

# A NONLOCAL VECTOR CALCULUS WITH APPLICATION TO NONLOCAL BOUNDARY VALUE PROBLEMS

MAX GUNZBURGER\* AND R. B. LEHOUCQ†

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**Abstract.** We develop a calculus for nonlocal operators that mimics Gauss’s theorem and the Green’s identities of the classical vector calculus. The operators we define do not involve derivatives. We then apply the nonlocal calculus to define weak formulations of nonlocal “boundary-value” problems that mimic the Dirichlet and Neumann problems for second-order scalar elliptic partial differential equations. For the nonlocal problems, we derive a fundamental solution and Green’s functions, demonstrate that weak formulations of the nonlocal “boundary-value” problems are well posed, and show how, under appropriate limits, the nonlocal problems reduce to their local analogs.

**Key words.** nonlocal operators, vector calculus, boundary-value problems, diffusion, heat conduction

**AMS subject classifications.** 35B40, 35J20, 35J25, 35Q99, 45A05, 45K05

**1. Introduction.** Gauss’s theorem and the Green’s identities are crucial for the analysis of the second-order scalar elliptic boundary-value problem

$$-\nabla \cdot (\mathbb{D}(\mathbf{x}) \cdot \nabla u(\mathbf{x})) = b(\mathbf{x}) \quad \text{in } \Omega \subset \mathbb{R}^d \quad (1.1)$$

augmented with Dirichlet or Neumann boundary conditions on the boundary  $\partial\Omega$ , where  $\mathbb{D}$  denotes a symmetric, positive definite, second-order tensor,  $b$  a scalar-valued data function, and  $d$  a positive integer. Gauss’s theorem and the ensuing Green’s identities provide compatibility relations, a solution operator, and a weak formulation for the boundary-value problem (1.1).

The nonlocal second-order scalar “elliptic boundary-value” problem is given by

$$\mathcal{L}(u)(\mathbf{x}) := 2 \int_{\Omega} (u(\mathbf{x}') - u(\mathbf{x})) \mu(\mathbf{x}, \mathbf{x}') d\mathbf{x}' = b(\mathbf{x}) \quad \text{in } \Omega \subset \mathbb{R}^d \quad (1.2)$$

augmented with nonlocal “Dirichlet” or “Neumann” “boundary” conditions, where  $\mu(\mathbf{x}, \mathbf{x}')$  denotes a symmetric function of its arguments. The “boundary-value” problem (1.2) characterizes the solution of the formal minimization problem

$$\int_{\Omega} \int_{\Omega} (u(\mathbf{x}') - u(\mathbf{x}))^2 \mu(\mathbf{x}, \mathbf{x}') d\mathbf{x}' d\mathbf{x} - \int_{\Omega} b(\mathbf{x}) u(\mathbf{x}) d\mathbf{x} \rightarrow \min! \quad (1.3)$$

augmented with nonlocal “boundary” conditions. The relationship between (1.3) and (1.2) is equivalent to the relationship between (1.1) and the minimization problem

$$\frac{1}{2} \int_{\Omega} \nabla u(\mathbf{x}) \cdot \mathbb{D}(\mathbf{x}) \cdot \nabla u(\mathbf{x}) d\mathbf{x} - \int_{\Omega} b(\mathbf{x}) u(\mathbf{x}) d\mathbf{x} \rightarrow \min!$$

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\*Department of Scientific Computing, Florida State University, Tallahassee FL 32306-4120; [gunzburg@fsu.edu](mailto:gunzburg@fsu.edu). Supported in part by the US Department of Energy under grant number DE-FG02-05ER25698 as part of the Office of Science’s “Multiscale Mathematics” program.

†Sandia National Laboratories, P.O. Box 5800, MS 1320, Albuquerque, NM 87185-1320; [rblehou@sandia.gov](mailto:rblehou@sandia.gov). Sandia is a multiprogram laboratory operated by Sandia Corporation, a Lockheed Martin Company, for the U.S. Department of Energy under contract DE-AC04-94AL85000.

The first contribution of this paper is the development of a calculus for nonlocal analogues of (1.1) that mimics Gauss’s theorem and the Green’s identities of the classical vector calculus. An important aspect of our analyses makes use of two relatively unknown lemmas established in [22, 24]. The lemmas lead to an exact expression for a flux such that the divergence of this flux is equal to  $\mathcal{L}(u)$ .

Our second contribution applies the nonlocal calculus to define weak formulations of nonlocal “boundary-value” problems that mimic the Dirichlet and Neumann problems for second-order scalar elliptic partial differential equations. In contrast to their local counterparts, e.g., (1.1), the nonlocal “Dirichlet” and “Neumann” data needed for (1.2) are defined on a *nonzero volume*.

Our third contribution demonstrates, under certain assumptions, the well posedness of the weak formulations by invoking the Lax-Milgram Theorem. In particular, under certain assumptions about  $\mu$ , given a square integrable data vector  $b$ , the solution  $u$  to (1.2) is also square integrable and depends continuously upon the data. The well posedness of the weak formulations also lays the basis for conforming finite element methods.<sup>1</sup> The well posedness crucially depends upon imposing nonlocal boundary conditions on nonzero volumes, given that a trace operator is not defined on the space of square integrable functions. We also demonstrate how, under appropriate limits, the nonlocal “Dirichlet” and “Neumann” problems reduce to their classical analogs, and derive a fundamental solution for the operator  $\mathcal{L}$ . This fundamental solution coincides with that of the Laplace operator when

$$\mu(\mathbf{x}' - \mathbf{x}) = \delta(\mathbf{x}' - \mathbf{x}) + \frac{\partial^2}{\partial x^2} \delta(\mathbf{x}' - \mathbf{x}),$$

where  $\delta(\cdot)$  denotes the Dirac delta function.

The second-order elliptic operator associated with (1.1) is local, e.g.,  $\nabla \cdot \mathbb{D}(\mathbf{x}) \cdot \nabla$  only depends on the point  $\mathbf{x}$  whenever  $\mathbb{D}(\mathbf{x})$  only depends upon  $\mathbf{x}$ . In contrast, the operator  $\mathcal{L}$  eschews the gradient of the scalar function  $u$ , and is nonlocal because points  $\mathbf{x}' \neq \mathbf{x}$  can interact with  $\mathbf{x}$ . The solution operator for (1.2) does not, in general, smooth the data  $b(\mathbf{x})$  in the same way as the conventional solution operator associated with (1.1). For example, given homogenous Dirichlet boundary conditions and appropriate conditions on the tensor  $\mathbb{D}(\mathbf{x})$ , the solution operator for the weak formulation of (1.1) maps  $H^{-1}(\Omega)$  to  $H_0^1(\Omega)$ ; see, for instance [7]. In contrast, given appropriate conditions [4] on  $\mu$ ,  $\mathcal{L}^{-1}$  maps a proper subspace of  $H^{-1}(\Omega)$  to  $H_0^s(\Omega)$ ,  $1/2 < s < 1$ . Note that, again given appropriate conditions on  $\mu$ ,  $\mathcal{L}^{-1}: H^{-1}(\Omega) \rightarrow H_0^1(\Omega)$  when  $u \in H_0^1(\Omega)$ . The papers [11, 12] considers the vector-valued formulation of  $\mathcal{L}$  and demonstrates that, depending on the choice for  $\mu$ , the associated solution operator maps the dual of  $[H^s(\mathbb{R}^d)]^d$  to  $[H^s(\mathbb{R}^d)]^d$  for  $0 \leq s \leq 1$ . Both these results, and the well posedness of the weak formulations established in this paper, imply that the solution of (1.2) exhibits multiscale character beyond that achieved by the solution of (1.1).

The nonlocal  $p$ -Laplacian diffusion equation with Dirichlet and Neumann boundary conditions (the operator  $\mathcal{L}$  corresponds to  $p = 2$ ) is investigated in [1, 2]. These papers provide mathematical analyses for the classical existence and uniqueness of solutions of the nonlocal  $p$ -Laplacian diffusion equation, including conditions under

<sup>1</sup>See [14] where piecewise-linear elements are used for a one-dimensional nonlocal equation on the real line and [10] where continuous and discontinuous piecewise linear as well as piecewise constant elements are considered in the same setting. Note that discontinuous finite element spaces are conforming for the nonlocal equation.

which its solution can approximate the solution to the classical  $p$ -Laplacian diffusion equation. In addition to citing the work of the authors and their collaborators, the papers [1, 2] also reference the use of the operator  $\mathcal{L}$  within a nonlocal diffusion equation for modeling phase transitions; see [6] and the survey [15] containing an excellent annotated reference section. The paper [18] also considers nonlocal versions of more general second-order elliptic boundary value problems.

The contrast between the local and nonlocal equations, respectively, is not simply mathematical, and for instance, the distinction between (1.1) and (1.2) can be explained in terms of the modeling of transport in heterogeneous media. The local equation assumes the validity of Fick's first law, e.g., the mass flux is given by  $\mathbb{D}(\mathbf{x}) \cdot \nabla u(\mathbf{x})$ . The nonlocal equation assumes that the underlying transport is non-Fickian and improves the fidelity of the modeling, especially given the heterogeneity of porous media or the ambient environment. See [23] for an extended discussion and many references to the literature.

In addition to nonlocal diffusion, the peridynamic continuum theory [8, 26, 27] postulates that internal heating and force density are given by an integral operator. In particular, (1.2) models the one-dimensional peridynamic equilibrium equation associated with the equation of motion considered in [28, 29] and also transient heat conduction [8]. The operator  $\mathcal{L}$  is also associated with the jump process of the master equation that generalizes Brownian motion; see [16, chap. 7] for a review. Such a jump process was considered by Einstein in his seminal paper on the origins of diffusion. The operator  $\mathcal{L}$  gives rise to nonlocal diffusion that enables improved multiscale modeling; see [3] and [5, chap. 3] for examples and citations to the literature. See also [16, chap. 7] for a review of the Kramers-Moyal and van Kampen asymptotic approximations of a jump process by a Fokker-Plank equation. The classical diffusion equation is a specialization of the Fokker-Plank equation under the assumption of no drift, i.e., no bias in the associated random walk.

