf{v}\; d\mathbf{x} \quad \forall\,\mathbf{v}\in U_0, \\ \mathbf{u}(\mathbf{x}) = \mathbf{h}_d(\mathbf{x}), \quad \mathbf{x}\in\Omega_c \end{cases} \tag{5.2}$$

for the linearized bond-based peridynamic solid. We then define the bilinear form

$$B(\mathbf{u},\mathbf{v}) = \int_\Omega\int_\Omega \psi_1(\mathbf{x},\mathbf{y})\mathrm{Tr}(\mathcal{D}_t^*\mathbf{u})\mathrm{Tr}(\mathcal{D}_t^*\mathbf{v})\; d\mathbf{y}\, d\mathbf{x} + \int_\Omega \psi_2(\mathbf{x})\mathrm{Tr}(\mathcal{D}_{t,\omega}^*\mathbf{u})\mathrm{Tr}(\mathcal{D}_{t,\omega}^*\mathbf{v})\; d\mathbf{x}, \tag{5.3}$$

where

$$\psi_1(\mathbf{x},\mathbf{y}) = \begin{cases} \gamma(\mathbf{x},\mathbf{y}) & \text{for the bond-based peridynamic solid,} \\ \eta(\mathbf{x})\underline{\omega}(\mathbf{x},\mathbf{y}) & \text{for the state-based peridynamic solid,} \end{cases} \tag{5.4a}$$

$$\psi_2(\mathbf{x}) = \begin{cases} 0 & \text{for the bond-based peridynamic solid,} \\ k(\mathbf{x}) - n(\mathbf{x})\eta(\mathbf{x})/d & \text{for the state-based peridynamic solid.} \end{cases} \quad (5.4b)$$

In the following, we focus on the well-posedness of the linearized state-based peridynamic model with a given volume-constraint. The corresponding theory for the bond-based material can be derived with simple modifications. To establish our findings, we need to make some assumptions which are listed separately below.

ASSUMPTION 5.1. *Ω is an open and connected domain in $\mathbb{R}^d$.*

ASSUMPTION 5.2. *Ω is bounded and satisfies the interior cone condition with parameters $r_0 > 0$ and $\theta_0 > 0$, as defined by the property that (see [1] for details) if for any point $\mathbf{x} \in \Omega$, the intersection between the ball centered at $\mathbf{x}$ with radius r_0 and the domain Ω contains a cone with an angle no smaller than θ_0.*

ASSUMPTION 5.3. *$\eta(\mathbf{x}) \geq \eta_0 > 0$ and $k(\mathbf{x}) \geq k_0 > 0$ for any $\mathbf{x} \in \Omega$, and $\underline{\omega}(\mathbf{x}, \mathbf{y})$ is non-degenerate and is nonnegative in $\Omega \times \Omega$, where the non-degeneracy condition is specified by the property that any function $\beta(\mathbf{x}, \mathbf{y})$ is said to be* non-degenerate *if there exist two positive constants $\delta_0 > 0$ and $\beta_0 > 0$ such that*

$$\beta(\mathbf{x}, \mathbf{y}) \geq \beta_0 > 0 \quad \forall \mathbf{x}, \mathbf{y} \in \Omega \text{ satisfying } \|\mathbf{x} - \mathbf{y}\| \leq \delta_0.$$

ASSUMPTION 5.4. *$k(\mathbf{x}) \leq k_1 < \infty$, $\eta(\mathbf{x}) \leq \eta_1 < \infty$ and $\underline{\omega} = \underline{\omega}(\mathbf{x}, \mathbf{y})$ is symmetric, i.e. $\underline{\omega}(\mathbf{x}, \mathbf{y}) = \underline{\omega}(\mathbf{y}, \mathbf{x})$, and for some constant $M > 0$,*

$$\int_\Omega \underline{\omega}^2(\mathbf{x}, \mathbf{y}) \, d\mathbf{y} \leq M \, , \quad \forall \mathbf{x} \in \Omega \, .$$

ASSUMPTION 5.5. *$\gamma = \gamma(\mathbf{x}, \mathbf{y})$ is non-degenerate and is nonnegative in $\Omega \times \Omega$.*

ASSUMPTION 5.6. *$\gamma = \gamma(\mathbf{x}, \mathbf{y})$ is symmetric, i.e. $\gamma(\mathbf{x}, \mathbf{y}) = \gamma(\mathbf{y}, \mathbf{x})$, and for some constant $M_\gamma > 0$,*

$$\int_\Omega \gamma^2(\mathbf{x}, \mathbf{y}) \, d\mathbf{y} \leq M_\gamma \, , \quad \forall \mathbf{x} \in \Omega \, .$$

REMARK 5.1. *This non-degeneracy condition in Assumption 5.3 is automatically satisfied if the function under consideration is bounded below uniformly by a positive constant over the domain Ω. In the special case for which $\beta(\mathbf{x}, \mathbf{y}) = \widetilde{\beta}(\mathbf{x} - \mathbf{y})$, we also have that β is non-degenerate if the support of $\widetilde{\beta}$ contains a small ball centered at the origin which is usually the case for peridynamic models* [11, 12, 13]. *In such a case, δ_0 is a parameter that is strictly smaller than the peridynamic horizon parameter.*

REMARK 5.2. *In the following discussions, the above assumptions are used in different places, but they all serve to establish the well-posedness of the PD models.*

For instance, we use the Assumption 5.1 to help us clarify, in Lemma 5.1, that the kernel of the operator from $\mathbf{u}$ *to* $\big(\mathbf{u}(\mathbf{y})-\mathbf{u}(\mathbf{x})\big)\cdot(\mathbf{y}-\mathbf{x})$ *is the rigid body motion. In addition, Assumptions 5.1 and 5.3 are used to prove the general well-poseness of the nonlocal variational problems associated with the PD-state models (as documented in Lemma 5.2 and Theorems 5.1 and 5.2), provided that the related energy space is a well-defined Hilbert space. The latter result on the energy space is established in the section 5.4 for the PD-state energy under the additional Assumptions 5.2 and 5.4, see Lemmas 5.3, 5.4, 5.5, 5.6 and Theorems 5.3, 5.4. The Assumption 5.5, 5.6 are used as alternatives of the Assumption 5.3 and 5.4 respectively when the bond-based PD models are concerned.*

Remark 5.3. *We also note that it is possible to relax the non-negativity assumptions on* $\eta(\mathbf{x})$, $k(\mathbf{x})$, $\underline{\omega}(\mathbf{x},\mathbf{y})$ *or* $\gamma(\mathbf{x},\mathbf{y})$ *to consider more general cases that allow for sign changes. Such generalizations are left to future studies.*

Setting $\mathbf{v}=\mathbf{u}$ in (5.3), we obtain

$$\begin{aligned} B(\mathbf{u},\mathbf{u}) &= \int_\Omega\int_\Omega \eta(\mathbf{x})\underline{\omega}(\mathbf{x},\mathbf{y})\mathrm{Tr}(\mathcal{D}_t^*\mathbf{u})\mathrm{Tr}(\mathcal{D}_t^*\mathbf{u})\ d\mathbf{y}\,d\mathbf{x} \\ &\quad + \int_\Omega \big(k(\mathbf{x})-n(\mathbf{x})\eta(\mathbf{x})/d\big)\mathrm{Tr}(\mathcal{D}_{t,\omega}^*\mathbf{u})\mathrm{Tr}(\mathcal{D}_{t,\omega}^*\mathbf{u})\ d\mathbf{x} \\ &= \int_\Omega \frac{k(\mathbf{x})\,\widehat{\theta}^2}{2}\ d\mathbf{x} + \int_\Omega\int_\Omega \frac{\eta(\mathbf{x})}{2}\underline{\omega}(\mathbf{x},\mathbf{y})(\underline{e}^d)^2\ d\mathbf{y}\,d\mathbf{x}, \end{aligned} \tag{5.5}$$

where we have used the definition (3.4c) and the derivation given in (4.1).

