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NEW INSIGHTS INTO CLASSICAL PARTICLE MECHANICS

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Abstract. Classical particle mechanics is developed from the precept that, kinematics and the notion of force aside, power is the most basic ingredient in any mechanical theory. The essential properties of forces between particles are derived by requiring that the net power expended within any subsystem of particles be frame-indifferent. Moreover, requiring that the net power expended on any subsystem of particles by external agencies be frame-indifferent yields force, moment, and power balances. These balances account for inertia but hold in any frame, inertial or noninertial. Assuming that each particle possesses an interaction energy that embodies the extent to which it is attracted or repelled by other particles leads to the proposition of an interaction-energy inequality that serves as a purely mechanical statement of the second law of thermodynamics. In combination with the power balance, this inequality provides an avenue to ensure that constitutive equations do not violate thermodynamics. This inequality is used to develop the simplest class of constitutive equations that accounts for both energetic and dissipative particle-particle interactions.

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1. **Introduction.** The conventional approach to developing continuum theories relies on distinguishing carefully between basic laws and constitutive equations. While basic laws apply to many different classes of materials, constitutive equations distinguish between particular classes of materials. Physically reasonable models for the properties and behavior of materials within any given class ensue upon requiring that the relevant constitutive equations be both thermodynamically compatible and frame-indifferent. An interesting alternative to this approach arises on applying frame-indifference at a more fundamental level. In particular, Noll [11] shows that the conventional postulates imposing the balance laws for forces and moments can be replaced in favor of a single postulate stipulating that the power be frame-indifferent. Importantly, this power incorporates inertial force as an environmental force that arises from interactions between objects in our solar system and all remaining objects in the universe. In a general frame of reference, the net external

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environmental force splits additively into noninertial and inertial components, and the axiomatic structure is independent of the notion of an inertial frame. Moreover, the customary law of inertia arises only as a peculiar constitutive postulate concerning the nature of the inertial force.

Noll's [11] seminal contributions are often confused with results appearing in a roughly concurrent work by Green and Rivlin [7], who derive the mass, linear momentum, and angular momentum balances from the energy balance. However, in addition to stipulating that the energy balance be invariant under constant, superimposed rigid-body motions, Green and Rivlin [7] enforce transformation rules for the mass density, specific internal energy, Cauchy stress, heat flux, specific external body force, and specific external heat supply under such motions. These rules are tantamount to demanding that these fields be frame-indifferent. Beatty [1] shows that the various assumptions imposed by Green and Rivlin [7] reduce the energy balance to a kinetic-energy balance. The linear and angular momentum balances ensue from requiring the kinetic-energy balance to be invariant under constant, superimposed rigid-body motions and assuming that the forces are frame-indifferent. As Beatty [1] remarks, the results of Green and Rivlin [7] do not actually require a consideration of the energy balance and, thus, actually hinge on Noll's [11] axiom of the frame-indifference of the power. The approach of Green and Rivlin [7] is therefore capable of yielding the same mechanics from multiple, contradictory, statements of energy balance. Notwithstanding this fundamental difficulty, Green and Rivlin [7] require more hypotheses to derive the mechanical balance laws than does Noll [11]. Specifically, whereas Green and Rivlin [7] assume that the Cauchy stress and external body force comply with certain transformation rules, Noll [11] assumes nothing of the sort. Instead, Noll [11] obtains the frame-indifference of the Cauchy stress and the specific external body force as consequences of stipulating that the power be frame-indifferent.

In this paper, Noll's [11] approach is adapted, without serious complication, to yield a simple formulation of classical particle mechanics. That this is possible confirms the primacy of power in mechanical theories. More importantly, because Noll [11] distinguishes carefully between kinematics, kinetics, and constitutive ingredients, his method reveals basic features of particle mechanics independent of constitutive considerations. This is antithetical to more conventional methods based on the provision of a Lagrangian, methods that make it difficult to discern to what extent any result might hold independent of the innate constitutive properties of the Lagrangian, regardless of how generic those properties might be. In particular, the approach taken here elucidates the quintessential attributes of particle-particle forces and moments and of environmental forces independent of constitutive considerations. In a similar fashion, it provides a sound basis for frame-free statements of the balance laws for forces and moments and clarifies the role of moment balance in the theory. Moreover, the approach delineates the importance of constitutive equations as the link between kinematics and kinetics. In particular, the resulting framework is sufficiently general to allow for the development of frame-indifferent and thermodynamically compatible constitutive equations that incorporate history dependence. Put differently, the constitutive equations describing particle-particle forces may account not only for interactions originating from a potential but also for dissipative interactions.

The treatment of particle systems presented here has a few important precedents. Gurtin and Williams [9] develop an axiomatic formulation of the energy balance for quite general material systems, including finite collections of particles or rigid bodies and continuous bodies. For any such system, they show that the mechanical balances and the laws of action and reaction for forces and moments follow from the energy balance and provide a detailed accounting of how their results specialize to the case of a particle system. Further developments along these lines are provided by Williams [18]. See also, Truesdell [17] who integrates Noll's [11] axiomatic treatment of the concepts of body, force, motion, and energy with the contributions of Gurtin and Williams [9] and Williams [18] and also discusses the implications for particle systems. In a more recent work that emulates the strategy of Green and Rivlin [7], Yavari and Marsden [19] show that the balance and conservation laws for particle systems can be obtained by postulating that the energy balance is invariant under arbitrary diffeomorphisms of Euclidean space. These results are also generalized to the case where the ambient space is a Riemannian manifold. In contrast to the works of Gurtin and Williams [9] and Williams [18], which are completely free of constitutive assumptions, the work of Yavari and Marsden [19] is predicated on providing a potential for each pair of particles as a function of the distance separating those particles and assuming that the force exerted on one particle by another particle is determined by the derivative of the underlying potential. Among other things, this restricts the applicability of the results to conservative particle systems.

One distinction between the strategy used here and that used by Gurtin and Williams [9] and Williams [18] relates to the treatment of thermodynamics. Here, the treatment is purely mechanical and particle subsystems are restricted by an interaction-energy inequality that serves as a purely mechanical statement of the second law of theremodynamics. In contrast, Gurtin and Williams [9] and Williams [18] endow each particle with an internal energy and heat flux and impose an energy balance involving these quantities. The view taken here resembles more closely that guiding the work of Pitteri [12, 13], where the underlying theory is a purely mechanical one and statistical mechanics is used to pass to a continuum limit involving an energy balance that incorporates field quantities identified as specific internal-energy and heat flux. Importantly, the system of differential equations that provides the foundation for Pitteri's [12, 13] analysis is identical to the system derived here when the frame is inertial and dissipative particle-particle interactions are neglected. Including dissipative particle-particle interactions would likely lead to a continuum limit different from that obtained by Pitteri [12, 13]. It would be particularly interesting to determine any continuum level ramifications ensuing from the thermodynamic restrictions placed by the interaction-energy inequality on the dissipative contributions to the particle-particle forces.

The insights on the structure of classical particle mechanics provided here might contribute to the design of improved multiscale simulation methods. Such methods are intended to enable simulations at computational costs lower than those incurred by purely particle-based simulations. Miller and Tadmor [10] define a partitioned-domain multiscale method to be a computational framework where atomistic and continuum models are synthesized. They distinguish between energy-based methods that stem from the provision of a global energy-functional and force-based methods that stem from imposing equilibrium equations. In either case, the most significant challenges are associated with matching conditions between the atomistic and continuum descriptions. A formulation based on matching power expenditures could have computational advantages.

Silling and Lehoucq [15] establish a formal connection between particle mechanics and Silling's [14] peridynamic theory. The starting point of peridynamic theory is a continuum-level field equation imposing linear-momentum balance with the essential difference that the conventional term involving the divergence of the stress is replaced by a nonlocal term in which a integral operator is applied to the displacement field. This operator determines not only the range over which material points interact but also the nature of allowed interactions. Silling's [14] formulation of peridynamic theory therefore mixes kinematics, kinetics, and constitutive equations from the outset. It seems likely that the conceptual basis underlying the approach to particle mechanics presented here can be adapted to provide an alternative formulation of peridynamic theory in which these ingredients are treated separately. In addition to clarifying the essential elements and features of peridynamics, such a formulation might provide currently unavailable constraints for peridynamic constitutive equations, including those constraints embodying dissipative interactions. This latter class of interactions has yet to be been considered in any peridynamic formulation.

The paper is organized as follows. The simple notions of particle systems and their motions are defined in §§2–3. Forces, moments, and power expenditures are introduced in §§4–6. A change-of-frame and the definition of frame-indifference are introduced in §§7–8. The laws of mutual-action and collinearity and the frame indifference of the particle-particle forces are derived, as consequences of assuming that the internal power be frame-indifferent, in §9. The laws of force balance and moment balance and the frame indifference of the environmental forces are derived, as consequences of assuming that the external power is frame-indifferent, in §10. Systemwise and particlewise force and moment balances are obtained as corollaries in §11, along with reciprocal balances for the forces and moments exerted between disjoint systems. The law of power balance, which equates the internal and the negative of external power expenditures corresponding to subsystems of particles, is derived in §12. The laws of mass, linear and angular momentum, and kinetic-energy balance in an inertial frame are derived in §13, where specialized results for the center-of-mass of a particle system are also presented. The principle of interaction-energy imbalance is introduced in §§14-15. Constitutive equations are discussed in §16, in which the focus is on determining the consequences of requiring consistency with frame-indifference and with restrictions imposed by requiring thermodynamic compatibility. Finally, a summary of results is provided in §17.

2. **Particle systems and subsystems.**

Let N be a natural number. A *particle system* is a finite set

$$\mathcal{B} = \{p_i : i = 1, \dots, N\}.$$

Consider a particle system $\mathcal{B}$. A subset $\mathcal{P}$ of $\mathcal{B}$ defines a *particle subsystem.* Given a particle subsystem $\mathcal{P}$ of $\mathcal{B}$, the complementary subsystem of all other particles contained in $\mathcal{B}$ is denoted by $\mathcal{P}' = \mathcal{B} \setminus \mathcal{P}$.[1] Thus,

$$\mathcal{P} \cup \mathcal{P}' = \mathcal{B} \qquad \text{and} \qquad \mathcal{P} \cap \mathcal{P}' = \emptyset.$$

[1] Throughout this work, $\mathcal{P}'$ denotes the complement of $\mathcal{P}$ with respect to $\mathcal{B}$. Thus, in particular, $\mathcal{B}' = \emptyset$.

3. **Motions of particle systems.** Let $\mathcal{E}$ denote three-dimensional Euclidean point space and let $\mathcal{V}$ be the associated vector, or translation, space. Elements of these spaces are referred to as *points* and *vectors*, respectively.

A *motion* of a particle system $\mathcal{B}$ is a mapping $\boldsymbol{\chi}$ that assigns to each particle p_i in $\mathcal{B}$ and each time t in some relevant interval I a point $\mathbf{x}_i(t)$, viz.

$$\boldsymbol{\chi}(p_i, t) = \mathbf{x}_i(t). \tag{3.1}$$

The mapping $\boldsymbol{\chi}(\cdot, t)$ is assumed to be one-to-one for each t in I. Thus, given distinct particles p_i and p_j and a time t,

$$\mathbf{x}_i(t) \neq \mathbf{x}_j(t) \qquad \text{unless } i = j. \tag{3.2}$$

The *configuration* (or *placement*) of a particle system $\mathcal{B}$ at time t is the collection

$$\{\boldsymbol{\chi}(p_1, t), \dots, \boldsymbol{\chi}(p_N, t)\} = \{\mathbf{x}_1(t), \dots, \mathbf{x}_N(t)\} \tag{3.3}$$

of points occupied at time t by all particles of $\mathcal{B}$. Moreover, the *trajectory* of particle p_i is denoted by

$$\mathcal{T}_i = \{\mathbf{y} \in \mathcal{E} : \mathbf{y} = \mathbf{x}_i(t), t \in I\}. \tag{3.4}$$

As is conventional, a superposed dot signifies differentiation with respect to time and, granted a sufficiently smooth motion, the *velocity* $\mathbf{v}_i$ and *acceleration* $\mathbf{a}_i$ of a particle p_i in $\mathcal{B}$ are vectors defined by

$$\mathbf{v}_i = \dot{\mathbf{x}}_i, \tag{3.5}$$

which ensures that $\mathbf{v}_i$ is tangent to the trajectory $\mathcal{T}_i$, and

$$\mathbf{a}_i = \dot{\mathbf{v}}_i = \ddot{\mathbf{x}}_i. \tag{3.6}$$

Given points $\mathbf{x}_i$ and $\mathbf{x}_j \neq \mathbf{x}_i$ corresponding to particles p_i and $p_j \neq p_i$ in $\mathcal{B}$,

$$\mathbf{r}_{ij} = \mathbf{x}_i - \mathbf{x}_j \qquad \text{and} \qquad r_{ij} = |\mathbf{r}_{ij}| = \sqrt{\mathbf{r}_{ij} \cdot \mathbf{r}_{ij}} \tag{3.7}$$

represent the vector directed from $\mathbf{x}_j$ to $\mathbf{x}_i$ and its length. Further,

$$\mathbf{v}_{ij} = \dot{\mathbf{r}}_{ij} \qquad \text{and} \qquad s_{ij} = \dot{r}_{ij} = \dot{\overline{|\mathbf{r}_{ij}|}} \tag{3.8}$$

represent the velocity $\mathbf{v}_i - \mathbf{v}_j$ of p_i relative to that of p_j and the rate at which the distance $|\mathbf{x}_i - \mathbf{x}_j|$ between p_i and p_j changes with respect to time. Since, by (3.2), $r_{ij} > 0$ for $i \neq j$, it is always possible to define the unit vector

$$\mathbf{n}_{ij} = \frac{\mathbf{r}_{ij}}{r_{ij}} \tag{3.9}$$

in the direction of $\mathbf{r}_{ij}$. Then, since $r_{ij}\dot{r}_{ij} = \mathbf{r}_{ij} \cdot \dot{\mathbf{r}}_{ij}$, (3.7) and (3.8) imply the useful identities

$$s_{ij} = \mathbf{n}_{ij} \cdot \mathbf{v}_{ij} \qquad \text{and} \qquad \mathbf{v}_{ij} = s_{ij}\mathbf{n}_{ij}. \tag{3.10}$$

4. **Forces.** Consider a particle system $\mathcal{B}$. Let p_i and $p_j \neq p_i$ belong to $\mathcal{B}$. The forces exerted on particle p_i by particle p_j and by entities external to $\mathcal{B}$ are vectors denoted respectively by $\mathbf{f}_{ij}$ and $\mathbf{f}_i^{\text{ext}}$. Whereas $\mathbf{f}_{ij}$ is referred to as a *particle-particle force*, $\mathbf{f}_i^{\text{ext}}$ is referred to as an *environmental force*. Environmental forces are assumed to include inertial forces and to admit decompositions of the form

$$\mathbf{f}_i^{\text{ext}} = \mathbf{f}_i^{\text{ni}} + \mathbf{f}_i^{\text{in}}, \tag{4.1}$$

where $\mathbf{f}^{\text{ni}}$ and $\mathbf{f}^{\text{in}}$ are the noninertial and inertial forces exerted on p_i by agencies external to the system $\mathcal{B}$.

Given disjoint subsystems $\mathcal{P}$ and $\mathcal{Q}$ of $\mathcal{B}$, the sums

$$\mathbf{f}(\mathcal{P},\mathcal{Q}) = \sum_{p_i\in\mathcal{P}}\sum_{p_j\in\mathcal{Q}} \mathbf{f}_{ij} \qquad \text{and} \qquad \mathbf{f}^{\text{ext}}(\mathcal{P}) = \sum_{p_i\in\mathcal{P}} \mathbf{f}_i^{\text{ext}} \tag{4.2}$$

represent the net forces exerted on the particles comprising subsystem $\mathcal{P}$ by the particles comprising subsystem $\mathcal{Q}$ and by agencies external to $\mathcal{B}$, respectively.

The definition $(4.2)_1$ is supplemented by the requirement that, for any particle subsystem $\mathcal{P}$,

$$\mathbf{f}(\mathcal{P},\emptyset) = \mathbf{f}(\emptyset,\mathcal{P}) = \mathbf{0}; \tag{4.3}$$

(4.3) embodies two reasonable notions: first, the net force exerted on a particle subsystem $\mathcal{P}$ by a subsystem consisting of no particles must vanish; second, and conversely, the net force exerted by any particle subsystem on a subsystem consisting of no particles must also vanish.

As a simple consequence of the definition $(4.2)_2$, the net force exerted on a particle subsystem by external agencies obeys

$$\mathbf{f}^{\text{ext}}(\mathcal{P}\cup\mathcal{Q}) = \mathbf{f}^{\text{ext}}(\mathcal{P}) + \mathbf{f}^{\text{ext}}(\mathcal{Q}) \tag{4.4}$$

for all disjoint particle subsystems $\mathcal{P}$ and $\mathcal{Q}$ and is, therefore, additive. Similarly, by $(4.2)_1$, the net force exerted on a particle subsystem by another disjoint particle subsystem is biadditive in the sense that it obeys both

$$\mathbf{f}(\mathcal{P}\cup\mathcal{Q},\mathcal{R}) = \mathbf{f}(\mathcal{P},\mathcal{R}) + \mathbf{f}(\mathcal{Q},\mathcal{R}) \tag{4.5}$$

and

$$\mathbf{f}(\mathcal{R},\mathcal{P}\cup\mathcal{Q}) = \mathbf{f}(\mathcal{R},\mathcal{P}) + \mathbf{f}(\mathcal{R},\mathcal{Q}) \tag{4.6}$$

for all particle subsystems $\mathcal{P}$, $\mathcal{Q}$, and $\mathcal{R}$ with $\mathcal{P}\cup\mathcal{Q}$ and $\mathcal{R}$ disjoint.