In [17], the nonlocal operator  $\mathcal{L}$  and its applications to image processing are considered and suggestions are made for its use in modeling physical phenomena. The nonlocal vector calculus presented in our paper extends the ideas introduced in [17] and applies this calculus to scalar nonlocal boundary-value problems. In follow-up papers, we consider a nonlocal calculus for vector-valued functions and apply it more general, three-dimensional linear peridynamic models for which deformation is given by a vector-valued function.

**2. A nonlocal Gauss's theorem.** Throughout, vectors are denoted by lower-case bold letters, e.g.,  $\mathbf{x}$  and  $\mathbf{q}$ , scalars by lower-case letters, e.g.,  $u$  and  $\alpha$ , and second-order tensors by upper-case blackboard letters, e.g.,  $\mathbb{K}$ . Mappings and products of mappings that appear throughout are assumed integrable with respect to the implied integration domains. Further, an *anti-symmetric mapping*  $p(\mathbf{x}', \mathbf{x})$  satisfies  $p(\mathbf{x}', \mathbf{x}) = -p(\mathbf{x}, \mathbf{x}')$  for all  $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^d$  whereas a *symmetric mapping*  $\mu(\mathbf{x}', \mathbf{x})$  satisfies  $\mu(\mathbf{x}', \mathbf{x}) = \mu(\mathbf{x}, \mathbf{x}')$ .

For any mapping  $r(\mathbf{x}, \mathbf{x}') : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathfrak{R}$ , it is easily seen that

$$\int_{\widehat{\Omega}} \int_{\widehat{\Omega}} r(\mathbf{x}, \mathbf{x}') d\mathbf{x}' d\mathbf{x} = \int_{\widehat{\Omega}} \int_{\widehat{\Omega}} r(\mathbf{x}', \mathbf{x}) d\mathbf{x}' d\mathbf{x} \quad \forall \widehat{\Omega} \subseteq \mathbb{R}^d. \quad (2.1)$$

If  $p(\mathbf{x}', \mathbf{x})$  denotes an anti-symmetric mapping, then (2.1) implies

$$\int_{\widehat{\Omega}} \int_{\widehat{\Omega}} p(\mathbf{x}, \mathbf{x}') d\mathbf{x}' d\mathbf{x} = 0 \quad \forall \widehat{\Omega} \subseteq \mathbb{R}^d. \quad (2.2)$$

Let  $\Omega$  denote an open bounded subset of  $\mathfrak{R}^d$ . If  $\Gamma \subseteq \mathfrak{R}^d \setminus \Omega$  denotes a nonzero volume and  $p(\mathbf{x}', \mathbf{x})$  is anti-symmetric, setting  $\widehat{\Omega} = \Omega \cup \Gamma$  in (2.2) implies

$$\int_{\Omega} \int_{\Omega \cup \Gamma} p(\mathbf{x}, \mathbf{x}') d\mathbf{x}' d\mathbf{x} = - \int_{\Gamma} \int_{\Omega \cup \Gamma} p(\mathbf{x}, \mathbf{x}') d\mathbf{x}' d\mathbf{x}. \quad (2.3)$$

Let  $\alpha(\mathbf{x}, \mathbf{x}') : \Omega \cup \Gamma \times \Omega \cup \Gamma \rightarrow \mathfrak{R}$  and  $f(\mathbf{x}, \mathbf{x}') : \Omega \cup \Gamma \times \Omega \cup \Gamma \rightarrow \mathfrak{R}$  denote scalar-valued mappings that are not necessarily symmetric or anti-symmetric. Define the linear operator  $\mathcal{D}$  that maps functions  $f(\mathbf{x}, \mathbf{x}')$  into functions defined over  $\Omega$  by

$$\mathcal{D}(f)(\mathbf{x}) := \int_{\Omega \cup \Gamma} (f(\mathbf{x}, \mathbf{x}')\alpha(\mathbf{x}, \mathbf{x}') - f(\mathbf{x}', \mathbf{x})\alpha(\mathbf{x}', \mathbf{x})) d\mathbf{x}' \quad \text{for } \mathbf{x} \in \Omega. \quad (2.4)$$

Similarly, let  $\mathcal{N}$  denote the linear operator that maps functions  $f(\mathbf{x}, \mathbf{x}')$  into functions defined over  $\Gamma$  given by

$$\mathcal{N}(f)(\mathbf{x}) := - \int_{\Omega \cup \Gamma} (f(\mathbf{x}, \mathbf{x}')\alpha(\mathbf{x}, \mathbf{x}') - f(\mathbf{x}', \mathbf{x})\alpha(\mathbf{x}', \mathbf{x})) d\mathbf{x}' \quad \text{for } \mathbf{x} \in \Gamma. \quad (2.5)$$

Then, setting  $p(\mathbf{x}, \mathbf{x}') = f(\mathbf{x}, \mathbf{x}')\alpha(\mathbf{x}, \mathbf{x}') - f(\mathbf{x}', \mathbf{x})\alpha(\mathbf{x}', \mathbf{x})$  in (2.3) results in the *non-local Gauss's theorem*

$$\int_{\Omega} \mathcal{D}(f) d\mathbf{x} = \int_{\Gamma} \mathcal{N}(f) d\mathbf{x}. \quad (2.6)$$

The results of the next section show that the term on the right-hand side of (2.6) corresponds to a flux so that  $\mathcal{D}(f)(\mathbf{x})$  is the nonlocal “divergence” of  $f$  at the point  $\mathbf{x}$ .

With  $\Gamma = \emptyset$ , we have from (2.6) (or directly from (2.2)) that  $\int_{\Omega} \mathcal{D}(f) d\mathbf{x} = 0$ . In [17, eq. (2.6)], this equation, is referred to as the “divergence theorem;” we see here that it is a special case of the nonlocal Gauss's theorem (2.6).

**2.1. Relation to the classical Gauss's theorem.** Let the vector-valued function  $\mathbf{q} : \mathfrak{R}^d \rightarrow \mathfrak{R}^d$  be defined by

$$\mathbf{q}(\mathbf{x}) := - \int_{\mathfrak{R}^d} (\mathbf{x}' - \mathbf{x})\varphi(\mathbf{x}, \mathbf{x}' - \mathbf{x}) d\mathbf{x}', \quad (2.7)$$

where, with  $p(\mathbf{x}, \mathbf{x}') = f(\mathbf{x}, \mathbf{x}')\alpha(\mathbf{x}, \mathbf{x}') - f(\mathbf{x}', \mathbf{x})\alpha(\mathbf{x}', \mathbf{x})$  and  $\mathbf{z} = \mathbf{x}' - \mathbf{x}$ , the function  $\varphi : \mathfrak{R}^d \times \mathfrak{R}^d \rightarrow \mathfrak{R}$  is given by

$$\varphi(\mathbf{x}, \mathbf{z}) = \int_0^1 p(\mathbf{x} + \lambda\mathbf{z}, \mathbf{x} - (1 - \lambda)\mathbf{z}) d\lambda. \quad (2.8)$$

We also suppose, for this subsection only,<sup>2</sup> that  $\Gamma = \mathfrak{R}^d \setminus \Omega$  so that  $\mathfrak{R}^d = \Omega \cup \Gamma$ . Then, a formal application of Lemma I in [22, 24] implies

$$\begin{aligned} \nabla \cdot \mathbf{q}(\mathbf{x}) &= \int_{\mathfrak{R}^d} (f(\mathbf{x}, \mathbf{x}')\alpha(\mathbf{x}, \mathbf{x}') - f(\mathbf{x}', \mathbf{x})\alpha(\mathbf{x}', \mathbf{x})) d\mathbf{x}' \\ &= \mathcal{D}(f)(\mathbf{x}) \quad \text{for } \mathbf{x} \in \Omega. \end{aligned} \quad (2.9)$$

<sup>2</sup>This assumption is made in [22, 24]; it is possible to extend the results of those papers to the case of convex domains  $\Omega$  and  $f(\cdot, \cdot)$  having compact support, in which case we need not assume that  $\mathfrak{R}^d = \Omega \cup \Gamma$ .