Note that $B(\mathbf{u},\mathbf{u})\geq 0$ for any $\mathbf{u}\in U(\Omega)$, we thus formally define an inner product and its associated norm by

$$((\mathbf{u},\mathbf{v})) = B(\mathbf{u},\mathbf{v}) \qquad \text{and} \qquad |||\mathbf{u}||| = (B(\mathbf{u},\mathbf{u}))^{1/2},$$

respectively. One readily sees that the function space $U(\Omega)$ can be expressed as

$$U(\Omega) = \{\mathbf{u}\colon |||\mathbf{u}||| < \infty\}.$$

We define the space $Z(\Omega)$ as

$$Z(\Omega) = \{\mathbf{u}\colon |||\mathbf{u}||| = 0\}.$$

To show that $((\cdot,\cdot))$ and $|||\cdot|||$ actually define an inner product and norm, respectively, in a suitable subspace, we need the following lemma that is analogous to a similar result stated for $\mathbb{R}^d$ in, for instance, in [14, Prop. 1.2]. The proof requires an argument for the nonlocal case which is substantially different from that given in [14] for the local counterpart.

Lemma 5.1. *Assume the domain* Ω *satisfies Assumption 5.1 and* $\mathbf{u}\in L^2(\Omega)$. *If for a.e.* $\mathbf{x}\in\Omega$,

$$\big(\mathbf{u}(\mathbf{y})-\mathbf{u}(\mathbf{x})\big)\cdot(\mathbf{y}-\mathbf{x}) = 0 \quad \text{for a.e. } \mathbf{y}\in\Omega\cap B_{\delta_0}(\mathbf{x}), \tag{5.6}$$

then $\mathbf{u}$ *is given by a rigid body motion in* Ω, *that is, there exists a constant-valued skew-symmetric matrix* A *and a constant-valued vector* $\mathbf{c}$ *such that*

$$\mathbf{u}(\mathbf{x}) = \mathbf{A}\mathbf{x}+\mathbf{c} \quad \text{a.e. } \mathbf{x}\in\Omega. \tag{5.7}$$

Proof. For any $\mathbf{x}_0 \in \Omega$, we may choose $\delta(\mathbf{x}_0) \in (0, \delta_0/2)$ such that

$$\cup_{\mathbf{y} \in B_{\delta(\mathbf{x}_0)}(\mathbf{x}_0)} B_{\delta(\mathbf{x}_0)}(\mathbf{y}) \subset \Omega \cap B_{\delta_0}(\mathbf{x}_0).$$

Let $\{\tilde{e}_i\}_{i=1,\cdots,d}$ be an orthonormal basis in $\mathbb{R}^d$. By the assumption of the lemma, we may find for some $\mathbf{x}_0 \in \Omega$ at which (5.6) holds for $\mathbf{y} \in \Omega \cap B_{\delta_0}(\mathbf{x}_0)$ except a measure zero set. Let $B_{\delta(\mathbf{x}_0)/N}(\mathbf{x}_0 + \delta(\mathbf{x}_0)\tilde{e}_i)$ be small balls centering at $\mathbf{x}_0 + \delta(\mathbf{x}_0)\tilde{e}_i$ with radii $\delta(\mathbf{x}_0)/N$ where the positive constant N is chosen such that the balls are small enough and do not intersect.

Since (5.6) holds for $\mathbf{x} \in \Omega$ almost everywhere, we may choose d points $\{\mathbf{x}_{0i}\}_{i=1}^d$ such that, $\mathbf{x}_{0i} \in B_{\delta(\mathbf{x}_0)/N}(\mathbf{x}_0 + \delta(\mathbf{x}_0)\tilde{e}_i)$ for $1 \leq i \leq d$. By the definitions of $B_{\delta(\mathbf{x}_0)/N}(\mathbf{x}_0 + \delta(\mathbf{x}_0)\tilde{e}_i)$, one readily sees that $\{\mathbf{x}_{0i} - \mathbf{x}_0\}_{i=1}^d$ form a basis. For notation simplicity, we set $\mathbf{e}_i = (\mathbf{x}_{0i} - \mathbf{x}_0)/\|\mathbf{x}_{0i} - \mathbf{x}_0\|$.

Then we have for $\mathbf{x}_0$ and some zero measure set $\mathfrak{N}_{\mathbf{x}_0}$,

$$\big(\mathbf{u}(\mathbf{y}) - \mathbf{u}(\mathbf{x}_0)\big) \cdot (\mathbf{y} - \mathbf{x}_0) = 0 \quad \forall\, \mathbf{y} \in \Omega_{\mathbf{x}_0}$$

where $\Omega_{\mathbf{x}_0} = \Omega \cap B_{\delta_0}(\mathbf{x}_0) \setminus \mathfrak{N}_{\mathbf{x}_0}$.

Thus, for any $\mathbf{x} \in B_{\delta(\mathbf{x}_0)}(\mathbf{x}_0) \cap \Omega_{\mathbf{x}_0}$, we have

$$\big(\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x}_0)\big) \cdot (\mathbf{x} - \mathbf{x}_0) = 0.$$

Moreover,

$$\big(\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x}_0 + \delta(\mathbf{x}_0)\mathbf{e}_i)\big) \cdot \big(\mathbf{x} - \mathbf{x}_0 - \delta(\mathbf{x}_0)\mathbf{e}_i\big) = 0.$$

This implies that $\mathbf{u}(\mathbf{x}) \cdot \mathbf{e}_i$ is linear in $\mathbf{x}$ for any i, which gives the linearity of $\mathbf{u}$ in $B_{\delta_0}(\mathbf{x}_0)$. Then, (5.6) implies that $\mathbf{u}$ has to take on the form given by (5.7).

Because Ω is a bounded connected open set, for any two points $\mathbf{x}_0$ and $\mathbf{x}_1$ in Ω, there exists a finite number of balls $\{B_{\delta(\mathbf{y}_k)}(\mathbf{y}_k)\}_{k=1}^K$ with sufficiently small radii $\delta(\mathbf{y}_k) > 0$ with the properties that their union is completely contained in Ω, that covers a connected path between $\mathbf{x}_0$ and $\mathbf{x}_1$, and $B_{\delta(\mathbf{y}_k)}(\mathbf{y}_k) \cap B_{\delta(\mathbf{y}_{k+1})}(\mathbf{y}_{k+1})$ has a non-empty interior. Then, we see that in each ball, $\mathbf{u}$ is given by a rigid body motion and their form must be the same in neighboring balls and thus it is a global rigid body in the whole domain Ω. ◻

We then have the following result.

LEMMA 5.2. *If the Assumptions 5.1, 5.3 and 5.5 hold,* $|||\mathbf{u}|||$ *and* $((\mathbf{u}, \mathbf{v}))$ *define a norm and an inner product, respectively, on both* $U_0(\Omega)$*, provided* Ω_c *has a non-empty interior, and* $U(\Omega) \setminus Z(\Omega)$.

