

















## 6. Nonlocal Green's functions and a nonlocal Green's third identity.

**6.1. Nonlocal fundamental solutions.** For each  $\mathbf{y} \in \mathbb{R}^d$ , let  $g(\mathbf{x}; \mathbf{y})$  denote the fundamental solution (or free-space Green's function) for the operator  $\mathcal{D}(\beta\mathcal{G}(\cdot))$ , formally defined as the solution of<sup>7</sup>

$$\mathcal{D}(\beta\mathcal{G}(g(\mathbf{x}; \mathbf{y}))) = \delta(\mathbf{x} - \mathbf{y}) \quad \forall \mathbf{x} \in \mathbb{R}^d.$$

We assume that  $\alpha$  and  $\beta$  are radial functions of  $\mathbf{x}$  and  $\mathbf{x}'$ , e.g.,  $\alpha(\mathbf{x}, \mathbf{x}') = \alpha(\mathbf{x}' - \mathbf{x})$ , and that  $g(\mathbf{x}; \mathbf{y}) = g(\mathbf{x} - \mathbf{y})$ . In this case, we can assume, without loss of generality, that  $\mathbf{y} = \mathbf{0}$ . Using (5.5), we then have

$$2 \int_{\mathbb{R}^d} (g(\mathbf{x}') - g(\mathbf{x})) \mu(\mathbf{x}' - \mathbf{x}) d\mathbf{x}' = \delta(\mathbf{x}) \quad \mathbf{x} \in \mathbb{R}^d, \quad (6.1)$$

where  $\mu = \alpha^2\beta$ . Assuming, again without loss of generality, that the radial function  $\mu$  satisfies  $\int_{\mathbb{R}^d} \mu d\mathbf{x} = 1$ , (6.1) can be expressed in the form

$$2 \int_{\mathbb{R}^d} g(\mathbf{x}') \mu(\mathbf{x}' - \mathbf{x}) d\mathbf{x}' - 2g(\mathbf{x}) = \delta(\mathbf{x}) \quad \forall \mathbf{x} \in \mathbb{R}^d,$$

so that

$$\hat{g} = \frac{(2\pi)^{-d/2}}{2} \left( \frac{1}{(2\pi)^{d/2} \hat{\mu} - 1} \right),$$

where the Fourier transforms of  $g$  and  $\mu$  are given by

$$\hat{g}(\mathbf{k}) := (2\pi)^{-d/2} \int_{\mathbb{R}^n} e^{-i\mathbf{k}\cdot\mathbf{x}} g(\mathbf{x}) d\mathbf{x} \quad \text{and} \quad \hat{\mu}(\mathbf{k}) := (2\pi)^{-d/2} \int_{\mathbb{R}^n} e^{-i\mathbf{k}\cdot\mathbf{x}} \mu(\mathbf{x}) d\mathbf{x},$$

respectively. Therefore,

$$g(\mathbf{x}) = \frac{(2\pi)^{-d}}{2} \int_{\mathbb{R}^n} e^{i\mathbf{k}\cdot\mathbf{x}} \frac{1}{(2\pi)^{d/2} \hat{\mu} - 1} d\mathbf{k}$$

so that, for general  $\mathbf{y}$ , the fundamental solution for (6.1) is given by

$$g(\mathbf{x}; \mathbf{y}) = \frac{(2\pi)^{-d}}{2} \int_{\mathbb{R}^n} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \frac{1}{(2\pi)^{d/2} \hat{\mu} - 1} d\mathbf{k}.$$

**6.2. Nonlocal Green's third identity.** For any  $\mathbf{y} \in \Omega$ , let  $G(\mathbf{x}; \mathbf{y})$  denote any function satisfying<sup>8</sup>

$$\mathcal{D}(\beta\mathcal{G}(G(\mathbf{x}; \mathbf{y}))) = \delta(\mathbf{x} - \mathbf{y}) \quad \forall \mathbf{x} \in \tilde{\Omega}. \quad (6.2)$$

Then, using the nonlocal Green's second identity (4.3) with  $v(\cdot) = G(\cdot; \mathbf{y})$ , we obtain the *nonlocal Green's third identity*

$$\begin{aligned} u(\mathbf{y}) &= \int_{\tilde{\Omega}} G(\mathbf{x}; \mathbf{y}) \mathcal{D}(\beta\mathcal{G}(u(\mathbf{x}))) d\mathbf{x} \\ &\quad - \int_{\Omega \setminus \tilde{\Omega}} \left( G(\mathbf{x}; \mathbf{y}) \mathcal{N}(\beta\mathcal{G}(u(\mathbf{x}))) - u(\mathbf{x}) \mathcal{N}(\beta\mathcal{G}(G(\mathbf{x}; \mathbf{y}))) \right) d\mathbf{x}. \end{aligned} \quad (6.3)$$

<sup>7</sup>In this section, we explicitly express the dependencies of functions.

<sup>8</sup>Note that the fundamental solution satisfies this equation.