Lemma II in [22, 24] implies

$$\int_{\partial\Omega} \mathbf{q}(\mathbf{x}) \cdot \mathbf{n} \, dA = \int_{\Omega} \int_{\Gamma} (f(\mathbf{x}, \mathbf{x}')\alpha(\mathbf{x}, \mathbf{x}') - f(\mathbf{x}', \mathbf{x})\alpha(\mathbf{x}', \mathbf{x})) \, d\mathbf{x}' \, d\mathbf{x}, \quad (2.10)$$

where  $\partial\Omega$  denotes the boundary of  $\Omega$ ,  $dA$  the surface element on  $\partial\Omega$ , and  $\mathbf{n}$  the outward pointing unit normal vector along  $\partial\Omega$ . Two successive applications of (2.2), first with  $\widehat{\Omega} = \Omega$  and then with  $\widehat{\Omega} = \Omega \cup \Gamma$ , yields

$$\begin{aligned} & \int_{\Omega} \int_{\Gamma} (f(\mathbf{x}, \mathbf{x}')\alpha(\mathbf{x}, \mathbf{x}') - f(\mathbf{x}', \mathbf{x})\alpha(\mathbf{x}', \mathbf{x})) \, d\mathbf{x}' \, d\mathbf{x} \\ &= \int_{\Omega} \int_{\Omega \cup \Gamma} (f(\mathbf{x}, \mathbf{x}')\alpha(\mathbf{x}, \mathbf{x}') - f(\mathbf{x}', \mathbf{x})\alpha(\mathbf{x}', \mathbf{x})) \, d\mathbf{x}' \, d\mathbf{x} \\ &= - \int_{\Gamma} \int_{\Omega \cup \Gamma} (f(\mathbf{x}, \mathbf{x}')\alpha(\mathbf{x}, \mathbf{x}') - f(\mathbf{x}', \mathbf{x})\alpha(\mathbf{x}', \mathbf{x})) \, d\mathbf{x}' \, d\mathbf{x}. \end{aligned}$$

Combining this result with (2.10) yields

$$\begin{aligned} \int_{\partial\Omega} \mathbf{q}(\mathbf{x}) \cdot \mathbf{n} \, dA &= - \int_{\Gamma} \int_{\Omega \cup \Gamma} (f(\mathbf{x}, \mathbf{x}')\alpha(\mathbf{x}, \mathbf{x}') - f(\mathbf{x}', \mathbf{x})\alpha(\mathbf{x}', \mathbf{x})) \, d\mathbf{x}' \\ &= \int_{\Gamma} \mathcal{N}(f) \, d\mathbf{x}. \end{aligned} \quad (2.11)$$

The left-hand side of (2.11) involves the normal flux  $\mathbf{q} \cdot \mathbf{n}$  of the vector  $\mathbf{q}$ ; thus, one is justified in referring to  $\mathcal{N}(f)$  in (2.11) as well as in (2.6) as the *nonlocal flux* of the vector  $f$ .

Substituting (2.9) and (2.11) into the nonlocal Gauss's theorem (2.6) results in

$$\int_{\Omega} \nabla \cdot \mathbf{q} \, d\mathbf{x} = \int_{\partial\Omega} \mathbf{n} \cdot \mathbf{q} \, dA,$$

i.e., the classical Gauss's theorem for the vector-valued function  $\mathbf{q}$ . Thus, we have shown that *the nonlocal Gauss's theorem (2.6) for the nonlocal scalar-valued mapping  $f$  formally implies the classical Gauss's theorem for the local vector-valued function  $\mathbf{q}$  derived from  $f$  through (2.7)*. Evidently, Gauss's theorem can be given a meaning without the notions of the divergence operator, unit normal vector, or surface.

Under appropriate assumptions, the results of [21] can be invoked to show that the vector field  $\mathbf{q}$  solves the minimization problem

$$\inf_{\widehat{\mathbf{q}} \in H_0(\operatorname{div}, \mathbb{R}^d)} \frac{1}{2} \int_{\mathbb{R}^d} |\widehat{\mathbf{q}}|^2 \, d\mathbf{x} \quad \text{subject to} \quad \nabla \cdot \widehat{\mathbf{q}} = \mathcal{D}(f) \in L_0^2(\mathbb{R}^d),$$

where  $H_0(\operatorname{div}, \mathbb{R}^d) := \{ \mathbf{q} \mid \nabla \cdot \mathbf{q} \in L_0^2(\mathbb{R}^d) \}$  and  $L_0^2(\mathbb{R}^d) := \{ \psi \in L^2(\mathbb{R}^d) \mid \int_{\mathbb{R}^d} \psi \, d\mathbf{x} = 0 \}$ .

**2.2. An application of the nonlocal Gauss's theorem.** In the sequel, for ease of notation, we define

$$\begin{aligned} v &:= v(\mathbf{x}), & v' &:= v(\mathbf{x}'), & \alpha &:= \alpha(\mathbf{x}, \mathbf{x}'), & \alpha' &:= \alpha(\mathbf{x}', \mathbf{x}), \\ f &:= f(\mathbf{x}, \mathbf{x}'), & f' &:= f(\mathbf{x}', \mathbf{x}), & s &:= s(\mathbf{x}, \mathbf{x}'), & s' &:= s(\mathbf{x}', \mathbf{x}), \end{aligned}$$

and analogously for other similar functions yet to be introduced.

We apply the nonlocal Gauss's theorem (2.6) to the product of two mappings. In particular, for mappings  $v(\mathbf{x}) : \Omega \cup \Gamma \rightarrow \mathfrak{R}$  and  $s(\mathbf{x}, \mathbf{x}') : \Omega \cup \Gamma \times \Omega \cup \Gamma \rightarrow \mathfrak{R}$ , set  $f = vs$  in (2.6) so that, along with (2.4) and (2.5), we obtain

$$\int_{\Omega} \int_{\Omega \cup \Gamma} (vs\alpha - v's'\alpha') d\mathbf{x}' d\mathbf{x} = - \int_{\Gamma} \int_{\Omega \cup \Gamma} (vs\alpha - v's'\alpha') d\mathbf{x}' d\mathbf{x}.$$

Then, setting  $vs\alpha - v's'\alpha' = vs\alpha - v's'\alpha' + vs'\alpha' - vs'\alpha' = v(s\alpha - s'\alpha') + (v - v')s'\alpha'$ , we obtain

$$\begin{aligned} \int_{\Omega} \int_{\Omega \cup \Gamma} v(s\alpha - s'\alpha') d\mathbf{x}' d\mathbf{x} + \int_{\Omega \cup \Gamma} \int_{\Omega \cup \Gamma} (v - v')s'\alpha' d\mathbf{x}' d\mathbf{x} \\ = - \int_{\Gamma} \int_{\Omega \cup \Gamma} v(s\alpha - s'\alpha') d\mathbf{x}' d\mathbf{x}. \end{aligned}$$

We use (2.4) and (2.5) for the first and third integrals, respectively, and (2.1) for the second term to obtain

$$\int_{\Omega} v\mathcal{D}(s) d\mathbf{x} + \int_{\Omega \cup \Gamma} \int_{\Omega \cup \Gamma} (v' - v)s\alpha d\mathbf{x}' d\mathbf{x} = \int_{\Gamma} v\mathcal{N}(s) d\mathbf{x}. \quad (2.12)$$

Let  $\mathcal{G}$  denote the linear operator mapping scalar-valued functions  $v : \Omega \cup \Gamma \rightarrow \mathfrak{R}$  into vector-valued functions defined over  $\Omega \cup \Gamma \times \Omega \cup \Gamma$  given by<sup>3</sup>

$$\mathcal{G}(v) := (v' - v)\alpha \quad \text{for } \mathbf{x}, \mathbf{x}' \in \Omega \cup \Gamma. \quad (2.13)$$

Then, using (2.13) in (2.12) results in

$$\int_{\Omega} v\mathcal{D}(s) d\mathbf{x} + \int_{\Omega \cup \Gamma} \int_{\Omega \cup \Gamma} s \cdot \mathcal{G}(v) d\mathbf{x}' d\mathbf{x} = \int_{\Gamma} v\mathcal{N}(s) d\mathbf{x}. \quad (2.14)$$

The particular choice  $v = \text{constant}$  in (2.14) yields the nonlocal Gauss' theorem (2.6) applied to  $f = s$ .

**3. Nonlinear, nonlocal boundary value problems.** Let  $U(\Omega, \Gamma)$  and  $V(\Omega, \Gamma)$  denote Banach spaces of scalar-valued functions defined over  $\Omega \cup \Gamma$ . In this section, we do not choose specific spaces because that choice may depend on the particular form of the possibly nonlinear operator  $\mathcal{W}(\cdot)$  introduced below. In Section 4, where we consider linear operators, a specific choice is made for these spaces. Let

$$\Gamma := \Gamma_e + \Gamma_n \quad \text{with} \quad \Gamma_e \cap \Gamma_n = \emptyset$$

and define

$$V_e(\Omega, \Gamma) := \{v \in V(\Omega, \Gamma) : v = 0 \text{ for } \mathbf{x} \in \Gamma_e\}.$$

Suppose the mappings

$$b : \Omega \rightarrow \mathfrak{R}, \quad h_e : \Gamma_e \rightarrow \mathfrak{R}, \quad \text{and} \quad h_n : \Gamma_n \rightarrow \mathfrak{R} \quad (3.1)$$

are given. Let  $\mathcal{W}$  denote a possibly nonlinear operator that acts on scalar-valued functions  $u \in U(\Omega, \Gamma)$ ;  $\mathcal{W}$  may also depend explicitly on  $\mathbf{x}$  and  $\mathbf{x}'$ . Then, consider the

<sup>3</sup>In [17, eq.(2.2)],  $\mathcal{G}$  is denoted by  $\nabla_w u$ .

problem

$$\left\{ \begin{array}{l} \text{seek } u \in U(\Omega, \Gamma) \text{ such that} \\ \quad u = h_e \quad \text{for } \mathbf{x} \in \Gamma_e \\ \text{and} \\ \int_{\Omega \cup \Gamma} \int_{\Omega \cup \Gamma} \mathcal{W}(u) \mathcal{G}(v) \, d\mathbf{x}' d\mathbf{x} \\ \quad = \int_{\Omega} v b \, d\mathbf{x} + \int_{\Gamma_n} v h_n \, d\mathbf{x} \quad \forall v \in V_e(\Omega, \Gamma). \end{array} \right. \quad (3.2)$$

Then, (2.14) with  $s = \mathcal{W}(u)$  and  $v = 0$  on  $\Gamma_e$  imply

$$- \int_{\Omega} v \mathcal{D}(\mathcal{W}(u)) \, d\mathbf{x} + \int_{\Gamma_n} v \mathcal{N}(\mathcal{W}(u)) \, d\mathbf{x} = \int_{\Omega} v b \, d\mathbf{x} + \int_{\Gamma_n} v h_n \, d\mathbf{x} \quad \forall v \in V_e(\Omega, \Gamma).$$

Hence, (3.2) can be viewed as a weak formulation of the “boundary-value” problem

$$\left\{ \begin{array}{ll} -\mathcal{D}(\mathcal{W}(u)) = b & \text{for } \mathbf{x} \in \Omega \\ \quad u = h_e & \text{for } \mathbf{x} \in \Gamma_e \\ \quad \mathcal{N}(\mathcal{W}(u)) = h_n & \text{for } \mathbf{x} \in \Gamma_n. \end{array} \right. \quad (3.3)$$

The second and third equations of (3.3) are the “Dirichlet boundary” and “Neumann boundary” conditions that are the *essential* and *natural* conditions,<sup>4</sup> respectively, for the weak formulation (3.2).