Proof. We establish the result for the state-based models first. We note that $|||\mathbf{u}|||$ defines a semi-norm on $U_0(\Omega)$. Thus, it suffices to prove that $B(\mathbf{u}, \mathbf{u}) = 0$ implies $\mathbf{u} = \mathbf{0}$. Because the Assumption 5.3 holds, from (5.5), we have

$$\int_\Omega \frac{k(\mathbf{x})\,\widehat{\theta}^2}{2}\, d\mathbf{x} = 0 \quad \text{and} \quad \int_\Omega \int_\Omega \frac{\eta(\mathbf{x})}{2} \underline{\omega}(\mathbf{x}, \mathbf{y}) (\underline{e}^d)^2\, d\mathbf{y}\, d\mathbf{x} = 0. \tag{5.8}$$

The first equation in (5.8) implies that $\mathrm{Tr}(\mathcal{D}^*_{t,\omega}\mathbf{u}) = 0$ by the first equation in (5.8) and thus, by the second equation in (5.8), $\mathrm{Tr}(\mathcal{D}^*_t\mathbf{u}) = \big(\mathbf{u}(\mathbf{y}) - \mathbf{u}(\mathbf{x})\big) \cdot (\mathbf{y} - \mathbf{x}) = 0$ in the support of $\omega(\mathbf{x}, \mathbf{y})$. Given the non-degeneracy condition given on ω, from Assumption 5.1 and Lemma 5.1 we can deduce that the kernel of the operator $\mathrm{Tr}(\mathcal{D}^*_t)$ is the set of rigid body motions $\mathbf{u}(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{c}$ for a constant vector $\mathbf{c}$ and constant skew-symmetric matrix $\mathbf{A}$. So, in either $U_0(\Omega)$ with Ω_c having a non-empty interior or in $U(\Omega) \setminus Z(\Omega)$, the only $\mathbf{u}$ satisfying (5.8) is $\mathbf{u} \equiv \mathbf{0}$. Thus we conclude that $|||\cdot|||$ defines a norm and $((\cdot,\cdot))$ defines an inner product on $U_0(\Omega)$ and $U(\Omega) \setminus Z(\Omega)$.

For the bond-based model, we can apply a similar argument and the Assumption 5.5 to obtain that, when $B(\mathbf{u},\mathbf{u}) = 0$ we have $\mathbf{u} = 0$ in either $U_0(\Omega)$ or $U(\Omega) \setminus Z(\Omega)$, as long as the non-degeneracy condition of γ is satisfied. ∎

5.1. Decomposition of the solution space. Let the space $S(\Omega)$ consist of functions $\mathbf{u} \in U(\Omega)$ that satisfy

$$\mathcal{D}_t\big(\psi_1\big(\mathcal{D}_t^*(\mathbf{u})\big)^T\big) + \mathcal{D}_{t,w}\big(\psi_2 \mathrm{Tr}\big(\mathcal{D}_{t,\omega}^*(\mathbf{u})\big)\mathrm{I}\big) = 0 \qquad \forall \mathbf{x} \in \Omega_s. \tag{5.9}$$

Then, for all $\mathbf{u} \in S(\Omega)$ and $\mathbf{v} \in U_0(\Omega)$ we have

$$B(\mathbf{u},\mathbf{v}) = 0.$$

Thus, we conclude that

$$U(\Omega) = U_0(\Omega) \oplus S(\Omega), \tag{5.10}$$

that is, any function in $U(\Omega)$ can be written as a direct sum of two functions that are orthogonal with respect to the inner product $((\cdot,\cdot))$, the first a function that vanishes on Ω_c whereas the second a function satisfying (5.9).

5.2. Nonlocal dual spaces and nonlocal trace spaces. Define the dual norm by

$$|||\mathbf{b}|||_o^* = \sup_{\mathbf{v}\in U_0(\Omega),\,\mathbf{v}\neq\mathbf{0}} \frac{\displaystyle\int_{\Omega_s} \mathbf{v}\cdot\mathbf{b}\,d\mathbf{x}}{|||\mathbf{v}|||}$$

and define the dual space of $U_0(\Omega)$ as

$$U_0^*(\Omega) = \{\mathbf{b} \;:\; |||\mathbf{b}|||_o^* < \infty\}.$$

Similarly, we can define the dual space of $U(\Omega)$ by

$$U^*(\Omega) = \{\mathbf{b} \;:\; |||\mathbf{b}|||^* < \infty\},$$

where

$$|||\mathbf{b}|||^* = \sup_{\mathbf{v}\in U(\Omega),\,\mathbf{v}\neq\mathbf{0}} \frac{\displaystyle\int_{\Omega} \mathbf{v}\cdot\mathbf{b}\,d\mathbf{x}}{|||\mathbf{v}|||}.$$

As for the nonlocal *trace* space, let $\mathbf{h}_d \colon \Omega_c \to \mathbb{R}^d$ denote a mapping; then the trace space is defined as

$$U_d = \{\mathbf{h}_d \;:\; |||\mathbf{h}_d|||_d < \infty\},$$

where

$$|||\mathbf{h}_d|||_d = \inf_{\mathbf{v}|_{\Omega_c}=\mathbf{h}_d} |||\mathbf{v}|||.$$

5.3. Well-posedness of variational problems. The variational problems discussed above can take on quite general forms with respect to the different constraints imposed. For example, a "*Dirichlet*" volume-constraint problem is formulated as

$$\begin{cases} \textit{given } \mathbf{b} \in U_0^* \textit{ and } \mathbf{h}_d \in U_d, \textit{ seek } \mathbf{u} \in U_0(\Omega) \textit{ such that} \\ \qquad B(\mathbf{u}, \mathbf{v}) = F_d(\mathbf{v}) \qquad \forall\, \mathbf{v} \in U_0(\Omega) \\ \text{and} \quad \mathbf{u}(\mathbf{x}) = \mathbf{h}_d(\mathbf{x}) \end{cases} \tag{5.11}$$

whereas the form of a "*Neumann*" volume-constraint problem is given by

$$\begin{cases} \textit{given } \mathbf{b} \in U^*, \textit{ seek } \mathbf{u} \in U(\Omega) \setminus Z(\Omega) \textit{ such that} \\ \qquad B(\mathbf{u}, \mathbf{v}) = F_n(\mathbf{v}) \qquad \forall\, \mathbf{v} \in U(\Omega) \setminus Z(\Omega), \end{cases} \tag{5.12}$$

where the linear functionals $F_d(\cdot)$ and $F_n(\cdot)$ are defined by

$$F_d(\mathbf{v}) = \int_\Omega \mathbf{v} \cdot \mathbf{b}\, d\mathbf{x} \qquad \forall\, \mathbf{v} \in U_0(\Omega)$$

and

$$F_n(\mathbf{v}) = \int_\Omega \mathbf{v} \cdot \mathbf{b}\, d\mathbf{x} \qquad \forall\, \mathbf{v} \in U(\Omega) \setminus Z(\Omega),$$

respectively.

We note that in the "Neumann" volume-constrained problem, the quotient space $U(\Omega) \setminus Z(\Omega)$ is considered which is to guarantee the uniqueness of the solution to the problem. Equivalently, we can instead pose a condition on the constraint domain Ω_c such as

$$\int_{\Omega_c} \mathbf{u}(\mathbf{x})\, d\mathbf{x} = 0.$$

Because $B(\cdot,\cdot)$ defines an inner product on both $U_0(\Omega)$ and $U(\Omega) \setminus Z(\Omega)$, it is a continuous and coercive bilinear form on those spaces. If we assume that $U(\Omega)$ is a complete space and the data are such that the functionals $F_d(\cdot)$ and $F_n(\cdot)$ are continuous, the Lax-Milgram theorem can be applied to show that both (5.11) and (5.12) have unique solutions and, moreover, those solutions satisfy

$$|||\mathbf{u}||| \le |||\mathbf{b}|||_o^* + C|||\mathbf{h}_d|||_d \qquad \text{and} \qquad |||\mathbf{u}||| \le |||\mathbf{b}|||^*,$$

respectively. Thus, for the "Dirichlet" and "Neumann" volume-constraint problems, we have the following pair of results where the completeness of the space $U(\Omega)$ is demonstrated in the next section.