Suppose that the constitutive function  $\beta = 1$  (see (4.4)) and

$$-\mathcal{D}(\mathcal{G}(u)) = 2 \int_{\Omega} (u' - u)\alpha^2 d\mathbf{x}' = 0 \quad \forall \mathbf{x} \in \tilde{\Omega}. \quad (6.4)$$

Then, the solution  $u(\mathbf{x})$  represents a nonlocal ‘‘harmonic’’ function and, from (6.3), we have

$$u(\mathbf{y}) = \int_{\Omega \setminus \tilde{\Omega}} \left( u(\mathbf{x})\mathcal{N}(\mathcal{G}(G(\mathbf{x}; \mathbf{y}))) - G(\mathbf{x}; \mathbf{y})\mathcal{N}(\mathcal{G}(u(\mathbf{x}))) \right) d\mathbf{x},$$

i.e., ‘‘harmonic’’ function are determined by their ‘‘boundary’’ values on  $\Omega \setminus \tilde{\Omega}$ . Non-local versions of the Poisson integral formula and Gauss’s law of the arithmetic mean can also be derived; see [8, Chap. 4] for the classical case.

**6.3. Nonlocal Green’s functions.** Let  $g(\mathbf{x}; \mathbf{y})$  denote the fundamental solution defined in Section 6.1. For each  $\mathbf{y}$  in  $\Omega$ , define the *nonlocal Green’s function*  $G_d(\mathbf{x}; \mathbf{y}) : \Omega \rightarrow \mathbb{R}$  as

$$G_d(\mathbf{x}; \mathbf{y}) = g(\mathbf{x}; \mathbf{y}) - g_d(\mathbf{x}; \mathbf{y}),$$

where  $g_d(\cdot; \cdot)$  is a solution of

$$\begin{cases} \mathcal{D}(\beta\mathcal{G}(g_d)) = 0 & \text{for } \mathbf{x} \in \tilde{\Omega} \\ g_d(\mathbf{x}; \mathbf{y}) = g(\mathbf{x}; \mathbf{y}) & \text{for } \mathbf{x} \in \Omega \setminus \tilde{\Omega}. \end{cases}$$

Then,  $G_d(\cdot; \cdot)$  satisfies (6.2) and the homogeneous Dirichlet ‘‘boundary condition’’  $G_d(\mathbf{x}; \mathbf{y}) = 0$  for  $\mathbf{x} \in \Omega \setminus \tilde{\Omega}$ . Applying (6.3), we have that the solution of the Dirichlet ‘‘boundary-value’’ problem (5.2) is given by

$$u(\mathbf{y}) = - \int_{\tilde{\Omega}} G_d(\mathbf{x}; \mathbf{y})b(\mathbf{x}) d\mathbf{x} + \int_{\Omega \setminus \tilde{\Omega}} h_d(\mathbf{x})\mathcal{N}(\beta\mathcal{G}(G_d(\mathbf{x}; \mathbf{y}))) d\mathbf{x} \quad \forall \mathbf{y} \in \tilde{\Omega}.$$

Alternately, for each  $\mathbf{y}$  in  $\Omega$ , we define the *nonlocal Green’s function*  $G_n(\mathbf{x}; \mathbf{y}) : \Omega \rightarrow \mathbb{R}$  as

$$G_n(\mathbf{x}; \mathbf{y}) = g(\mathbf{x}; \mathbf{y}) - g_n(\mathbf{x}; \mathbf{y}),$$

where  $g_n(\cdot; \cdot)$  is a solution of

$$\begin{cases} \mathcal{D}(\beta\mathcal{G}(g_n)) = 0 & \text{for } \mathbf{x} \in \tilde{\Omega} \\ \mathcal{N}(\beta\mathcal{G}(g_n)) = \mathcal{N}(\beta\mathcal{G}(g)) & \text{for } \mathbf{x} \in \Omega \setminus \tilde{\Omega}. \end{cases}$$

Then,  $G_n(\cdot; \cdot)$  satisfies (6.2) and the homogeneous Neumann ‘‘boundary condition’’  $\mathcal{N}(\beta\mathcal{G}(g)) = 0$  for  $\mathbf{x} \in \Omega \setminus \tilde{\Omega}$ . Applying (6.3), we have that the solution of the Neumann ‘‘boundary-value’’ problem (5.4) is given by

$$u(\mathbf{y}) = - \int_{\tilde{\Omega}} G_n(\mathbf{x}; \mathbf{y})b(\mathbf{x}) d\mathbf{x} - \int_{\Omega \setminus \tilde{\Omega}} G_n(\mathbf{x}; \mathbf{y})h_n(\mathbf{x}) d\mathbf{x} \quad \forall \mathbf{y} \in \tilde{\Omega}.$$

**7. Local smooth limits.** In this section, we connect the linear nonlocal “boundary” value problems of Section 5 to the classical Dirichlet and Neumann problems for second-order elliptic partial differential equations. To do so, we make two assumptions: solutions of the nonlocal “boundary” value problems are smooth and operators are asymptotically local.<sup>9</sup>