5. **Moments.** Let p_i and $p_j \neq p_i$ belong to particle system $\mathcal{B}$. Let $\mathbf{x}_i$ and $\mathbf{x}_j$ be the points occupied by p_i and p_j in some configuration $\{\mathbf{x}_i,\dots,\mathbf{x}_N\}$ of $\mathcal{B}$ and let $\mathbf{y}$ be an arbitrary point. The respectively moments, about $\mathbf{y}$, of the force exerted by particle p_j on particle p_i and the force exerted on particle p_i by agencies external to $\mathcal{B}$ are vectors given by

$$\mathbf{m}_{ij}(\mathbf{y}) = (\mathbf{x}_i - \mathbf{y})\times\mathbf{f}_{ij} \qquad \text{and} \qquad \mathbf{m}_i^{\text{ext}}(\mathbf{y}) = (\mathbf{x}_i - \mathbf{y})\times\mathbf{f}_i^{\text{ext}}. \tag{5.1}$$

The sums

$$\mathbf{m}(\mathcal{P},\mathcal{Q};\mathbf{y}) = \sum_{p_i\in\mathcal{P}}\sum_{p_j\in\mathcal{Q}} \mathbf{m}_{ij}(\mathbf{y}) \qquad \text{and} \qquad \mathbf{m}^{\text{ext}}(\mathcal{P};\mathbf{y}) = \sum_{p_i\in\mathcal{P}} \mathbf{m}_i^{\text{ext}}(\mathbf{y}) \tag{5.2}$$

therefore represent the net moments, about $\mathbf{y}$, on the particles comprising subsystem $\mathcal{P}$ by the particles comprising subsystem $\mathcal{Q}$ and by agencies external to $\mathcal{B}$, respectively.

By (4.3), the definition $(5.2)_1$ is supplemented by the requirement that, for any particle subsystem $\mathcal{P}$ and any point $\mathbf{y}$,

$$\mathbf{m}(\mathcal{P},\emptyset;\mathbf{y}) = \mathbf{m}(\emptyset,\mathcal{P};\mathbf{y}) = \mathbf{0}. \tag{5.3}$$

Remarks similar to those following (4.3) apply to the stipulation (5.3).

Analogous to the results (4.4) and (4.5)–(4.6) obtained for the net forces $\mathbf{f}^{\text{ext}}(\mathcal{P})$ and $\mathbf{f}(\mathcal{P},\mathcal{Q})$ exerted on a particle subsystem $\mathcal{P}$ by the external agencies and and for those forces by a disjoint particle subsystem $\mathcal{Q}$, the definitions (5.2) imply that the net external moment, about $\mathbf{y}$, exerted on a particle subsystem obeys

$$\mathbf{m}^{\text{ext}}(\mathcal{P}\cup\mathcal{Q};\mathbf{y}) = \mathbf{m}^{\text{ext}}(\mathcal{P};\mathbf{y}) + \mathbf{m}^{\text{ext}}(\mathcal{Q};\mathbf{y}) \tag{5.4}$$

for all disjoint particle subsystems $\mathcal{P}$ and $\mathcal{Q}$ and is, therefore, additive, while the net moment, about $\mathbf{y}$, exerted on a particle system by another disjoint particle subsystem is biadditive in the sense that it obeys both

$$\mathbf{m}(\mathcal{P}\cup\mathcal{Q},\mathcal{R};\mathbf{y}) = \mathbf{m}(\mathcal{P},\mathcal{R};\mathbf{y}) + \mathbf{m}(\mathcal{Q},\mathcal{R};\mathbf{y}) \tag{5.5}$$

and

$$\mathbf{m}(\mathcal{R},\mathcal{P}\cup\mathcal{Q};\mathbf{y}) = \mathbf{m}(\mathcal{R},\mathcal{P};\mathbf{y}) + \mathbf{m}(\mathcal{R},\mathcal{Q};\mathbf{y}) \tag{5.6}$$

for all particle subsystems $\mathcal{P}$, $\mathcal{Q}$, and $\mathcal{R}$ with $\mathcal{P}\cup\mathcal{Q}$ and $\mathcal{R}$ disjoint.

Let $\mathbf{y}$ and $\mathbf{z}$ be points. Then,

$$\begin{aligned}\mathbf{m}_{ij}(\mathbf{y}) &= [\mathbf{x}_i - \mathbf{z} - (\mathbf{y}-\mathbf{z})]\times\mathbf{f}_{ij}\\ &= (\mathbf{x}_i-\mathbf{z})\times\mathbf{f}_{ij} - (\mathbf{y}-\mathbf{z})\times\mathbf{f}_{ij}\\ &= \mathbf{m}_{ij}(\mathbf{z}) + (\mathbf{z}-\mathbf{y})\times\mathbf{f}_{ij}\end{aligned}$$

and $(5.2)_1$ implies that

$$\mathbf{m}(\mathcal{P},\mathcal{Q};\mathbf{y}) = \mathbf{m}(\mathcal{P},\mathcal{Q};\mathbf{z}) + (\mathbf{z}-\mathbf{y})\times\mathbf{f}(\mathcal{P},\mathcal{Q}). \tag{5.7}$$

Similarly, from $(5.2)_2$

$$\mathbf{m}^{\mathrm{ext}}(\mathcal{P};\mathbf{y}) = \mathbf{m}^{\mathrm{ext}}(\mathcal{P};\mathbf{z}) + (\mathbf{z}-\mathbf{y})\times\mathbf{f}^{\mathrm{ext}}(\mathcal{P}). \tag{5.8}$$

6. **Power.** Let p_i and $p_j \neq p_i$ belong to a particle system $\mathcal{B}$. The inner products

$$\mathbf{f}_{ij}\cdot\mathbf{v}_i \qquad\text{and}\qquad \mathbf{f}_i^{\mathrm{ext}}\cdot\mathbf{v}_i \tag{6.1}$$

then respectively represent power expenditures associated with moving p_i at velocity $\mathbf{v}_i$ under the action of forces $\mathbf{f}_{ij}$ exerted on p_i by p_j and $\mathbf{f}_i^{\mathrm{ext}}$ by agencies external to $\mathcal{B}$.

The power expenditures (6.1) account implicitly for the actions of the moments $\mathbf{m}_{ij}(\mathbf{y})$ and $\mathbf{m}_i^{\mathrm{ext}}(\mathbf{y})$ of the forces $\mathbf{f}_{ij}$ and $\mathbf{f}_i^{\mathrm{ext}}$, about $\mathbf{y}$, exerted on p_i. To see this, choose a point $\mathbf{y}\neq\mathbf{x}_i$, introduce the projection

$$\boldsymbol{\Pi}_i(\mathbf{y}) = \frac{\mathbf{x}_i-\mathbf{y}}{|\mathbf{x}_i-\mathbf{y}|}\otimes\frac{\mathbf{x}_i-\mathbf{y}}{|\mathbf{x}_i-\mathbf{y}|},$$

and observe that the velocity $\mathbf{v}_i$ of p_i can be expressed as

$$\begin{aligned}\mathbf{v}_i &= \boldsymbol{\Pi}_i(\mathbf{y})\mathbf{v}_i + [\mathbf{1}-\boldsymbol{\Pi}_i(\mathbf{y})]\mathbf{v}_i\\ &= \boldsymbol{\Pi}_i(\mathbf{y})\mathbf{v}_i + \left(\frac{\mathbf{x}_i-\mathbf{y}}{|\mathbf{x}_i-\mathbf{y}|}\times\mathbf{v}_i\right)\times\frac{\mathbf{x}_i-\mathbf{y}}{|\mathbf{x}_i-\mathbf{y}|}\\ &= \boldsymbol{\Pi}_i(\mathbf{y})\mathbf{v}_i + \boldsymbol{\omega}_i(\mathbf{y})\times(\mathbf{x}_i-\mathbf{y}),\end{aligned}$$

where

$$\boldsymbol{\omega}_i = \frac{(\mathbf{x}_i-\mathbf{y})\times\mathbf{v}_i}{|\mathbf{x}_i-\mathbf{y}|^2}.$$

is the angular velocity of p_i about $\mathbf{y}$. Thus, by $(5.1)_1$,

$$\begin{aligned}\mathbf{f}_{ij}\cdot\mathbf{v}_i &= \mathbf{f}_{ij}\cdot\boldsymbol{\Pi}_i(\mathbf{y})\mathbf{v}_i + \mathbf{f}_{ij}\cdot[\boldsymbol{\omega}_i(\mathbf{y})\times(\mathbf{x}_i-\mathbf{y})]\\ &= \mathbf{f}_{ij}\cdot\boldsymbol{\Pi}_i(\mathbf{y})\mathbf{v}_i + [(\mathbf{x}_i-\mathbf{y})\times\mathbf{f}_{ij}]\cdot\boldsymbol{\omega}_i(\mathbf{y})\\ &= \mathbf{f}_{ij}\cdot\boldsymbol{\Pi}_i(\mathbf{y})\mathbf{v}_i + \mathbf{m}_{ij}(\mathbf{y})\cdot\boldsymbol{\omega}_i(\mathbf{y})\end{aligned}$$

while, by $(5.1)_2$,

$$\mathbf{f}_i^{\mathrm{ext}}\cdot\mathbf{v}_i = \mathbf{f}_i^{\mathrm{ext}}\cdot\boldsymbol{\Pi}_i(\mathbf{y})\mathbf{v}_i + \mathbf{m}_i^{\mathrm{ext}}(\mathbf{y})\cdot\boldsymbol{\omega}_i(\mathbf{y}).$$

Since $\mathbf{m}_{ij}(\mathbf{y})\cdot\boldsymbol{\omega}_i(\mathbf{y})$ and $\mathbf{m}_i^{\rm ext}(\mathbf{y})\cdot\boldsymbol{\omega}_i(\mathbf{y})$ reckon the power expenditures associated with the moments $\mathbf{m}_{ij}(\mathbf{y})$ and $\mathbf{m}_i^{\rm ext}(\mathbf{y})$, it therefore follows that it is unnecessary to account separately for these power expenditures.[2]

In view of $(6.1)_1$ and the previous discussion, the power expended on a particle p_i belonging to subsystem $\mathcal{P}$ by all particles $p_j \neq p_i$ also belonging to $\mathcal{P}$ is given by

$$\sum_{p_j\in\mathcal{P}\setminus\{p_i\}} \mathbf{f}_{ij}\cdot\mathbf{v}_i,$$

which, when summed over all particles in $\mathcal{P}$ then yields the *internal power* $w_{\rm int}(\mathcal{P})$ *expended on* $\mathcal{P}$:

$$w_{\rm int}(\mathcal{P}) = \sum_{p_i\in\mathcal{P}}\sum_{p_j\in\mathcal{P}\setminus\{p_i\}} \mathbf{f}_{ij}\cdot\mathbf{v}_i. \tag{6.2}$$

Similarly, in view of $(6.1)_2$, the power expended on a particle p_i belonging to a subsystem $\mathcal{P}$ by all particles belonging to the remainder $\mathcal{P}'$ of the system and agencies external to $\mathcal{B}$ is given by

$$\sum_{p_j\in\mathcal{P}'} \mathbf{f}_{ij}\cdot\mathbf{v}_i + \mathbf{f}_i^{\rm ext}\cdot\mathbf{v}_i,$$

which, when summed over the particles in $\mathcal{P}$, yields the *external power* $w_{\rm ext}(\mathcal{P})$ *expended on* $\mathcal{P}$:

$$w_{\rm ext}(\mathcal{P}) = \sum_{p_i\in\mathcal{P}}\sum_{p_j\in\mathcal{P}'} \mathbf{f}_{ij}\cdot\mathbf{v}_i + \sum_{p_i\in\mathcal{P}} \mathbf{f}_i^{\rm ext}\cdot\mathbf{v}_i. \tag{6.3}$$

7. **Changes of frame.** A *change-of-frame* is defined, at each t in I, by a point $\mathbf{z}(t)$ and a rotation $\mathbf{Q}(t)$ whereby each point $\mathbf{x}_i(t)$ corresponding to a particle p_i in $\mathcal{B}$ is transformed into a point[3]

$$\mathbf{x}_i^\star(t) = \mathbf{z}(t) + \mathbf{Q}(t)\left(\mathbf{x}_i(t) - \mathbf{y}\right), \tag{7.1}$$

with $\mathbf{y}$ an arbitrary point. Being a rotation, $\mathbf{Q}$ obeys $\mathbf{Q}^\top\mathbf{Q} = \mathbf{1}$, from which it follows that $\mathbf{Q}^\top\dot{\mathbf{Q}} = -\dot{\mathbf{Q}}^\top\mathbf{Q} = -(\mathbf{Q}^\top\dot{\mathbf{Q}})^\top$; thus, $\mathbf{Q}^\top\dot{\mathbf{Q}}$ is skew and there exists a vector $\boldsymbol{\omega}$, the *angular velocity of the change-of-frame*, such that

$$\boldsymbol{\omega}\times\mathbf{r} = \mathbf{Q}^\top\dot{\mathbf{Q}}\mathbf{r} \tag{7.2}$$

for all vectors $\mathbf{r}$.

Let $\mathbf{v}_i^\star = \dot{\mathbf{x}}_i^\star$ be the velocity of particle p_i in the new frame defined by (7.1). Computing the time derivative of (7.1) and using (7.2) then yields

$$\mathbf{v}_i^\star = \mathbf{Q}\mathbf{v}_i + \dot{\mathbf{z}} + \dot{\mathbf{Q}}(\mathbf{x}_i - \mathbf{y}) = \mathbf{Q}\mathbf{v}_i + \mathbf{c} + \mathbf{Q}[\boldsymbol{\omega}\times(\mathbf{x}_i - \mathbf{y})], \tag{7.3}$$

with $\mathbf{c} = \dot{\mathbf{z}}$ the *translational velocity of the change-of-frame*. Consider the last term $\mathbf{Q}[\boldsymbol{\omega}\times(\mathbf{x}_i - \mathbf{y})]$ on the rightmost side of (7.3). Since, for any invertible tensor $\mathbf{A}$ and any two vectors $\mathbf{r}_1$ and $\mathbf{r}_2$,

$$(\det\mathbf{A})\mathbf{A}^{-\top}(\mathbf{r}_1\times\mathbf{r}_2) = (\mathbf{A}\mathbf{r}_1)\times(\mathbf{A}\mathbf{r}_2) \tag{7.4}$$

and since $\mathbf{Q}$ is a rotation,

$$\mathbf{Q}[\boldsymbol{\omega}\times(\mathbf{x}_i - \mathbf{y})] = (\det\mathbf{Q})\mathbf{Q}^{-\top}[\boldsymbol{\omega}\times(\mathbf{x}_i - \mathbf{y})] = (\mathbf{Q}\boldsymbol{\omega})\times[\mathbf{Q}(\mathbf{x}_i - \mathbf{y})]. \tag{7.5}$$

[2] This is no longer true when moments arising from actions other than torques, such as couples, are taken into account. When such moments are present, it is necessary to account for the associated power expenditues.

[3] The sum of a point and a vector is *defined* to be a point. See, for example, Gurtin [8].

By (7.1) and (7.5), $\mathbf{Q}[\boldsymbol{\omega}\times(\mathbf{x}_i-\mathbf{y})]=(\mathbf{Q}\boldsymbol{\omega})\times(\mathbf{x}_i^\star-\mathbf{z})$ and the transformation rule (7.3) for $\mathbf{v}_i$ under a change-of-frame becomes

$$\mathbf{v}_i^\star=\mathbf{Q}\mathbf{v}_i+\mathbf{c}+(\mathbf{Q}\boldsymbol{\omega})\times(\mathbf{x}_i^\star-\mathbf{z}). \tag{7.6}$$

8. **Frame-indifferent objects.** A scalar φ and a vector $\boldsymbol{\varphi}$ are *frame-indifferent* if the transformation rules

$$\varphi^\star=\varphi \qquad \text{and} \qquad \boldsymbol{\varphi}^\star=\mathbf{Q}\boldsymbol{\varphi} \tag{8.1}$$

hold under all changes of frame.