THEOREM 5.1. *If Ω satisfies Assumption 5.1 and $U_0(\Omega)$ is a Hilbert space, $\mathbf{b} \in U_0^*(\Omega)$, and $\mathbf{h}_d \in U_d(\Omega)$, the Dirichlet volume-constraint problem* (5.11) *for the linear state-based peridynamic material* (*see* (4.4) *and* (4.5)) *has a unique solution $\mathbf{u} \in U_0(\Omega)$ provided that the Assumption 5.3 is satisfied. Similarly, the nonlocal Dirichlet problem* (5.11) *for the linear bond-based peridynamic material* (*see* (4.9) *and* (4.10)) *has a unique solution $\mathbf{u} \in U_0(\Omega)$ provided that $\gamma = \gamma(\mathbf{x}, \mathbf{y})$ satisfies the Assumption 5.5.*

THEOREM 5.2. *If Ω satisfies Assumption 5.1 and $U(\Omega)/Z(\Omega)$ is a Hilbert space and $\mathbf{b} \in U^*(\Omega)$, the Neumann volume-constraint problem* (5.12) *for the linear state-based peridynamic material* (*see* (4.4) *and* (4.5)) *has a unique solution $\mathbf{u} \in U(\Omega)/Z(\Omega)$*

provided that the Assumption 5.3 is satisfied. Similarly, the nonlocal Neumann problem (5.12) *for the linear bond-based peridynamic material* (*see* (4.9) *and* (4.10)) *has a unique solution* $\mathbf{u} \in U(\Omega)/Z(\Omega)$ *provided that* $\gamma = \gamma(\mathbf{x}, \mathbf{y})$ *satisfies the Assumption 5.5.*

REMARK 5.4. *The above theorems represent the first available results on the rigorous well-posedness of the linear bond-based and state-based peridynamic elasticity models with volume-constraints, though it remains to verify the energy space is indeed a Hilbert space. In the next subsection, we provide an illustrative example as a demonstration.*

5.4. An example of the peridynamic energy space. A technique for proving that $U(\Omega)$ is a Hilbert space is to establish, under appropriate conditions on the influence function $\underline{\omega}(\mathbf{x}, \mathbf{y})$, that $U(\Omega)$ is in fact equivalent to a well-known Sobolev space. In the classical (local) elasticity theory, this is established using some useful facts such as the Korn inequality that generally relies on a compactness argument. In [15], the equivalence is established for special volume-constraints that makes the peridynamic operators commute with the differential operators and allows for a precise characterization of the energy spaces in terms of the eigenspaces of the associated differential operators. While that technique is special, the results/conclusions are expected to remain valid in general. Here, we present another illustrative example showing the equivalence of spaces via a nonlocal Korn inequality combined with compactness results associated with Hilbert-Schmidt operators.

In this section, for much of the discussions here, we consider in details a state-based peridynamic material which is possibly inhomogeneous and anisotropic. In addition to the Assumptions 5.1 and 5.3, we need to further use the Assumptions 5.2 and 5.4. For completeness, we also briefly state the analogous results for the bond-based model which can be similarly derived if the Assumptions 5.3 and 5.4 are replaced by the Assumptions 5.5 and 5.6.

We now define the tensor field

$$P(\mathbf{x}) = \int_{\Omega} \big(K_1(\mathbf{x}, \mathbf{y}) + C_0(\mathbf{x}, \mathbf{y})\big)\; d\mathbf{y} \tag{5.13}$$

where K_1 and C_0 are defined in (4.8a) and (4.8b). Under the interior cone condition given in the Assumption 5.2 on the domain, for any $\mathbf{x} \in \bar{\Omega}$, we may assume that a cone $B_{r_0,\theta_0}(\mathbf{x})$, centered at $\mathbf{x}$ with radius r_0 and angle θ_0 is contained in the domain Ω. Let us define

$$\pi(\mathbf{x}) = \frac{1}{d} \int_{B_{r_0,\theta_0}(\mathbf{x})} |\mathbf{y} - \mathbf{x}|^2 \underline{\omega}(\mathbf{x}, \mathbf{y})\, d\mathbf{y}\ .$$

We now show that, under the Assumptions 5.1-5.4, the energy space $U(\Omega)$ is in fact equivalent to $L^2(\Omega)$. First, we note that $n(\mathbf{x})$ and $c(\mathbf{x})$, defined in (3.2) and (4.6) respectively, have the following upper and lower bounds.

LEMMA 5.3. *If the domain and coefficients in the peridynamic model* (4.4) *satisfy Assumptions 5.1-5.4, we have that*

$$0 < \pi_0(r_0, \theta_0) \le n(\mathbf{x}) \le \pi_1(\Omega) = |\mathrm{diam}(\Omega)|^2 |\Omega|^{1/2} M^{1/2}/d\,, \tag{5.14a}$$

$$|c(\mathbf{x})| \le (k_1 + \eta_1 \pi_1(\Omega)/d)/\pi_0(r_0, \theta_0)\ , \tag{5.14b}$$

where $\pi_0(r_0,\theta_0) = \min_{\mathbf{x}\in\overline{\Omega}} \pi(\mathbf{x})$.

Proof. We first prove the inequality for $n(\mathbf{x})$.

$$\begin{aligned} n(\mathbf{x}) &= \frac{1}{d}\int_\Omega |\mathbf{y}-\mathbf{x}|^2 \underline{\omega}(\mathbf{x},\mathbf{y})\,d\mathbf{y} \\ &\le \frac{1}{d}\Big(\int_\Omega |\mathbf{y}-\mathbf{x}|^4\,d\mathbf{y}\Big)^{1/2}\Big(\int_\Omega \underline{\omega}^2(\mathbf{x},\mathbf{y})\,d\mathbf{y}\Big)^{1/2} \\ &\le \frac{1}{d}|\mathrm{diam}(\Omega)|^2|\Omega|^{1/2}M^{1/2} \\ &= \pi_1(\Omega). \end{aligned}$$

By the interior cone condition of the domain Ω, we have $n(\mathbf{x}) \ge \pi(\mathbf{x})$. On the other hand, since $|\mathbf{y}-\mathbf{x}|^2\underline{\omega}(\mathbf{x},\mathbf{y}) > 0$ on any nonzero measure set in $B_{r_0,\theta_0}(\mathbf{x})$ by the non-degeneracy condition, we have $\pi(\mathbf{x}) > 0$ for any $\mathbf{x}$ in $\bar{\Omega}$. Given the integrability condition of $\underline{\omega}(\mathbf{x},\mathbf{y})$, we have that $\pi(\mathbf{x})$ is continuous in $\bar{\Omega}$, thus we have

$$n(\mathbf{x}) \ge \min_{\mathbf{x}\in\overline{\Omega}} \pi(\mathbf{x}) = \pi_0(r_0,\theta_0) > 0.$$

Then one readily sees that

$$|c(\mathbf{x})| \le |k(\mathbf{x}) - \eta(\mathbf{x})n(\mathbf{x})/d|/n^2(\mathbf{x}) < (k_1 + \eta_1\pi_1(\Omega)/d)/\pi_0^2(r_0,\theta_0).$$

This proves the lemma. ∎

We next show that the tensor field $P(\mathbf{x})$ is uniformly positive definite.

Lemma 5.4. *If the domain and coefficients in the peridynamic model* (4.4) *satisfy Assumptions 5.1-5.4, we have $P(\mathbf{x})$ is uniformly positive definite, i.e. $P(\mathbf{x}) \ge P_0 > 0$, where*

$$P_0 = \eta_0 \min_{\mathbf{x}\in\overline{\Omega}} \int_{B_{r_0,\theta_0}(\mathbf{x})} \frac{\underline{\omega}(\mathbf{x},\mathbf{y})}{|\mathbf{y}-\mathbf{x}|^2}(\mathbf{y}-\mathbf{x})\otimes(\mathbf{y}-\mathbf{x})\,d\mathbf{y}, \tag{5.15}$$

and $B_{r_0,\theta_0}(\mathbf{x})$ represents the intersection between a ball with the radius r_0 centered at $\mathbf{x}$ and a cone with vertex $\mathbf{x}$ and angle θ_0.