We assume that  $\mathbf{K}$ ,  $\alpha$ , and  $\gamma$  are radial functions, e.g.,

$$\mathbf{K}_{ij}(\mathbf{x}, \mathbf{x}') = \mathbf{K}_{ij}(\mathbf{x}' - \mathbf{x}), \quad i, j = 1, \dots, d.$$

We assume that the radial function  $\gamma(\mathbf{x}' - \mathbf{x})$  satisfies, for a specified  $\varepsilon > 0$ ,

$$\gamma_\varepsilon(\mathbf{x}' - \mathbf{x}) \neq 0 \quad \text{only if } \mathbf{x}' \in B_\varepsilon(\mathbf{x}) := \{\mathbf{y} \in \Omega \mid |\mathbf{y} - \mathbf{x}| < \varepsilon\}, \quad (7.1)$$

where the dependence on  $\varepsilon$  is explicitly denoted through the subscript. Then, the “constitutive” function  $\beta_\varepsilon$  introduced in (4.4) is the radial function given by

$$\beta_\varepsilon = \gamma_\varepsilon(\mathbf{x}' - \mathbf{x}) \cdot \mathbf{K} \cdot (\mathbf{x}' - \mathbf{x}). \quad (7.2)$$

Referring to Figure 3.1, we assume that the thickness of the “boundary” domain  $\Omega \setminus \tilde{\Omega}$  is of  $O(\varepsilon)$  so that

$$|\Omega| - |\tilde{\Omega}| = O(\varepsilon),$$

where  $|\cdot|$  denotes the volume. For  $\mathbf{x} \in \Omega$ , let  $\Omega_\varepsilon(\mathbf{x}) = B_\varepsilon(\mathbf{x}) \cap \Omega$ . We further assume that

$$\text{trial functions } u \in U(\Omega) \text{ and test functions } v \in U(\Omega) \text{ are smooth.} \quad (7.3)$$

Note that no assumptions are made about the smoothness of the functions  $\alpha$ ,  $\gamma_\varepsilon$ , and the elements of the matrix function  $\mathbf{K}$  other than those needed to ensure that the integrals encountered are finite.

The definition (2.13) of the operator  $\mathcal{G}$  and assumption (7.3) imply

$$\mathcal{G}(v) = (v' - v)\alpha = \left( (\mathbf{x}' - \mathbf{x}) \cdot \nabla v(\mathbf{x}) + O(\varepsilon^2) \right) \alpha \quad \forall \mathbf{x}' \in \Omega_\varepsilon(\mathbf{x}) \quad (7.4)$$

so that, using (7.2),

$$\begin{aligned} & \int_{\Omega} \int_{\Omega} \beta_\varepsilon \mathcal{G}(v) \mathcal{G}(u) \, d\mathbf{x}' \, d\mathbf{x} \\ &= \int_{\Omega} \int_{\Omega_\varepsilon(x)} \left( (\mathbf{x}' - \mathbf{x}) \cdot \nabla v \right) \left( (\mathbf{x}' - \mathbf{x}) \cdot \mathbf{K} \cdot (\mathbf{x}' - \mathbf{x}) \right) \left( (\mathbf{x}' - \mathbf{x}) \cdot \nabla u \right) \gamma_\varepsilon \alpha^2 \, d\mathbf{x}' \, d\mathbf{x} \\ & \quad + \text{h.o.t.} \\ &= \int_{\Omega} \nabla v \cdot \left( \int_{\Omega_\varepsilon(x)} (\mathbf{x}' - \mathbf{x}) \otimes (\mathbf{x}' - \mathbf{x}) \mathbf{K} (\mathbf{x}' - \mathbf{x}) \otimes (\mathbf{x}' - \mathbf{x}) \gamma_\varepsilon \alpha^2 \, d\mathbf{x}' \right) \cdot \nabla u \, d\mathbf{x} \\ & \quad + \text{h.o.t.} \end{aligned}$$

<sup>9</sup>It is important to emphasize that these assumptions are made only to make the connection to classical problems for partial differential equations and are not required for the well posedness of the nonlocal “boundary” value problems; see Section 8. In addition, the nonlocal “boundary” value problems admit solutions that are not solutions, even in the usual sense of weak solutions, of the partial differential equations. Thus, one can view solutions of the nonlocal “boundary” value problems as further generalizations of solutions of the partial differential equations, generalized in two ways: they are nonlocal and they lack the smoothness needed for them to be standard weak solutions.

Then,

$$\int_{\Omega} \int_{\Omega} \beta_{\varepsilon} \mathcal{G}(v) \mathcal{G}(u) d\mathbf{x}' d\mathbf{x} = \int_{\Omega} \nabla v \cdot (\mathbf{D}_{\varepsilon} \cdot \nabla u) d\mathbf{x} + \text{h.o.t.}, \quad (7.5)$$

where the second-order tensor  $\mathbf{D}_{\varepsilon}$  is given by

$$\mathbf{D}_{\varepsilon}(\mathbf{x}) := \int_{\Omega_{\varepsilon}(x)} (\mathbf{x}' - \mathbf{x}) \otimes (\mathbf{x}' - \mathbf{x}) \mathbf{K}(\mathbf{x}' - \mathbf{x}) \otimes (\mathbf{x}' - \mathbf{x}) \gamma_{\varepsilon} \alpha^2 d\mathbf{x}' \quad (7.6)$$

and where we assume that the integral and its derivatives are finite valued for all  $\mathbf{x} \in \Omega$  and all  $\varepsilon > 0$ .