If  $\Gamma_e = \emptyset$ , then the space of test functions  $V_e(\Omega, \Gamma)$  in (3.2) is replaced by the quotient space  $V(\Omega, \Gamma)/\mathfrak{R}$  and the data must satisfy the compatibility condition

$$\int_{\Omega} b \, d\mathbf{x} + \int_{\Gamma} h_n \, d\mathbf{x} = 0. \quad (3.4)$$

Alternately, one can use the space  $\{v \in V(\Omega, \Gamma) : \int_{\Omega \cup \Gamma} v \, d\mathbf{x} = 0\}$  instead of  $V(\Omega, \Gamma)/\mathfrak{R}$ . If  $\Gamma_e = \emptyset$ , (3.4) is a necessary condition for the existence of solutions of the problem (3.2) because, in this case, the left-hand side of (3.2) vanishes whenever  $v$  is constant. Note that (3.4) states that the data  $b$  and  $h_n$  have to be orthogonal to the one-dimensional null space of the operator  $\mathcal{G}(v) = (v' - v)\alpha$ . Correspondingly, we exclude the constant functions from the space of test functions in (3.2). This is all entirely analogous to the situation for classical, local Neumann boundary-value problems.

**4. Linear, nonlocal operators and nonlocal Green’s identities.** In this section, we specialize the nonlocal Gauss’s theorem discussed in Section 2 and, in particular, (2.14) to the case of  $U(\Omega, \Gamma) = V(\Omega, \Gamma) = L^2(\Omega \cup \Gamma)$  and to linear operators. To this end, for  $u \in L^2(\Omega \cup \Gamma)$ , let

$$s = \mathcal{W}(u) = \beta \mathcal{G}(u) = (u' - u)\alpha\beta, \quad (4.1)$$

<sup>4</sup>To show the correspondence between the weak problem (3.2) and the “boundary-value” problem (3.3), we must impose the condition  $u = h_e$  on candidate solutions. Such conditions are often referred to as being “essential;” see, e.g., [9]. On the other hand, showing the correspondence between (3.2) and (3.3) does not require the imposition of  $\mathcal{N}(\mathcal{W}(u)) = h_n$  on candidate solutions; this condition is satisfied without further constraining candidate solutions. Such conditions are often referred to as being “natural.”

where, at this time, no assumption is made about the symmetry or anti-symmetry of the scalar-valued mapping  $\beta(\mathbf{x}, \mathbf{x}') : \Omega \cup \Gamma \times \Omega \cup \Gamma \rightarrow \mathfrak{R}$ . Then, substitution into (2.14) results in the *nonlocal Green's first identity*<sup>5</sup>

$$\int_{\Omega} v \mathcal{D}(\beta \mathcal{G}(u)) \, d\mathbf{x} + \int_{\Omega \cup \Gamma} \int_{\Omega \cup \Gamma} \mathcal{G}(v) \beta \mathcal{G}(u) \, d\mathbf{x}' \, d\mathbf{x} = \int_{\Gamma} v \mathcal{N}(\beta \mathcal{G}(u)) \, d\mathbf{x}. \quad (4.2)$$

Reversing the roles of  $u$  and  $v$  in (4.2) and then subtracting the result from (4.2) results in the *nonlocal Green's second identity*

$$\int_{\Omega} v \mathcal{D}(\beta \mathcal{G}(u)) \, d\mathbf{x} - \int_{\Omega} u \mathcal{D}(\beta \mathcal{G}(v)) \, d\mathbf{x} = \int_{\Gamma} \left( v \mathcal{N}(\beta \mathcal{G}(u)) - u \mathcal{N}(\beta \mathcal{G}(v)) \right) \, d\mathbf{x}. \quad (4.3)$$

From (4.2) we have, by setting  $v = \text{constant}$ ,

$$\int_{\Omega} \mathcal{D}(\beta \mathcal{G}(u)) \, d\mathbf{x} = \int_{\Gamma} \mathcal{N}(\beta \mathcal{G}(u)) \, d\mathbf{x}$$

and, by setting  $v = u$ , the “energy” identity

$$\int_{\Omega} u \mathcal{D}(\beta \mathcal{G}(u)) \, d\mathbf{x} + \int_{\Omega \cup \Gamma} \int_{\Omega \cup \Gamma} \mathcal{G}(u) \beta \mathcal{G}(u) \, d\mathbf{x}' \, d\mathbf{x} = \int_{\Gamma} u \mathcal{N}(\beta \mathcal{G}(u)) \, d\mathbf{x}.$$

See, e.g., [20, Chap. 4], for the analogous identities in the classical linear elliptic operator case.

We defer discussion of a nonlocal Green's third identity until after we discuss linear, nonlocal boundary-value problems.

The relation (4.1) is a “constitutive” relation. To define a general form of the constitutive function  $\beta$ , we let  $\mathbb{K}(\mathbf{x}, \mathbf{x}') : \Omega \times \Omega \rightarrow \mathfrak{R}^{d \times d}$  denote a symmetric tensor each of whose elements is a symmetric function of  $\mathbf{x}$  and  $\mathbf{x}'$ . Then, a general constitutive function  $\beta$  is given by

$$\beta = (\mathbf{x}' - \mathbf{x}) \cdot \mathbb{K} \cdot (\mathbf{x}' - \mathbf{x}), \quad (4.4)$$

where, for ease of notation, we have suppressed the dependence of  $\beta$  and  $\mathbb{K}$  on  $\mathbf{x}$  and  $\mathbf{x}'$ . Here,  $\mathbb{K}$  is a constitutive tensor that serves to define the physical setting being studied.

**5. Linear, nonlocal boundary-value problems.** With  $s$  given by (4.1) and  $U(\Omega, \Gamma) = V(\Omega, \Gamma) = L^2(\Omega \cup \Gamma)$ , the problem (3.2) reduces to

$$\left\{ \begin{array}{l} \text{seek } u \in L^2(\Omega \cup \Gamma) \text{ such that} \\ \quad u = h_e \quad \text{for } \mathbf{x} \in \Gamma_e \\ \text{and} \\ \quad \int_{\Omega \cup \Gamma} \int_{\Omega \cup \Gamma} \mathcal{G}(v) \beta \mathcal{G}(u) \, d\mathbf{x}' \, d\mathbf{x} \\ \quad \quad = \int_{\Omega} v b \, d\mathbf{x} + \int_{\Gamma_n} v h_n \, d\mathbf{x} \quad \forall v \in L_e^2(\Omega \cup \Gamma), \end{array} \right. \quad (5.1)$$

where

$$L_e^2(\Omega \cup \Gamma) = \{v \in L^2(\Omega \cup \Gamma) : v = 0 \text{ for } \mathbf{x} \in \Gamma_e\}.$$

<sup>5</sup>See also the “integration by parts” formula given in [19, Lemma 2.1].

The corresponding nonlocal “boundary-value” problem (3.3) reduces to the linear problem

$$\begin{cases} -\mathcal{D}(\beta\mathcal{G}(u)) = b & \text{for } \mathbf{x} \in \Omega \\ u = h_e & \text{for } \mathbf{x} \in \Gamma_e \\ \mathcal{N}(\beta\mathcal{G}(u)) = h_n & \text{for } \mathbf{x} \in \Gamma_n, \end{cases} \quad (5.2)$$

where again the second equation is a “Dirichlet boundary” condition that is *essential* for the problem (5.1) and the third equation is a “Neumann boundary” condition that is *natural* for that problem.

Substituting the definitions (2.4), (2.5), and (2.13) for  $\mathcal{D}$ ,  $\mathcal{N}$ , and  $\mathcal{G}$ , respectively, we have the explicit relations

$$\begin{aligned} \int_{\Omega \cup \Gamma} \int_{\Omega \cup \Gamma} \mathcal{G}(v)\beta\mathcal{G}(u) \, d\mathbf{x}' \, d\mathbf{x} &= \int_{\Omega \cup \Gamma} \int_{\Omega \cup \Gamma} (v' - v)\alpha\beta\alpha(u' - u) \, d\mathbf{x}' \, d\mathbf{x} \\ \mathcal{D}(\beta\mathcal{G}(u)) &= 2 \int_{\Omega \cup \Gamma} (u' - u)\alpha\beta\alpha \, d\mathbf{x}' & \text{for } \mathbf{x} \in \Omega \\ \mathcal{N}(\beta\mathcal{G}(u)) &= -2 \int_{\Omega \cup \Gamma} (u' - u)\alpha\beta\alpha \, d\mathbf{x}' & \text{for } \mathbf{x} \in \Gamma_n. \end{aligned} \quad (5.3)$$

The problem (5.1) may be expressed in the form:

$$\begin{cases} \text{seek } u \in L^2(\Omega \cup \Gamma) \text{ such that } u = h_e \text{ for } \mathbf{x} \in \Gamma_e \text{ and} \\ B(u, v) = F(v) \quad \forall v \in L_e^2(\Omega \cup \Gamma), \end{cases} \quad (5.4)$$

where, for all  $u, v \in L^2(\Omega \cup \Gamma)$ , we have the bilinear form

$$\begin{aligned} B(u, v) &:= \int_{\Omega \cup \Gamma} \int_{\Omega \cup \Gamma} \mathcal{G}(v)\beta\mathcal{G}(u) \, d\mathbf{x}' \, d\mathbf{x} \\ &= \int_{\Omega \cup \Gamma} \int_{\Omega \cup \Gamma} (v' - v)\alpha\beta\alpha(u' - u) \, d\mathbf{x}' \, d\mathbf{x} \end{aligned} \quad (5.5)$$

and the linear functional

$$F(v) := \int_{\Omega} v b \, d\mathbf{x} + \int_{\Gamma_n} v h_n \, d\mathbf{x}. \quad (5.6)$$