Proof. First of all, for any unit vector $\mathbf{u}\in\mathbb{R}^d$, we may use a similar argument as in the proof of the Lemma 5.3 to get that

$$\mathbf{u}\cdot\left[\int_{B_{r_0,\theta_0}(\mathbf{x})} \frac{\underline{\omega}(\mathbf{x},\mathbf{y})}{|\mathbf{y}-\mathbf{x}|^2}(\mathbf{y}-\mathbf{x})\otimes(\mathbf{y}-\mathbf{x})\,d\mathbf{y}\right]\mathbf{u} = \int_{B_{r_0,\theta_0}(\mathbf{x})} \frac{\underline{\omega}(\mathbf{x},\mathbf{y})}{|\mathbf{y}-\mathbf{x}|^2}[\mathbf{u}(\mathbf{y}-\mathbf{x})]^2\,d\mathbf{y}$$

is uniformly bounded below by a positive constant, independent of $\mathbf{x}\in\Omega$. Taking the minimum over all $\mathbf{u}$ in the unit sphere in $\mathbb{R}^d$, we see that P_0 is a positive definite matrix.

From the definitions (4.8b) and (4.6), we get

$$\begin{aligned} \int_\Omega C_0(\mathbf{x},\mathbf{y})\,d\mathbf{y} = \int_\Omega\int_\Omega \Big(& c(\mathbf{z})\underline{\omega}(\mathbf{x},\mathbf{z})\underline{\omega}(\mathbf{z},\mathbf{y})(\mathbf{x}-\mathbf{z})\otimes(\mathbf{z}-\mathbf{y}) \\ & - c(\mathbf{y})\underline{\omega}(\mathbf{x},\mathbf{y})\underline{\omega}(\mathbf{y},\mathbf{z})(\mathbf{x}-\mathbf{y})\otimes(\mathbf{y}-\mathbf{z}) \\ & + c(\mathbf{x})\underline{\omega}(\mathbf{x},\mathbf{z})\underline{\omega}(\mathbf{x},\mathbf{y})(\mathbf{x}-\mathbf{z})\otimes(\mathbf{x}-\mathbf{y})\Big)\,d\mathbf{z}\,d\mathbf{y}. \end{aligned}$$

By changing the order of integrals, we obtain

$$\begin{aligned}\int_\Omega C_0(\mathbf{x},\mathbf{y})\;d\mathbf{y} &= \int_\Omega\int_\Omega \Big(c(\mathbf{z})\underline{\omega}(\mathbf{x},\mathbf{z})\underline{\omega}(\mathbf{z},\mathbf{y})(\mathbf{x}-\mathbf{z})\otimes(\mathbf{z}-\mathbf{y}) \\ &\quad - c(\mathbf{z})\underline{\omega}(\mathbf{x},\mathbf{z})\underline{\omega}(\mathbf{z},\mathbf{y})(\mathbf{x}-\mathbf{z})\otimes(\mathbf{z}-\mathbf{y}) \\ &\quad + c(\mathbf{x})\underline{\omega}(\mathbf{x},\mathbf{z})\underline{\omega}(\mathbf{x},\mathbf{y})(\mathbf{x}-\mathbf{z})\otimes(\mathbf{x}-\mathbf{y})\Big)\;d\mathbf{z}\,d\mathbf{y} \\ &= \int_\Omega\int_\Omega c(\mathbf{x})\underline{\omega}(\mathbf{x},\mathbf{z})\underline{\omega}(\mathbf{x},\mathbf{y})(\mathbf{x}-\mathbf{z})\otimes(\mathbf{x}-\mathbf{y})\;d\mathbf{z}\,d\mathbf{y} \\ &= c(\mathbf{x})\Big(\int_\Omega \underline{\omega}(\mathbf{x},\mathbf{y})(\mathbf{x}-\mathbf{y})\;d\mathbf{y}\Big)\otimes\Big(\int_\Omega \underline{\omega}(\mathbf{x},\mathbf{y})(\mathbf{x}-\mathbf{y})\;d\mathbf{y}\Big) \\ &= \frac{k(\mathbf{x})}{n^2(\mathbf{x})}\Big(\int_\Omega \underline{\omega}(\mathbf{x},\mathbf{y})(\mathbf{x}-\mathbf{y})\;d\mathbf{y}\Big)\otimes\Big(\int_\Omega \underline{\omega}(\mathbf{x},\mathbf{y})(\mathbf{x}-\mathbf{y})\;d\mathbf{y}\Big) \\ &\quad - \frac{\eta(\mathbf{x})}{dn(\mathbf{x})}\Big(\int_\Omega \underline{\omega}(\mathbf{x},\mathbf{y})(\mathbf{x}-\mathbf{y})\;d\mathbf{y}\Big)\otimes\Big(\int_\Omega \underline{\omega}(\mathbf{x},\mathbf{y})(\mathbf{x}-\mathbf{y})\;d\mathbf{y}\Big) \\ &= A_1 - A_2.\end{aligned}$$

Since $k(\mathbf{x}) > k_0 > 0$ and $n(\mathbf{x}) \geq \pi_0(r_0,\theta_0) > 0$, we conclude that $A_1 \geq 0$. By the definition of K_1 in (4.8a), we get

$$\begin{aligned}\int_\Omega K_1(\mathbf{x},\mathbf{y}) &= 2\eta(\mathbf{x})\int_\Omega \frac{\underline{\omega}(\mathbf{x},\mathbf{y})}{|\mathbf{y}-\mathbf{x}|^2}(\mathbf{y}-\mathbf{x})\otimes(\mathbf{y}-\mathbf{x})\,d\mathbf{y} \\ &= 2A_3\end{aligned}$$

First, because of the interior cone condition of the domain Ω, we have

$$\begin{aligned}A_3 &= \eta(\mathbf{x})\int_\Omega \frac{\underline{\omega}(\mathbf{x},\mathbf{y})}{|\mathbf{y}-\mathbf{x}|^2}(\mathbf{y}-\mathbf{x})\otimes(\mathbf{y}-\mathbf{x})\,d\mathbf{y} \\ &\geq P_0\;.\end{aligned}$$

We can rewrite $P(\mathbf{x})$ as $P(\mathbf{x}) = 2A_3 + A_1 - A_2 = A_3 + A_1 + (A_3 - A_2)$. Since $A_3 \geq P_0 > 0$ and $A_1 \geq 0$, we only need to show that $A_3 - A_2$ is semi-positive definite, i.e. $A_3 - A_2 \geq 0$, which is equivalent to show that $d\,n(\mathbf{x})(A_3 - A_2) \geq 0$. Let $\mathbf{u}$ is an arbitrary nonzero vector. By the definition of $n(\mathbf{x})$, we have

$$\begin{aligned}&\mathbf{u}\cdot d\,n(\mathbf{x})(A_3 - A_2)\mathbf{u} \\ &\quad = \mathbf{u}\cdot\Big(\eta(\mathbf{x})\big(\int_\Omega \underline{\omega}(\mathbf{x},\mathbf{y})|\mathbf{x}-\mathbf{y}|^2\,d\mathbf{y}\int_\Omega \frac{\underline{\omega}(\mathbf{x},\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^2}(\mathbf{y}-\mathbf{x})\otimes(\mathbf{y}-\mathbf{x})\,d\mathbf{y}\big) \\ &\qquad - \eta(\mathbf{x})\big(\int_\Omega \underline{\omega}(\mathbf{x},\mathbf{y})(\mathbf{x}-\mathbf{y})\,d\mathbf{y}\big)\otimes\big(\int_\Omega \underline{\omega}(\mathbf{x},\mathbf{y})(\mathbf{x}-\mathbf{y})\,d\mathbf{y}\big)\Big)\mathbf{u} \\ &\quad = \eta(\mathbf{x})\Big(\int_\Omega \underline{\omega}(\mathbf{x},\mathbf{y})|\mathbf{x}-\mathbf{y}|^2\,d\mathbf{y}\int_\Omega \frac{\underline{\omega}(\mathbf{x},\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^2}\big(\mathbf{u}(\mathbf{y}-\mathbf{x})\big)^2\,d\mathbf{y} \\ &\qquad - \big(\int_\Omega \underline{\omega}(\mathbf{x},\mathbf{y})\big(\mathbf{u}(\mathbf{y}-\mathbf{x})\big)\,d\mathbf{y}\big)^2\Big)\end{aligned}$$

where d is space dimension.

By Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\Big(\int_\Omega \underline{\omega}(\mathbf{x},\mathbf{y})\big(\mathbf{u}(\mathbf{y}-\mathbf{x})\big)\,d\mathbf{y}\Big)^2\\
&\quad= \Big(\int_\Omega \underline{\omega}^{1/2}(\mathbf{x},\mathbf{y})|\mathbf{x}-\mathbf{y}|\frac{\underline{\omega}^{1/2}(\mathbf{x},\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\big(\mathbf{u}(\mathbf{y}-\mathbf{x})\big)\,d\mathbf{y}\Big)^2\\
&\quad\le \int_\Omega \underline{\omega}(\mathbf{x},\mathbf{y})|\mathbf{x}-\mathbf{y}|^2\,d\mathbf{y}\int_\Omega \frac{\underline{\omega}(\mathbf{x},\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^2}\big(\mathbf{u}(\mathbf{y}-\mathbf{x})\big)^2\,d\mathbf{y}
\end{aligned}$$

which implies $\mathbf{u}\cdot d\,n(\mathbf{x})(A_3-A_2)\mathbf{u}\ge 0$ for any nonzero $\mathbf{u}$, i.e. $d\,n(\mathbf{x})(A_3-A_2)\ge 0$. The result of the Lemma then follows. ∎

We then have the following lemma that is a nonlocal Korn inequality for the state-based model.

LEMMA 5.5. *Let the domain and coefficients in the bilinear form* (5.5) *satisfy the Assumptions 5.1-5.4. Then, there exists a constant $c_1 > 0$ such that for $\mathbf{u}$ in $U_0(\Omega)$, provided that Ω_c has a non-empty interior, we have*

$$|||\mathbf{u}||| \ge c_1\|\mathbf{u}\|_{L^2(\Omega)}. \tag{5.16}$$

An inequality similar to the above also holds for $\mathbf{u}\in U(\Omega)\setminus Z(\Omega)$.

Proof. We show $K_1(\mathbf{x},\mathbf{y})$ and $C_0(\mathbf{x},\mathbf{y})$ are square integrable which in turn implies $K_1(\mathbf{x},\mathbf{y})+C_0(\mathbf{x},\mathbf{y})$ is square integrable. First,

$$\int_\Omega\int_\Omega |K_1(\mathbf{x},\mathbf{y})|^2\ d\mathbf{y}\,d\mathbf{x} = 4\eta^2(\mathbf{x})\int_\Omega\int_\Omega \underline{\omega}^2(\mathbf{x},\mathbf{y})\ d\mathbf{y}\,d\mathbf{x} \le 4\eta_1^2|\Omega|M<\infty. \tag{5.17}$$

Similarly,

$$\begin{aligned}
\int_\Omega\int_\Omega |C_0(\mathbf{x},\mathbf{y})|^2\ d\mathbf{x}\,d\mathbf{y} \le |\Omega| \int_\Omega\int_\Omega\int_\Omega &\big|c(\mathbf{z})\underline{\omega}(\mathbf{x},\mathbf{z})\underline{\omega}(\mathbf{z},\mathbf{y})|\mathbf{x}-\mathbf{z}||\mathbf{z}-\mathbf{y}|\\
&+c(\mathbf{y})\underline{\omega}(\mathbf{x},\mathbf{y})\underline{\omega}(\mathbf{y},\mathbf{z})|\mathbf{x}-\mathbf{y}||\mathbf{y}-\mathbf{z}|\\
&+c(\mathbf{x})\underline{\omega}(\mathbf{x},\mathbf{z})\underline{\omega}(\mathbf{x},\mathbf{y})|\mathbf{x}-\mathbf{z}||\mathbf{x}-\mathbf{y}|\big|^2\ d\mathbf{z}\,d\mathbf{y}\,d\mathbf{x}\\
\le 3|\Omega|\big(\int_\Omega\int_\Omega\int_\Omega &\big|c(\mathbf{z})\underline{\omega}(\mathbf{x},\mathbf{z})\underline{\omega}(\mathbf{z},\mathbf{y})|\mathbf{x}-\mathbf{z}||\mathbf{z}-\mathbf{y}|\big|^2\ d\mathbf{z}\,d\mathbf{y}\,d\mathbf{x}\\
+\int_\Omega\int_\Omega\int_\Omega &\big|c(\mathbf{y})\underline{\omega}(\mathbf{x},\mathbf{y})\underline{\omega}(\mathbf{y},\mathbf{z})|\mathbf{x}-\mathbf{y}||\mathbf{y}-\mathbf{z}|\big|^2\ d\mathbf{z}\,d\mathbf{y}\,d\mathbf{x}\\
+\int_\Omega\int_\Omega\int_\Omega &\big|c(\mathbf{x})\underline{\omega}(\mathbf{x},\mathbf{z})\underline{\omega}(\mathbf{x},\mathbf{y})|\mathbf{x}-\mathbf{z}||\mathbf{x}-\mathbf{y}|\big|^2\ d\mathbf{z}\,d\mathbf{y}\,d\mathbf{x}\big).
\end{aligned}$$

By a change of variables, we have

$$\begin{aligned}
\int_\Omega\int_\Omega |C_0(\mathbf{x},\mathbf{y})|^2\ d\mathbf{x}\,d\mathbf{y} &\le 9|\Omega|\int_\Omega\int_\Omega\int_\Omega \big|c(\mathbf{x})\underline{\omega}(\mathbf{x},\mathbf{z})\underline{\omega}(\mathbf{x},\mathbf{y})|\mathbf{x}-\mathbf{z}||\mathbf{x}-\mathbf{y}|\big|^2\ d\mathbf{z}\,d\mathbf{y}\,d\mathbf{x}\\
&= 9|\Omega|\int_\Omega |c(\mathbf{x})|^2\big(\int_\Omega \underline{\omega}^2(\mathbf{x},\mathbf{z})|\mathbf{x}-\mathbf{z}|^2\ d\mathbf{z}\big)^2\ d\mathbf{x}\\
&\le 9|\mathrm{diam}(\Omega)|^4M^2|\Omega|\int_\Omega |c(\mathbf{x})|^2\ d\mathbf{x}
\end{aligned}$$

where $|\text{diam}(\Omega)|$ denotes the diameter of the domain Ω. The final inequality holds by the integrability condition on $\underline{\omega}(\mathbf{y}, \mathbf{x})$.

By the inequality (5.14b), we can see that

$$\int_\Omega \int_\Omega |C_0(\mathbf{x}, \mathbf{y})|^2 \; d\mathbf{x}\, d\mathbf{y} < \infty.$$

For simplicity, we separate the PD operator $\mathcal{L}$, as the following

$$-\mathcal{L}\mathbf{u} = -\int_\Omega \mathbb{C}(\mathbf{x}, \mathbf{y})\mathbf{u}(\mathbf{y})\, d\mathbf{y} + \int_\Omega \mathbb{C}(\mathbf{x}, \mathbf{y})\, d\mathbf{y}\mathbf{u}(\mathbf{x}) = A\mathbf{u} + B\mathbf{u}.$$

We can then invoke the Hilbert-Schmidt theory to conclude that the operator A has a sequence of real eigenvalues λ_i such that $|\lambda_i|$ is monotonically non-increasing and $\lambda_i \to 0$ as $i \to N$. Then 0 is the only accumulation point of the spectrum of A.

