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ON STABILIZED FINITE ELEMENT METHODS FOR TRANSIENT PROBLEMS WITH VARYING TIME SCALES

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Abstract. *Many engineering and scientific applications require a detailed analysis of strongly coupled continuum transport and chemical reaction physics. These systems are characterized by the co-existence of advection and diffusion time scales similar to those occurring in typical flow problems, with very short time scales for non-equilibrium chemical reactions. Accurate resolution of the reaction phase may require time-steps orders of magnitude smaller than those normally used in the flow solver. In this paper we investigate the impact of such time-steps on stabilized mixed methods. We show that spatial stabilization in conjunction with finite difference discretization in time leads to couplings that may cause unstable and/or inaccurate approximations. A careful examination of the fully discrete equations reveals that instability is caused by changes in the incompressibility equation by terms that are needed to fulfill the consistency requirement.*

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1 INTRODUCTION

Stabilized mixed finite element methods are in widespread use for the discretization of the Navier-Stokes and related systems in both steady-state and time-dependent settings [1, 2, 3, 9, 10, 15, 17, 18]. They are designed to circumvent the onerous inf-sup (or LBB) condition [6, 11, 14] required of mixed finite element discretizations and, as a result, they allow for the use of standard, piecewise continuous, finite element spaces of equal-order and defined with respect to a single grid for both the pressure and velocity approximations. For time-dependent problems, stabilized mixed finite element methods are commonly defined through a process wherein the spatial and temporal discretization steps are separated. Temporal discretization is effected by an implicit time discretization algorithm, e.g., the backward Euler method. To discretize in space, a stabilized finite element formulation that is designed to relax the continuity equation is applied. This relaxation is accomplished either through residuals, leading to the class of *consistently stabilized methods*; see [1, 10, 15, 16], or by changing this equation directly as in [2, 3, 7, 9, 17, 18]. Here we focus on consistently stabilized methods that have proven to be among the most popular stabilization techniques.

There is one situation in which consistently stabilized methods do not perform as well as one would expect (e.g., relative to their performance for steady-state problems), namely whenever the time-step is much smaller than the spatial grid size. There are at least two settings in which one makes such a choice. First, in problems involving chemical reactions, the size of the time-step is governed by the reaction rates and not by other considerations such as accuracy. Thus, accuracy considerations would suggest the use of a relatively large spatial grid size but a relatively smaller time-step is needed in order to account for the stiffness due to the reactions. Second, temporal and spatial discretization algorithms of disparate orders of accuracy require that the errors due to the two discretization steps be balanced by choosing correspondingly disparate time-step and spatial grid sizes. The following observations are very relevant to our study.

- Even for time-dependent problems, the stabilization methods in common use are designed to relax a spatial constraint, namely the continuity equation. In fact, the stabilization strategies employed in these methods are directly inherited from successful stabilization strategies used for steady-state problems.

- Although the use of relatively small time-steps can result from the need to properly resolve transients due to fast reactions, the resulting problems encountered with stabilized methods are not directly due to the appearance of the reaction terms. As a result, here, we need not consider problems that actually include such terms.

- Likewise, the nonlinear terms appearing in the Navier-Stokes system are not the cause of the problems one encountered with relatively small time-steps. As a result of this an the previous observation, it suffices for us to consider the time-dependent

Stokes problem for the velocity $\mathbf{u}(\mathbf{x},t)$ and the pressure $p(\mathbf{x},t)$:

$$\begin{aligned}
\frac{\partial \mathbf{u}}{\partial t} - \triangle \mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega \times (0,T) && (1)\\
\nabla \cdot \mathbf{u} &= 0 \quad \text{in } \Omega \times (0,T) && (2)\\
\mathbf{u} &= \mathbf{0} \quad \text{on } \Gamma \times (0,T) && (3)\\
\mathbf{u}|_{t=0} &= \mathbf{u}_0 \quad \text{in } \Omega, && (4)
\end{aligned}$$

where Ω denotes a simply connected bounded region in $\mathbf{R}^n$, $n = 2, 3$, with a sufficiently smooth boundary Γ, $(0,T)$ with $T > 0$ the time interval of interest, and $\mathbf{f}(\mathbf{x},t)$ and $\mathbf{u}_0(\mathbf{x})$ are given functions.

One of our goals is to document, through computational experiments, the difficulties that arise when one uses relatively small time-steps in consistently stabilized finite element methods for the time-dependent Stokes problem. These experiments strongly suggest that, in this setting, the velocity approximation is unaffected but the pressure approximation is severely compromised.

2 NOTATIONS

We let $H^d(\Omega)$, $\|\cdot\|_d$, and $(\cdot,\cdot)_d$ with $d \geq 0$ denote the Sobolev spaces consisting of all functions with square integrable derivatives up to order d with respect to Ω, and the standard Sobolev norm and inner product, respectively. Whenever $d = 0$, we will write $L^2(\Omega)$ instead of $H^0(\Omega)$ and drop the index from the inner product designation. As usual, $H^1_0(\Omega) = \{v \in H^1(\Omega) \,|\, v = 0 \text{ on } \Gamma\}$ and $L^2_0(\Omega) = \{q \in L^2(\Omega) \,|\, \int_\Omega q d\Omega = 0\}$. Spaces of vector valued functions are denoted by bold-face notations, e.g., $\mathbf{H}^1(\Omega)$ is the space of vector-valued functions each of whose components belong to $H^1(\Omega)$.