Let  $\mathbf{z} := \mathbf{x}' - \mathbf{x}$ . Then, the assumption that  $\alpha$ ,  $\gamma_{\varepsilon}$ , and the elements of  $\mathbf{K}$  are radial functions results in

$$\alpha^2(\mathbf{x} - (1 - \lambda)\mathbf{z}, \mathbf{x} + \lambda\mathbf{z}) = \alpha^2(\mathbf{x} + \lambda\mathbf{z} - \mathbf{x} + (1 - \lambda)\mathbf{z}) = \alpha^2(\mathbf{z})$$

and

$$\beta_{\varepsilon}(\mathbf{x} - (1 - \lambda)\mathbf{z}, \mathbf{x} + \lambda\mathbf{z}) = \beta_{\varepsilon}(\mathbf{x} + \lambda\mathbf{z} - \mathbf{x} + (1 - \lambda)\mathbf{z}) = \beta_{\varepsilon}(\mathbf{z}) = \gamma_{\varepsilon}(\mathbf{z}) \mathbf{z} \cdot \mathbf{K}(\mathbf{z}) \cdot \mathbf{z}.$$

We also have

$$u(\mathbf{x} - (1 - \lambda)\mathbf{z}) - u(\mathbf{x} + \lambda\mathbf{z}) = -\nabla u(\mathbf{x}) \cdot \mathbf{z} + O(\varepsilon^2).$$

Now, in (2.7), let  $f = \beta_{\varepsilon} \mathcal{G}(u) = (u' - u) \beta_{\varepsilon} \alpha$  so that

$$\begin{aligned} \varphi(\mathbf{x}, \mathbf{z}) &= - \int_0^1 \left[ \mathbf{z} \cdot \mathbf{K}(\mathbf{z}) \cdot \mathbf{z} \right] \left[ \mathbf{z} \cdot \nabla u(\mathbf{x}) \right] \gamma_{\varepsilon}(\mathbf{z}) \alpha^2(\mathbf{z}) d\lambda + \text{h.o.t.} \\ &= - \left[ \mathbf{z} \cdot \mathbf{K}(\mathbf{z}) \cdot \mathbf{z} \right] \left[ \mathbf{z} \cdot \nabla u(\mathbf{x}) \right] \gamma_{\varepsilon}(\mathbf{z}) \alpha^2(\mathbf{z}) + \text{h.o.t.} \end{aligned}$$

so that

$$-\mathbf{z} \varphi(\mathbf{x}, \mathbf{z}) = \mathbf{z} \left[ \mathbf{z} \cdot \mathbf{K}(\mathbf{z}) \cdot \mathbf{z} \right] \left[ \mathbf{z} \cdot \nabla u(\mathbf{x}) \right] \gamma_{\varepsilon}(\mathbf{z}) \alpha^2(\mathbf{z}) + \text{h.o.t.}$$

Then, from (2.6), we have

$$\begin{aligned} \mathbf{q}_{\varepsilon}(\mathbf{x}) &= - \int_{\Omega} (\mathbf{x}' - \mathbf{x}) \varphi(\mathbf{x}, \mathbf{x}' - \mathbf{x}) d\mathbf{x}' \\ &= \int_{\Omega_{\varepsilon}(x)} (\mathbf{x}' - \mathbf{x}) \left[ (\mathbf{x}' - \mathbf{x}) \cdot \mathbf{K} \cdot (\mathbf{x}' - \mathbf{x}) \right] \left[ (\mathbf{x}' - \mathbf{x}) \cdot \nabla u \right] \gamma_{\varepsilon} \alpha^2 d\mathbf{x}' + \text{h.o.t.} \\ &= \left( \int_{\Omega_{\varepsilon}(x)} \left[ (\mathbf{x}' - \mathbf{x}) \otimes (\mathbf{x}' - \mathbf{x}) \mathbf{K}(\mathbf{x}' - \mathbf{x}) \otimes (\mathbf{x}' - \mathbf{x}) \right] \gamma_{\varepsilon} \alpha^2 d\mathbf{x}' \right) \cdot \nabla u + \text{h.o.t.} \\ &= \mathbf{D}_{\varepsilon} \cdot \nabla u + \text{h.o.t.} \end{aligned}$$

so that, from (2.8),

$$\mathcal{D}(\beta_{\varepsilon} \mathcal{G}(u)) = \nabla \cdot \mathbf{q}_{\varepsilon} = \nabla \cdot (\mathbf{D}_{\varepsilon} \cdot \nabla u) + \text{h.o.t.} \quad (7.7)$$