Let p_i and p_j be distinct particles in $\mathcal{B}$. Consider the vector $\mathbf{r}_{ij}=\mathbf{x}_i-\mathbf{x}_j$ directed from the point $\mathbf{x}_j$ occupied by p_j to the point $\mathbf{x}_i$ occupied by p_i. By (7.1), $\mathbf{r}_{ij}$ transforms under a change-of-frame to

$$\mathbf{r}_{ij}^\star=\mathbf{x}_i^\star-\mathbf{x}_j^\star=\mathbf{Q}(\mathbf{x}_i-\mathbf{x}_j)=\mathbf{Q}\mathbf{r}_{ij} \tag{8.2}$$

and, thus, is frame-indifferent. By (8.2), $|\mathbf{r}_{ij}^\star|=|\mathbf{Q}\mathbf{r}_{ij}|=|\mathbf{r}_{ij}|$. The distance $r_{ij}=|\mathbf{r}_{ij}|$ separating $\mathbf{x}_i$ and $\mathbf{x}_j$ therefore transforms under a change-of-frame to

$$r_{ij}^\star=|\mathbf{r}_{ij}^\star|=r_{ij} \tag{8.3}$$

and, thus, is frame-indifferent. Further, by (8.2) and (8.3), the unit vector $\mathbf{n}_{ij}=\mathbf{r}_{ij}/r_{ij}$ parallel to $\mathbf{r}_{ij}$ transforms according to

$$\mathbf{n}_{ij}^\star=\frac{\mathbf{r}_{ij}^\star}{r_{ij}^\star}=\mathbf{Q}\mathbf{n}_{ij} \tag{8.4}$$

and, thus, is frame-indifferent. Consider next, the velocity $\mathbf{v}_{ij}=\mathbf{v}_i-\mathbf{v}_j$ of p_i relative to p_j. By (7.6) and (8.2), $\mathbf{v}_{ij}$ transforms under a change-of-frame to

$$\mathbf{v}_{ij}^\star=\mathbf{v}_i^\star-\mathbf{v}_j^\star=\mathbf{Q}(\mathbf{v}_i-\mathbf{v}_j)+\mathbf{Q}\boldsymbol{\omega}\times\mathbf{Q}(\mathbf{x}_i^\star-\mathbf{x}_j^\star)=\mathbf{Q}\mathbf{v}_{ij}+\mathbf{Q}\boldsymbol{\omega}\times\mathbf{Q}\mathbf{r}_{ij} \tag{8.5}$$

and, thus, is not frame-indifferent. However, since, by $(3.10)_1$, $s_{ij}^\star=\mathbf{n}_{ij}^\star\cdot\mathbf{v}_{ij}^\star$, (8.4) and (8.5) imply that the time-derivative $s_{ij}=\dot{\overline{r_{ij}}}$ of the distance r_{ij} between $\mathbf{x}_i$ and $\mathbf{x}_j$ transforms under a change-of-frame to

$$s_{ij}^\star=\mathbf{Q}\mathbf{n}_{ij}\cdot(\mathbf{Q}\mathbf{v}_{ij}+\mathbf{Q}\boldsymbol{\omega}\times\mathbf{Q}\mathbf{r}_{ij})=\mathbf{n}_{ij}\cdot\mathbf{v}_{ij}=s_{ij}, \tag{8.6}$$

and, thus, is frame-indifferent.

9. **Consequences of assuming that the internal power is frame-indifferent.** The fundamental consequences of assuming that the internal power be frame-indifferent, for all particle subsystems, are now developed. In view of the definitions (6.2) and $(8.1)_1$, this assumption requires that

$$w_{\text{int}}^\star(\mathcal{P})=w_{\text{int}}(\mathcal{P}) \tag{9.1}$$

for any particle subsystem $\mathcal{P}$ of $\mathcal{B}$, where

$$w_{\text{int}}^\star(\mathcal{P})=\sum_{p_i\in\mathcal{P}}\sum_{p_j\in\mathcal{P}\setminus\{p_i\}}\mathbf{f}_{ij}^\star\cdot\mathbf{v}_i^\star \tag{9.2}$$

is the power expended within $\mathcal{P}$ under a change-of-frame. Here, $\mathbf{f}_{ij}^\star$ is the particle-particle force exerted on p_i by p_j under the change-of-frame and $\mathbf{v}_i^\star$ is the velocity of p_i under the change-of-frame.

9.1. **Preliminary calculations.** By (7.6) and elementary properties of the cross-product, the power expenditure associated with moving p_i at velocity $\mathbf{v}_i^\star$ under the action of the force $\mathbf{f}_{ij}^\star$ is given by

$$\mathbf{f}_{ij}^\star \cdot \mathbf{v}_i^\star = \mathbf{f}_{ij}^\star \cdot \mathbf{Q}\mathbf{v}_i + \mathbf{f}_{ij}^\star \cdot \mathbf{c} + [(\mathbf{x}_i - \mathbf{y})^\star \times \mathbf{f}_{ij}^\star] \cdot \mathbf{Q}\boldsymbol{\omega}, \tag{9.3}$$

where $(8.1)_2$ has been employed to write $\mathbf{x}_i^\star - \mathbf{z} = \mathbf{Q}(\mathbf{x}_i - \mathbf{y})$ as $(\mathbf{x}_i - \mathbf{y})^\star$. Using (9.3) in (9.2) then yields an expression for the power (9.2) expended within $\mathcal{P}$ under a change-of-frame:

$$w_{\text{int}}^\star(\mathcal{P}) = \sum_{p_k \in \mathcal{P}} \sum_{p_l \in \mathcal{P}\setminus\{p_k\}} \mathbf{f}_{kl}^\star \cdot \mathbf{Q}\mathbf{v}_k + \Bigg\{ \sum_{p_k \in \mathcal{P}} \sum_{p_l \in \mathcal{P}\setminus\{p_k\}} \mathbf{f}_{kl}^\star \Bigg\} \cdot \mathbf{c} + \Bigg\{ \sum_{p_k \in \mathcal{P}} \sum_{p_l \in \mathcal{P}\setminus\{p_k\}} (\mathbf{x}_k - \mathbf{y})^\star \times \mathbf{f}_{kl}^\star \Bigg\} \cdot \mathbf{Q}\boldsymbol{\omega}. \tag{9.4}$$

Since for any rotation $\mathbf{Q}$ and any two vectors $\mathbf{r}_1$ and $\mathbf{r}_2$, $\mathbf{Q}\mathbf{r}_1 \cdot \mathbf{Q}\mathbf{r}_2 = \mathbf{r}_1 \cdot \mathbf{Q}^\top \mathbf{Q}\mathbf{r}_2 = \mathbf{r}_1 \cdot \mathbf{r}_2$, from (6.2), the power $w_{\text{int}}(\mathcal{P})$ expended within $\mathcal{P}$ can be expressed as

$$w_{\text{int}}(\mathcal{P}) = \sum_{p_k \in \mathcal{P}} \sum_{p_l \in \mathcal{P}\setminus\{p_k\}} \mathbf{Q}\mathbf{f}_{kl} \cdot \mathbf{Q}\mathbf{v}_k. \tag{9.5}$$

Using (9.4) and (9.5) in (9.1) shows that the internal power is frame-indifferent for all particle subsystems if and only if

$$\begin{aligned} 0 &= w_{\text{int}}^\star(\mathcal{P}) - w_{\text{int}}(\mathcal{P}) \\ &= \sum_{p_k \in \mathcal{P}} \sum_{p_l \in \mathcal{P}\setminus\{p_k\}} (\mathbf{f}_{kl}^\star - \mathbf{Q}\mathbf{f}_{kl}) \cdot \mathbf{Q}\mathbf{v}_k \\ &\quad + \Bigg\{ \sum_{p_k \in \mathcal{P}} \sum_{p_l \in \mathcal{P}\setminus\{p_k\}} \mathbf{f}_{kl}^\star \Bigg\} \cdot \mathbf{c} + \Bigg\{ \sum_{p_k \in \mathcal{P}} \sum_{p_l \in \mathcal{P}\setminus\{p_k\}} (\mathbf{x}_k - \mathbf{y})^\star \times \mathbf{f}_{kl}^\star \Bigg\} \cdot \mathbf{Q}\boldsymbol{\omega} \end{aligned} \tag{9.6}$$

holds for all choices of the rotation $\mathbf{Q}$, translational velocity $\mathbf{c}$, and angular velocity $\boldsymbol{\omega}$ associated with a change-of-frame (7.1) and for all $\mathcal{P}$.

9.2. **Mutual-action and collinearity.** With reference to (9.6), consider a change-of-frame for which the translational velocity $\mathbf{c}$ and the angular velocity $\boldsymbol{\omega}$ vanish. For such a change-of-frame, (9.6) implies that

$$\sum_{p_k \in \mathcal{P}} \sum_{p_l \in \mathcal{P}\setminus\{p_k\}} (\mathbf{f}_{kl}^\star - \mathbf{Q}\mathbf{f}_{kl}) \cdot \mathbf{Q}\mathbf{v}_l = 0 \tag{9.7}$$

must hold for all rotations $\mathbf{Q}$ and particle subsystems $\mathcal{P}$.

Next, consider a change-of-frame for which $\mathbf{c} \neq \mathbf{0}$ and $\boldsymbol{\omega} = \mathbf{0}$. Granted (9.7), for such a change of frame, (9.6) implies that

$$\Bigg\{ \sum_{p_k \in \mathcal{P}} \sum_{p_l \in \mathcal{P}\setminus\{p_k\}} \mathbf{f}_{kl}^\star \Bigg\} \cdot \mathbf{c} = 0 \tag{9.8}$$

must hold for all vectors $\mathbf{c}$ and all particle subsystems $\mathcal{P}$.

Next, consider a change-of-frame for which $\mathbf{c} = \mathbf{0}$ and $\boldsymbol{\omega} \neq \mathbf{0}$. Granted (9.7) and (9.8), for such a change-of-frame (9.6) implies that

$$\Bigg\{ \sum_{p_k \in \mathcal{P}} \sum_{p_l \in \mathcal{P}\setminus\{p_k\}} (\mathbf{x}_k - \mathbf{y})^\star \times \mathbf{f}_{kl}^\star \Bigg\} \cdot \mathbf{Q}\boldsymbol{\omega} = 0 \tag{9.9}$$

must hold for all rotations $\mathbf{Q}$, all vectors $\boldsymbol{\omega}$, and all particle subsystems $\mathcal{P}$.

Select arbitrarily two points p_i and $p_j \neq p_j$ in $\mathcal{B}$ and consider the doubleton $\mathcal{P} = \{p_i, p_j\}$. For this choice of $\mathcal{P}$, (9.7)–(9.9) yield the conditions

$$
\begin{gathered}
(\mathbf{f}_{ij}^\star - \mathbf{Q}\mathbf{f}_{ij}) \cdot \mathbf{Q}\mathbf{v}_i + (\mathbf{f}_{ji}^\star - \mathbf{Q}\mathbf{f}_{ji}) \cdot \mathbf{Q}\mathbf{v}_j = 0, \\
\mathbf{f}_{ij}^\star + \mathbf{f}_{ji}^\star = \mathbf{0}, \qquad (\mathbf{x}_i - \mathbf{y})^\star \times \mathbf{f}_{ij}^\star + (\mathbf{x}_j - \mathbf{y})^\star \times \mathbf{f}_{ji}^\star = \mathbf{0},
\end{gathered} \tag{9.10}
$$

the first of which must hold for all rotations $\mathbf{Q}$. By $(9.10)_2$,

$$
(\mathbf{x}_i - \mathbf{y})^\star \times \mathbf{f}_{ij}^\star + (\mathbf{x}_j - \mathbf{y})^\star \times \mathbf{f}_{ji}^\star = (\mathbf{x}_i - \mathbf{y})^\star \times \mathbf{f}_{ij}^\star - (\mathbf{x}_j - \mathbf{y})^\star \times \mathbf{f}_{ij}^\star = \mathbf{r}_{ij}^\star \times \mathbf{f}_{ij}^\star,
$$

where, analogous to $(3.7)_1$, $\mathbf{r}_{ij}^\star = \mathbf{x}_i^\star - \mathbf{x}_j^\star$ is the vector directed from p_j to p_i with respect to the starred frame, so that $(9.10)_3$ is equivalent to

$$
\mathbf{r}_{ij}^\star \times \mathbf{f}_{ij}^\star = \mathbf{0}. \tag{9.11}
$$

Bearing in mind that p_i and $p_j \neq p_i$ are generic particles in $\mathcal{B}$, $(9.10)_{1,2}$ and (9.11) must hold for all $i, j = 1, \dots, N$ with $j \neq i$.

Observe that $(9.10)_2$ and (9.11) involve only quantities referred to in the starred frame; they must therefore be valid in any frame. The stars can therefore be dropped from $(9.10)_2$ and (9.11). Doing so, $(9.10)_2$ implies that the forces $\mathbf{f}_{ij}$ and $\mathbf{f}_{ji}$ exerted between any two particles p_i and p_j of $\mathcal{B}$ must obey the *law of mutual-action*

$$
\mathbf{f}_{ij} = -\mathbf{f}_{ji}, \tag{9.12}
$$

while (9.11) implies that the force exterted $\mathbf{f}_{ij}$ on any particle p_i of $\mathcal{B}$ by any other particle p_j of $\mathcal{B}$ must satisfy $\mathbf{r}_{ij} \times \mathbf{f}_{ij} = \mathbf{0}$ and, thus, obey the *law of collinearity*

$$
\mathbf{f}_{ij} = f_{ij}\mathbf{n}_{ij}, \tag{9.13}
$$

where f_{ij} necessarily obeys

$$
f_{ij} = \mathbf{f}_{ij} \cdot \mathbf{n}_{ij} \qquad \text{and} \qquad f_{ij}^2 = |\mathbf{f}_{ij}|^2. \tag{9.14}
$$

While (9.12) requires that $\mathbf{f}_{ij}$ and $\mathbf{f}_{ji}$ be of equal magnitude and of opposite direction, (9.13) requires that $\mathbf{f}_{ij}$ be either parallel or antiparallel to $\mathbf{n}_{ij}$.

9.3. **Frame-indifference of the particle-particle forces.** By (3.9), $\mathbf{n}_{ji} = -\mathbf{n}_{ij}$. Thus, the mutual-action and collinearity laws (9.12) and (9.13) combine to yield $(f_{ij} - f_{ji})\mathbf{n}_{ij} = \mathbf{0}$ and, since $\mathbf{n}_{ij}$ is a unit vector, f_{ij} and f_{ji} must obey[4]

$$
f_{ji} = f_{ij}. \tag{9.15}
$$

Consider $(9.10)_1$. By the law (9.12) of mutual action, $(9.10)_1$ simplifies to

$$
(\mathbf{f}_{ij}^\star - \mathbf{Q}\mathbf{f}_{ij}) \cdot \mathbf{Q}\mathbf{v}_{ij} = 0. \tag{9.16}
$$

Defining $f_{ij}^\star = \mathbf{f}_{ij}^\star \cdot \mathbf{Q}\mathbf{n}_{ij}$ and using the consequences

$$
\mathbf{f}_{ij} = f_{ij}\mathbf{n}_{ij} \qquad \text{and} \qquad \mathbf{f}_{ij}^\star = f_{ij}^\star \mathbf{n}_{ij}^\star = f_{ij}^\star \mathbf{Q}\mathbf{n}_{ij}, \tag{9.17}
$$

of $(3.10)_1$, (8.4), (9.11), and (9.13) in (9.16) yields

$$
(f_{ij}^\star - f_{ij})\mathbf{Q}\mathbf{n}_{ij} \cdot \mathbf{Q}\mathbf{v}_{ij} = (f_{ij}^\star - f_{ij})\mathbf{n}_{ij} \cdot \mathbf{v}_{ij} = (f_{ij}^\star - f_{ij})s_{ij} = 0, \tag{9.18}
$$

which holds trivially for $s_{ij} = 0$ and for all choices of $s_{ij} \neq 0$ only if

$$
f_{ij}^\star = f_{ij}, \tag{9.19}
$$

[4]Pitteri [12, 13] credits Bressan [2] with the derivation of the conditions (9.13) and (9.15) but states that the result requires the assumption that all inertial frames are physically indistinguishable. The developments presented here demonstrate that (9.13) and (9.15) hold regardless of the nature of the underlying frame.

so that f_{ij} is frame-indifferent. Granted (9.12) and (9.13), the relation $(9.10)_1$ implies that f_{ij} is frame-indifferent. Conversely, granted (9.12) and (9.13), if f_{ij} is frame-indifferent, (9.16) and (9.17) imply $(9.10)_1$. Hence, (9.19) is necessary and sufficient to ensure satisfaction of $(9.10)_1$. By (8.4), (9.13), and (9.19), the force $\mathbf{f}_{ij}$ exerted by a particle p_j in $\mathcal{B}$ on a particle $p_j \neq p_i$ in $\mathcal{B}$ transforms under a change-of-frame to

$$\mathbf{f}^{\star}_{ij} = \mathbf{Q}\mathbf{f}_{ij} \tag{9.20}$$

and, thus, by $(8.1)_2$, is frame-indifferent. Since (9.19) is necessary and sufficient to ensure satisfaction of $(9.10)_1$ so also is (9.20).

Consider the moment $\mathbf{m}^{\star}_{ij}(\mathbf{y})$, about $\mathbf{y}$, exerted on a p_i under a change-of-frame. By $(5.1)_1$, (7.4) with $\mathbf{A} = \mathbf{Q}$, and (9.20),

$$\begin{aligned} \mathbf{m}^{\star}_{ij}(\mathbf{y}) &= (\mathbf{x}_i - \mathbf{y})^{\star} \times \mathbf{f}^{\star}_{ij} \\ &= \mathbf{Q}(\mathbf{x}_i - \mathbf{y}) \times \mathbf{Q}\mathbf{f}_{ij} \\ &= \mathbf{Q}[(\mathbf{x}_i - \mathbf{y}) \times \mathbf{f}_{ij}] \\ &= \mathbf{Q}\mathbf{m}_{ij}(\mathbf{y}). \end{aligned} \tag{9.21}$$

Thus, by $(8.1)_2$, $\mathbf{m}^{\star}_{ij}(\mathbf{y})$ is, like $\mathbf{f}_{ij}$, frame-indifferent.