Note that, under the assumptions on  $\mathbb{K}$  made in Section 6, the solution of the “boundary-value” problem (5.4) corresponds to the solution of the optimization problem

$$\arg \min_{\{v \in L^2(\Omega \cup \Gamma), v = h_e \text{ for } \mathbf{x} \in \Gamma_e\}} \left( \frac{1}{2} B(v, v) - F(v) \right).$$

## 6. Well posedness of the linear nonlocal “boundary-value” problem.

We now demonstrate, in the case of  $\Gamma_e = \Gamma$  and  $h_e = 0$ , that, under appropriate assumptions about  $\alpha$  and  $\beta$ , the “boundary-value” problem (5.1), or equivalently (5.4), is well posed.<sup>6</sup> We assume that  $U(\Omega, \Gamma) = V(\Omega, \Gamma) = L^2(\Omega \cup \Gamma)$ .<sup>7</sup>

<sup>6</sup>The case  $\Gamma_e = \Gamma$  and  $h_e = 0$  corresponds to the homogeneous Dirichlet problem in the classical elliptic partial differential equation setting. Note that the well posedness of the nonlocal “Neumann” problem can also be demonstrated.

<sup>7</sup>See [4] for a well-posedness result when  $U(\Omega, \Gamma) = H_0^s(\Omega)$ ,  $1/2 < s < 1$ , the papers [11, 12, 13] for results on the linear peridynamic model for which  $u$  in (1.2) is a vector field, and [1, 2] for results about the strong form of the nonlocal boundary value problem (5.2).

We assume that the data function  $b \in L^2(\Omega)$  and that, for all  $\mathbf{x}, \mathbf{x}' \in \Omega \cup \Gamma$ ,  $\alpha^2$  is a symmetric mapping. Thus, necessarily,  $\alpha(\mathbf{x}, \mathbf{x}')$  is either a symmetric or anti-symmetric mapping. We also assume that, for all  $\mathbf{x}, \mathbf{x}' \in \Omega \cup \Gamma$ ,  $\beta(\mathbf{x}, \mathbf{x}')$  is a positive, symmetric mapping; sufficient conditions for these assumptions to hold are that, in (4.4), the constitutive tensor  $\mathbb{K}$  has a positive definite symmetric part and has elements that are symmetric in the mapping sense, i.e.,  $\mathbb{K}(\mathbf{x}, \mathbf{x}') = \mathbb{K}(\mathbf{x}', \mathbf{x})$ . We also assume that, for some positive constants  $K_1$  and  $K_2$

$$\int_{\Omega \cup \Gamma} \beta \alpha^2 d\mathbf{x}' \leq K_1 \quad \text{and} \quad \int_{\Gamma_e} \beta \alpha^2 d\mathbf{x}' \geq K_2 \quad \forall \mathbf{x} \in \Omega. \quad (6.1)$$

One easily sees that, because  $v = 0$  on  $\Gamma_e = \Gamma$ ,

$$|F(v)| \leq \|b\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \leq \|b\|_{L^2(\Omega)} \|v\|_{L^2(\Omega \cup \Gamma)} \quad \forall v \in L^2(\Omega \cup \Gamma), \quad (6.2)$$

so that  $F(v)$  is a bounded linear functional on  $L^2_e(\Omega \cup \Gamma)$ .

It is clear that the bilinear form  $B(\cdot, \cdot)$  is symmetric, i.e.,  $B(u, v) = B(v, u)$  for all  $u, v \in L^2_e(\Omega \cup \Gamma)$ . We now show that, with respect to  $L^2_e(\Omega \cup \Gamma)$ ,  $B(\cdot, \cdot)$  is continuous. From (5.5),

$$\begin{aligned} B(u, v) &= \int_{\Omega \cup \Gamma} \int_{\Omega \cup \Gamma} (u'v' + uv) \beta \alpha^2 d\mathbf{x}' d\mathbf{x} - \int_{\Omega \cup \Gamma} \int_{\Omega \cup \Gamma} (u'v + uv') \beta \alpha^2 d\mathbf{x}' d\mathbf{x} \\ &= 2 \int_{\Omega \cup \Gamma} uv \left( \int_{\Omega \cup \Gamma} \beta \alpha^2 d\mathbf{x}' \right) d\mathbf{x} - 2 \int_{\Omega \cup \Gamma} \int_{\Omega \cup \Gamma} uv' \beta \alpha^2 d\mathbf{x}' d\mathbf{x}. \end{aligned}$$

Using the first inequality in (6.1) and  $u(\mathbf{x}) = v(\mathbf{x}) = 0$  for  $\mathbf{x} \in \Gamma_e = \Gamma$ , we have

$$\begin{aligned} \left| \int_{\Omega \cup \Gamma} uv \left( \int_{\Omega \cup \Gamma} \beta \alpha^2 d\mathbf{x}' \right) d\mathbf{x} \right| &= \left| \int_{\Omega} uv \left( \int_{\Omega \cup \Gamma} \beta \alpha^2 d\mathbf{x}' \right) d\mathbf{x} \right| \\ &\leq K_1 \int_{\Omega \cup \Gamma} |u| |v| d\mathbf{x} \leq K_1 \|u\|_{L^2(\Omega \cup \Gamma)} \|v\|_{L^2(\Omega \cup \Gamma)} \end{aligned}$$

and

$$\begin{aligned} &\left| \int_{\Omega \cup \Gamma} \int_{\Omega \cup \Gamma} uv' \beta \alpha^2 d\mathbf{x}' d\mathbf{x} \right| \\ &\leq \left( \int_{\Omega \cup \Gamma} \int_{\Omega \cup \Gamma} u^2 \beta \alpha^2 d\mathbf{x}' d\mathbf{x} \right)^{1/2} \left( \int_{\Omega \cup \Gamma} \int_{\Omega \cup \Gamma} (v')^2 \beta \alpha^2 d\mathbf{x}' d\mathbf{x} \right)^{1/2} \\ &\leq \left( \int_{\Omega \cup \Gamma} \int_{\Omega \cup \Gamma} u^2 \beta \alpha^2 d\mathbf{x}' d\mathbf{x} \right)^{1/2} \left( \int_{\Omega \cup \Gamma} \int_{\Omega \cup \Gamma} v^2 \beta \alpha^2 d\mathbf{x}' d\mathbf{x} \right)^{1/2} \\ &\leq \left( \int_{\Omega} u^2 \int_{\Omega \cup \Gamma} \beta \alpha^2 d\mathbf{x}' d\mathbf{x} \right)^{1/2} \left( \int_{\Omega} v^2 \int_{\Omega \cup \Gamma} \beta \alpha^2 d\mathbf{x}' d\mathbf{x} \right)^{1/2} \\ &\leq K_1 \|u\|_{L^2(\Omega \cup \Gamma)} \|v\|_{L^2(\Omega \cup \Gamma)} \end{aligned}$$

so that

$$|B(u, v)| \leq 4K_1 \|u\|_{L^2(\Omega \cup \Gamma)} \|v\|_{L^2(\Omega \cup \Gamma)},$$

i.e., the bilinear form  $B(\cdot, \cdot)$  is continuous on  $L^2_e(\Omega \cup \Gamma) \times L^2_e(\Omega \cup \Gamma)$ .

Next, we show that the bilinear form  $B(\cdot, \cdot)$  is coercive on  $L_e^2(\Omega \cup \Gamma)$ . Using the second inequality in (6.1) and  $u(\mathbf{x}') = 0$  for  $\mathbf{x}' \in \Gamma_e = \Gamma$ ,

$$\begin{aligned}
B(u, u) &= \int_{\Omega \cup \Gamma} \int_{\Omega \cup \Gamma} (u' - u)^2 \beta \alpha^2 d\mathbf{x}' d\mathbf{x} \\
&= \int_{\Omega \cup \Gamma} \int_{\Gamma_e} (u' - u)^2 \beta \alpha^2 d\mathbf{x}' d\mathbf{x} + \int_{\Omega \cup \Gamma} \int_{\Omega} (u' - u)^2 \beta \alpha^2 d\mathbf{x}' d\mathbf{x} \\
&\geq \int_{\Omega \cup \Gamma} \int_{\Gamma_e} (u' - u)^2 \beta \alpha^2 d\mathbf{x}' d\mathbf{x} = \int_{\Omega \cup \Gamma} u^2 \int_{\Gamma_e} \beta \alpha^2 d\mathbf{x}' d\mathbf{x} \quad (6.3) \\
&= \int_{\Omega} u^2 \left( \int_{\Gamma_e} \beta \alpha^2 d\mathbf{x}' \right) d\mathbf{x} \\
&\geq K_2 \|u\|_{L^2(\Omega)}^2 = K_2 \|u\|_{L^2(\Omega \cup \Gamma)}^2 \quad \forall u \in L_e^2(\Omega \cup \Gamma),
\end{aligned}$$

i.e., the bilinear form  $B(\cdot, \cdot)$  is coercive on  $L_e^2(\Omega \cup \Gamma)$ .