The symbol S^h_k denotes the space of continuous, piecewise polynomial functions of degree k defined with respect to a regular triangulation $\mathcal{T}_h$ of the domain Ω into finite elements $\mathcal{K}$; see [11]. For example, $\mathcal{K}$ can be a hexahedron or tetrahedron in three dimensions or a triangle or quadrilateral in two dimensions.

3 MIXED AND STABILIZED GALERKIN DISCRETIZATION OF THE TIME DEPENDENT STOKES PROBLEM

To effect discretization in the spatial variable, we first choose conforming finite element subspaces $\mathbf{V}^h \subset \mathbf{H}^1_0(\Omega)$ and $P^h \subset L^2_0(\Omega)$ for the velocity and pressure approximations, respectively. Then, an (unstabilized) mixed finite element semi-discretization of (1)–(4) is defined as follows [11, 14]: seek $\mathbf{u}^h(\cdot,t) \in \mathbf{V}^h$ and $p^h(\cdot,t) \in P^h$ such that

$$\begin{aligned}
(\dot{\mathbf{u}}^h, \mathbf{v}^h) + G(\{\mathbf{u}^h, p^h\}, \{\mathbf{v}^h, q^h\}) &= (\mathbf{f}, \mathbf{v}^h) && (5)\\
\big(\mathbf{u}^h(\cdot,0), \mathbf{v}^h\big) &= \big(\mathbf{u}_0, \mathbf{v}^h\big) && (6)
\end{aligned}$$

for all $\mathbf{v}^h \in \mathbf{V}^h$, $q^h \in P^h$, and $t \in (0,T)$, where

$$G(\{\mathbf{u}^h, p^h\}, \{\mathbf{v}^h, q^h\}) = (\nabla \mathbf{u}^h, \nabla \mathbf{v}^h) - (p^h, \nabla \cdot \mathbf{v}^h) - (q^h, \nabla \cdot \mathbf{u}^h)\,. \tag{7}$$

If $(\mathbf{V}^h, P^h)$ represents a pair of finite element spaces that satisfies the inf-sup condition, then (5)–(6) is a stable problem. This problem is equivalent to the differential algebraic equation (DAE) problem

$$\begin{pmatrix} \mathbb{M}\dot{U} \\ 0 \end{pmatrix} + \begin{pmatrix} \mathbb{A} & \mathbb{B}^T \\ \mathbb{B} & 0 \end{pmatrix} \begin{pmatrix} U \\ P \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix} \tag{8}$$

along with the initial condition determined from

$$\mathbb{M}U(0) = U_0\,, \tag{9}$$

where $U(t) = (\alpha_1(t), \ldots, \alpha_N(t))$ and $P(t) = (\beta_1(t), \ldots, \beta_M(t))$ are the vectors of unknown coefficients corresponding to $\mathbf{u}^h(\mathbf{x}, t)$ and $p^h(\mathbf{x}, t)$, respectively. The matrices $\mathbb{M}$, $\mathbb{A}$, and $\mathbb{B}$ are defined in the usual manner from the terms in (5) and represent the (consistent) mass, stiffness, and divergence matrices, the vectors $F(t)$ and U_0 are defined from the source term $\mathbf{f}$ and the initial data $\mathbf{u}_0$, respectively.

The second equation of (8) ($\mathbb{B}U = 0$) implies that the velocity is discretely divergence free (or discretely solenoidal). The saddle-point system (8) is stable if and only if the finite element spaces $\mathbf{V}^h$ and P^h satisfy the inf-sup condition

$$\inf_{q^h \in P^h,\, q^h \neq 0} \ \sup_{\mathbf{v}^h \in \mathbf{V}^h,\, \mathbf{v}^h \neq \mathbf{0}} \frac{(q^h, \nabla \cdot \mathbf{v}^h)}{\|\mathbf{v}^h\|_1 \|q^h\|_0} \geq \kappa\,, \tag{10}$$

where $\kappa > 0$ is independent of the grid size h; see, e.g., [6, 11, 14].

There are many pairs of finite element spaces $(\mathbf{V}^h, P^h)$ for which it is known that the inf-sup condition (10) holds. However, it is also known that piecewise polynomial velocity and pressure approximating spaces of the same degree defined with respect to the same grid do not satisfy (10); see [11, 14] for details. This is the main driving motivation for developing stabilized finite element methods that are stable even when equal-order finite element spaces defined with respect to the same grid are used.

3.1 Stabilized finite element method for the steady-state Stokes problem

Before we introduce stabilized spatial discretization for the time-dependent Stokes problem (1)–(4), we review consistent spatial stabilization for the steady-state case:

$$-\triangle \mathbf{u} + \nabla p = \mathbf{f} \quad \text{and} \quad \nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega \qquad \text{and} \qquad \mathbf{u} = \mathbf{0} \quad \text{on } \Gamma\,. \tag{11}$$

We choose a pair of (conforming) subspaces $\mathbf{V}^h \subset \mathbf{H}^1_0(\Omega)$ and $P^h \subset L^2_0(\Omega)$, a pair of weighting functions $W_m(\mathbf{v}^h, q^h)$ and $W_c(\mathbf{v}^h, q^h)$, and a pair of discrete inner products