The relations (7.5) and (7.7) together imply that

$$\begin{aligned} \int_{\tilde{\Omega}} v \nabla \cdot (\mathbf{D}_{\varepsilon} \cdot \nabla u) d\mathbf{x} + \int_{\Omega} \nabla v \cdot (\mathbf{D}_{\varepsilon} \cdot \nabla u) d\mathbf{x} \\ = \int_{\tilde{\Omega}} v \mathcal{D}(\beta_{\varepsilon} \mathcal{G}(u)) d\mathbf{x} + \int_{\Omega} \int_{\Omega} \beta_{\varepsilon} \mathcal{G}(v) \mathcal{G}(u) d\mathbf{x}' d\mathbf{x} + \text{h.o.t.} \end{aligned} \quad (7.8)$$

However,

$$\begin{aligned} \int_{\tilde{\Omega}} v \nabla \cdot (\mathbf{D}_\varepsilon \cdot \nabla u) \, d\mathbf{x} &= \int_{\Omega} v \nabla \cdot (\mathbf{D}_\varepsilon \cdot \nabla u) \, d\mathbf{x} - \int_{\Omega \setminus \tilde{\Omega}} v \nabla \cdot (\mathbf{D}_\varepsilon \cdot \nabla u) \, d\mathbf{x} \\ &= \int_{\Omega} v \nabla \cdot (\mathbf{D}_\varepsilon \cdot \nabla u) \, d\mathbf{x} + \text{h.o.t.}, \end{aligned}$$

where the last term on the right-hand side of the first line is of higher order by virtue of the boundedness of its integrand and because  $|\Omega \setminus \tilde{\Omega}| = O(\varepsilon)$ . Then, (7.8) can be rewritten as

$$\begin{aligned} \int_{\Omega} v \nabla \cdot (\mathbf{D}_\varepsilon \cdot \nabla u) \, d\mathbf{x} + \int_{\Omega} \nabla v \cdot (\mathbf{D}_\varepsilon \cdot \nabla u) \, d\mathbf{x} \\ = \int_{\tilde{\Omega}} v \mathcal{D}(\beta_\varepsilon \mathcal{G}(u)) \, d\mathbf{x} + \int_{\Omega} \int_{\Omega} \beta_\varepsilon \mathcal{G}(v) \mathcal{G}(u) \, d\mathbf{x}' \, d\mathbf{x} + \text{h.o.t.} \end{aligned} \quad (7.9)$$

Application of the classical Green's first identity and the nonlocal Green's first identity (4.2) to the left-hand side and right-hand sides of (7.9), respectively, results in

$$\int_{\partial\Omega} v (\mathbf{D}_\varepsilon \cdot \nabla u) \cdot \mathbf{n} \, dA = \int_{\Omega \setminus \tilde{\Omega}} v \mathcal{N}(\beta_\varepsilon \mathcal{G}(u)) \, d\mathbf{x} + \text{h.o.t.} \quad (7.10)$$

Let

$$\mathbf{D} = \lim_{\varepsilon \rightarrow 0} \mathbf{D}_\varepsilon, \quad (7.11)$$

where we assume that the integrand in (7.6) is sufficiently well-behaved so that this limit exists and that, on the other hand, this limit is not the zero tensor. Hence, from (7.5),

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} \int_{\Omega} \beta_\varepsilon \mathcal{G}(v) \mathcal{G}(u) \, d\mathbf{x}' \, d\mathbf{x} = \int_{\Omega} \nabla v \cdot (\mathbf{D} \cdot \nabla u) \, d\mathbf{x},$$

from (7.7)

$$\lim_{\varepsilon \rightarrow 0} \mathcal{D}(\beta_\varepsilon \mathcal{G}(u)) = \nabla \cdot (\mathbf{D} \cdot \nabla u),$$

and from (7.10)

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega \setminus \tilde{\Omega}} v \mathcal{N}(\beta_\varepsilon \mathcal{G}(u)) \, d\mathbf{x} = \int_{\partial\Omega} v (\mathbf{D} \cdot \nabla u) \cdot \mathbf{n} \, dA.$$

We can use these results to show that the nonlocal Green's identities (4.2) and (4.3) reduce to the corresponding classical Green's identities. Moreover, the variational problem (5.1) reduces to

$$\begin{cases} \int_{\Omega} \nabla v \cdot (\mathbf{D} \cdot \nabla u) = \int_{\Omega} v b \, d\mathbf{x} & \text{in } \Omega \\ u = h_d & \text{on } \partial\Omega \end{cases}$$

so that the corresponding "Dirichlet boundary" value problem (5.2) reduces to the classical Dirichlet problem

$$\begin{cases} -\nabla \cdot (\mathbf{D} \cdot \nabla u) = b & \text{in } \Omega \\ u = h_d & \text{on } \partial\Omega. \end{cases}$$