The process of arriving at the results of §§9.2–9.3 stands in sharp contrast to standard practice. Unlike conventional developments of particle mechanics, where the laws (9.12) and (9.13) of mutual-action and collinearity and the frame-indifference (9.20) of the particle-particle forces are introduced as postulates,[5] here these properties of the particle-particle forces arise from the premise that the internal power $w_{\rm int}(\mathcal{P})$, as defined in (6.2), is frame-indifferent for all particle subsystems $\mathcal{P}$ and are, therefore, derived consequences of the theory. This distinction cannot be overemphasized.

9.4. **Frame-indifferent expression for the internal power.** Using the law (9.12) of mutual action in the expression (6.2) defining the internal power yields

$$\begin{aligned} w_{\rm int}(\mathcal{P}) &= \sum_{p_i \in \mathcal{P}} \sum_{p_j \in \mathcal{P}\setminus\{p_i\}} \mathbf{f}_{ij} \cdot \mathbf{v}_i \\ &= \tfrac{1}{2} \sum_{p_i \in \mathcal{P}} \sum_{p_j \in \mathcal{P}} (\mathbf{f}_{ij} - \mathbf{f}_{ji}) \cdot \mathbf{v}_i \\ &= \tfrac{1}{2} \sum_{p_i \in \mathcal{P}} \sum_{p_j \in \mathcal{P}} \mathbf{f}_{ij} \cdot \mathbf{v}_{ij}. \end{aligned}$$

Thus, on noting that, by (3.10) and the law (9.13) of collinearity,

$$\mathbf{f}_{ij} \cdot \mathbf{v}_{ij} = f_{ij}\mathbf{n}_{ij} \cdot \mathbf{v}_{ij} = f_{ij} s_{ij},$$

the power expended within $\mathcal{P}$ admits the alternative expression

$$w_{\rm int}(\mathcal{P}) = \tfrac{1}{2} \sum_{p_i \in \mathcal{P}} \sum_{p_j \in \mathcal{P}} f_{ij} s_{ij}. \tag{9.22}$$

[5] See, for instance, Goldstein [6]. In such treatments, (9.12) on its own is often referred to as the "*weak form of Newton's third law*" while (9.12) and (9.13) are together referred to as the "*strong form of Newton's third law.*"

Since, by (8.6) and (9.19), s_{ij} and f_{ij} are frame-indifferent, so also is the product $f_{ij}s_{ij}$. The representation (9.22) of the internal power $w_{\text{int}}(\mathcal{P})$ is therefore intrinsically frame-indifferent.

10. **Consequences of assuming the the external power is frame-indifferent.** The fundamental consequences of assuming that the external power be frame-indifferent, for all particle subsystems, are now developed. In view of the definitions (6.3) and $(8.1)_1$, this assumption requires that

$$w^{\star}_{\text{ext}}(\mathcal{P}) = w_{\text{ext}}(\mathcal{P}) \tag{10.1}$$

for any particle subsystem $\mathcal{P}$ of $\mathcal{B}$, where

$$w^{\star}_{\text{ext}}(\mathcal{P}) = \sum_{p_i\in\mathcal{P}}\sum_{p_j\in\mathcal{P}'} \mathbf{f}^{\star}_{ij}\cdot\mathbf{v}^{\star}_i + \sum_{p_i\in\mathcal{P}} (\mathbf{f}^{\text{ext}}_i)^{\star}\cdot\mathbf{v}^{\star}_i, \tag{10.2}$$

is the external power expended on $\mathcal{P}$ under a change-of-frame.

10.1. **Preliminary calculations.** Calculations analogous to those leading to the difference (9.6) show that

$$\begin{aligned} 0 &= w^{\star}_{\text{ext}}(\mathcal{P}) - w_{\text{ext}}(\mathcal{P}) \\ &= \sum_{p_k\in\mathcal{P}} [(\mathbf{f}^{\text{ext}}_k)^{\star} - \mathbf{Q}\mathbf{f}^{\text{ext}}_k]\cdot\mathbf{Q}\mathbf{v}_k + \Big\{\sum_{p_i\in\mathcal{P}}\sum_{p_j\in\mathcal{P}'}\mathbf{f}^{\star}_{ij} + \sum_{p_i\in\mathcal{P}}(\mathbf{f}^{\text{ext}}_i)^{\star}\Big\}\cdot\mathbf{c} \\ &\qquad + \Big\{\sum_{p_i\in\mathcal{P}}\sum_{p_j\in\mathcal{P}'}(\mathbf{x}_i-\mathbf{y})^{\star}\times\mathbf{f}^{\star}_{ij} + \sum_{p_i\in\mathcal{P}}(\mathbf{x}_i-\mathbf{y})^{\star}\times(\mathbf{f}^{\text{ext}}_i)^{\star}\Big\}\cdot\mathbf{Q}\boldsymbol{\omega}. \end{aligned} \tag{10.3}$$

10.2. **Laws of balance for forces and moments.** An argument analogous to that employed in deriving (9.10) from (9.6) shows that the terms on the right-hand side of (10.3) must vanish separately. This yields three conditions, the first of which,

$$[(\mathbf{f}^{\text{ext}}_i)^{\star} - \mathbf{Q}\mathbf{f}^{\text{ext}}_i]\cdot\mathbf{Q}\mathbf{v}_i = 0, \tag{10.4}$$

must hold for each particle p_i in $\mathcal{B}$ and all rotations $\mathbf{Q}$, and the second and third of which,

$$\begin{aligned} \sum_{p_i\in\mathcal{P}}\sum_{p_j\in\mathcal{P}'}\mathbf{f}^{\star}_{ij} + \sum_{p_i\in\mathcal{P}}(\mathbf{f}^{\text{ext}}_i)^{\star} &= \mathbf{0} \\ \sum_{p_i\in\mathcal{P}}\sum_{p_j\in\mathcal{P}'}(\mathbf{x}_i-\mathbf{y})^{\star}\times\mathbf{f}^{\star}_{ij} + \sum_{p_i\in\mathcal{P}}(\mathbf{x}_i-\mathbf{y})^{\star}\times(\mathbf{f}^{\text{ext}}_i)^{\star} &= \mathbf{0}, \end{aligned} \tag{10.5}$$

must hold for all particle subsystems $\mathcal{P}$. Since (10.5) involve only quantities referred to the starred frame, they must be valid in any frame. The stars may therefore be dropped from (10.5). Doing so and invoking the definitions (4.2) and (5.2) of the forces and moments, with the particular choice $\mathcal{Q} = \mathcal{P}'$, yields the *law of force balance*

$$\mathbf{f}(\mathcal{P},\mathcal{P}') + \mathbf{f}^{\text{ext}}(\mathcal{P}) = \mathbf{0} \tag{10.6}$$

and the *law of moment balance*

$$\mathbf{m}(\mathcal{P},\mathcal{P}';\mathbf{y}) + \mathbf{m}^{\text{ext}}(\mathcal{P};\mathbf{y}) = \mathbf{0} \tag{10.7}$$

for any particle subsystem $\mathcal{P}$.

10.3. **Frame-indifference of the environmental forces.** By the frame-indifference (9.20) of the force $\mathbf{f}_{ij}$ exerted on particle p_i by particle $p_j \neq p_i$ and $(4.2)_1$, again with the particular choice $\mathcal{Q} = \mathcal{P}'$, it follows that, under a change-of-frame involving a rotation $\mathbf{Q}$,

$$\sum_{p_i \in \mathcal{P}} \sum_{p_j \in \mathcal{P}'} \mathbf{f}_{ij}^{\star} = \sum_{p_i \in \mathcal{P}} \sum_{p_j \in \mathcal{P}'} \mathbf{Q}\mathbf{f}_{ij} = \mathbf{Q}\Big\{ \sum_{p_i \in \mathcal{P}} \sum_{p_j \in \mathcal{P}'} \mathbf{f}_{ij} \Big\} = \mathbf{Q}\mathbf{f}(\mathcal{P}, \mathcal{P}');$$

$(10.5)_1$ and (10.6) therefore imply that

$$\sum_{p_k \in \mathcal{P}} (\mathbf{f}_k^{\text{ext}})^{\star} = \mathbf{Q}\mathbf{f}^{\text{ext}}(\mathcal{P}) \tag{10.8}$$

for any particle subsystem $\mathcal{P}$. For the special case where the subsystem $\mathcal{P}$ is a singleton $\{p_i\}$, (10.8) specializes to yield

$$(\mathbf{f}_i^{\text{ext}})^{\star} = \mathbf{Q}\mathbf{f}_i^{\text{ext}}, \tag{10.9}$$

whereby the force $\mathbf{f}_i^{\text{ext}}$ exerted on a generic particle p_i of $\mathcal{B}$ by agencies external to $\mathcal{B}$ is frame-indifferent. Importantly, (10.9) ensures satisfaction of the first condition (10.4) ensuing from the requirement that (10.3) vanish. Recall that $\mathbf{f}_i^{\text{ext}}$ incorporates external forces.

In view of (10.9), a calculation analogous to (9.21) shows that

$$[\mathbf{m}_i^{\text{ext}}(\mathbf{y})]^{\star} = \mathbf{Q}\mathbf{m}_i^{\text{ext}}(\mathbf{y}). \tag{10.10}$$

Thus, the moment, about $\mathbf{y}$, exerted on p_i by external agencies is frame-indifferent.

The process leading to the results of §§10.2–10.3 stands in sharp contrast to standard practice in the development of theories for particle systems. For instance (10.6) and (10.9) are commonly imposed as postulates and (10.7) is commonly derived under the assumption of collinearity.[6] Instead, the balance laws (10.6) and (10.7) for forces and moments and the frame-indifference (10.9) of the environmental forces result here from the premise (10.1) that the internal power $w_{\text{int}}(\mathcal{P})$ is frame-indifferent for all particle subsystems $\mathcal{P}$ and, therefore, are derived consequences of the theory. As with the laws of mutual-action and collinearity and the frame-indifference of the particle-particle forces, this distinction cannot be overemphasized.

11. Corollaries of the balance laws.

11.1. **Balances for the system.** By (4.3) and (5.3), choosing $\mathcal{P} = \mathcal{B}$ in the laws (10.6) and (10.7) of force balance and moment balance yields balances

$$\mathbf{f}^{\text{ext}}(\mathcal{B}) = \mathbf{0} \tag{11.1}$$

and

$$\mathbf{m}^{\text{ext}}(\mathcal{B}) = \mathbf{0} \tag{11.2}$$

of forces and moments for the entire particle system $\mathcal{B}$. The balance laws for forces and moments for arbitrary particle subsystems therefore imply that the net external forces and moments exerted on the complete particle system must vanish. When

[6] See, for instance, Goldstein [6]. Conventional treatments generally impose force (and the ensuing moment) balances for individual particles. When summed over a particle subsystem, such particle-wise balances yield the system-wise balances (10.6) and (10.7). A simple argument, provided here in Section 11.4, demonstrates that the system-wise balances (10.6) and (10.7) encompass the conventional particle-wise as special cases. See also the essay "Whence the law of the moment of momentum" in [16] explaining that (10.7) is to be postulated alongside (10.6), and cannot, in general, be derived from (10.6).

interpreting the balances (11.1) and (11.2) it is important to bear in mind that $\mathbf{f}^{\rm ext}(\mathcal{B})$ and, therefore (by $(5.1)_2$ and $(5.2)_2$), $\mathbf{m}^{\rm ext}(\mathcal{B})$ generally include inertial forces and inertial moments, respectively.

11.2. **Reciprocal balance of forces for disjoint particle subsystems.** Consider the force balance (10.6). Since $(\mathcal{P}')' = \mathcal{P}$, (10.6) implies that

$$\mathbf{f}(\mathcal{P}', \mathcal{P}) + \mathbf{f}^{\rm ext}(\mathcal{P}') = \mathbf{0}. \tag{11.3}$$

Thus, since $\mathcal{P} \cup \mathcal{P}' = \mathcal{B}$, the additivity property (4.4), the force balance (10.6) for a generic particle subsystem $\mathcal{P}$, and the force balance (11.1) for the entire particle system $\mathcal{B}$ imply that

$$\mathbf{f}(\mathcal{P}, \mathcal{P}') + \mathbf{f}(\mathcal{P}', \mathcal{P}) = \mathbf{0} \tag{11.4}$$

for all particle subsystems $\mathcal{P}$. The resultant force exerted on subsystem $\mathcal{P}$ by the remainder $\mathcal{P}'$ of the system must therefore be equal in magnitude but of opposite direction to the resultant force exerted on $\mathcal{P}'$ by $\mathcal{P}$.

Next, since, for any disjoint pair $\mathcal{P}$ and $\mathcal{Q}$ of particle subsystems,

$$(\mathcal{P} \cup \mathcal{Q})' \cup \mathcal{Q} = \mathcal{P}' \qquad \text{and} \qquad (\mathcal{P} \cup \mathcal{Q})' \cup \mathcal{P} = \mathcal{Q}', \tag{11.5}$$

the biadditivity properties (4.5) and (4.6) imply that

$$\begin{aligned} \mathbf{f}(\mathcal{P}, \mathcal{P}') + \mathbf{f}(\mathcal{Q}, \mathcal{Q}') &= \mathbf{f}(\mathcal{P}, (\mathcal{P} \cup \mathcal{Q})' \cup \mathcal{Q}) + \mathbf{f}(\mathcal{Q}, (\mathcal{P} \cup \mathcal{Q})' \cup \mathcal{P}) \\ &= \mathbf{f}(\mathcal{P}, (\mathcal{P} \cup \mathcal{Q})') + \mathbf{f}(\mathcal{P}, \mathcal{Q}) + \mathbf{f}(\mathcal{Q}, (\mathcal{P} \cup \mathcal{Q})') + \mathbf{f}(\mathcal{Q}, \mathcal{P}) \\ &= \mathbf{f}(\mathcal{P} \cup \mathcal{Q}, (\mathcal{P} \cup \mathcal{Q})') + \mathbf{f}(\mathcal{P}, \mathcal{Q}) + \mathbf{f}(\mathcal{Q}, \mathcal{P}). \end{aligned}$$

Thus, by the force balance (10.6) and the additivity property (4.4),

$$\begin{aligned} \mathbf{f}(\mathcal{P}, \mathcal{Q}) + \mathbf{f}(\mathcal{Q}, \mathcal{P}) &= -\mathbf{f}(\mathcal{P} \cup \mathcal{Q}, (\mathcal{P} \cup \mathcal{Q})') + \mathbf{f}(\mathcal{P}, \mathcal{P}') + \mathbf{f}(\mathcal{Q}, \mathcal{Q}') \\ &= \mathbf{f}^{\rm ext}(\mathcal{P} \cup \mathcal{Q}) - \mathbf{f}^{\rm ext}(\mathcal{P}) - \mathbf{f}^{\rm ext}(\mathcal{Q}) \\ &= \mathbf{f}^{\rm ext}(\mathcal{P} \cup \mathcal{Q}) - \mathbf{f}^{\rm ext}(\mathcal{P} \cup \mathcal{Q}) \\ &= \mathbf{0}. \end{aligned}$$

For any pair $\mathcal{P}$ and $\mathcal{Q}$ of disjoint particle subsystems of $\mathcal{B}$, $\mathbf{f}$ must therefore obey the *reciprocal balance of forces*

$$\mathbf{f}(\mathcal{P}, \mathcal{Q}) = -\mathbf{f}(\mathcal{Q}, \mathcal{P}). \tag{11.6}$$

This result, which is substantially stronger than (11.4), implies that the resultant force exerted on any subsystem $\mathcal{P}$ by some other disjoint subsystem $\mathcal{Q}$ be equal in magnitude but of opposite direction to the resultant force exerted on $\mathcal{Q}$ by $\mathcal{P}$. The choice $\mathcal{Q} = \mathcal{P}'$ reduces (11.6) to (11.4), whereby (11.4) is but a special case of (11.6).

The reciprocal balance of forces (11.6) encompasses the law (9.12) of mutual-action as a special case. To confirm this, consider two elements p_i and $p_j \neq p_j$ of a particle system $\mathcal{B}$. On defining the singleton sets $\mathcal{P} = \{p_i\}$ and $\mathcal{Q} = \{p_j\}$, it follows from $(4.2)_1$ that $\mathbf{f}(\mathcal{P}, \mathcal{Q}) = \mathbf{f}(\{p_i\}, \{p_j\}) = \mathbf{f}_{ij}$ and that $\mathbf{f}(\mathcal{Q}, \mathcal{P}) = \mathbf{f}(\{p_j\}, \{p_i\}) = \mathbf{f}_{ji}$, whereby (11.6) specializes to give the law (9.12) of mutual-action.

11.3. **Reciprocal balance of moments for disjoint particle subsystems.** An argument analogous to that leading from (10.6) to (11.4), but using (5.4), (10.7), and (11.2), yields

$$\mathbf{m}(\mathcal{P},\mathcal{P}';\mathbf{y})+\mathbf{m}(\mathcal{P}',\mathcal{P};\mathbf{y})=\mathbf{0} \tag{11.7}$$

for all particle subsystems $\mathcal{P}$ of $\mathcal{B}$ and all points $\mathbf{y}$. Further, an argument analogous to that leading from (11.4) to (11.6), but using the biadditivity properties (5.5)–(5.6) of $\mathbf{m}(\cdot,\cdot;\mathbf{y})$ for each point $\mathbf{y}$ and invoking (11.7), yields the *reciprocal balance of moments*

$$\mathbf{m}(\mathcal{P},\mathcal{Q};\mathbf{y})=-\mathbf{m}(\mathcal{Q},\mathcal{P};\mathbf{y}) \tag{11.8}$$

for all disjoint particle subsystems $\mathcal{P}$ and $\mathcal{Q}$ of $\mathcal{B}$ and each point $\mathbf{y}$. The choice $\mathcal{Q}=\mathcal{P}'$ reduces (11.8) to (11.7), whereby (11.7) is but a special case of (11.8).