$< \cdot,\cdot >_m$ and $< \cdot,\cdot >_c$. Then, consistently stabilized finite element discretizations of the steady-state Stokes problem (11) are defined as follows: seek $\mathbf{u}^h \in \mathbf{V}^h$ and $p^h \in P^h$ such that

$$\begin{aligned} G(\{\mathbf{u}^h,p^h\},\{\mathbf{v}^h,q^h\}) + \left\langle R_m(\mathbf{u}^h,p^h), W_m(\mathbf{v}^h,q^h)\right\rangle_m & \\ + \left\langle R_c(\mathbf{u}^h,p^h), W_c(\mathbf{v}^h,q^h)\right\rangle_c &= (\mathbf{f},\mathbf{v}^h) \end{aligned} \tag{12}$$

for all $\mathbf{v}^h \in \mathbf{V}^h$ and $q^h \in P^h$, where

$$R_m(\mathbf{u}^h,p^h)\Big|_{\mathcal{K}} = -\triangle\mathbf{u}^h + \nabla p^h - \mathbf{f} \qquad \text{and} \qquad R_c(\mathbf{u}^h,p^h)\Big|_{\mathcal{K}} = \nabla\cdot\mathbf{u}^h$$

are the momentum and continuity equation residuals; see (11). We restrict our attention to the popular class of consistently stabilized methods for which

$$W_c(\mathbf{v}^h,q^h)=0, \quad W_m(\mathbf{v}^h,q^h) = \gamma\triangle\mathbf{v}^h - \nabla q^h, \quad \text{and} \quad \left\langle\mathbf{u}^h,\mathbf{v}^h\right\rangle_m = \sum_{\mathcal{K}\in\mathcal{T}_h}\tau_{\mathcal{K}}(\mathbf{u}^h,\mathbf{v}^h)_{\mathcal{K}}\,,$$

where γ can take on the values ± 1 or 0. Note that $< \cdot,\cdot >_m$ is a "broken" L^2 inner product (broken into a sum of inner products over the individual elements) weighted by the parameters $\tau_{\mathcal{K}}$. This results in the class of stabilized methods: seek $\mathbf{u}^h \in \mathbf{V}^h$ and $p^h \in P^h$ such that

$$G(\{\mathbf{u}^h,p^h\},\{\mathbf{v}^h,q^h\}) - \sum_{\mathcal{K}\in\mathcal{T}_h}\tau_{\mathcal{K}}\left(-\triangle\mathbf{u}^h+\nabla p^h-\mathbf{f}, -\gamma\triangle\mathbf{v}^h+\nabla q^h\right)_{\mathcal{K}} = (\mathbf{f},\mathbf{v}^h) \tag{13}$$

for all $\mathbf{v}^h \in \mathbf{V}^h$ and $q^h \in P^h$. For $\gamma = 1, 0, -1$, the method (13) is respectively known as the the *Galerkin-least-squares* [15], the *pressure-Poisson stabilized Galerkin* [16], and the *Douglas-Wang* [10] methods; see also [1]. A typical definition for the parameters $\tau_{\mathcal{K}}$ is

$$\tau_{\mathcal{K}} = \delta h_{\mathcal{K}}^2\,, \tag{14}$$

where $h_{\mathcal{K}}$ is a measure of the element size and $\delta > 0$ is a stabilization parameter that is independent of $h_{\mathcal{K}}$ but whose values may be restricted in order to guarantee the stability of the discrete problem (13); see [1, 15, 16].

3.2 Spatially stabilized discretizations of the time-dependent Stokes problem

To stabilize (5)–(6) spatially, we may modify $G(\cdot,\cdot)$ by adding the same terms as in (13). However, if $\mathbf{u}$ is an unsteady solution of (5)–(6), then $-\triangle\mathbf{u}+\nabla p-\mathbf{f} = -\dot{\mathbf{u}} \neq 0$ and so the modified equation will no longer be consistent. This difficulty can be easily avoided by simply changing the stabilization term to

$$-\sum_{\mathcal{K}\in\mathcal{T}_h}\tau_{\mathcal{K}}\left(\dot{\mathbf{u}}^h - \triangle\mathbf{v}^h + \nabla p^h - \mathbf{f}, -\gamma\triangle\mathbf{v}^h + \nabla q^h\right)_{\mathcal{K}}.$$

The modified problem is: seek $\mathbf{u}^h(\cdot,t) \in \mathbf{V}^h$ and $p^h(\cdot,t) \in P^h$ such that

$$\begin{aligned}(\dot{\mathbf{u}}^h, \mathbf{v}^h) - \sum_{\mathcal{K}\in\mathcal{T}_h} \tau_{\mathcal{K}} \left(\dot{\mathbf{u}}^h, -\gamma\triangle\mathbf{v}^h + \nabla q^h\right)_{\mathcal{K}} + G(\{\mathbf{u}^h, p^h\}, \{\mathbf{v}^h, q^h\}) \\ - \sum_{\mathcal{K}\in\mathcal{T}_h} \tau_{\mathcal{K}} \left(-\triangle\mathbf{u}^h + \nabla p^h - \mathbf{f}, -\gamma\triangle\mathbf{v}^h + \nabla q^h\right)_{\mathcal{K}} = (\mathbf{f}, \mathbf{v}^h)\end{aligned} \tag{15}$$

$$\left(\mathbf{u}^h(\cdot,0), \mathbf{v}^h\right) = \left(\mathbf{u}_0, \mathbf{v}^h\right) \tag{16}$$

for all $\mathbf{v}^h \in \mathbf{V}^h$, $q^h \in P^h$, and $t \in (0,T)$. The semi-discrete problem (15)–(16) is consistent and stable whenever the discrete steady-state problem (13) is stable [13].