As we have verified that the PD operator $-\mathcal{L} = A + B$ is non-negative, and under the Assumptions 5.1-5.4, , the kernel of $|||\cdot|||$ only contains the zero element in $U_0(\Omega)$ or $U(\Omega) \setminus Z(\Omega)$, we have $-\mathcal{L} = A + B > 0$. Since operator $B \geq P_0 > 0$ and 0 is the only accumulation point of A, we conclude that the smallest eigenvalue of the state-based peridynamic operator under study, which becomes a self-adjoint and compact operator in $L^2(\Omega)$, is strictly positive. This gives (5.16). ◻

We can also prove the upper bound of the energy norm.

Lemma 5.6. *Under the condition of Lemma* 5.5, *we have*

$$|||\mathbf{u}||| \leq c_2 \|\mathbf{u}\|_{L^2(\Omega)}, \tag{5.18}$$

where c_2 is a constant.

Proof. We prove the two relations

$$\int_\Omega \int_\Omega \eta(\mathbf{x})\underline{\omega}(\mathbf{x}, \mathbf{y})\text{Tr}(\mathcal{D}_t^*\mathbf{u})\text{Tr}(\mathcal{D}_t^*\mathbf{u}) \; d\mathbf{y}\, d\mathbf{x} \leq c\|\mathbf{u}\|_{L^2}^2 \tag{5.19a}$$

and

$$\int_\Omega \left|k(\mathbf{x}) - n(\mathbf{x})\eta(\mathbf{x})/d\right| \text{Tr}(\mathcal{D}_{t,\omega}^*\mathbf{u})\text{Tr}(\mathcal{D}_{t,\omega}^*\mathbf{u}) \; d\mathbf{x} \leq c\|\mathbf{u}\|_{L^2}^2, \tag{5.19b}$$

wheret c is a generic constant. First,

$$\begin{aligned}
&\int_\Omega \int_\Omega \eta(\mathbf{x})\underline{\omega}(\mathbf{x}, \mathbf{y})\text{Tr}(\mathcal{D}_t^*\mathbf{u})\text{Tr}(\mathcal{D}_t^*\mathbf{u}) \; d\mathbf{y}\, d\mathbf{x} \\
&\quad \leq \eta_1 \int_\Omega \int_\Omega \underline{\omega}(\mathbf{x}, \mathbf{y})|\mathbf{u}(\mathbf{y}) - \mathbf{u}(\mathbf{x})|^2 |\boldsymbol{\alpha}(\mathbf{x}, \mathbf{y})|^2 \; d\mathbf{x}\, d\mathbf{y} \\
&\quad \leq 2\eta_1 \int_\Omega \int_\Omega \underline{\omega}(\mathbf{x}, \mathbf{y})\left(|\mathbf{u}(\mathbf{y})|^2 + |\mathbf{u}(\mathbf{x})|^2\right) \; d\mathbf{x}\, d\mathbf{y} \\
&\quad = 2\eta_1 \int_\Omega \int_\Omega \underline{\omega}(\mathbf{x}, \mathbf{y})|\mathbf{u}(\mathbf{y})|^2 \; d\mathbf{x}\, d\mathbf{y} + 2\eta_1 \int_\Omega \int_\Omega \underline{\omega}(\mathbf{x}, \mathbf{y})|\mathbf{u}(\mathbf{x})|^2 \; d\mathbf{y}\, d\mathbf{x} \\
&\quad = 2\eta_1 \int_\Omega |\mathbf{u}(\mathbf{y})|^2 \int_\Omega \underline{\omega}(\mathbf{x}, \mathbf{y}) \; d\mathbf{x}\, d\mathbf{y} + 2\eta_1 \int_\Omega |\mathbf{u}(\mathbf{x})|^2 \int_\Omega \underline{\omega}(\mathbf{x}, \mathbf{y}) \; d\mathbf{y}\, d\mathbf{x} \\
&\quad \leq 4\eta_1 |\Omega|^{1/2} M^{1/2} \|\mathbf{u}\|_{L^2(\Omega)}^2.
\end{aligned}$$

The final inequality holds because

$$\int_\Omega \underline{\omega}(\mathbf{x},\mathbf{y})\ d\mathbf{y} < |\Omega|^{1/2}\big(\int_\Omega \underline{\omega}^2(\mathbf{x},\mathbf{y})\ d\mathbf{y}\big)^{1/2} < |\Omega|^{1/2}M^{1/2}.$$

By the fact that $|k(\mathbf{x})-\eta(\mathbf{x})n(\mathbf{x})/d| \le k_1+\eta_1\pi_1(\Omega)/d$, we then have

$$\begin{aligned}
&\int_\Omega \big|k(\mathbf{x})-n(\mathbf{x})\eta(\mathbf{x})/d\big|\mathrm{Tr}(\mathcal{D}^*_{t,\omega}\mathbf{u})\mathrm{Tr}(\mathcal{D}^*_{t,\omega}\mathbf{u})\ d\mathbf{x}\\
&\quad= \int_\Omega \big|k(\mathbf{x})-n(\mathbf{x})\eta(\mathbf{x})/d\big|\big(\int_\Omega (\mathbf{u}(\mathbf{y})-\mathbf{u}(\mathbf{x}))\cdot\boldsymbol{\alpha}(\mathbf{x},\mathbf{y})\underline{\omega}(\mathbf{x},\mathbf{y})|\mathbf{y}-\mathbf{x}|\ d\mathbf{y}\big)^2\ d\mathbf{x}\\
&\quad\le \big(k_1+\eta_1\pi_1(\Omega)/d\big)\int_\Omega\left(\int_\Omega |\mathbf{u}(\mathbf{y})-\mathbf{u}(\mathbf{x})|^2\ d\mathbf{y}\int_\Omega \underline{\omega}^2(\mathbf{x},\mathbf{y})|\mathbf{y}-\mathbf{x}|^2\ d\mathbf{y}\right)d\mathbf{x}\\
&\quad\le 2\big(k_1+\eta_1\pi_1(\Omega)/d\big)\int_\Omega\left(\int_\Omega (|\mathbf{u}(\mathbf{x})|^2+|\mathbf{u}(\mathbf{y})|^2)\ d\mathbf{y}\int_\Omega \underline{\omega}^2(\mathbf{x},\mathbf{y})|\mathbf{y}-\mathbf{x}|^2\ d\mathbf{y}\right)d\mathbf{x}\\
&\quad\le 2|\mathrm{diam}(\Omega)|^2M(k_1+\eta_1\pi_1(\Omega)/d)\int_\Omega\int_\Omega\big(|\mathbf{u}(\mathbf{x})|^2+|\mathbf{u}(\mathbf{y})|^2\big)\ d\mathbf{y}\ d\mathbf{x}\\
&\quad\le 4|\Omega||\mathrm{diam}(\Omega)|^2M\big(k_1+\eta_1 d_1(\Omega)/d\big)\|\mathbf{u}\|^2_{L^2(\Omega)}.
\end{aligned}$$

Combining the two inequalities, the result of the lemma follows. □

Thus, the energy space for the homogeneous and anisotropic state-based peridynamic model is a Hilbert Space, in fact, is $L^2(\Omega)$. This in turn implies that the model equation for the state-based peridynamic material is well-posed.

As a conclusion, we have the following theorem.

THEOREM 5.3. *The Dirichlet volume-constrained problem for the state-based peridynamic model* (5.11) *is well-posed provided that the assumptions 5.1-5.4 are satisfied. Moreover, the energy space, in this case, is equivalent to* $L^2(\Omega)$.

Similar results hold for the "*Neumann*" volume-constraint problem of the state-based peridynamic model (5.12) as well as for bond-based models. We state the results and omit the derivations.