We have shown that the linear functional  $F(\cdot)$  is continuous on  $L_e^2(\Omega \cup \Gamma)$ , the bilinear form is symmetric and continuous on  $L_e^2(\Omega \cup \Gamma) \times L_e^2(\Omega \cup \Gamma)$ , and that that form is coercive on  $L_e^2(\Omega \cup \Gamma)$ . Then, by the the Lax-Milgram theorem [9], (5.4), or equivalently (5.1), has a unique solution  $u \in L_e^2(\Omega \cup \Gamma)$  and, moreover, using (5.5), (6.2), and (6.3), that solution satisfies

$$\|u\|_{L^2(\Omega \cup \Gamma)} \leq \frac{1}{K_2} \|b\|_{L^2(\Omega)},$$

i.e., the solution depends continuously upon the data. The case  $h_e \neq 0$  can be treated in a similar manner after rendering the essential boundary condition in (5.1) homogeneous by subtracting from  $u$  any function  $\tilde{u}$  satisfying  $\tilde{u} = h_e$  for  $\mathbf{x} \in \Gamma_e$ .

We have assumed that  $\beta$  is positive over  $\Omega \cup \Gamma$ . For the special choice of  $\beta \alpha^2$  a nonnegative, radial, continuous function such that  $\int_{\mathbb{R}^d} \beta \alpha^2 d\mathbf{x}' > 0$ , a Poincaré inequality has been established; see [Proposition 2.5][2]. Such an inequality may be used to show that the bilinear form  $B(\cdot, \cdot)$  is coercive on  $L_e^2(\Omega \cup \Gamma)$  without assuming that  $\beta$  is strictly positive, so that the Lax-Milgram theorem can be invoked in this more general case as well.

**6.1. Decomposition of the solution space.** Choosing  $\beta = 1$ , the identity tensor, we let the space  $K(\Omega \cup \Gamma)$  consist of functions  $u \in L^2(\Omega \cup \Gamma)/\mathfrak{R}$  that satisfy

$$\begin{cases} \mathcal{D}(\mathcal{G}(u)) = 2 \int_{\Omega \cup \Gamma} (u' - u) \alpha^2 d\mathbf{x}' = 0 & \forall \mathbf{x} \in \Omega \\ \mathcal{N}(\mathcal{G}(u)) = -2 \int_{\Omega \cup \Gamma} (u' - u) \alpha^2 d\mathbf{x}' = 0 & \forall \mathbf{x} \in \Gamma, \end{cases} \quad (6.4)$$

where, as always, we assume that the integrals exist. Then, from (4.2), we have that, for all  $u \in K(\Omega \cup \Gamma)$  and  $v \in L_e^2(\Omega \cup \Gamma)$ ,

$$\int_{\Omega \cup \Gamma} \int_{\Omega \cup \Gamma} \mathcal{G}(v) \mathcal{G}(u) d\mathbf{x}' d\mathbf{x} = 0. \quad (6.5)$$

Next, suppose (6.5) holds for all  $v \in L_e^2(\Omega \cup \Gamma)$ . Then, from (4.2),

$$\int_{\Omega} v \mathcal{D}(\mathcal{G}(u)) d\mathbf{x} = \int_{\Gamma_n} v \mathcal{N}(\mathcal{G}(u)) d\mathbf{x} \quad \forall v \in L_e^2(\Omega \cup \Gamma). \quad (6.6)$$

By choosing  $v$  such that  $v = 0$  on  $\Gamma_n$  but that is otherwise arbitrary, we conclude that  $u$  satisfies the first equation in (6.4). Then, (6.6) reduces to  $\int_{\Gamma_n} v \mathcal{N}(\mathcal{G}(u)) \, d\mathbf{x} = 0$  for arbitrary  $v \in L_e^2(\Omega \cup \Gamma)$  and we conclude that  $u$  satisfies the second equation in (6.4). Thus, if (6.5) holds for all  $v \in L_e^2(\Omega \cup \Gamma)$ , we have that  $u \in K(\Omega \cup \Gamma)$ .

Thus, we conclude that

$$L^2(\Omega \cup \Gamma)/\mathfrak{R} = L_e^2(\Omega \cup \Gamma) \oplus K(\Omega \cup \Gamma), \quad (6.7)$$

i.e., any function in  $L^2(\Omega \cup \Gamma)/\mathfrak{R}$  can be written as a sum of two functions that are orthogonal with respect to the inner product induced by the bilinear form defined by the left-hand side of (6.5), the first a function that vanishes on  $\Gamma_e$  and the second a function satisfying (6.4). We then conclude that any function in  $L^2(\Omega \cup \Gamma)$  can then be expressed as the sum of a function in  $K(\Omega \cup \Gamma)$ , a function that vanishes on  $\Gamma_e$ , and a constant.

The space  $K(\Omega \cup \Gamma)$  consists of nonlocal ‘‘harmonic’’ functions; see (7.5). Then, the decomposition (6.7) is entirely analogous to the decomposition of the Sobolev space  $H^1(\Omega)$  into functions belonging to  $H_0^1(\Omega)$  and harmonic functions.

## 7. Nonlocal Green’s functions and a nonlocal Green’s third identity.

**7.1. Nonlocal fundamental solutions.** For each  $\mathbf{y} \in \mathfrak{R}^d$ , let  $g(\mathbf{x}; \mathbf{y})$  denote the fundamental solution (or free-space Green’s function) for the operator  $\mathcal{D}(\beta\mathcal{G}(\cdot))$ , formally defined as the solution of

$$\mathcal{D}(\beta\mathcal{G}(g(\mathbf{x}; \mathbf{y}))) = \delta(|\mathbf{x} - \mathbf{y}|) \quad \forall \mathbf{x} \in \mathfrak{R}^d,$$

where  $\delta(\cdot)$  denotes the Dirac delta function. We assume that  $\alpha(\mathbf{x}, \mathbf{x}') = \alpha(\mathbf{x}' - \mathbf{x})$  and  $\mathbb{K}(\mathbf{x}, \mathbf{x}') = \mathbb{K}(\mathbf{x}' - \mathbf{x})$  and then seek a translation invariant solution  $g(\mathbf{x}; \mathbf{y}) = g(\mathbf{x} - \mathbf{y})$ . Using (5.3), we then have

$$2 \int_{\mathfrak{R}^d} (g(\mathbf{x}') - g(\mathbf{x})) \mu(\mathbf{x}' - \mathbf{x}) \, d\mathbf{x}' = \delta(|\mathbf{x}|) \quad \mathbf{x} \in \mathfrak{R}^d, \quad (7.1)$$

where  $\mu(\mathbf{x}, \mathbf{x}') = \mu(\mathbf{x}' - \mathbf{x}) = \alpha^2 \beta$ . Assuming the first inequality in (6.1), we can then assume, without loss of generality, that the function  $\mu$  satisfies  $\int_{\mathfrak{R}^d} \mu \, d\mathbf{x} = 1$ . Then, (7.1) can be expressed in the form

$$2 \int_{\mathfrak{R}^d} g(\mathbf{x}') \mu(\mathbf{x}' - \mathbf{x}) \, d\mathbf{x}' - 2g(\mathbf{x}) = \delta(|\mathbf{x}|) \quad \forall \mathbf{x} \in \mathfrak{R}^d$$

so that

$$\widehat{g} = \frac{(2\pi)^{-d/2}}{2} \left( \frac{1}{(2\pi)^{d/2} \widehat{\mu} - 1} \right),$$

where the Fourier transforms of  $g$  and  $\mu$  are given by

$$\widehat{g}(\mathbf{k}) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-i\mathbf{k} \cdot \mathbf{x}} g(\mathbf{x}) \, d\mathbf{x} \quad \text{and} \quad \widehat{\mu}(\mathbf{k}) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-i\mathbf{k} \cdot \mathbf{x}} \mu(\mathbf{x}) \, d\mathbf{x},$$

respectively. Therefore,

$$g(\mathbf{x}) = \frac{(2\pi)^{-d}}{2} \int_{\mathbb{R}^d} e^{i\mathbf{k} \cdot \mathbf{x}} \frac{1}{(2\pi)^{d/2} \widehat{\mu} - 1} \, d\mathbf{k}$$

so that, for general  $\mathbf{x}, \mathbf{y} \in \mathfrak{R}^d$ , the fundamental solution for (7.1) is given by

$$g(\mathbf{x}; \mathbf{y}) = \frac{(2\pi)^{-d}}{2} \int_{\mathbb{R}^d} e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{y})} \frac{1}{(2\pi)^{d/2} \widehat{\mu} - 1} d\mathbf{k}. \quad (7.2)$$

The special choice

$$\mu(\mathbf{x}' - \mathbf{x}) = \delta(\mathbf{x}' - \mathbf{x}) + \frac{\partial^2}{\partial x^2} \delta(\mathbf{x}' - \mathbf{x})$$

leads to the same fundamental solution<sup>8</sup> as that for the Laplace operator, i.e.,

$$-\frac{(2\pi)^{-d}}{2} \int_{\mathbb{R}^d} e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{y})} |\mathbf{k}|^{-2} d\mathbf{k} = \begin{cases} \frac{1}{2} |x - y|, & d = 1, \\ \frac{1}{2\pi} \ln |\mathbf{x} - \mathbf{y}|, & d = 2, \\ \frac{1}{2\omega_d} \frac{|\mathbf{x} - \mathbf{y}|^{2-d}}{2-d}, & d \geq 3, \end{cases}$$

where  $\omega_d$  denotes the volume of the unit ball in  $\mathbb{R}^d$ .