Compared to the standard mixed Galerkin semi-discrete problem (5), the spatially stabilized problem (15) contains additional terms. The role of the terms

$$-\sum_{\mathcal{K}\in\mathcal{T}_h} \tau_{\mathcal{K}} \left(-\triangle\mathbf{u}^h + \nabla p^h - \mathbf{f}, -\gamma\triangle\mathbf{v}^h + \nabla q^h\right)_{\mathcal{K}} \tag{17}$$

is to stabilize the discretization with respect to the spatial variable, while the terms

$$-\sum_{\mathcal{K}\in\mathcal{T}_h} \tau_{\mathcal{K}} \left(\dot{\mathbf{u}}^h, -\gamma\triangle\mathbf{v}^h + \nabla q^h\right)_{\mathcal{K}} \tag{18}$$

are introduced to preserve consistency for transient solutions. We may write (15)–(16) as

$$\begin{pmatrix} (\mathbb{M} + \gamma\widetilde{\mathbb{C}})\,\dot{U} \\ \widetilde{\mathbb{B}}\,\dot{U} \end{pmatrix} + \begin{pmatrix} \mathbb{A} - \gamma\widetilde{\mathbb{A}} & \mathbb{B}^T + \gamma\widetilde{\mathbb{S}}^T \\ -\mathbb{B} - \widetilde{\mathbb{S}} & \widetilde{\mathbb{K}} \end{pmatrix} \begin{pmatrix} U \\ P \end{pmatrix} = \begin{pmatrix} F + \gamma\widetilde{H} \\ \widetilde{G} \end{pmatrix} \tag{19}$$

along with the initial condition determined from

$$\mathbb{M}U(0) = U_0\,. \tag{20}$$

The matrices $\mathbb{M}$, $\mathbb{A}$, and $\mathbb{B}$ have already been described; all $\widetilde{(\cdot)}$ terms appearing in (19) result from the stabilizing term (17) and the consistency term (18). Note that if we do not add any stabilization, i.e., if $\tau_{\mathcal{K}} = 0$ for all $\mathcal{K}$, then all $\widetilde{(\cdot)}$ terms in (19) vanish and that equation reduces, as it should, to (8).

4 TEMPORAL DISCRETIZATION

Here, for the sake of simplicity of exposition, we apply the θ-method (also called the *generalized trapezoidal rule*) for this purpose. The interval $(0,T)$ is subdivided into K intervals $[t_k, t_{k+1}]$, $k = 0,\ldots,K-1$, having lengths Δt_k. We define $F^k = F(t_k)$ and likewise for $\widetilde{G}$ and $\widetilde{H}$; $\mathbf{u}^{h,k}$, $p^{h,k}$, U^k, and P^k respectively denote approximations to $\mathbf{u}^h(\mathbf{x},t_k)$, $p^h(\mathbf{x},t_k)$, $U(t_k)$, and $P(t_k)$. The θ-method is then defined as follows: choose $\theta \in [0,1]$

and let $U^0 = U(0)$, i.e., U^0 is the solution of $\mathbb{M}U^0 = U_0$; then, for $k = 0, 1, \ldots, K-1$, determine U^{k+1} and P^{k+1} from

$$\frac{1}{\Delta t_k}(\mathbb{M} + \gamma\widetilde{\mathbb{C}})\ \left(U^{k+1} - U^k\right) + \left(\mathbb{A} - \gamma\widetilde{\mathbb{A}}\right)U^k_\theta + \left(\mathbb{B}^T + \gamma\widetilde{\mathbb{S}}^T\right)P^k_\theta = F^k_\theta + \gamma\widetilde{H}^k_\theta \tag{21}$$

$$\frac{1}{\Delta t_k}\widetilde{\mathbb{B}}\ \left(U^{k+1} - U^k\right) - \mathbb{B}U^k_\theta - \widetilde{\mathbb{S}}U^k_\theta + \widetilde{\mathbb{K}}P^k_\theta = \widetilde{G}^k_\theta\,, \tag{22}$$

where $U^k_\theta = \theta U^{k+1} + (1-\theta)U^k$ and likewise for F^k_θ, $\widetilde{G}^k_\theta$, and $\widetilde{H}^k_\theta$. There are other possible generalizations of the θ-method for the heat equations to the case begin considered here. For example, we could replace P^k_θ in (21) and U^k_θ in (22) by P^{k+1} and U^{k+1}, respectively. Altering the definition of the method given by (21)–(22) will not in any way change the results of interest in this paper, so that it suffices to consider just one generalization of the heat equation case.

For $\theta = 0$, (21)–(22) reduces to the explicit forward-Euler method; for $\theta = 1$, (21)–(22) reduces to the implicit backward-Euler method; for $\theta = 1/2$, (21)–(22) is a Crank-Nicholson method. Clearly, (21)–(22) is a system of linear algebraic equations for the unknown vectors of coefficients U^{k+1} and P^{k+1}:

$$\begin{pmatrix} \frac{1}{\Delta t_k}(\mathbb{M} + \gamma\widetilde{\mathbb{C}}) + \theta\mathbb{A} - \gamma\theta\widetilde{\mathbb{A}} & \theta\mathbb{B}^T + \gamma\theta\widetilde{\mathbb{S}}^T \\ \frac{1}{\Delta t_k}\widetilde{\mathbb{B}} - \theta\mathbb{B} - \theta\widetilde{\mathbb{S}} & \theta\widetilde{\mathbb{K}} \end{pmatrix} \begin{pmatrix} U^{k+1} \\ P^{k+1} \end{pmatrix} = \begin{pmatrix} F^k_\theta + \gamma\widetilde{H}^k_\theta \\ \widetilde{G}^k_\theta \end{pmatrix}$$
$$+ \begin{pmatrix} \frac{1}{\Delta t_k}(\mathbb{M} + \gamma\widetilde{\mathbb{C}}) - (1-\theta)\mathbb{A} + \gamma(1-\theta)\widetilde{\mathbb{A}} & -(1-\theta)\mathbb{B}^T - \gamma(1-\theta)\widetilde{\mathbb{S}}^T \\ \frac{1}{\Delta t_k}\widetilde{\mathbb{B}} + (1-\theta)\mathbb{B} + (1-\theta)\widetilde{\mathbb{S}} & -(1-\theta)\widetilde{\mathbb{K}} \end{pmatrix} \begin{pmatrix} U^k \\ P^k \end{pmatrix}. \tag{23}$$

4.1 Simplifications and solution of the discrete equations

We now introduce several simplifications into (23) that will allow us to not only more easily examine the behaviors of that system in the limit of $\Delta t_k \to 0$, but that will more readily expose those behaviors. We remark that the same behaviors occur even without these simplifications so that we do not loose any essential information by treating the simplified system. First, because we are only considering single-step methods, we can set $\Delta t_k = \Delta t$ with the understanding that the value of Δt may change from one time-step to the next. Second, we choose $\gamma = 0$ so that we restrict considerations to the pressure-Poisson stabilized Galerkin method. This, of course, results in being able to ignore several terms in (23). Next, we set the stabilization parameter to the same value in all elements $\mathcal{K}$, i.e., we set $\tau_\mathcal{K} = \tau = \delta h^2$, where h is some overall measure of the grid size. This allows several terms in (23) that involve the "broken" inner product $\sum_{\mathcal{K}\in\mathcal{T}_h} \tau_\mathcal{K}(\cdot,\cdot)_\mathcal{K}$ to simplify to $\tau(\cdot,\cdot)$. Next, we set $\theta = 1$ so that we only consider the backward Euler method. Finally, we use the same continuous piecewise polynomials of the same degree for both the approximation of the pressure and velocity components. This is justified because one of the main reasons for using stabilized methods is to be able to use such combinations of finite element spaces that are unstable for the Galerkin mixed methods. As a result of these simplifications, (23) reduces to

$$\begin{pmatrix} \mathbb{M}+\Delta t\mathbb{A} & \Delta t\mathbb{B}^T \\ (\tau-\Delta t)\mathbb{B}-\tau\Delta t\mathbb{S} & \tau\Delta t\mathbb{K} \end{pmatrix}\begin{pmatrix} U^{k+1} \\ P^{k+1} \end{pmatrix} = \begin{pmatrix} \mathbb{M} & 0 \\ \tau\mathbb{B} & 0 \end{pmatrix}\begin{pmatrix} U^{k} \\ P^{k} \end{pmatrix} + \Delta t\begin{pmatrix} F^{k+1} \\ \tau G^{k+1} \end{pmatrix}, \tag{24}$$

where $\mathbb{K}_{ij} = \left(\nabla\chi_j^h, \nabla\chi_i^h\right)$, $\mathbb{S}_{ij} = \sum_{\mathcal{K}\in\mathcal{T}_h}\left(\Delta\boldsymbol{\xi}_j^h, \nabla\chi_i^h\right)_{\mathcal{K}}$, and $(G)_i = \left(\mathbf{f}, \nabla\xi_i^h\right)$. A simple but tedious calculation [4] shows that

$$\begin{aligned} U^{k+1} &= (\mathbb{M}+\Delta t\mathbb{A})^{-1}\Big[\mathbb{M}-\mathbb{B}^T\Big(\tau\mathbb{K}+\widehat{\mathbb{G}}\mathbb{B}^T\Big)^{-1}\Big(\widehat{\mathbb{G}}\mathbb{M}+\tau\mathbb{B}\Big)\Big]U^k \\ &\quad +\Delta t(\mathbb{M}+\Delta t\mathbb{A})^{-1}\Big[F^{k+1}-\mathbb{B}^T\Big(\tau\mathbb{K}+\widehat{\mathbb{G}}\mathbb{B}^T\Big)^{-1}\Big(\widehat{\mathbb{G}}F^{k+1}+\tau G^{k+1}\Big)\Big], \end{aligned} \tag{25}$$

for the velocity component, and by

$$\Delta t P^{k+1} = \Big(\tau\mathbb{K}+\widehat{\mathbb{G}}\mathbb{B}^T\Big)^{-1}\Big[\Big(\widehat{\mathbb{G}}\mathbb{M}+\tau\mathbb{B}\Big)U^k+\Delta t\widehat{\mathbb{G}}F^{k+1}+\tau\Delta t G^{k+1}\Big], \tag{26}$$

for the pressure component, where

$$\widehat{\mathbb{A}} = \mathbb{M}+\Delta t\mathbb{A}\,, \qquad \widehat{\mathbb{G}} = \Delta t\mathbb{B}\widehat{\mathbb{A}}^{-1}-\tau(\mathbb{B}-\Delta t\mathbb{S})\widehat{\mathbb{A}}^{-1}\,, \qquad \widehat{\mathbb{D}} = \tau\Delta t\mathbb{K}+\Delta t\widehat{\mathbb{G}}\mathbb{B}^T\,. \tag{27}$$

The symmetric, positive definite matrices $\mathbb{M}$ and $\mathbb{A}$ are invertible for any standard choice of finite element spaces for the velocity approximation. As a result, the matrix $\widehat{\mathbb{A}} = \mathbb{M}+\Delta t\mathbb{A}$ is likewise invertible. Therefore, the well posedness of the solution (26)–(25) is totally reliant on the invertibility of the matrix $(\tau\mathbb{K}+\widehat{\mathbb{G}}\mathbb{B}^T)$.