THEOREM 5.4. *If the Assumptions 5.1-5.4 are satisfied, the Neumann volume-constrained problem for the state-based peridynamic model* (5.12) *is well-posed in* $U(\Omega)\setminus Z(\Omega)$ *and the energy space is equivalent to* $L^2(\Omega)$. *Similarly, the nonlocal Dirichlet and Neumann volume-constraint problems for the bond-based peridynamic models are also well-posed in* $U_0(\Omega)$ *and* $U(\Omega)\setminus Z(\Omega)$, *respectively, and the energy spaces are equivalent to* $L^2(\Omega)$, *if the Assumptions 5.1-5.2 and 5.5-5.6 are all satisfied.*

REMARK 5.5. *In many practical applications of peridynamic models,* $\underline{\omega}=\underline{\omega}(\mathbf{x},\mathbf{y})$ *has compact support for* $\mathbf{x}-\mathbf{y}\in B_\delta(0)$ *which is a special case of the above general discussion. In this case, the integrability condition in the Assumption 5.4 becomes*

$$\int_{B_\delta(\mathbf{x})}\underline{\omega}^2(\mathbf{x},\mathbf{y})\ d\mathbf{y} < M.$$

This is only one of the sufficient conditions to assure the well-posedness. It is commonly believed that a weaker condition requiring only the L^1 *integrability of* $\underline{\omega}$ *would suffice for the well-posedness of solutions in* $L^2(\Omega)$, *as shown in* [15] *for a system of bond-based models with special constraints. Yet, the stronger conditions used in the theorems here allow the immediate application of the standard Hilbert-Schmidt theory. In addition, as studied in* [15], *for a special bond-based peridynamic system and more* *recently for scalar nonlocal equations in* [7]*, if we change the integrability condition on* $\underline{\omega}(\mathbf{x}, \mathbf{y})$*, one may also derive a more regular energy space* $U(\Omega)$*, say, that equivalent to a fractional Sobolev space.*

6. Relation between classical elasticity theory and peridynamic models. In this section, we state a rigorous result relating the linear isotropic state-based peridynamic model with the linear Navier equation of classical elasticity with a general Poisson ratio. Similar results have been presented before for linear bond-based models, demonstrating that they converge to the Navier equation with Poisson ratio one-fourth as the peridynamic horizon approaches zero [6, 15]. The discussions here generalize the earlier findings to the state-based model.

To be able to readily use the framework of the nonlocal vector calculus developed in [6], we consider the case in which the peridynamic state-based solid is homogeneous and isotropic, that is,

$$k(\mathbf{x}) = k > 0, \eta(\mathbf{x}) = \eta > 0, \tag{6.1}$$

and

$$\begin{cases} \underline{\omega}(\mathbf{x}, \mathbf{y}) = \underline{\omega}(|\mathbf{x} - \mathbf{y}|), \\ \underline{\omega}(|\mathbf{z}|) \text{ is nonnegtive function of } \mathbf{z}\text{, and has compact support } B_\delta(0). \end{cases} \tag{6.2}$$

And we also assume that the model is defined on $\mathbb{R}^d$ which implies the normalization function $n(\mathbf{x})$ is also constant due to the condition (6.2).

First, we recall some convergence results in the nonlocal vector calculus in [6, Corollary 11].

Lemma 6.1. *Let* $\mathbf{u} \in [H^1(\mathbb{R}^d)]^d$ *and* $c : \mathbb{R}^d \to \mathbb{R}$ *in* $L^\infty(\mathbb{R}^d)$*. If* $\underline{\omega}(\mathbf{x}, \mathbf{y})$ *satisfies the condition* (6.2)*, then*

$$\mathcal{D}_{t,\omega}\big(c\,\mathrm{Tr}\big(\mathcal{D}^*_{t,\omega}(\mathbf{u})\big)\mathrm{I}\big) \to -\nabla(c\nabla \cdot \mathbf{u}),$$

where the convergence as $\delta \to 0$ *is in* $H^{-1}(\mathbb{R}^d)$*.*

Following arguments similar to those in the proof of [8, Theorem 2.19], we also have the following result (a detailed proof is omitted).

Lemma 6.2. *Let* $\mathbf{u} \in H^1(\mathbb{R}^d)$ *and* $\underline{\omega}(\mathbf{x}, \mathbf{y})$ *satisfy the condition* (6.2)*. If*

$$\eta \int_{B_\delta(\mathbf{0})} |\mathbf{x}|^2 \underline{\omega}(|\mathbf{x}|)\, d\mathbf{y} \to d(d+2)\mu \qquad \text{as } \delta \to 0,$$

then, as $\delta \to 0$*,*

$$\mathcal{D}_t\big(\eta\underline{\omega}(|\mathbf{x}-\mathbf{y}|)\big(\mathcal{D}^*_t(\mathbf{u})\big)^T\big) \to -\mu\nabla\cdot(\nabla\mathbf{u}) - 2\mu\nabla(\nabla\cdot\mathbf{u}) \quad \text{in } H^{-1}(\mathbb{R}^d).$$

We then readily see that under conditions (6.1) and (6.2), the state-based peridynamic operator defined in (4.5),

$$-\mathcal{L}\mathbf{u} = \mathcal{D}_t\big(\eta\underline{\omega}(|\mathbf{x}-\mathbf{y}|)\big(\mathcal{D}^*_t(\mathbf{u})\big)^T\big) + \mathcal{D}_{t,\omega}\big((k - n\eta/d)\mathrm{Tr}\big(\mathcal{D}^*_{t,\omega}(\mathbf{u})\big)\mathrm{I}\big)$$

converges, in $H^{-1}(\mathbb{R}^d)$ in the local limit, to the Navier operator

$$\mathcal{N}\mathbf{u} = -\mu\nabla\cdot(\nabla\mathbf{u}) - (\mu+\lambda)\nabla(\nabla\cdot\mathbf{u}),$$

where

$$\mu = \lim_{\delta \to 0} \int_{B_\delta(\mathbf{0})} |\mathbf{x}|^2 \eta \omega(|\mathbf{x}|)\, d\mathbf{x}/(d(d+2)) \quad \text{and} \quad \lambda = \mu + k - n\eta/d. \tag{6.3}$$

Therefore, the state-based homogeneous and isotropic peridynamic material corresponds to the material, in the local case, having Poisson ratio

$$\nu = \frac{d\mu + dk - n\eta}{4d\mu + 2dk - 2n\eta} \tag{6.4}$$

that could be any reasonable value.

We summarize these result in the following theorem.

THEOREM 6.1. *If the conditions* (6.1) *and* (6.2) *are satisfied, for any function* $\mathbf{u} \in H^1(\mathbb{R}^d)$, *as* $\delta \to 0$ *the state-based peridynamic operator* (4.5)

$$-\mathcal{L}\mathbf{u} = \mathcal{D}_t\big(\eta\underline{\omega}(|\mathbf{x}-\mathbf{y}|)\big(\mathcal{D}_t^*(\mathbf{u})\big)^T\big) + \mathcal{D}_{t,\omega}\big((k - n\eta/d)\mathrm{Tr}\big(\mathcal{D}_{t,\omega}^*(\mathbf{u})\big)\mathrm{I}\big)$$

converges to Navier operator

$$\mathcal{N}\mathbf{u} = -\mu\nabla\cdot(\nabla\mathbf{u}) - (\mu+\lambda)\nabla(\nabla\cdot\mathbf{u})$$

in $H^{-1}(\mathbb{R}^d)$ *with* μ *and* λ *given by* (6.3) *and Poisson ratio given by* (6.4).

7. Concluding remarks. In this study, we apply the nonlocal vector calculus developed in [6] to peridynamic models. We express the constitutive relations of the peridynamic models in terms of nonlocal calculus operators. We rewrite the peridynamic equation in terms of a variational principle using the nonlocal calculus operators and prove that the both the bond-based and state-based peridynamic models are well-posed in their intrinsic energy norm. Moreover, the convergence, in an appropriate limit, of the state-based peridynamic operator to the Navier operator with any Poisson ratio is also demonstrated. Discussions on more general peridynamics state-based models, that is, for anisotropic and/or non-ordinary models, will be pursued in the future.

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