Similarly, the variational principle (3.5) reduces to

$$\int_{\Omega} \nabla v \cdot (\mathbf{D} \cdot \nabla u) = \int_{\Omega} v b \, d\mathbf{x} + \int_{\partial\Omega} v h_n \, dA \quad \text{in } \Omega$$

so that the corresponding ‘‘Neumann boundary’’ value problem (5.4) reduces to the classical Neumann problem

$$\begin{cases} -\nabla \cdot (\mathbf{D} \cdot \nabla u) = b & \text{in } \Omega \\ (\mathbf{D} \cdot \nabla u) \cdot \mathbf{n} = h_n & \text{on } \partial\Omega. \end{cases}$$

**8. Well posedness of the linear nonlocal boundary value problems.** We restrict attention to the case  $U(\Omega) = V(\Omega)$  and demonstrate that the variational problems (5.1) and (5.3) are well posed. We assume that  $\beta(\mathbf{x}, \mathbf{x}') > 0$  for all  $\mathbf{x}, \mathbf{x}' \in \Omega$ .

**8.1. Bilinear forms, norms, and inner products.** Define the symmetric bilinear form

$$B(u, v) = \int_{\Omega} \int_{\Omega} (v' - v)(u' - u) \beta \alpha^2 \, d\mathbf{x}' \, d\mathbf{x} \quad \forall u, v \in V(\Omega). \quad (8.1)$$

Note that  $B(u, u) \geq 0$  and let

$$((u, v)) = B(u, v) \quad \text{and} \quad |||u||| = (B(u, u))^{1/2}.$$

We now define the function space  $V(\Omega)$  as

$$V(\Omega) = \{u : |||u||| < \infty\}.$$

We now show that  $|||\cdot|||$  and  $((\cdot, \cdot))$  define a norm and an inner product, respectively, on both  $V_0(\Omega)$  and  $V(\Omega) \setminus \mathbb{R}$ . Note that  $|||\cdot|||$  only defines a semi-norm on  $V(\Omega)$ .

Let  $\Omega \setminus \tilde{\Omega} \subset \Omega$  have finite measure and let  $u \in V_0(\Omega)$  so that

$$u(\mathbf{x}) = 0 \quad \forall \mathbf{x} \in \Omega \setminus \tilde{\Omega}.$$

Then,

$$\begin{aligned} B(u, u) &= \int_{\Omega} \int_{\Omega} (u' - u)^2 \beta \alpha^2 \, d\mathbf{x}' \, d\mathbf{x} \\ &= \int_{\Omega} \int_{\Omega \setminus \tilde{\Omega}} (u' - u)^2 \beta \alpha^2 \, d\mathbf{x}' \, d\mathbf{x} + \int_{\Omega} \int_{\tilde{\Omega}} (u' - u)^2 \beta \alpha^2 \, d\mathbf{x}' \, d\mathbf{x} \\ &\geq \int_{\Omega} \int_{\Omega \setminus \tilde{\Omega}} (u' - u)^2 \beta \alpha^2 \, d\mathbf{x}' \, d\mathbf{x} = \int_{\Omega} u^2 \int_{\Omega \setminus \tilde{\Omega}} \beta \alpha^2 \, d\mathbf{x}' \, d\mathbf{x} \\ &= \int_{\tilde{\Omega}} u^2 \left( \int_{\Omega \setminus \tilde{\Omega}} \beta \alpha^2 \, d\mathbf{x}' \right) \, d\mathbf{x}. \end{aligned}$$

Assuming that  $0 < \int_{\Omega \setminus \tilde{\Omega}} \beta \alpha^2 \, d\mathbf{x}' < \infty$  for all  $\mathbf{x} \in \tilde{\Omega}$ , we have that

$$B(u, u) = 0 \quad \text{implies that} \quad u = 0 \quad \forall \mathbf{x} \in \tilde{\Omega}.$$

But,  $u = 0$  in  $\Omega \setminus \tilde{\Omega}$  as well so that we have that

$$B(u, u) = 0 \quad \text{implies that} \quad u = 0 \quad \forall \mathbf{x} \in \Omega.$$

Thus, we have that  $||| \cdot |||$  defines a norm and  $((\cdot, \cdot))$  defines an inner product on  $V_0(\Omega)$ .

Also, note that  $B(u, u) = 0$  only if  $(u' - u)^2 \beta \alpha^2 = 0$  for all  $\mathbf{x}, \mathbf{x}' \in \Omega$ , i.e., only if  $u = \text{constant}$  for all  $\mathbf{x} \in \Omega$ . Thus, we again conclude that  $||| \cdot |||$  defines a norm and  $((\cdot, \cdot))$  defines an inner product on  $V(\Omega) \setminus \mathbb{R} \subset V(\Omega)$ .