The reciprocal balance of moments encompasses the law of collinearity as a special case. Namely, an argument analogous to leading from (11.4) to (11.6) applied to (11.8) yields $\mathbf{m}_{ij}(\mathbf{y})=-\mathbf{m}_{ji}(\mathbf{y})$ for all $i\neq j$ and all points $\mathbf{y}$. Thus, by $(5.1)_1$ and (11.6), $(\mathbf{x}_i-\mathbf{y})\times\mathbf{f}_{ij}=-(\mathbf{x}_j-\mathbf{y})\times\mathbf{f}_{ji}=(\mathbf{x}_j-\mathbf{y})\times\mathbf{f}_{ij}$ for all $i\neq j$ and for each point $\mathbf{y}$, from which the law (9.13) of collinearity ensues.

11.4. **Laws of force and moment balance for individual particles.** For the special case where subsystem $\mathcal{P}$ is a singleton $\{p_i\}$, the laws (10.6) and (10.7) of force balance and moment balance specialize to yield the more familiar relations

$$\sum_{p_j\in\mathcal{B}\setminus\{p_i\}}\mathbf{f}_{ij}+\mathbf{f}_i^{\rm ext}=\mathbf{0} \tag{11.9}$$

and, for any point $\mathbf{y}$,

$$(\mathbf{x}_i-\mathbf{y})\times\Big\{\sum_{p_j\in\mathcal{B}\setminus\{p_i\}}\mathbf{f}_{ij}+\mathbf{f}_i^{\rm ext}\Big\}=\mathbf{0}. \tag{11.10}$$

Computing the moment of each term in (11.9) with $\mathbf{x}_i-\mathbf{y}$ obviously yields (11.10). Moment balance for an individual particle is therefore a consequence of force balance for that particle.[7]

Summing the *particlewise* force and moment balances (11.9) and (11.10) over the particles belonging to a subsystem $\mathcal{P}$ leads directly to the laws (10.6) and (10.7) of force and moment balance for $\mathcal{P}$. Hence, the requirement that (11.9) and (11.10) hold for each p_i in $\mathcal{B}$ is necessary and sufficient for (10.6) and (10.7) to hold for all $\mathcal{P}$ in $\mathcal{B}$. In this sense, the particle-based statements of balance are equivalent to the subsystem-based statements.

12. **Law of power balance.** Let p_i belong to a particle subsystem $\mathcal{P}$ of $\mathcal{B}$. Then, since

$$\mathcal{B}\setminus\{p_i\}=(\mathcal{P}\cup\mathcal{P}')\setminus\{p_i\}=(\mathcal{P}\setminus\{p_i\})\cup\mathcal{P}',$$

[7]For systems of structured particles—e.g., systems of spherical particles with spin and systems of nonspherical particles (both with nonnegligible moments of inertia)—the moment balance includes additional terms and yields a richer condition leading to a particle-based statement of moment balance that *cannot* simply be derived from the local consequence of the particle-based statement of force balance.

the definition (6.2) of the internal power expended within $\mathcal{P}$ and the force balance (11.9) for a generic particle p_i imply that

$$\begin{aligned} w_{\text{int}}(\mathcal{P}) &= \sum_{p_i \in \mathcal{P}} \sum_{p_j \in \mathcal{P}\setminus\{p_i\}} \mathbf{f}_{ij} \cdot \mathbf{v}_i \\ &= -\sum_{p_i \in \mathcal{P}} \sum_{p_j \in \mathcal{P}'} \mathbf{f}_{ij} \cdot \mathbf{v}_i + \sum_{p_i \in \mathcal{P}} \sum_{p_j \in \mathcal{B}\setminus\{p_i\}} \mathbf{f}_{ij} \cdot \mathbf{v}_i \\ &= -\sum_{p_i \in \mathcal{P}} \sum_{p_j \in \mathcal{P}'} \mathbf{f}_{ij} \cdot \mathbf{v}_i - \sum_{p_i \in \mathcal{P}} \mathbf{f}_i^{\text{ext}} \cdot \mathbf{v}_i, \end{aligned}$$

which, with the definition (6.3) of the power expended on $\mathcal{P}$ by external agencies, yields the *law of power balance*

$$w_{\text{int}}(\mathcal{P}) = -w_{\text{ext}}(\mathcal{P}) \tag{12.1}$$

for any particle subsystem $\mathcal{P}$ of $\mathcal{B}$. Hence, within the framework developed here, power balance for a particle subsystem arises as a consequence of the assumed frame-indifference of the internal and external power expenditures.

13. Digression on inertia.

13.1. Mass. Linear and angular momenta. Inertial frames. Consider a particle system $\mathcal{B}$. The definition of an inertial frame is based on the existence of a relation that assigns to each particle p_i in $\mathcal{B}$ a (constant) *mass* $m_i > 0$. It is assumed that m_i is frame-indifferent, so that, under a change-of-frame,

$$m_i^{\star} = m_i. \tag{13.1}$$

Given m_i, the *linear momentum* $\mathbf{p}_i$ of p_i and the *angular momentum* $\mathbf{l}_i$ of p_i, about a point $\mathbf{y}$, are defined by

$$\mathbf{p}_i = m_i \mathbf{v}_i \qquad \text{and} \qquad \mathbf{l}_i = (\mathbf{x}_i - \mathbf{y}) \times \mathbf{p}_i. \tag{13.2}$$

An *inertial frame* is one in which the inertial force $\mathbf{f}_i^{\text{in}}$ entering the decomposition (4.1) of the force $\mathbf{f}_i^{\text{ext}}$ exerted on a particle p_i by all agencies external to $\mathcal{B}$ takes the particular form

$$\mathbf{f}_i^{\text{in}} = -\dot{\mathbf{p}}_i. \tag{13.3}$$

Since the mass m_i of p_i is constant, it follows trivially from (13.2) and (13.3) that $\mathbf{f}_i^{\text{in}} = -m_i \dot{\mathbf{v}}_i = -m_i \mathbf{a}_i$.

13.2. Net mass and momenta for a subsystem. The net mass $m(\mathcal{P})$ of particle subsystem $\mathcal{P}$ is simply the sum

$$m(\mathcal{P}) = \sum_{p_i \in \mathcal{P}} m_i \tag{13.4}$$

of the masses of the particles comprising $\mathcal{P}$. Similarly, by (13.2), the sums

$$\mathbf{p}(\mathcal{P}) = \sum_{p_i \in \mathcal{P}} \mathbf{p}_i \tag{13.5}$$

and

$$\mathbf{l}(\mathcal{P}; \mathbf{y}) = \sum_{p_i \in \mathcal{P}} (\mathbf{x}_i - \mathbf{y}) \times \mathbf{p}_i, \tag{13.6}$$

represent the net linear momentum of particle subsystem $\mathcal{P}$ and the net angular momentum of $\mathcal{P}$ about $\mathbf{y}$.

As direct consequences of (13.4)–(13.6), the net mass, linear momentum, and angular momentum (about $\mathbf{y}$) of a particle subsystem obey

$$\left.\begin{aligned} m(\mathcal{P}\cup\mathcal{Q}) &= m(\mathcal{P})+m(\mathcal{Q}),\\ \mathbf{p}(\mathcal{P}\cup\mathcal{Q}) &= \mathbf{p}(\mathcal{P})+\mathbf{p}(\mathcal{Q}),\\ \mathbf{l}(\mathcal{P}\cup\mathcal{Q};\mathbf{y}) &= \mathbf{l}(\mathcal{P};\mathbf{y})+\mathbf{l}(\mathcal{Q};\mathbf{y}), \end{aligned}\right\} \tag{13.7}$$

for all disjoint particle subsystems $\mathcal{P}$ and $\mathcal{Q}$ and are, therefore, additive.

In view of (4.1) and (13.3), in an inertial frame, the resultant environmental force $\mathbf{f}^{\text{ext}}(\mathcal{P})$ exerted on $\mathcal{P}$ by agencies external to $\mathcal{B}$ is given by

$$\mathbf{f}^{\text{ext}}(\mathcal{P}) = \sum_{p_i\in\mathcal{P}} \mathbf{f}_i^{\text{ext}} = \sum_{p_i\in\mathcal{P}} \mathbf{f}_i^{\text{ni}} - \sum_{p_i\in\mathcal{P}} \dot{\mathbf{p}}_i = \mathbf{f}^{\text{ni}}(\mathcal{P}) - \dot{\overline{\mathbf{p}(\mathcal{P})}}, \tag{13.8}$$

where

$$\mathbf{f}^{\text{ni}}(\mathcal{P}) = \sum_{p_i\in\mathcal{P}} \mathbf{f}_i^{\text{ni}} \tag{13.9}$$

is the net noninertial force exerted on $\mathcal{P}$ by agencies external to $\mathcal{B}$. Similarly, the resultant external moment exerted on $\mathcal{P}$, about $\mathbf{y}$, by agencies external to $\mathcal{B}$ is given by

$$\mathbf{m}^{\text{ext}}(\mathcal{P};\mathbf{y}) = \mathbf{m}^{\text{ni}}(\mathcal{P};\mathbf{y}) - \dot{\overline{\mathbf{l}(\mathcal{P};\mathbf{y})}}, \tag{13.10}$$

where

$$\mathbf{m}^{\text{ni}}(\mathcal{P}) = \sum_{p_i\in\mathcal{P}} (\mathbf{x}_i-\mathbf{y})\times\mathbf{f}_i^{\text{ni}} \tag{13.11}$$

is the net noninertial moment exerted on $\mathcal{P}$, about $\mathbf{y}$, by agencies external to $\mathcal{B}$.

By (13.6), for any two points $\mathbf{y}$ and $\mathbf{z}$,

$$\mathbf{l}(\mathcal{P};\mathbf{y}) = \mathbf{l}(\mathcal{P};\mathbf{z}) + (\mathbf{z}-\mathbf{y})\times\mathbf{p}(\mathcal{P}). \tag{13.12}$$

13.3. **Law of mass balance.** Choosing $\mathbf{Q}=\mathbf{1}$ and $\mathbf{z}$ such that $\dot{\mathbf{z}}=\mathbf{c}$, with $\mathbf{c}$ constant, in the definition (7.1) of a general change-of-frame yields a *Galilean change-of-frame*. For such a change-of-frame, $\mathbf{Q}^{\top}\dot{\mathbf{Q}}=\mathbf{0}$, so that, by (7.2), the angular velocity $\boldsymbol{\omega}$ vanishes, and the transformation rule (7.6) specializes to

$$\mathbf{v}_i^{\star} = \mathbf{v}_i + \mathbf{c}. \tag{13.13}$$

Bearing in mind the assumed frame-indifference (13.1) of m_i and that $\mathbf{c}$ in (13.13) is constant, (7.1), $(13.2)_1$, (13.4), and (13.5) imply that the time-derivative of the difference between the linear momentum of a particle subsystem $\mathcal{P}$ under a Galilean change-of-frame and the linear momentum of $\mathcal{P}$ equals

$$\dot{\overline{\sum_{p_i\in\mathcal{P}} m_i\mathbf{v}_i^{\star}}} - \dot{\overline{\mathbf{p}(\mathcal{P})}} = \dot{\overline{m(\mathcal{P})}}\mathbf{c}, \tag{13.14}$$

with $\mathbf{v}_i^{\star}$ as given in (13.13).

By (13.14), requiring the time-rate of the linear momentum of a generic particle subsystem $\mathcal{P}$ to be invariant under Galilean changes-of-frame implies the *law of mass balance*

$$\dot{\overline{m(\mathcal{P})}} = 0 \tag{13.15}$$

for all $\mathcal{P}$. Conversely, requiring that (13.15) hold for all $\mathcal{P}$ implies that the time-rate of the linear momentum of all subsystems $\mathcal{P}$ is invariant under Galilean changes-of-frame.

13.4. **Laws of linear and angular momentum balance.** In an inertial frame, (13.8) and (13.10) may be used in the balances (10.6) and (10.7) for forces and moments to yield the *law of linear momentum balance*

$$\overline{\mathbf{p}(\mathcal{P})}^{\boldsymbol{\cdot}} = \mathbf{f}(\mathcal{P},\mathcal{P}') + \mathbf{f}^{\rm ni}(\mathcal{P}) \tag{13.16}$$

and the *law of angular momentum balance*

$$\overline{\mathbf{l}(\mathcal{P};\mathbf{y})}^{\boldsymbol{\cdot}} = \mathbf{m}(\mathcal{P},\mathcal{P}';\mathbf{y}) + \mathbf{m}^{\rm ni}(\mathcal{P};\mathbf{y}) \tag{13.17}$$

for an arbitrary particle subsystem $\mathcal{P}$ of $\mathcal{B}$.

The counterparts of the balances (11.1) and (11.2) of forces and moments for $\mathcal{B}$ are

$$\overline{\mathbf{p}(\mathcal{B})}^{\boldsymbol{\cdot}} = \mathbf{f}^{\rm ni}(\mathcal{B}) \qquad \text{and} \qquad \overline{\mathbf{l}(\mathcal{B};\mathbf{y})}^{\boldsymbol{\cdot}} = \mathbf{m}^{\rm ni}(\mathcal{B};\mathbf{y}), \tag{13.18}$$

and follow either on using (13.8) and (13.10) in (11.1) and (11.2) or on setting $\mathcal{P}=\mathcal{B}$ in (13.16) and (13.17) and employing (4.3) and (5.3). Similarly, the counterparts of the balances (11.9) and (11.10) for a generic particle p_i of $\mathcal{B}$ are

$$m_i\mathbf{a}_i = \sum_{p_j\in\mathcal{B}\setminus\{p_i\}} \mathbf{f}_{ij} + \mathbf{f}_i^{\rm ni} \tag{13.19}$$

and

$$(\mathbf{x}_i-\mathbf{y})\times m_i\mathbf{a}_i = (\mathbf{x}_i-\mathbf{y})\times\bigg\{\sum_{p_j\in\mathcal{B}\setminus\{p_i\}} \mathbf{f}_{ij} + \mathbf{f}_i^{\rm ni}\bigg\}, \tag{13.20}$$

and follow either on using (13.8) and (13.10) in (11.9) and (11.10) or on choosing $\mathcal{P}$ in (13.16) and (13.17) to be the singleton $\{p_i\}$. A discussion completely analogous to that following the force and moment balances (11.9) and (11.10) for a particle p_i applies regarding the relationship between the linear and angular momentum balances (13.19) and (13.20), respectively.

13.5. **Kinetic energy.** Granted an inertial frame, the *kinetic energy* k_i of particle p_i belonging to $\mathcal{B}$ is defined by

$$k_i = \tfrac{1}{2}m_i|\mathbf{v}_i|^2, \tag{13.21}$$

and the sum

$$k(\mathcal{P}) = \sum_{p_i\in\mathcal{P}} k_i \tag{13.22}$$

represents the kinetic energy of particle subsystem $\mathcal{P}$. By (13.22), the kinetic energy of a particle subsystem obeys

$$k(\mathcal{P}\cup\mathcal{Q}) = k(\mathcal{P}) + k(\mathcal{Q}) \tag{13.23}$$

for all disjoint subsystems $\mathcal{P}$ and $\mathcal{Q}$ and is, therefore, additive.

13.6. **Law of kinetic-energy balance.** Since, by (13.21),

$$\dot{k}_i = \mathbf{v}_i\cdot(m_i\mathbf{a}_i), \tag{13.24}$$

using (4.1) and (13.3) in the expression (6.3) for the power expended on $\mathcal{P}$ by agencies external to $\mathcal{P}$ yields

$$\begin{aligned} w_{\rm ext}(\mathcal{P}) &= \sum_{p_i\in\mathcal{P}}\sum_{p_j\in\mathcal{P}'} \mathbf{f}_{ij}\cdot\mathbf{v}_i + \sum_{p_i\in\mathcal{P}} \mathbf{f}_i^{\rm ni}\cdot\mathbf{v}_i - \sum_{p_i\in\mathcal{P}} m_i\mathbf{a}_i\cdot\mathbf{v}_i \\ &= w_{\rm ext}^{\rm ni}(\mathcal{P}) - \overline{k(\mathcal{P})}^{\boldsymbol{\cdot}}, \end{aligned} \tag{13.25}$$

with

$$w^{\text{ni}}_{\text{ext}}(\mathcal{P}) = \sum_{p_i\in\mathcal{P}}\sum_{p_j\in\mathcal{P}'}\mathbf{f}_{ij}\cdot\mathbf{v}_i + \sum_{p_i\in\mathcal{P}}\mathbf{f}^{\text{ni}}_i\cdot\mathbf{v}_i. \tag{13.26}$$

An immediate, but not unimportant, consequence of (6.3) and (13.25) is that the balances (13.16) and (13.17) of linear and angular momentum can be derived by requiring that the difference

$$\dot{\overline{k(\mathcal{P})}} - w^{\text{ni}}_{\text{ext}}(\mathcal{P}) = -w_{\text{ext}}(\mathcal{P}) = w_{\text{int}}(\mathcal{P})$$

be frame-indifferent.