**7.2. Nonlocal Green's third identity.** For any  $\mathbf{y} \in \Omega \cup \Gamma$ , let  $G(\mathbf{x}; \mathbf{y}) : \Omega \cup \Gamma \rightarrow \mathfrak{R}$  denote any function satisfying

$$\mathcal{D}(\beta \mathcal{G}(G(\mathbf{x}; \mathbf{y}))) = \delta(|\mathbf{x} - \mathbf{y}|) \quad \forall \mathbf{x} \in \Omega. \quad (7.3)$$

Then, using the nonlocal Green's second identity (4.3) with  $v(\cdot) = G(\cdot; \mathbf{y})$ , we obtain the *nonlocal Green's third identity*

$$\begin{aligned} u(\mathbf{y}) &= \int_{\Omega} G(\mathbf{x}; \mathbf{y}) \mathcal{D}(\beta \mathcal{G}(u(\mathbf{x}))) d\mathbf{x} \\ &\quad - \int_{\Gamma} \left( G(\mathbf{x}; \mathbf{y}) \mathcal{N}(\beta \mathcal{G}(u(\mathbf{x}))) - u(\mathbf{x}) \mathcal{N}(\beta \mathcal{G}(G(\mathbf{x}; \mathbf{y}))) \right) d\mathbf{x} \quad \forall \mathbf{y} \in \Omega. \end{aligned} \quad (7.4)$$

Suppose that the constitutive function  $\beta = 1$ . Further suppose

$$\mathcal{D}(\mathcal{G}(u)) = 2 \int_{\Omega \cup \Gamma} (u' - u) \alpha^2 d\mathbf{x}' = 0 \quad \forall \mathbf{x} \in \Omega. \quad (7.5)$$

Then, the solution  $u(\mathbf{x})$  represents a nonlocal ‘‘harmonic’’ function that, from (7.4), is given by

$$u(\mathbf{y}) = \int_{\Gamma} \left( u(\mathbf{x}) \mathcal{N}(\mathcal{G}(G(\mathbf{x}; \mathbf{y}))) - G(\mathbf{x}; \mathbf{y}) \mathcal{N}(\mathcal{G}(u(\mathbf{x}))) \right) d\mathbf{x},$$

i.e., ‘‘harmonic’’ functions are determined by their ‘‘boundary’’ values on  $\Gamma$ . Nonlocal versions of the Poisson integral formula and Gauss's law of the arithmetic mean can also be derived; see [20, Chap. 4] for the classical case.

<sup>8</sup>See also [28, § 2] for a derivation of the result in one dimension.

**7.3. Nonlocal Green's functions.** Let  $g(\mathbf{x}; \mathbf{y})$  denote the fundamental solution defined in Section 7.1. For each  $\mathbf{y} \in \Omega \cup \Gamma$ , define the *nonlocal Green's function*  $G(\mathbf{x}; \mathbf{y}) : \Omega \cup \Gamma \rightarrow \mathfrak{R}$  as

$$G(\mathbf{x}; \mathbf{y}) = g(\mathbf{x}; \mathbf{y}) - \tilde{g}(\mathbf{x}; \mathbf{y}),$$

where  $\tilde{g}(\cdot; \cdot)$  is a solution of

$$\begin{cases} \mathcal{D}(\beta\mathcal{G}(\tilde{g})) = 0 & \text{for } \mathbf{x} \in \Omega \\ \mathcal{N}(\beta\mathcal{G}(\tilde{g})) = \mathcal{N}(\beta\mathcal{G}(g)) & \text{for } \mathbf{x} \in \Gamma_n \\ \tilde{g}(\mathbf{x}; \mathbf{y}) = g(\mathbf{x}; \mathbf{y}) & \text{for } \mathbf{x} \in \Gamma_e. \end{cases}$$

Then,  $G(\cdot; \cdot)$  satisfies (7.3) and the homogeneous “boundary” conditions  $G(\mathbf{x}; \mathbf{y}) = 0$  for  $\mathbf{x} \in \Gamma_e$  and  $\mathcal{N}(\beta\mathcal{G}(G)) = 0$  for  $\mathbf{x} \in \Gamma_n$ . Applying (7.4), we then have that the solution of the “boundary-value” problem (5.2) is given by

$$\begin{aligned} u(\mathbf{y}) = & - \int_{\Omega} G(\mathbf{x}; \mathbf{y}) b(\mathbf{x}) \, d\mathbf{x} \\ & + \int_{\Gamma_e} h_e(\mathbf{x}) \mathcal{N}(\beta\mathcal{G}(G(\mathbf{x}; \mathbf{y}))) \, d\mathbf{x} - \int_{\Gamma_n} G(\mathbf{x}; \mathbf{y}) h_n(\mathbf{x}) \, d\mathbf{x} \quad \forall \mathbf{y} \in \Omega. \end{aligned}$$

Because the operators  $\mathcal{D}$  and  $\mathcal{N}$  differ only in their signs and domains, it follows that this formula also holds for  $\mathbf{y} \in \Gamma_n$ .

**8. Local smooth limits.** By closely following relevant material from [25], we now connect the linear nonlocal “boundary-value” problem of Section 5 to weak and classical formulations of the Dirichlet and Neumann problems for linear, second-order elliptic partial differential equations. To do so, we make two central assumptions, one about solutions and the other about the constitutive model. We emphasize that these assumptions are made only to make the connection to classical problems for partial differential equations and are not required for the well posedness of the nonlocal “boundary-value” problems. We make additional geometric assumptions and also adopt the assumptions made in Section 6 for the well posedness of the nonlocal “boundary-value” problem.

First, we assume that *solutions of the nonlocal “boundary-value” problems are smooth*; specifically, we assume that,<sup>9</sup> for  $\varepsilon > 0$ ,

$$u(\mathbf{x}') = u(\mathbf{x}) + \nabla u(\mathbf{x}) \cdot (\mathbf{x}' - \mathbf{x}) + o(\varepsilon) \quad \text{if } |\mathbf{x}' - \mathbf{x}| \leq \varepsilon, \quad (8.1)$$

where, for  $n \geq 0$ ,  $\varepsilon^{-n} o(\varepsilon^n) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . Second, we assume that *the nonlocal operators are asymptotically local*; specifically, we assume that, for given  $\varepsilon > 0$ ,

$$\mathbb{K}(\mathbf{x}, \mathbf{x}') = \frac{1}{\varepsilon^{d+2}} \mathbb{C}_\varepsilon(\mathbf{x}, \mathbf{x}') = 0 \quad \text{whenever } |\mathbf{x}' - \mathbf{x}| \geq \varepsilon, \quad (8.2)$$

where we assume that  $\widehat{\mathbb{C}}(\mathbf{x}) = \lim_{\mathbf{x}' \rightarrow \mathbf{x}} \mathbb{C}_\varepsilon(\mathbf{x}, \mathbf{x}')$  exists so that

$$\mathbb{C}_\varepsilon(\mathbf{x}, \mathbf{x}') = \widehat{\mathbb{C}}(\mathbf{x}) + o(1) \quad \text{if } |\mathbf{x}' - \mathbf{x}| \leq \varepsilon. \quad (8.3)$$

<sup>9</sup>The nonlocal “boundary-value” problems admit solutions that belong to  $L^2(\Omega \cap \Gamma)$ , e.g., they can be discontinuous; however, classical second-order elliptic equations do not admit such solutions, even in a weak sense. Thus, to connect with classical equations, we have to assume additional smoothness for solutions of the nonlocal equations. We do note that we need only assume that (8.1) holds weakly i.e., it suffices for  $u \in H^1(\Omega \cup \Gamma)$ . Note also that (8.1) does not hold uniformly in  $u$ .

The  $\varepsilon^{d+2}$  scaling introduced in (8.2) is needed to balance the left- and right-hand sides of the nonlocal “boundary-value” problem (5.1) in the limit  $\varepsilon \rightarrow 0$ . Note that  $\widehat{\mathbb{C}}(\mathbf{x})$  is not necessarily the same as  $\mathbb{C}_\varepsilon(\mathbf{x}, \mathbf{x})$  and, in fact, the latter need not be defined; thus, we do not assume that  $\mathbb{C}_\varepsilon(\mathbf{x}, \mathbf{x}')$  is continuous at  $\mathbf{x}' = \mathbf{x}$ . Also, from (4.4) and (8.2), we have

$$\beta_\varepsilon(\mathbf{x}, \mathbf{x}') = \frac{1}{\varepsilon^{d+2}}(\mathbf{x}' - \mathbf{x}) \cdot \mathbb{C}_\varepsilon \cdot (\mathbf{x}' - \mathbf{x}) = 0 \quad \text{whenever } |\mathbf{x}' - \mathbf{x}| \geq \varepsilon, \quad (8.4)$$

where we have added a subscript to  $\beta$  to highlight its explicit dependence on  $\varepsilon$  for the purposes of this section.

We also set

$$\alpha(\mathbf{x}, \mathbf{x}') = \frac{1}{|\mathbf{x}' - \mathbf{x}|}. \quad (8.5)$$

The geometric assumptions we make are that, if  $\partial\Omega_\varepsilon := \partial\Omega \cap \partial\Gamma_\varepsilon$  and  $\partial\Omega_n := \partial\Omega \cap \partial\Gamma_n$ , then  $\partial\Omega = \partial\Omega_\varepsilon \cup \partial\Omega_n$  and

$$\partial\Omega_\varepsilon \neq \emptyset \quad \text{whenever } \Gamma_\varepsilon \neq \emptyset \quad \text{and} \quad \partial\Omega_n \neq \emptyset \quad \text{whenever } \Gamma_n \neq \emptyset.$$

These assumptions merely state that both  $\Gamma_\varepsilon$  and  $\Gamma_n$  abut  $\Omega$  and that the common boundaries of both  $\Gamma_\varepsilon$  and  $\Gamma_n$  with  $\Omega$  comprise the whole boundary of  $\Omega$ . Other than this section and Section 2.1, the results presented do not require such assumptions, e.g., neither  $\Gamma_\varepsilon$  or  $\Gamma_n$  need abut  $\Omega$ .