**8.2. Decomposition of the solution space.** Let the space  $S(\Omega)$  consist of functions  $u \in V(\Omega)$  that satisfy

$$-\mathcal{D}(\beta \mathcal{G}(u)) = 2 \int_{\Omega} (u' - u) \beta \alpha^2 d\mathbf{x}' = 0 \quad \forall \mathbf{x} \in \tilde{\Omega}. \tag{8.2}$$

Then, from (4.2), we have that, for all  $u \in S$  and  $v \in V_0(\Omega)$ ,

$$\int_{\Omega} \int_{\Omega} \mathcal{G}(v) \mathcal{G}(u) \beta d\mathbf{x}' d\mathbf{x} = 0$$

so that, by (8.1), for all  $u \in S$  and  $v \in V_0(\Omega)$ ,

$$B(u, v) = ((u, v)) = \int_{\Omega} \int_{\Omega} (v' - v)(u' - u) \beta \alpha^2 d\mathbf{x}' d\mathbf{x} = \int_{\Omega} \int_{\Omega} \mathcal{G}(v) \mathcal{G}(u) \beta d\mathbf{x}' d\mathbf{x} = 0.$$

Thus, we conclude that

$$V(\Omega) = V_0(\Omega) \oplus S(\Omega), \tag{8.3}$$

i.e., any function in  $V(\Omega)$  can be written as a sum of two functions that are orthogonal with respect to the inner product  $((\cdot, \cdot))$ , the first a function that vanishes on  $\Omega \setminus \tilde{\Omega}$  and the second a function satisfying (8.2).<sup>10</sup>

**8.3. Nonlocal dual spaces and nonlocal trace spaces.** Let

$$|||b|||_* = \sup_{v \in V_0(\Omega), v \neq 0} \frac{\int_{\tilde{\Omega}} v(\mathbf{x}) b(\mathbf{x}) d\mathbf{x}}{|||v|||}$$

and define the “dual” space

$$V_0^*(\Omega) = \{b : |||b|||_* < \infty\}.$$

Next, define the “trace” space

$$V_d = \{\chi_{\Omega \setminus \tilde{\Omega}} u : u \in V(\Omega)\},$$

where  $\chi_{(\cdot)}$  denotes the characteristic function, along with the norm

$$|||u|||_d = |||\chi_{\Omega \setminus \tilde{\Omega}} u|||.$$

Finally, define the norm

$$|||h|||_n = \sup_{v \in V_d, v \neq 0} \frac{\int_{\Omega \setminus \tilde{\Omega}} v(\mathbf{x}) h(\mathbf{x}) d\mathbf{x}}{|||v|||_d}$$

and the second “trace” space

$$V_n = \{h : |||h|||_n < \infty\}.$$

<sup>10</sup>If, in (8.2), we set  $\beta = 1$ , then the space  $S(\Omega)$  consists of “harmonic” functions; see (6.4). Of course, then the decomposition (8.3) is entirely analogous to the decomposition of the Sobolev space  $H^1(\Omega)$  into functions belonging to  $H_0^1(\Omega)$  and harmonic functions.

**8.4. Well-posedness of variational problems.** The variational problems (5.1) and (5.3), respectively, take the form of the homogeneous “Dirichlet” problem

$$\begin{cases} \text{given } b \in V_0^* \text{ and } h_d \in V_d, \text{ seek } u \in V_0(\Omega) \text{ such that} \\ B(u, v) = F_d(v) \quad \forall v \in V_0(\Omega) \end{cases} \quad (8.4)$$

and the “Neumann” problem

$$\begin{cases} \text{given } b \in V^* \text{ and } h_n \in V_n \text{ satisfying (3.4), seek } u \in V(\Omega) \setminus \mathbb{R} \text{ such that} \\ B(u, v) = F_n(v) \quad \forall v \in V(\Omega) \setminus \mathbb{R}, \end{cases} \quad (8.5)$$

where the linear functionals  $F_d(\cdot)$  and  $F_n(\cdot)$  are defined by

$$F_d(v) = \int_{\tilde{\Omega}} vb \, d\mathbf{x} \quad \forall v \in V_0(\Omega)$$

and

$$F_n(v) = \int_{\tilde{\Omega}} vb \, d\mathbf{x} + \int_{\Omega \setminus \tilde{\Omega}} vh_n \, d\mathbf{x} \quad \forall v \in V(\Omega) \setminus \mathbb{R},$$

respectively.<sup>11</sup>

Because  $B(\cdot, \cdot)$  defines an inner product on  $V_0(\Omega)$  and  $V(\Omega) \setminus \mathbb{R}$ , it is continuous and coercive on those spaces. Then, if we assume that the data are such that the functionals  $F_d(\cdot)$  and  $F_n(\cdot)$  are continuous, the Lax-Milgram theorem can be applied to show that both (8.4) and (8.5) have unique solutions and, moreover, those solutions satisfy

$$|||u||| \leq |||b|||_* \quad \text{and} \quad |||u||| \leq |||b|||_* + |||h_n|||_n,$$

respectively.

**9. Concluding remarks.** We have developed a nonlocal vector calculus that consists of a nonlocal Gauss’s theorem and nonlocal Green’s identities that mimic the corresponding theorem and identities of the classical vector calculus. We defined some nonlocal variational principles and used the vector calculus to show that the principles correspond to nonlocal “boundary” value problems that mimic the classical Dirichlet and Neumann problems for second-order elliptic partial differential equations. In fact, we showed that, in an appropriate limit, the nonlocal variational principles and the nonlocal boundary-value problems reduce to their classical counterparts. We also derived fundamental solutions and showed how one can derive existence and uniqueness results for the nonlocal boundary-value problems.