Using (13.25) in the power balance (12.1) yields the *law of kinetic-energy balance*

$$\dot{\overline{k(\mathcal{P})}} = w_{\text{int}}(\mathcal{P}) + w^{\text{ni}}_{\text{ext}}(\mathcal{P}), \tag{13.27}$$

which states that the rate at which the kinetic energy of a particle subsystem $\mathcal{P}$ changes with respect to time is equal to the sum of the power expended within $\mathcal{P}$ and the noninertial contribution of the power expended on $\mathcal{P}$ by agencies external to the system $\mathcal{B}$. Integrating (13.27) with respect to time yields a generalization of the classical work-energy theorem. Whereas the classical version of theorem is generally stated for the system as a whole, the generalization arising from (13.27) is valid for any subsystem $\mathcal{P}$.

13.7. **Results for the center-of-mass of a particle system.**

13.7.1. *Definition of the center-of-mass.* The *center-of-mass* of a particle system $\mathcal{B}$ is the point $\mathbf{x}_c$ satisfying

$$\sum_{p_i\in\mathcal{B}} m_i(\mathbf{x}_i - \mathbf{x}_c) = \mathbf{0}, \tag{13.28}$$

where $\mathbf{x}_i - \mathbf{x}_c$ is the vector directed from the center-of-mass to the point $\mathbf{x}_i$. Given a generic time-independent point $\mathbf{y}$, it follows from (13.28) that

$$\sum_{p_i\in\mathcal{B}} m_i(\mathbf{x}_i - \mathbf{x}_c) = \sum_{p_i\in\mathcal{B}} m_i[(\mathbf{x}_i - \mathbf{y}) - (\mathbf{x}_c - \mathbf{y})] \tag{13.29}$$

and, thus, that the vector directed from $\mathbf{y}$ to the center-of-mass $\mathbf{x}_c$ is given by the mass-weighted average

$$\mathbf{x}_c - \mathbf{y} = \frac{1}{m(\mathcal{B})}\sum_{p_i\in\mathcal{P}} m_i(\mathbf{x}_i - \mathbf{y}) \tag{13.30}$$

of the vectors directed from $\mathbf{y}$ to each of the points $\mathbf{x}_i$ occupied by particles in $\mathcal{B}$.

The velocity $\mathbf{v}_c$ and the acceleration $\mathbf{a}_c$ of the center-of-mass $\mathbf{x}_c$ of a particle system $\mathcal{B}$ are given by

$$\mathbf{v}_c = \dot{\mathbf{x}}_c \qquad \text{and} \qquad \mathbf{a}_c = \dot{\mathbf{v}}_c = \ddot{\mathbf{x}}_c, \tag{13.31}$$

where, since the point $\mathbf{y}$ in (13.30) is time-independent,

$$\mathbf{v}_c = \frac{1}{m(\mathcal{B})}\sum_{p_i\in\mathcal{B}} m_i\mathbf{v}_i = \frac{1}{m(\mathcal{B})}\sum_{p_i\in\mathcal{B}} m_i\dot{\mathbf{x}}_i \tag{13.32}$$

and

$$\mathbf{a}_c = \frac{1}{m(\mathcal{B})}\sum_{p_i\in\mathcal{B}} m_i\dot{\mathbf{v}}_i = \frac{1}{m(\mathcal{B})}\sum_{p_i\in\mathcal{B}} m_i\ddot{\mathbf{x}}_i. \tag{13.33}$$

13.7.2. *Laws of balance for forces and moments exerted on the center-of-mass.* By (13.2)$_1$, (13.5), and (13.32), the net linear momentum of particle system $\mathcal{B}$ is simply

$$\mathbf{p}(\mathcal{B}) = m(\mathcal{B})\mathbf{v}_c. \tag{13.34}$$

Thus, $\dot{\overline{\mathbf{p}(\mathcal{B})}} = m(\mathcal{B})\dot{\mathbf{v}}_c = m(\mathcal{B})\mathbf{a}_c$ and, since, by (4.3), $\mathbf{f}(\mathcal{B},\emptyset) = \mathbf{0}$, setting $\mathcal{P} = \mathcal{B}$ in (13.16) yields

$$m(\mathcal{B})\mathbf{a}_c = \mathbf{f}^{\rm ni}(\mathcal{B}), \tag{13.35}$$

which expresses linear momentum balance for $\mathcal{B}$.

Next, choosing $\mathcal{P} = \mathcal{B}$ in (13.17) yields

$$\dot{\overline{\mathbf{l}(\mathcal{B};\mathbf{y})}} = \mathbf{m}^{\rm ni}(\mathcal{B};\mathbf{y}), \tag{13.36}$$

which expresses angular momentum balance, about any point $\mathbf{y}$, for $\mathcal{B}$.[8] To obtain a balance closer in spirit to (13.35), note that, by (5.8) and (13.12) with $\mathcal{P} = \mathcal{B}$, (13.31), and (13.35),

$$\begin{aligned}
\dot{\overline{\mathbf{l}(\mathcal{B};\mathbf{y})}} - \mathbf{m}^{\rm ni}(\mathcal{B};\mathbf{y}) &= \dot{\overline{\mathbf{l}(\mathcal{B};\mathbf{x}_c)}} + \mathbf{v}_c \times m(\mathcal{B})\mathbf{v}_c + (\mathbf{x}_c - \mathbf{y}) \times m(\mathcal{B})\mathbf{a}_c - \mathbf{m}^{\rm ni}(\mathcal{B};\mathbf{y}) \\
&= \dot{\overline{\mathbf{l}(\mathcal{B};\mathbf{x}_c)}} + (\mathbf{x}_c - \mathbf{y}) \times \mathbf{f}^{\rm ni}(\mathcal{B}) - \mathbf{m}^{\rm ni}(\mathcal{B};\mathbf{y}) \\
&= \dot{\overline{\mathbf{l}(\mathcal{B};\mathbf{x}_c)}} + \mathbf{m}^{\rm ni}(\mathcal{B};\mathbf{y}) - \mathbf{m}^{\rm ni}(\mathcal{B};\mathbf{x}_c) - \mathbf{m}^{\rm ni}(\mathcal{B};\mathbf{y}) \\
&= \dot{\overline{\mathbf{l}(\mathcal{B};\mathbf{x}_c)}} - \mathbf{m}^{\rm ni}(\mathcal{B};\mathbf{x}_c),
\end{aligned}$$

from which it follows that the angular momentum balance (13.36) for $\mathcal{B}$ about any time-independent point $\mathbf{y}$ is actually equivalent to an angular momentum balance

$$\dot{\overline{\mathbf{l}(\mathcal{B};\mathbf{x}_c)}} = \mathbf{m}^{\rm ni}(\mathcal{B};\mathbf{x}_c) \tag{13.37}$$

for $\mathcal{B}$ about the, generally time-dependent, center-of-mass $\mathbf{x}_c$ of $\mathcal{B}$.

13.7.3. *Power relative to the center-of-mass.* Let $\mathbf{z}$ be a time-dependent point. The *net power expended on particle system $\mathcal{B}$ relative to* $\mathbf{z}$, excluding contributions related to inertial forces, is then defined by

$$w(\mathcal{B};\mathbf{z}) = \tfrac{1}{2}\sum_{p_i\in\mathcal{B}}\ \sum_{p_j\in\mathcal{B}\setminus\{p_i\}} \mathbf{f}_{ij}\cdot(\mathbf{v}_{ij} - 2\dot{\mathbf{z}}) + \sum_{p_i\in\mathcal{B}} \mathbf{f}_i^{\rm ni}\cdot(\mathbf{v}_i - \dot{\mathbf{z}}). \tag{13.38}$$

Using (9.22), (13.9), and (13.26), all with $\mathcal{P} = \mathcal{B}$, thus results in

$$\begin{aligned}
w(\mathcal{B};\mathbf{z}) &= w_{\rm int}(\mathcal{B}) + w_{\rm ext}^{\rm ni}(\mathcal{B}) - \Bigg\{\sum_{p_i\in\mathcal{B}}\ \sum_{p_j\in\mathcal{B}\setminus\{p_i\}} \mathbf{f}_{ij} + \sum_{p_i\in\mathcal{B}} \mathbf{f}_i^{\rm ni}\Bigg\}\cdot\dot{\mathbf{z}} \\
&= w_{\rm int}(\mathcal{B}) + w_{\rm ext}^{\rm ni}(\mathcal{B}) - \mathbf{f}^{\rm ni}(\mathcal{B})\cdot\dot{\mathbf{z}},
\end{aligned}$$

from which it follows that

$$w_{\rm int}(\mathcal{B}) + w_{\rm ext}^{\rm ni}(\mathcal{B}) = w(\mathcal{B};\mathbf{z}) + \mathbf{f}^{\rm ni}(\mathcal{B})\cdot\dot{\mathbf{z}}. \tag{13.39}$$

In particular, choosing $\mathbf{z} = \mathbf{x}_c$ in (13.39) yields

$$w_{\rm int}(\mathcal{B}) + w_{\rm ext}^{\rm ni}(\mathcal{B}) = w(\mathcal{B};\mathbf{x}_c) + \mathbf{f}_{\rm ext}^{\rm ni}(\mathcal{B})\cdot\mathbf{v}_c. \tag{13.40}$$

[8] An alternative derivation of (13.36) arises on choosing $\mathcal{P} = \mathcal{B}$ in (13.6), using the local balance (13.19) of linear momentum for a particle p_i to replace the linear momentum $\mathbf{p}_i = m_i\mathbf{a}_i$ and the definition (5.2)$_2$ of the moment, about a point $\mathbf{y}$, exerted on a particle subsystem by entities external to $\mathcal{B}$.

The net noninertial power expended on a particle system is therefore equal to the net power relative to the center-of-mass of the system plus the power expended on moving the center-of-mass under the action of the net force exerted on the system by external entities. Finally, by (13.39) and (13.40),

$$w(\mathcal{B};\mathbf{z}) = w^{\rm ni}_{\rm ext}(\mathcal{B};\mathbf{x}_c) + \mathbf{f}^{\rm ni}(\mathcal{B})\cdot(\mathbf{v}_c - \dot{\mathbf{z}}). \tag{13.41}$$

13.7.4. *Kinetic energy relative to the center-of-mass.* Given a time-dependent point $\mathbf{z}$, the *kinetic energy of a particle system $\mathcal{B}$ relative to* $\mathbf{z}$ is then defined by

$$k(\mathcal{B};\mathbf{z}) = \tfrac{1}{2}\sum_{p_i\in\mathcal{B}} m_i|\mathbf{v}_i - \dot{\mathbf{z}}|^2.$$

Since $|\mathbf{v}_i - \dot{\mathbf{z}}|^2 = |\mathbf{v}_i|^2 - 2\mathbf{v}_i\cdot\dot{\mathbf{z}} + |\dot{\mathbf{z}}|^2$, (13.4), $(13.2)_1$, (13.21), (13.22), and (13.34) imply that

$$\begin{aligned} k(\mathcal{B};\mathbf{z}) &= \tfrac{1}{2}\sum_{p_i\in\mathcal{B}} |\mathbf{v}_i|^2 - \sum_{p_i\in\mathcal{B}} m_i\mathbf{v}_i\cdot\dot{\mathbf{z}} + \tfrac{1}{2}\sum_{p_i\in\mathcal{B}} m_i|\dot{\mathbf{z}}|^2 \\ &= k(\mathcal{B}) - m(\mathcal{B})\mathbf{v}_c\cdot\dot{\mathbf{z}} + \tfrac{1}{2}m(\mathcal{B})|\dot{\mathbf{z}}|^2 \\ &= k(\mathcal{B}) + \tfrac{1}{2}m(\mathcal{B})|\mathbf{v}_c - \dot{\mathbf{z}}|^2 - \tfrac{1}{2}m(\mathcal{B})|\mathbf{v}_c|^2. \end{aligned} \tag{13.42}$$

In particular, choosing $\mathbf{z} = \mathbf{x}_c$ in (13.42) yields

$$k(\mathcal{B}) = k(\mathcal{B};\mathbf{x}_c) + \tfrac{1}{2}m(\mathcal{B})|\mathbf{v}_c|^2. \tag{13.43}$$

The kinetic energy of a particle system $\mathcal{B}$ is thus equal to the kinetic energy of $\mathcal{B}$ relative to its center-of-mass plus the kinetic energy of its center-of-mass. Finally, by (13.42) and (13.43),

$$k(\mathcal{B};\mathbf{z}) = k(\mathcal{B};\mathbf{x}_c) + \tfrac{1}{2}m(\mathcal{B})|\mathbf{v}_c - \dot{\mathbf{z}}|^2. \tag{13.44}$$

13.7.5. *Law of balance for the kinetic-energy relative to the center-of-mass.* In view of the relations (13.40) and (13.43),

$$\begin{aligned} \overline{k(\mathcal{B};\mathbf{x}_c)}^{\boldsymbol\cdot} - w(\mathcal{B};\mathbf{x}_c) &= \dot{k}(\mathcal{B}) - m(\mathcal{B})\mathbf{a}_c\cdot\mathbf{v}_c - w_{\rm int}(\mathcal{B}) - w^{\rm ni}_{\rm ext}(\mathcal{B}) + \mathbf{f}^{\rm ni}(\mathcal{B})\cdot\mathbf{v}_c \\ &= \dot{k}(\mathcal{B}) - w_{\rm int}(\mathcal{B}) - w^{\rm ni}_{\rm ext}(\mathcal{B}) - [m(\mathcal{B})\mathbf{a}_c - \mathbf{f}^{\rm ni}(\mathcal{B})]\cdot\mathbf{v}_c, \end{aligned}$$

whereby (13.27), with $\mathcal{P} = \mathcal{B}$, and (13.35) yield the kinetic-energy balance

$$\overline{k(\mathcal{B};\mathbf{x}_c)}^{\boldsymbol\cdot} = w(\mathcal{B};\mathbf{x}_c) \tag{13.45}$$

for a system $\mathcal{B}$ with respect to its center-of-mass.

14. **Principle of interaction-energy imbalance.** In continuum theories, restricting attention to isothermal processes reduces the energy balance and entropy imbalance to a free-energy imbalance, which expresses the requirement that the time-rate of the net free-energy of any material subset of the body not exceed the rate at which power is expended on the convecting subset by all external forces, inertial forces included.

Because temperature is not germane to the description of particle mechanics presented here, it seems reasonable to impose a notion of imbalance that, similar to the notion of free-energy imbalance mentioned above, serves as a purely mechanical statement of the second law of thermodynamics. This perspective differs sharply from that underlying the works of Gurtin and Williams [9] and Williams [18], where each particle is endowed with an internal energy and heat flux. The view taken here resembles more closely that underlying the work of Pitteri [12, 13], where the underlying theory is a purely mechanical one and statistical mechanics is used to pass to a continuum limit in which an energy balance involving field quantities identified as specific internal-energy and heat flux arise.

As with the force, moment, and power balances, a law of energy imbalance should apply to arbitrary particle subsystems. Toward stating such a law, assume that each pair of particles p_i and $p_j \neq p_i$ possesses an *interaction-energy* ψ_{ij} satisfying

$$\psi_{ij} = \psi_{ji}, \tag{14.1}$$

that embodies the extent to which p_i and p_j are attracted or repelled. The interaction energy between a particle and itself is taken to vanish, so that

$$\psi_{ii} = 0. \tag{14.2}$$

Assume that ψ_{ij} is frame-indifferent, so that, in view of $(7.1)_1$,

$$\psi^{\star}_{ij} = \psi_{ij}, \tag{14.3}$$

where $\psi^{\star}_{ij}$ is the interaction energy between p_i and p_j under the change-of-frame. Bearing in mind (14.2), the net interaction-energy $\psi(\mathcal{P})$ of a particle subsystem $\mathcal{P}$ is given by the sum

$$\psi(\mathcal{P}) = \tfrac{1}{2} \sum_{p_i \in \mathcal{P}} \sum_{p_j \in \mathcal{P}} \psi_{ij}. \tag{14.4}$$

The *principle of interaction-energy imbalance* postulates that, throughout a motion of a particle system, the time-derivative of the net interaction-energy of any particle subsystem cannot exceed the power expended by external agencies acting on the subsystem. In view of definitions (6.3), (13.22), and (14.4), this postulate is equivalent to the requirement that[9]

$$\overline{\psi(\mathcal{P})}^{\boldsymbol{\cdot}} \leq w_{\mathrm{ext}}(\mathcal{P}) \tag{14.5}$$

for all particle subsystems $\mathcal{P}$ and all instants of time.

15. **Interaction-energy inequality.** Using the power balance (12.1) in (14.5) yields the interaction-energy inequality[10]

$$\overline{\psi(\mathcal{P})}^{\boldsymbol{\cdot}} + w_{\mathrm{int}}(\mathcal{P}) \leq 0, \tag{15.1}$$

which holds, of course, for all particle subsystems $\mathcal{P}$ and all instants of time.

Select arbitrarily two points p_i and $p_j \neq p_j$ in $\mathcal{B}$ and consider the doubleton $\mathcal{P} = \{p_i, p_j\}$. For this choice of $\mathcal{P}$, (15.1) specializes to

$$\dot{\psi}_{ij} + f_{ij} s_{ij} \leq 0, \tag{15.2}$$

Recall, $(3.10)_2$ and (9.14), that $s_{ij} = \mathbf{v}_{ij} \cdot \mathbf{n}_{ij}$ and $f_{ij} = \mathbf{f}_{ij} \cdot \mathbf{n}_{ij}$, where, by (3.9), $\mathbf{n}_{ij} = \mathbf{r}_{ij}/r_{ij}$.