4.2 The $\tau\to 0$ limit: the Galerkin mixed method

We first examine (26) and (25) in the limit $\tau\to 0$ or, more precisely, for Δt and h fixed, we let $\delta\to 0$. The discrete problem (24) reduces to

$$\begin{pmatrix} \mathbb{M}+\Delta t\mathbb{A} & \Delta t\mathbb{B}^T \\ \Delta t\mathbb{B} & 0 \end{pmatrix}\begin{pmatrix} U^{k+1} \\ P^{k+1} \end{pmatrix} = \begin{pmatrix} \mathbb{M} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} U^{k} \\ P^{k} \end{pmatrix} + \Delta t\begin{pmatrix} F^{k+1} \\ 0 \end{pmatrix} \tag{28}$$

that, of course, is the discrete problem for the backward Euler/mixed Galerkin finite element method. We have that $\widehat{\mathbb{G}} = \Delta t\mathbb{B}\widehat{\mathbb{A}}^{-1} = \Delta t\mathbb{B}(\mathbb{M}+\Delta t\mathbb{A})^{-1}$ so that the solution relations (25) and (26) respectively reduce to

$$\begin{aligned} U^{k+1} &= (\mathbb{M}+\Delta t\mathbb{A})^{-1}\Big[\mathbb{M}-\Delta t^{-1}\mathbb{B}^T\Big(\mathbb{B}(\mathbb{M}+\Delta t\mathbb{A})^{-1}\mathbb{B}^T\Big)^{-1}\Delta t\mathbb{B}(\mathbb{M}+\Delta t\mathbb{A})^{-1}\mathbb{M}\Big]U^k \\ &\quad +\Delta t(\mathbb{M}+\Delta t\mathbb{A})^{-1}\Big[\mathbb{I}-\Delta t^{-1}\mathbb{B}^T\Big(\mathbb{B}(\mathbb{M}+\Delta t\mathbb{A})^{-1}\mathbb{B}^T\Big)^{-1}\Big(\Delta t\mathbb{B}(\mathbb{M}+\Delta t\mathbb{A})^{-1}\Big)\Big]F^{k+1} \end{aligned} \tag{29}$$

and

$$P^{k+1} = \Delta t^{-1}\Big(\mathbb{B}(\mathbb{M}+\Delta t\mathbb{A})^{-1}\mathbb{B}^T\Big)^{-1}\Big[\mathbb{B}(\mathbb{M}+\Delta t\mathbb{A})^{-1}\mathbb{M}+U^k \\ +\Delta t\mathbb{B}(\mathbb{M}+\Delta t\mathbb{A})^{-1}F^{k+1}\Big]\,. \tag{30}$$

We see that, in the mixed Galerkin case, the well posedness of the solution (29)–(30) depends upon the uniform invertibility of the matrix $\mathbb{B}(\mathbb{M}+\Delta t\mathbb{A})^{-1}\mathbb{B}^T$. Given the invertibility of $\mathbb{M}+\Delta t\mathbb{A}$, we conclude that $\mathbb{B}(\mathbb{M}+\Delta t\mathbb{A})^{-1}\mathbb{B}^T$ is invertible if and only if the matrix $\mathbb{B}$ is of full row rank. This property is exactly implied by the inf-sup condition (10) for the bilinear form $b(\cdot,\cdot)$. The uniform invertibility of $\mathbb{B}$ requires that the constant in the inf-sup condition for $b(\cdot,\cdot)$ be bounded away from zero independently of the spatial grid size h.

We now let $\Delta t \ll 1$, subsequent to taking the $\tau \to 0$ limit. We then obtain

$$\left\{\begin{array}{l} U^{k+1} = \Big(\mathbb{I}-\mathbb{M}^{-1}\mathbb{B}^T(\mathbb{B}\mathbb{M}^{-1}\mathbb{B}^T)^{-1}\mathbb{B}\Big)U^k + O(\Delta t) \\ P^{k+1} = \Delta t^{-1}\Big((\mathbb{B}\mathbb{M}^{-1}\mathbb{B}^T)^{-1}\mathbb{B}\mathbb{M}^{-1}\Big)\mathbb{M}U^k + O(1) \end{array}\right. . \tag{31}$$

4.3 The role of stabilization

Let us return to the exact solution (26)–(25) of the discrete stabilized equations. The well posedness of that solution is dependent on the invertibility of the matrix

$$\tau\mathbb{K}+\widehat{\mathbb{G}}\mathbb{B}^T = \\ \Delta t\mathbb{B}(\mathbb{M}+\Delta t\mathbb{A})^{-1}\mathbb{B}^T + \tau\mathbb{K} - \tau\mathbb{B}(\mathbb{M}+\Delta t\mathbb{A})^{-1}\mathbb{B}^T + \tau\Delta t\mathbb{S}(\mathbb{M}+\Delta t\mathbb{A})^{-1}\mathbb{B}^T\,. \tag{32}$$

From (29)–(30), we see that the first term $\Delta t\mathbb{B}(\mathbb{M}+\Delta t\mathbb{A})^{-1}\mathbb{B}^T$ in (32) is inherited from the Galerkin mixed formulation. If one uses an unstable pair of finite element spaces, e.g., equal-order spaces defined with respect to the same grid for the velocity and pressure approximations, then this matrix, although symmetric and positive semi-definite, is not uniformly (with respect to h) invertible.