The nonlocal variational problems (5.1) and (5.3) and the corresponding nonlocal “boundary” value problems (5.2) and (5.4), respectively, mimic the classical setting described by (1.1) along with Dirichlet and Neumann boundary conditions, respectively.

<sup>11</sup>Note that solutions of the variational “boundary” value problems (8.4) and (8.5) correspond to solutions of the optimization problems

$$\arg \min_{V_0(\Omega)} \left( \frac{1}{2} B(v, v) - F_d(v) \right) \quad \text{and} \quad \arg \min_{V(\Omega) \setminus \mathbb{R}} \left( \frac{1}{2} B(v, v) - F_n(v) \right),$$

respectively.

Nonlocal versions of more general second-order elliptic boundary value problems can also be defined. For example, consider the nonlocal variational principle<sup>12</sup>

$$\left\{ \begin{array}{l} \text{seek } u \in V(\Omega) \text{ such that} \\ u = h_d \text{ for } \mathbf{x} \in \Omega \setminus \tilde{\Omega} \\ \text{and} \\ \int_{\Omega} \int_{\Omega} \beta \mathcal{G}(v) \mathcal{G}(u) \, d\mathbf{x}' d\mathbf{x} + \int_{\Omega} v \int_{\Omega} \sigma \mathcal{G}(u) \, d\mathbf{x}' d\mathbf{x} \\ + \int_{\Omega} v \int_{\Omega} \omega(u' + u) \, d\mathbf{x}' d\mathbf{x} = \int_{\tilde{\Omega}} v b \, d\mathbf{x} \quad \forall v \in V_0(\Omega), \end{array} \right. \quad (9.1)$$

where  $\sigma(\mathbf{x}, \mathbf{x}')$  and  $\omega(\mathbf{x}, \mathbf{x}')$  are skew-symmetric and symmetric functions, respectively. The corresponding nonlocal “Dirichlet” boundary-value problem is given by

$$\left\{ \begin{array}{ll} -\mathcal{D}(\beta \mathcal{G}(u)) + \sigma \mathcal{G}(u) + \omega(u' + u) = b & \text{for } \mathbf{x} \in \tilde{\Omega} \\ u = h_d & \text{for } \mathbf{x} \in \Omega \setminus \tilde{\Omega}. \end{array} \right. \quad (9.2)$$

General problems may be defined by setting  $\beta$  as in (4.4) and setting  $\sigma = \xi \mathbf{a} \cdot (\mathbf{x}' - \mathbf{x})$  and  $\omega = \eta r$ , where  $\mathbf{a}(\mathbf{x}, \mathbf{x}')$  is a symmetric vector-valued function and  $\xi(\mathbf{x}, \mathbf{x}')$ ,  $\eta(\mathbf{x}, \mathbf{x}')$ , and  $r(\mathbf{x}, \mathbf{x}')$  are symmetric functions. Let  $\mathbf{D}$  be given by (7.11). Analogously, let

$$\mathbf{w}(\mathbf{x}) = \lim_{\varepsilon \rightarrow 0} \int_{\Omega_{\varepsilon}(x)} (\mathbf{x}' - \mathbf{x}) \otimes (\mathbf{x}' - \mathbf{x}) \cdot \mathbf{a} \xi_{\varepsilon} \alpha \, d\mathbf{x}'$$

and

$$c(\mathbf{x}) = \lim_{\varepsilon \rightarrow 0} \int_{\Omega_{\varepsilon}(x)} r \eta_{\varepsilon} \, d\mathbf{x}'$$

where  $\xi_{\varepsilon}(\mathbf{x}, \mathbf{x}')$  and  $\eta_{\varepsilon}(\mathbf{x}, \mathbf{x}')$  are localized in the same way as  $\gamma_{\varepsilon}(\mathbf{x}, \mathbf{x}')$ . We may then proceed as in Section 7 to show that, for smooth solutions  $u$  and for asymptotically local operators, (9.2) corresponds to the general linear convection-diffusion-reaction problem

$$-\nabla \cdot (\mathbf{D} \cdot \nabla u) + \mathbf{w} \cdot \nabla u + cu = b$$

along with a Dirichlet boundary condition.

Current work focuses on further refining and extending the results of this paper. In particular, we are

- developing functional analytic characterizations of the solution, trace, and data spaces used in Section 8;
- developing the equivalent multidomain formulations for the linear boundary value problems introduced in Section 5;
- developing and analyzing finite element discretization methods, including discontinuous Galerkin methods, for nonlocal variational problems such as (5.1) and (5.3);
- extending the nonlocal vector calculus to vector-valued functions and developing nonlocal variational problems and their corresponding nonlocal “boundary” value problems for vector-valued functions; of particular interest is the application of the nonlocal vector calculus to the peridynamic [12, 13] model for materials.

<sup>12</sup>We only examine nonlocal “Dirichlet” problems; clearly, their “Neumann” counterparts can be treated in a similar manner.

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