[9] For a motion with respect to an inertial frame, (14.5) is replaced by

$$\overline{\psi(\mathcal{P}) + k(\mathcal{P})}^{\boldsymbol{\cdot}} \leq w^{\mathrm{ni}}_{\mathrm{ext}}(\mathcal{P}), \tag{\dagger}$$

with the kinetic energy $k(\mathcal{P})$ and the noninertial component $w^{\mathrm{ni}}_{\mathrm{ext}}(\mathcal{P})$ of $\mathcal{P}$ as defined in (13.22) and (13.26).

[10] Importantly, utilizing the kinetic-energy balance (13.27) to simplify (†) of Footnote 9 leads directly to the interaction-energy inequality (15.1). Unsurprisingly, results obtained on the basis of (15.1) therefore apply *also* for motions with respect to inertial frames.

16. **Constitutive equations.** Constitutive equations that characterize possibly dissipative particle-particle interactions are now developed. This is achieved in the general context where inertial forces are included in the environmental forces as expressed by the decomposition (4.1). All results are therefore independent of the particular nature of the underlying frame.

16.1. **Basic considerations.** A constitutive equation gives the value of a physical field in terms of the values of other fields. Common examples include equations giving stress in terms of strain and heat flux in terms of temperature gradient. In continuum theories, constitutive equations are generally required to be frame-indifferent and thermodynamically compatible. Whereas frame-indifference is assured by imposing the requirement that constitutive equations transform properly under changes of observer, thermodynamic compatibility is assured by ruling out constitutive equations that allow processes in which an inequality akin to (15.2) is violated. It would be somewhat surprising if analogous requirements did not arise and were not valuable for developing constitutive equations in the context of particle mechanics.

Typically, the independent variables entering constitutive equations are kinematical. Consider a particle p_i in $\mathcal{B}$. Among the various kinematical descriptors introduced in §3, consulting (8.2), (8.3), (8.4), and (8.6) shows that $\mathbf{r}_{ij}$, r_{ij}, $\mathbf{n}_{ij}$, and s_{ij} are frame-indifferent and, thus, are, for each $j \neq i$, potential choices for independent constitutive variables. Since ψ_{ij} accounts for interactions strictly between p_i and $p_j \neq p_i$, a dependence upon r_{ij} seems reasonable. Moreover, since the time-rate s_{ij} of r_{ij} appears as a factor in the second term on the left-hand side of (15.2), it seems reasonable to allow for the possibility that, for each $j \neq i$, f_{ij} might also depend on s_{ij}. However, dissipative particle-particle interactions are precluded within a constitutive framework for which ψ_{ij} and f_{ij} depend at most on the instantaneous value of r_{ij}. A comprehensive approach to developing constitutive equations encompassing dissipative interactions would allow the values $\psi_{ij}(t)$ and $f_{ij}(t)$ of ψ_{ij} and f_{ij} at any time t in I terms of the collection $\{r_{ij}(s) : s \in I, s \leq t\}$ of values of r_{ij} at all times in I up to and including t, much as in Coleman's [3] theory for simple materials with memory.[11] Here, the complex dependence on time-history embodied in such constitutive equations is ceded in favor a more modest rate-dependence. Specifically, constitutive equations of the relatively uncomplicated form[12]

$$\psi_{ij} = \hat{\psi}_{ij}(r_{ij}, s_{ij}), \qquad f_{ij} = \hat{f}_{ij}(r_{ij}, s_{ij}), \tag{16.1}$$

are considered, where, to ensure satisfaction of the combined consequence (9.15) of the laws (9.12) and (9.13) of mutual action and collinearity and the symmetry requirement (14.1), $\hat{\psi}_{ij}$ and $\hat{f}_{ij}$ are assumed to be consistent with

$$\hat{\psi}_{ij}(r_{ij}, s_{ij}) = \hat{\psi}_{ji}(s_{ij}, r_{ij}) \qquad \text{and} \qquad \hat{f}_{ij}(r_{ij}, s_{ij}) = \hat{f}_{ji}(r_{ij}, s_{ij}). \tag{16.2}$$

Here, $\hat{\psi}_{ij}$ and $\hat{f}_{ij}$ are called (*constitutive*) *response functions.*

[11]Since Coleman's [3] theory allows for dependence upon the entire time-history, including all times prior to the initial time, as a means to account for processing prior to the initial state, a strict analogy between the type of constitutive theory alluded to here and the theory of simple materials with memory is valid only for the specialization of that theory to materials with virginal initial states.

[12]To clarify, the constitutive equations (16.1) embody the notion that, at any instant t in I, the values $\psi_{ij}(t)$ and $f_{ij}(t)$ of ψ_{ij} and f_{ij} are determined by the values $r_{ij}(t)$ and $s_{ij}(t)$ of the lists r_{ij} and s_{ij} through $\hat{\psi}_{ij}(r_{ij}(t), s_{ij}(t))$ and $\hat{f}_{ij}(r_{ij}(t), s_{ij}(t))$.

Although it might seem contrived to include the relative speeds as independent variables in the constitutive equation (16.1) for the interaction-energy, it is more sensible to allow for such a dependence at the outset rather than to rule it out peremptorily.[13] In the present context, this comprises Truesdell's *principle of equipresence*.

16.2. **Consistency with frame-indifference.** As noted above, the independent constitutive variables r_{ij} and s_{ij} appearing in (16.1) are frame-indifferent; that is, $(r^{\star}_{ij}, s^{\star}_{ij}) = (r_{ij}, s_{ij})$. Hence,

$$\psi^{\star}_{ij} = \hat{\psi}_{ij}(r^{\star}_{ij}, s^{\star}_{ij}) = \hat{\psi}_{ij}(r_{ij}, s_{ij}) = \psi_{ij},$$

$$f^{\star}_{ij} = \hat{f}_{ij}(r^{\star}_{ij}, s^{\star}_{ij}) = \hat{f}_{ij}(r_{ij}, s_{ij}) = f_{ij},$$

and the constitutive equations (16.1) are intrinsically consistent with the requirements (14.3) and (9.19) that ψ_{ij} and f_{ij} be frame-indifferent.

16.3. **Restrictions imposed by the requirement of thermodynamic compatibility.** The constitutive equations (16.1) must not lead to violations of the interaction-energy inequality (15.2). The restrictions imposed by this requirement are now explored. Observe that the function $\hat{f}_{ij}$ admits a decomposition of the form

$$\hat{f}_{ij}(r_{ij}, s_{ij}) = \hat{f}^{\rm eq}_{ij}(r_{ij}) + \hat{f}^{\rm dis}_{ij}(r_{ij}, s_{ij}), \tag{16.3}$$

where $\hat{f}^{\rm eq}_{ij}$ and $\hat{f}^{\rm dis}_{ij}$ are defined, respectively, by

$$\hat{f}^{\rm eq}_{ij}(r_{ij}) = \hat{f}_{ij}(r_{ij}, 0) \tag{16.4}$$

and

$$\hat{f}^{\rm dis}_{ij}(r_{ij}, s_{ij}) = \hat{f}_{ij}(r_{ij}, s_{ij}) - \hat{f}_{ij}(r_{ij}, 0). \tag{16.5}$$

By (16.3), the force $\mathbf{f}_{ij}$ exerted on particle p_i by particle p_j decomposes into the sum

$$\mathbf{f}_{ij} = \mathbf{f}^{\rm eq}_{ij} + \mathbf{f}^{\rm dis}_{ij} \tag{16.6}$$

of equilibrium and dissipative contributions, where, by (9.13),

$$\mathbf{f}^{\rm eq}_{ij} = \hat{f}^{\rm eq}_{ij}(r_{ij})\mathbf{n}_{ij} \qquad \text{and} \qquad \mathbf{f}^{\rm dis}_{ij} = \hat{f}^{\rm dis}_{ij}(r_{ij}, s_{ij})\mathbf{n}_{ij}. \tag{16.7}$$

Consider a process consisting of an arbitrary motion $\boldsymbol{\chi}$ together with interaction energies ψ_{ij} $(i, j = 1, \dots, N)$ and forces $\mathbf{f}_{ij}$ $(i, j = 1, \dots, N,\ i \neq j)$ determined by $\boldsymbol{\chi}$ through (3.1) and the constitutive equations (16.1). The force balance (11.9) for p_i then determines the external forces $\mathbf{f}^{\rm ext}_i$ $(i = 1, \dots, N)$ needed to support the process. As such, assume that $\mathbf{f}^{\rm ext}_i$ $(i = 1, \dots, N)$ is *arbitrarily assignable*. This hypothesis is important: because of it, the law of force balance in no way restricts the class of constitutive equations under consideration. On the other hand, unless these constitutive equations are suitably restricted, not all constitutive processes will be compatible with thermodynamics as embodied by the interaction-energy inequality (15.2). The subclass of constitutive equations of the form (16.1) that are compatible with thermodynamics is determined by requiring that all constitutive processes be consistent with the interaction-energy inequality (15.2). This approach is a simple adaptation of a procedure developed within the context of continuum thermomechanics by Coleman and Noll [4].

[13] Energetic dependence upon the symmetric component of the velocity gradient arises, for instance, in the theory of second-grade fluids, as shown by Dunn and Fosdick [5].

The interaction-energy inequality severely restricts the constitutive equations. To see this, consider an arbitrary constitutive process. Then, by (16.1)$_1$,

$$\dot{\psi}_{ij} = \frac{\partial\hat{\psi}_{ij}(r_{ij},s_{ij})}{\partial r_{ij}} s_{ij} + \frac{\partial\hat{\psi}_{ij}(r_{ij},s_{ij})}{\partial s_{ij}} \dot{s}_{ij}; \tag{16.8}$$

hence, granted constitutive equations of the form (16.1), the interaction-energy inequality (15.2) is equivalent to the requirement that

$$\left\{\frac{\partial\hat{\psi}_{ij}(r_{ij},s_{ij})}{\partial r_{ij}} + \hat{f}^{\rm eq}_{ij}(r_{ij})\right\} s_{ij} + \hat{f}^{\rm dis}_{ij}(r_{ij},s_{ij}) s_{ij} + \frac{\partial\hat{\psi}_{ij}(r_{ij},s_{ij})}{\partial s_{ij}} \dot{s}_{ij} \;\le\; 0 \tag{16.9}$$

for all motions. Given a time t in I, it is possible to construct a motion such that r_{ij}, s_{ij}, and $\dot{s}_{ij}$ take arbitrarily prescribed, and therefore independent, values at t for all $j \ne i$. The coefficients of the rates $\dot{s}_{ij}$, which appear linearly in (16.9), must therefore vanish for each $j \ne i$. Thus, $\partial\hat{\psi}_{ij}(r_{ij},s_{ij})/\partial s_{ij} = 0$ for each $j \ne i$ and there must exist a response function $\tilde{\psi}_i$ of r_{ij} such that

$$\psi_{ij} = \hat{\psi}_{ij}(r_{ij},s_{ij}) = \tilde{\psi}_{ij}(r_{ij}). \tag{16.10}$$

In view of (16.10), the interaction-inequality (16.9) simplifies to

$$\left\{\frac{\partial\tilde{\psi}_{ij}(r_{ij})}{\partial r_{ij}} + \hat{f}^{\rm eq}_{ij}(r_{ij})\right\} s_{ij} + \hat{f}^{\rm dis}_{ij}(r_{ij},s_{ij}) s_{ij} \;\le\; 0. \tag{16.11}$$

Next, fix r_{ij} and replace s_{ij} in (16.11) by αs_{ij} for some $\alpha > 0$. Dividing the resulting inequality by α then yields

$$\left\{\frac{\partial\tilde{\psi}_{ij}(r_{ij})}{\partial r_{ij}} + \hat{f}^{\rm eq}_{ij}(r_{ij})\right\} s_{ij} + \hat{f}^{\rm dis}_{ij}(r_{ij},\alpha s_{ij}) s_{ij} \;\le\; 0. \tag{16.12}$$

By (16.5), $\tilde{f}^{\rm dis}_{ij}(r,0) = 0$ for all choices of r and for all $j \ne i$. Letting $\alpha \to 0$ in (16.12) therefore yields the inequality

$$\left\{\frac{\partial\tilde{\psi}_{ij}(r_{ij})}{\partial r_{ij}} + \hat{f}^{\rm eq}_{ij}(r_{ij})\right\} s_{ij} \;\le\; 0. \tag{16.13}$$

Since, for each $j \ne i$, (16.13) must hold for all s_{ij}, it follows that, for each $j \ne i$,

$$\hat{f}^{\rm eq}_{ij}(r_{ij}) = -\frac{\partial\tilde{\psi}_{ij}(r_{ij})}{\partial r_{ij}}. \tag{16.14}$$

Importantly, (16.14) is consistent with the symmetry constraints embodied by (16.2). An immediate, but significant, consequence of (16.14) is that the first term on the left-hand side of (16.12) vanishes identically. In addition, the residual dissipation inequality

$$\hat{f}^{\rm dis}_{ij}(r_{ij},s_{ij}) s_{ij} \le 0 \tag{16.15}$$

must therefore hold for each $j \ne i$.

In summary, requiring that the constitutive equations (16.1) be compatible with the interaction-energy inequality (15.2) in all processes yields the *thermodynamic restrictions*:

- The interaction energy ψ_{ij} between particles p_i and p_j depends only upon r_{ij}, the distance between p_i and p_j:

$$\psi_{ij} = \tilde{\psi}_{ij}(r_{ij}). \tag{16.16}$$

- The interaction energy determines the equilibrium contribution $\mathbf{f}_{ij}^{\text{eq}}$ to the force exerted on p_i by another particle p_j through the relation

$$\mathbf{f}_{ij}^{\text{eq}} = -\frac{\partial \tilde{\psi}_{ij}(r_{ij})}{\partial r_{ij}} \mathbf{n}_{ij}, \tag{16.17}$$

- The dissipative contribution $\mathbf{f}^{\text{dis}}$ to the force exerted on p_i by another particle p_j has the form

$$\mathbf{f}_{ij}^{\text{dis}} = \hat{f}_{ij}^{\text{dis}}(r_{ij}, s_{ij}) \mathbf{n}_{ij}, \tag{16.18}$$

where $\hat{f}_{ij}^{\text{dis}}$ must obey the inequality

$$\hat{f}_{ij}^{\text{dis}}(r_{ij}, s_{ij}) s_{ij} \leq 0. \tag{16.19}$$

On combining (16.17) and (16.18) with the decomposition (16.6), it follows that the force $\mathbf{f}_{ij}$ exerted on p_i by another particle p_j has the specific form

$$\mathbf{f}_{ij} = -\frac{\partial \tilde{\psi}_{ij}(r_{ij})}{\partial r_{ij}} \mathbf{n}_{ij} + \hat{f}_{ij}^{\text{dis}}(r_{ij}, s_{ij}) \mathbf{n}_{ij}. \tag{16.20}$$

16.4. **Final differential equations.** The final differential equation for a generic particle p_i ensues on using the general constitutive equation (16.20) for the force $\mathbf{f}_{ij}$ exerted on p_i by another particle p_j into the statement (11.9) of force balance for p_i to yield

$$\sum_{p_j \in \mathcal{B} \setminus \{p_i\}} \left\{ \frac{\partial \tilde{\psi}_{ij}(r_{ij})}{\partial r_{ij}} \mathbf{n}_{ij} - \hat{f}_{ij}^{\text{dis}}(r_{ij}, s_{ij}) \mathbf{n}_{ij} \right\} = \mathbf{f}_i^{\text{ext}}. \tag{16.21}$$

The specialization of (16.21) to the case when the underlying frame is inertial, which follows directly from the decomposition (4.1) and the relations $(13.2)_1$ and (13.3) determining the inertial force, is simply

$$m_i \ddot{\mathbf{x}} + \sum_{p_j \in \mathcal{B} \setminus \{p_i\}} \left\{ \frac{\partial \tilde{\psi}_{ij}(r_{ij})}{\partial r_{ij}} \mathbf{n}_{ij} - \hat{f}_{ij}^{\text{dis}}(r_{ij}, s_{ij}) \mathbf{n}_{ij} \right\} = \mathbf{f}_i^{\text{ni}}. \tag{16.22}$$

where m_i denotes the mass of p_i. When dissipative particle-particle interactions are neglected, so that $\hat{f}_{ij}^{\text{dis}}(r_{ij}, s_{ij}) = 0$ for $i, j = 1, \ldots, N$, $i \neq j$, (16.22) is classical. In particular, the system of equations arising in this case is identical to that used by Pitteri [12, 13] in his statistical-mechanical derivation of continuum equations of balance from particle mechanics. Moreover, the system is identical to that obtained by Yavari and Marsden [19] for the case when the ambient space is Euclidean.