The second term $\tau\mathbb{K}$ is the stabilizing term; it arises from the pressure-Poisson type stabilization term $(\nabla p^h, \nabla q^h)$ in (15). Note that the matrix $\mathbb{K}$ is symmetric and positive definite for any standard choice of finite element spaces for the pressure. As a result, the combination

$$\Delta t\mathbb{B}(\mathbb{M}+\Delta t\mathbb{A})^{-1}\mathbb{B}^T + \tau\mathbb{K}$$

is symmetric and positive definite for any choice of spaces for the velocity and pressure approximations, including equal-order, same grid pairs. Thus, the $\tau\mathbb{K}$ term clearly effects stabilization. This is the whole story with regards to stabilization that is not based on residuals and only involves the addition of some kind of a pressure term directly to the weak continuity constraint; see [2, 3, 7, 9, 17, 18]. For instance, [7] considers a method where exactly the same pressure-Poisson term $(\nabla p^h, \nabla q^h)$ is used to effect the stabilization. However, these scheme are inconsistent in the sense that the exact solution of the Stokes problem does not satisfy the resulting equations.

The remaining terms in (32) appear in order to fulfill the consistency requirement. In particular, the third term $-\tau\mathbb{B}(\mathbb{M}+\Delta t\mathbb{A})^{-1}\mathbb{B}^T$ arises from the consistency term $(\dot{\mathbf{u}}^h, \nabla q^h)$ in (15) and the fourth term $\tau\Delta t\mathbb{S}(\mathbb{M}+\Delta t\mathbb{A})^{-1}\mathbb{B}^T$ arises from the other consistency term $(\Delta\mathbf{u}^h, \nabla q^h)$ in (15).

The term $-\tau\mathbb{B}(\mathbb{M}+\Delta t\mathbb{A})^{-1}\mathbb{B}^T$ is a *destabilizing* term and is of the same form, except for having an opposite sign, as is the term arising from the mixed Galerkin method. Hence, this term is a symmetric, negative semidefinite matrix. In fact, a closer inspection of (32); see [4], reveals that one might get into trouble if τ is large compared to Δt; since $\tau = \delta h^2$, this indicates that trouble might occur if h^2 is large with respect to Δt. Thus, to be safe, we would want to require that $\Delta t > Ch^2$; similar conclusions were drawn by the authors in [5], and by other authors [2, 3], although the justification for this requirement is much less transparent in these references that the one given here. It is important to note that the destabilization term $-\tau\mathbb{B}(\mathbb{M}+\Delta t\mathbb{A})^{-1}\mathbb{B}^T$ only appears in the time-dependent setting (as noted above, it arises from the term $(\dot{\mathbf{u}}^h, \nabla q^h)$ so that in steady-state settings destabilization effects due to this term to not appear.

The effect of the $\tau\Delta t\mathbb{S}(\mathbb{M}+\Delta t\mathbb{A})^{-1}\mathbb{B}^T$ term in (32) is not crucial since it is of higher order compared to $-\tau\mathbb{B}(\mathbb{M}+\Delta t\mathbb{A})^{-1}\mathbb{B}^T$. We do note that it corresponds to a (nonsymmetric) discretization of $-\Delta^2$ (a negative semidefinite operators) so that it certainly does not provide any additional stabilization effects.

5 THE $\Delta t \to 0$ LIMIT FOR FIXED h

We now examine in more detail the topic of most interest to us: the behavior of consistently stabilized methods as $\Delta t \to 0$ with the spatial grid size h remaining fixed so that $\tau = \delta h^2$ remains fixed as well. As indicated in the discussion of the matrix in (32), we can expect some difficulties in this situation. We now have that

$$\tau\mathbb{K} + \widehat{\mathbb{G}}\mathbb{B}^T = \tau(\mathbb{K} - \mathbb{B}\mathbb{M}^{-1}\mathbb{B}^T) + O(\Delta t)$$

and

$$\widehat{\mathbb{G}}\mathbb{M} + \tau\mathbb{B} = \Delta t\Big(\mathbb{B} + \tau\mathbb{S} + \tau\mathbb{B}\mathbb{M}^{-1}\mathbb{A} + O(\Delta t)\Big)$$

so that (26) and (25) respectively yield

$$U^{k+1} = U^k + O(\Delta t) \tag{33}$$

and

$$\begin{aligned} P^{k+1} &= \tau^{-1}(\mathbb{K} - \mathbb{B}\mathbb{M}^{-1}\mathbb{B}^T)^{-1}\mathbb{B}U^k \\ &\quad + (\mathbb{K} - \mathbb{B}\mathbb{M}^{-1}\mathbb{B}^T)^{-1}\Big[\Big(\mathbb{S} + \mathbb{B}\mathbb{M}^{-1}\mathbb{A}\Big)U^k - \mathbb{B}\mathbb{M}^{-1}F^{k+1} + G^{k+1}\Big] + O(\Delta t)\,. \end{aligned} \tag{34}$$