Let  $\Gamma(\varepsilon) := \cup_{\mathbf{x} \in \Omega} \text{supp}(\beta_\varepsilon) \setminus \Omega$ . Then, with  $|\Gamma|$  denoting the volume of  $\Gamma$ , we have from (8.4) that

$$|\Gamma(\varepsilon)| = O(\varepsilon) \quad \text{and} \quad \Gamma_\varepsilon \rightarrow \partial\Omega_\varepsilon \quad \text{and} \quad \Gamma_n \rightarrow \partial\Omega_n \quad \text{as } \varepsilon \rightarrow 0. \quad (8.6)$$

Let  $S_\varepsilon(\mathbf{x}) := \{\mathbf{x}' \in \Omega \cup \Gamma(\varepsilon) \mid |\mathbf{x} - \mathbf{x}'| \leq \varepsilon\}$ . Then, using (2.13) and (8.1)–(8.5), we have that, with  $\mathbf{z} = \mathbf{x}' - \mathbf{x}$ ,

$$\begin{aligned} B(u, v) &= \int_{\Omega \cup \Gamma(\varepsilon)} \int_{\Omega \cup \Gamma(\varepsilon)} \mathcal{G}(v) \beta_\varepsilon \mathcal{G}(u) \, d\mathbf{x}' \, d\mathbf{x} \\ &= \frac{1}{\varepsilon^{d+2}} \int_{\Omega \cup \Gamma(\varepsilon)} \int_{S_\varepsilon(\mathbf{x})} \left( (\nabla v(\mathbf{x}) \cdot \mathbf{z})(\nabla u(\mathbf{x}) \cdot \mathbf{z}) + o(\varepsilon^2) \right) \left( \mathbf{z} \cdot \widehat{\mathbb{C}}(\mathbf{x}) \cdot \mathbf{z} + o(\varepsilon^2) \right) \alpha^2 \, d\mathbf{x}' \, d\mathbf{x} \\ &= \int_{\Omega \cup \Gamma(\varepsilon)} \nabla v(\mathbf{x}) \cdot \mathbb{D}(\mathbf{x}) \cdot \nabla u(\mathbf{x}) \, d\mathbf{x} + o(1), \end{aligned}$$

where

$$\mathbb{D}(\mathbf{x}) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^{d+2}} \int_{S_\varepsilon(0)} \frac{(\mathbf{z} \otimes \mathbf{z}) \widehat{\mathbb{C}}(\mathbf{z} \otimes \mathbf{z})}{|\mathbf{z}|^2} \, d\mathbf{z}.$$

Note that

$$\mathbb{D}_{ij} = \widehat{\mathbb{C}}_{k\ell} \lim_{\varepsilon \rightarrow 0} \left( \frac{1}{\varepsilon^{d+2}} \int_{S_\varepsilon(0)} \frac{z_k z_i z_j z_\ell}{|\mathbf{z}|^2} \, d\mathbf{z} \right),$$

where repeated indices imply summation. It is shown in [25] that,<sup>10</sup> if  $d = 3$ ,

$$\int_{S_\varepsilon(0)} \frac{z_k z_i z_j z_\ell}{|\mathbf{z}|^2} \, d\mathbf{z} = \frac{4\pi\varepsilon^5}{75} (\delta_{ij}\delta_{k\ell} + \delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk})$$

<sup>10</sup>In [25], similar results are obtained for  $d = 1$  and  $2$  but with an  $\varepsilon^{d+2}$  factor.

so that

$$\mathbb{D}(\mathbf{x}) = \frac{4\pi}{75} \left( \widehat{\mathbb{C}}(\mathbf{x}) + \widehat{\mathbb{C}}^T(\mathbf{x}) + \text{tr}(\widehat{\mathbb{C}}(\mathbf{x})) \right),$$

where  $\text{tr}(\widehat{\mathbb{C}})$  denotes the trace of  $\widehat{\mathbb{C}}$ . Note that if  $\mathbb{K}$  and therefore  $\widehat{\mathbb{C}}$  have a positive definite symmetric part, then  $\mathbb{D}$  is symmetric and positive definite. In particular,  $\mathbb{D}$  is symmetric even if  $\mathbb{K}$  is not. Of course, this is reflected in (4.4) that implies that only the symmetric part of  $\mathbb{K}$  enters into the definition of  $\beta$ .

Combining the results found above, we have

$$\lim_{\varepsilon \rightarrow 0} B(u, v) = \int_{\Omega} \nabla v \cdot \mathbb{D} \cdot \nabla u \, d\mathbf{x} \quad (8.7)$$

and that the nonlocal problem (5.1) reduces to the local problem

$$\begin{cases} \int_{\Omega} \nabla v \cdot \mathbb{D} \cdot \nabla u = \int_{\Omega} v b \, d\mathbf{x} + \int_{\partial\Omega_n} v \tilde{h}_n \, d\mathbf{x} & \text{in } \Omega, \\ u = \tilde{h}_e & \text{on } \partial\Omega_e, \end{cases}$$

where  $\tilde{h}_e$  and  $\tilde{h}_n$  denote traces of the nonlocal data  $h_e$  and  $h_n$  on  $\partial\Omega_e$  and  $\partial\Omega_n$ , respectively. The corresponding “boundary-value” problem (5.2) reduces to the classical boundary-value problem

$$\begin{cases} -\nabla \cdot (\mathbb{D} \cdot \nabla u) = b & \text{in } \Omega \\ u = \tilde{h}_e & \text{on } \partial\Omega_e \\ (\mathbb{D} \cdot \nabla u) \cdot \mathbf{n} = \tilde{h}_n & \text{on } \partial\Omega_n. \end{cases}$$

Note that for the above traces to exist, the data  $h_e$  and  $h_n$  have to be smoother than previously assumed. This, of course, is consistent with the assumption that the solution is smoother than what is needed in the other sections. Again, assuming that the data is smoother is only needed to connect the nonlocal model to classical models.

**9. Concluding remarks.** We developed a nonlocal vector calculus that consists of a nonlocal Gauss’s theorem and nonlocal Green’s identities that mimic the corresponding theorem and identities of the classical vector calculus. We defined a nonlocal weak problem and used the nonlocal vector calculus to show that (1.3) corresponds to a nonlocal “boundary-value” problems that mimics the classical Dirichlet and Neumann problems for second-order elliptic partial differential equations. For the nonlocal problems, the well posedness of the weak formulation was demonstrated, and, under appropriate limits, it was shown that the nonlocal problems reduce to their local analogs; we also derived a fundamental solution and Green’s functions for the nonlocal operator. One important observation is that, unlike weak solutions of second-order elliptic partial differential equations, the nonlocal analogues are well posed with respect to  $L^2$  norms so that the nonlocal problems admit, e.g, solutions with jump discontinuities. An additional observation is that the well posedness crucially depends upon imposing nonlocal boundary conditions on nonzero volumes, given that a trace operator is not defined on the space of  $L^2$  functions.

The nonlocal weak problem (5.1) and the corresponding nonlocal “boundary-value” problem (5.2) mimic the setting described by (1.1) along with Dirichlet and

Neumann boundary conditions. Nonlocal versions of more general second-order elliptic boundary value problems can also be defined; see [18]. For example, restricting attention to “Dirichlet” problems, consider the nonlocal weak problem

$$\left\{ \begin{array}{l} \text{seek } u \in L^2(\Omega \cup \Gamma) \text{ such that} \\ u = h_e \quad \text{for } \mathbf{x} \in \Gamma \\ \text{and} \\ \int_{\Omega \cup \Gamma} \int_{\Omega \cup \Gamma} \mathcal{G}(v)\beta\mathcal{G}(u) \, d\mathbf{x}'d\mathbf{x} + \int_{\Omega \cup \Gamma} v \int_{\Omega \cup \Gamma} a\mathcal{G}(u) \, d\mathbf{x}'d\mathbf{x} \\ + \int_{\Omega \cup \Gamma} v \int_{\Omega \cup \Gamma} \omega\mathcal{A}(u) \, d\mathbf{x}'d\mathbf{x} = \int_{\Omega} vb \, d\mathbf{x} \quad \forall v \in L^2_e(\Omega \cup \Gamma), \end{array} \right. \quad (9.1)$$

where  $a(\mathbf{x}, \mathbf{x}')$  and  $\omega(\mathbf{x}, \mathbf{x}')$  are anti-symmetric and symmetric functions, respectively, and where  $\mathcal{A}(u) = u' + u$ . The corresponding nonlocal “Dirichlet” boundary-value problem is given by

$$\left\{ \begin{array}{ll} -\mathcal{D}(\beta\mathcal{G}(u)) + a\mathcal{G}(u) + \omega\mathcal{A}(u) = b & \text{for } \mathbf{x} \in \Omega \\ u = h_e & \text{for } \mathbf{x} \in \Gamma, \end{array} \right. \quad (9.2)$$

We may then proceed as in Section 8 to show that, for smooth solutions  $u$  and for asymptotically local operators, (9.2) corresponds to the general linear convection-diffusion-reaction problem

$$-\nabla \cdot (\mathbb{D} \cdot \nabla u) + \mathbf{w} \cdot \nabla u + cu = b$$

along with a Dirichlet boundary condition, where  $\mathbf{w}$  and  $c$  are related to  $a$  and  $\omega$ , respectively, through a limit process analogous to that relating  $\mathbb{D}$  to  $\beta$ .

Current work focuses on further refining and extending the results of this paper. In particular, we are

- developing the equivalent multidomain formulations for the linear boundary value problems introduced in Section 5;
- developing and analyzing finite element discretization methods, including discontinuous Galerkin methods, for nonlocal variational problems such as (5.1);
- extending the nonlocal vector calculus to vector-valued functions and developing nonlocal weak problems and their corresponding nonlocal “boundary-value” problems for vector-valued functions; of particular interest is the application of the nonlocal vector calculus to the peridynamic [26, 27] model for materials.

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