16.5. **Sample constitutive equations.**

16.5.1. *Interaction energies and associated equilibrium forces.* The classical law of universal gravitation constitutively determines the interaction energy ψ_{ij} between two particles p_i and p_j of respective mass m_i and m_j and the associated force $\mathbf{f}_{ij}^{\text{eq}}$ exerted on p_i by p_j, viz.,

$$\psi_{ij} = \frac{G m_i m_j}{r_{ij}} \qquad \text{and} \qquad \mathbf{f}_{ij}^{\text{eq}} = -\frac{G m_i m_j}{r_{ij}^2} \mathbf{n}_{ij}, \tag{16.23}$$

where $G > 0$ is the universal gravitational constant. Since the notion of particle mass is tied to the notion of an inertial frame, these relations implicitly assume that the underlying frame is inertial. By (16.23), the net interaction energy $\psi(\mathcal{P})$ of a subsystem $\mathcal{P}$ corresponding to (16.23) is

$$\psi(\mathcal{P}) = \tfrac{1}{2} \sum_{p_i \in \mathcal{P}} \sum_{p_j \in \mathcal{P} \setminus \{p_i\}} \frac{Gm_i m_j}{r_{ij}}, \tag{16.24}$$

which, modulo a sign, is simply the classical gravitational potential energy.

Similarly, the interaction energy ψ_{ij} between particles p_i and p_j connected by a linear spring with constant $k_{ij} > 0$ and reference length $\ell_{ij} > 0$ and the associated force $\mathbf{f}^{\mathrm{eq}}_{ij}$ exerted on p_j by p_j are given by constitutive equations of the form

$$\psi_{ij} = \tfrac{1}{2} k_{ij}(r_{ij} - \ell_{ij})^2 \qquad \text{and} \qquad \mathbf{f}^{\mathrm{eq}}_{ij} = -k_{ij}(r_{ij} - \ell_{ij})\mathbf{n}_{ij}. \tag{16.25}$$

16.5.2. *Dissipative forces.* Provided $\tilde{f}^{\mathrm{dis}}_{ij}$ is a continuous function of s_{ij}, the residual dissipation inequality (16.19) admits a solution of the form

$$\tilde{f}^{\mathrm{dis}}_{ij}(r_{ij}, s_{ij}) = -M_{ij}(r_{ij}, s_{ij})s_{ij}, \qquad M_{ij}(r_{ij}, s_{ij}) \geq 0, \tag{16.26}$$

where M_{ij} is the mobility of particle p_i with respect to particle p_j. Since $r_{ij} = r_{ji}$ and $s_{ji} = s_{ij}$,

$$f_{ji} = -M_{ji}(r_{ji}, s_{ji})s_{ji} = -M_{ji}(r_{ij}, s_{ij})s_{ij}.$$

On recalling from (9.15) that $f_{ij} = f_{ji}$, it follows that

$$M_{ji} = M_{ij}, \tag{16.27}$$

whereby the mobilities of p_i with respect to p_j and of p_j with respect to p_i must be equal. The dependence of M_{ij} on s_{ij} allows for general viscous damping.

The residual dissipation inequality also admits discontinuous solutions. The simplest such solution,

$$\tilde{f}^{\mathrm{dis}}_{ij}(r_{ij}, s_{ij}) = \bar{f}_{ij}(r_{ij}) \frac{s_{ij}}{|s_{ij}|}, \qquad \bar{f}_{ij}(r_{ij}) > 0, \tag{16.28}$$

for all r_{ij}, corresponds to classical Coulomb damping. More generally, the general class of response functions $\tilde{f}^{\mathrm{dis}}_{ij}$ defined so that

$$\left.\begin{aligned} \tilde{f}^{\mathrm{dis}}_{ij}(r_{ij}, s_{ij}) \leq 0 \quad \text{for all } s_{ij} \leq 0, \\ \tilde{f}^{\mathrm{dis}}_{ij}(r_{ij}, s_{ij}) \geq 0 \quad \text{for all } s_{ij} \geq 0, \end{aligned}\right\} \tag{16.29}$$

for all r_{ij}, is consistent with the residual dissipation inequality, incorporates both (16.26) and (16.28) as special cases, and allows for dissipative particle-particle interactions where viscous and frictional damping are combined.

On returning to the example of particles p_i and p_j connected by linear springs, if, in addition to springs, p_i by p_j are connected by a linearly viscous dashpot with damping coefficient γ_{ij}, then the dissipative contribution $\mathbf{f}^{\mathrm{dis}}_{ij}$ to the force exerted on p_i by p_j is given by

$$\mathbf{f}^{\mathrm{dis}}_{ij} = -\gamma_{ij} s_{ij} \mathbf{n}_{ij}, \tag{16.30}$$

where, by $(16.26)_2$ and (16.27), $\gamma_{ij} = \gamma_{ji} \geq 0$.

17. **Summary.** The general framework for classical particle mechanics developed here is based on five simple hypotheses:

H_1: Each particle p_i in a system $\mathcal{B}$ is subject to forces $\mathbf{f}_{ij}$ and $\mathbf{f}^{\rm ext}$ due, respectively, to interactions with other particles in the system and with external entities, inertia included.

H_2: The moments $\mathbf{m}_{ij}(\mathbf{y})$ and $\mathbf{m}_i^{\rm ext}(\mathbf{y})$ exerted on a particle p_i in a system $\mathcal{B}$ are confined to torques associated with the forces $\mathbf{f}_{ij}$ and $\mathbf{f}^{\rm ext}$.

H_3: For any particle subsystem $\mathcal{P}$ of a system $\mathcal{B}$, the net internal and external power expenditures

$$w_{\rm int}(\mathcal{P}) = \sum_{p_i\in\mathcal{P}} \sum_{p_j\in\mathcal{P}\backslash\{p_i\}} \mathbf{f}_{ij}\cdot\mathbf{v}_i$$

and

$$w_{\rm ext}(\mathcal{P}) = \sum_{p_i\in\mathcal{P}} \sum_{p_j\in\mathcal{P}'} \mathbf{f}_{ij}\cdot\mathbf{v}_i + \sum_{p_i\in\mathcal{P}} \mathbf{f}_i^{\rm ext}\cdot\mathbf{v}_i$$

are frame-indifferent.

H_4: Each pair of particles in a system possesses a frame-indifferent interaction energy $\psi_{ij} = \psi_{ji}$ (with $\psi_{ii} = 0$).

H_5: For any particle subsystem $\mathcal{P}$ of a system $\mathcal{B}$, the rate at which the frame-indifferent net interaction-energy

$$\psi(\mathcal{P}) = \tfrac{1}{2} \sum_{p_i\in\mathcal{P}} \sum_{p_j\in\mathcal{P}} \psi_{ij}$$

changes with respect to time cannot exceed the net external power $w_{\rm ext}(\mathcal{P})$ expended on that subsystem.

Most, if not all, of the ingredients encountered in conventional developments of classical particle mechanics are missing from this set of hypotheses. Force and moment balances are not imposed. A choice of frame, inertial or otherwise, is not made. Neither the forces between particles nor the forces exerted on particles by external agencies are presumed to be frame-indifferent. The forces exerted between two particles are not assumed to be of equal magnitude and opposite direction. The force between two particles is not assumed to be collinear with the vector connecting the points occupied by those particles. A constitutive structure determining the particle-particle forces or the interaction energies in terms of kinematical descriptors such as the distances between the points is not invoked. The consequences of the hypotheses stated above are, nevertheless, significant.

Of these hypotheses, H_3 is the most consequential. In particular, the portion of H_3 concerning the internal power $w_{\rm int}(\mathcal{P})$ implies that:

C_1: The forces $\mathbf{f}_{ij}$ and $\mathbf{f}_{ji}$ exerted by p_j on p_i and by p_i on p_j are of equal magnitude and opposite direction, so that

$$\mathbf{f}_{ij} = -\mathbf{f}_{ij}.$$

C_2: Given points $\mathbf{x}_i$ and $\mathbf{x}_j$ corresponding to p_i and p_j, the force $\mathbf{f}_{ij}$ exerted on p_i by p_j is either parallel or antiparallel to $\mathbf{r}_{ij} = \mathbf{x}_i - \mathbf{x}_j$, so that

$$\mathbf{f}_{ij} = f_{ij}\mathbf{n}_{ij},$$

with

$$f_{ij} = \mathbf{f}_{ij}\cdot\mathbf{n}_{ij}, \qquad \mathbf{n}_{ij} = \frac{\mathbf{r}_{ij}}{r_{ij}}, \qquad \text{and} \qquad r_{ij} = |\mathbf{r}_{ij}|.$$

C_3: The force $\mathbf{f}_{ij}$ exerted on p_i by p_j and the associated moment $\mathbf{m}_{ij}$, about an arbitrary point $\mathbf{y}$, are frame-indifferent, so that, for a change-of-frame involving a rotation $\mathbf{Q}$,

$$\mathbf{f}^\star_{ij} = \mathbf{Q}\mathbf{f}_{ij} \qquad \text{and} \qquad \mathbf{m}^\star_{ij}(\mathbf{y}) = \mathbf{Q}\mathbf{m}_{ij}(\mathbf{y}).$$

Consequences C_1 and C_2 are here referred to as the laws of mutual action and collinearity, respectively. C_2 and the first of C_3 imply that f_{ij} is frame-indifferent and that f_{ij} and f_{ji} must be equal. Moreover, these results combine to yield an alternative expression,

$$w_{\rm int}(\mathcal{P}) = \tfrac{1}{2}\sum_{p_i\in\mathcal{P}}\sum_{p_j\in\mathcal{P}} f_{ij}s_{ij}, \qquad s_{ij} = \dot{r}_{ij},$$

for the internal power. Since s_{ij} is frame-indifferent, this expression for $w_{\rm int}(\mathcal{P})$ is intrinsically frame-indifferent.

Next, the most salient consequences of H_3 concerning the external power $w_{\rm ext}(\mathcal{P})$ are:

C_4: The resultant forces $\mathbf{f}(\mathcal{P},\mathcal{P}')$ and $\mathbf{f}^{\rm ext}(\mathcal{P})$ exerted on the particles in any particle subsystem $\mathcal{P}$ of a system $\mathcal{B}$ by the particles in the remainder $\mathcal{P}'$ of $\mathcal{B}$ and by external agencies obey the balance

$$\mathbf{f}(\mathcal{P},\mathcal{P}') + \mathbf{f}^{\rm ext}(\mathcal{P}) = \mathbf{0}.$$

C_5: The resultant moments $\mathbf{m}(\mathcal{P},\mathcal{P}')$ and $\mathbf{m}^{\rm ext}(\mathcal{P})$ exerted, about a point $\mathbf{y}$, on the particles in $\mathcal{P}$ by the remaining particles in $\mathcal{B}$ and by external agencies obey the balance

$$\mathbf{m}(\mathcal{P},\mathcal{P}';\mathbf{y}) + \mathbf{m}^{\rm ext}(\mathcal{P};\mathbf{y}) = \mathbf{0}$$

for any point $\mathbf{y}$.

C_6: The force $\mathbf{f}^{\rm ext}_i$ exerted on p_i by external agencies the associated moment $\mathbf{m}^{\rm ext}_i$, about an arbitrary point $\mathbf{y}$, are frame-indifferent, so that, for a change-of-frame involving a rotation $\mathbf{Q}$,

$$(\mathbf{f}^{\rm ext}_i)^\star = \mathbf{Q}\mathbf{f}^{\rm ext}_i \qquad \text{and} \qquad [\mathbf{m}^{\rm ext}_i(\mathbf{y})]^\star = \mathbf{Q}\mathbf{m}^{\rm ext}_i(\mathbf{y}).$$

C_4 and C_5 constitute force and moment balances for a particle subsystem $\mathcal{P}$. Choosing $\mathcal{P}$ to be a singleton in these subsystemwise balances yields the more familiar particlewise force and moment balances as special cases. Moreover, choosing $\mathcal{P} = \mathcal{B}$ yields force and balances for the system. In an inertial frame, the force and moment balances for arbitrary particle subsystem $\mathcal{P}$ of $\mathcal{B}$ specialize to the linear and angular momentum balances,

$$\overline{\mathbf{p}(\mathcal{P})}^{\boldsymbol{\cdot}} = \mathbf{f}(\mathcal{P},\mathcal{P}') + \mathbf{f}^{\rm ni}(\mathcal{P})$$

and

$$\overline{\mathbf{l}(\mathcal{P};\mathbf{y})}^{\boldsymbol{\cdot}} = \mathbf{m}(\mathcal{P},\mathcal{P}';\mathbf{y}) + \mathbf{m}^{\rm ni}(\mathcal{P};\mathbf{y}),$$

respectively.

An important corollary of C_4 is that the power balance

$$w_{\rm int}(\mathcal{P}) + w_{\rm ext}(\mathcal{P}) = 0$$

must hold for any particle subsystem $\mathcal{P}$. Using the power balance in hypothesis H_5 and restricting attention to the case where $\mathcal{P}$ is a doubleton yields an interaction-energy inequality

$$\dot{\psi}_{ij} + f_{ij}s_{ij} \le 0,$$

for each particle p_i. Along with hypothesis H_4 and the derived frame-indifference of f_{ij}, the interaction-energy inequality forms the basis for the development of thermodynamically consistent constitutive theories along the lines developed in the context of continuum mechanics by Coleman and Noll [4].

While it is certainly possible to develop a general constitutive framework in which the values of ψ_{ij} and f_{ij} at any time t in the interval I depends upon terms of the values of r_{ij} up to and including t, only the simplest possible class of rate-dependent constitutive equations is considered here. In this case, dependence upon r_{ij} is augmented by dependence on s_{ij}. Constitutive equations of the form

$$\psi_{ij} = \hat{\psi}_{ij}(r_{ij}, s_{ij}) \qquad \text{and} \qquad f_{ij} = \hat{f}_{ij}(r_{ij}, s_{ij})$$

are therefore considered. A simple calculation shows that such constitutive equations are inherently frame-indifferent. Since, by C_1 and C_2, $f_{ij} = f_{ji}$, a simple argument shows that f_{ij} can depend at most on r_{ij} and s_{ij}. With this result in mind, stipulating that the constitutive equations be consistent with the interaction-energy inequality in all processes requires that the interaction energy ψ_{ij} be determined strictly by a constitutive equation involving a response function $\tilde{\psi}$ that depends only on r_{ij}. The force $\mathbf{f}_{ij}$ exerted on p_i by another particle p_j therefore admits an additive decomposition

$$\mathbf{f}_{ij} = -\frac{\partial \tilde{\psi}_{ij}(r_{ij})}{\partial r_{ij}}\mathbf{n}_{ij} + \hat{f}_{ij}^{\text{dis}}(r_{ij}, s_{ij})\mathbf{n}_{ij}$$

into equilibrium and dissipative components, both collinear with the unit vector

$$\mathbf{n}_{ij} = \frac{\mathbf{r}_{ij}}{|\mathbf{r}_{ij}|} = \frac{\mathbf{x}_i - \mathbf{x}_j}{|\mathbf{x}_i - \mathbf{x}_j|}.$$

directed from the point $\mathbf{x}_j$ occupied by p_j to the point $\mathbf{x}_i$ occupied by p_i, where the magnitude $\hat{f}_{ij}^{\text{dis}}$ of the dissipative contribution must obey

$$\hat{f}_{ij}^{\text{dis}}(r_{ij}, s_{ij})s_{ij} \le 0.$$

The final differential equation for a generic particle p_i ensues on using the general constitutive equation for the force $\mathbf{f}_{ij}$ exerted on p_i by another particle p_j into the statement of force balance for p_i to yield

$$\sum_{p_j \in \mathcal{B}\backslash\{p_i\}} \left\{ \frac{\partial \tilde{\psi}_{ij}(r_{ij})}{\partial r_{ij}}\mathbf{n}_{ij} - \hat{f}_{ij}^{\text{dis}}(r_{ij}, s_{ij})\mathbf{n}_{ij} \right\} = \mathbf{f}_i^{\text{ext}}.$$

For an inertial frame, $\mathbf{f}_i^{\text{ext}} = \mathbf{f}_i^{\text{ni}} + \mathbf{f}_i^{\text{in}} = \mathbf{f}_i^{\text{ni}} - m_i\mathbf{a}_i$, with m_i the mass of p_i and $\mathbf{a}_i = \ddot{\mathbf{x}}_i$ the acceleration. Thus, when the underlying frame is inertial, the differential equation for p_i becomes

$$m_i\ddot{\mathbf{x}} + \sum_{p_j \in \mathcal{B}\backslash\{p_i\}} \left\{ \frac{\partial \tilde{\psi}_{ij}(r_{ij})}{\partial r_{ij}}\mathbf{n}_{ij} - \hat{f}_{ij}^{\text{dis}}(r_{ij}, s_{ij})\mathbf{n}_{ij} \right\} = \mathbf{f}_i^{\text{ni}}.$$

Except for the presence of the dissipative contributions $\hat{f}_{ij}^{\text{dis}}(r_{ij}, s_{ij})\mathbf{n}_{ij}$, $i, j = 1, \dots, N$, to the particle-particle forces, this equation is classical.

Thus, when attention is restricted to inertial frames and dissipative particle-particle interactions are neglected, the differential equations for a system of particles obtained here specialize to the classical system of evolution equations to those used by Pitteri [12, 13] to derive continuum equations of balance from statistical mechanics. Assuming that Pitteri's [12, 13] arguments can be extended, perhaps by using the tools of nonequilibrium statistical mechanics, to yield a distinguished continuum limit, it is interesting to speculate on the nature of any restrictions that the residual dissipation inequalities

$$\hat{f}_{ij}^{\rm dis}(r_{ij}, s_{ij})s_{ij} \le 0, \qquad i,j = 1,\dots,N, \qquad i \ne j,$$

might impose at the continuum level.

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