Table 1: The $\tau \to 0$ limit. Errors after one implicit Euler step with $\Delta t = (\Delta t)_1$.

	velocity		pressure
δ	L^2	H^1	H^1
0.0005	0.41047E-04	0.40044E-02	0.56666E+00
0.00005	0.53300E-04	0.44985E-02	0.18235E+01
0.000005	0.57916E-04	0.46987E-02	0.12433E+02

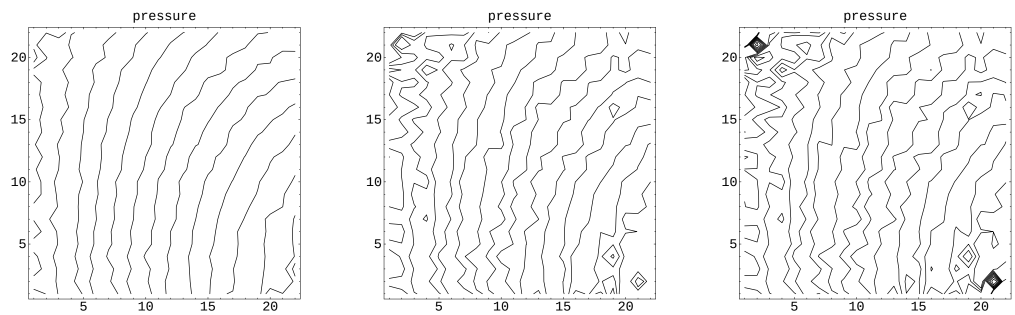

Figure 1: The $\tau \to 0$ limit: Stabilized method; $\Delta t = (\Delta t)_1$.

Equation (33) indicates that as $\Delta t \to 0$ with τ fixed, the velocity approximations are stable, i.e., $U^{k+1} \to U^k$ as $\Delta t \to 0$.

Equation (34) provides some questions concerning the stability and convergence of pressure approximations. The first important question is the invertibility of the matrix $\mathbb{K} - \mathbb{B}\mathbb{M}^{-1}\mathbb{B}^T$. As has already being pointed out, $\mathbb{K}$ is the symmetric, positive definite Poisson (i.e., stiffness) matrix corresponding the pressure finite element space; $\mathbb{B}\mathbb{M}^{-1}\mathbb{B}^T$ is the matrix arising in the mixed Galerkin method and, if we use an unstable pair of approximating spaces (e.g., of equal-order and based on the same grid), it is a positive, semidefinite (at least in an asymptotic sense as $h \to 0$) matrix. Thus, it is totally unclear, given a pair of finite element spaces for the velocity and pressure approximations, what are the properties of the matrix $\mathbb{K} - \mathbb{B}\mathbb{M}^{-1}\mathbb{B}^T$. We note that this issue does not seem to affect the stability of the velocity approximation, but may be relevant to the accuracy of those approximations.

The first term on the right-hand side of (34) also deserves notice. Note the dependence on the stabilization parameter $\tau = \delta h^2$. Thus, for a fixed grid size h, unless $\mathbb{B}U^k = 0$, the solution for the discrete pressure will change if one changes the stabilization parameter. In fact, even if $\mathbb{B}U^k = O(h^2)$, the discrete pressure will undergo an $O(1)$ change as we vary δ. In the next section, we explore these questions and observations through a series of computational experiments.

Table 2: The $\Delta t \to 0$ limit: Taylor-Hood element.

	velocity		pressure
Δt	L^2	H^1	L^2
0.1	0.39334D-03	0.30349D-01	0.67770D-03
0.01	0.39244D-03	0.30349D-01	0.69915D-03
0.001	0.39239D-03	0.30352D-01	0.90321D-03
0.0001	0.39477D-03	0.30390D-01	0.15369D-02
0.00001	0.39665D-03	0.30439D-01	0.18965D-02
0.000001	0.39698D-03	0.30450D-01	0.66562D-01

Table 3: The $\Delta t \to 0$ limit: Stabilized method with $\delta = 0.05$.

	velocity		pressure
Δt	L^2	H^1	L^2
0.1	0.54734E-04	0.36186E-02	0.44419E-02
0.01	0.54967E-04	0.36184E-02	0.49387E-02
0.001	0.55565E-04	0.36191E-02	0.99227E-02
0.0001	0.58893E-04	0.36983E-02	0.69373E-01
0.00001	0.55531E-04	0.39168E-02	0.47687E+00
0.000001	0.46465E-04	0.40049E-02	0.10385E+01

6 COMPUTATIONAL EXPERIMENTS

The main goal of this section is to determine to what extent very small time-steps can cause numerical instabilities, or other anomalies in the stabilized mixed method. According to (33) we can expect stable velocity approximations for all time-steps, while pressure approximations may suffer because of the degradation of $\mathbb{K} - \mathbb{B}\mathbb{M}^{-1}\mathbb{B}^T$. We also want to test pressure behavior for very small time-steps and a varying stabilization parameter δ. Here, the goal is to test whether or not the discrete pressure undergoes a change when δ is varied. We use piecewise cubic Lagrangian elements for the velocity, and so $\mathbb{B}U^k = O(h^3)$. Pressure is approximated by a finite element space of the same order, defined with respect to the same triangulation of the domain Ω into finite elements. We recall that the resulting equal-order $P_3 - P_3$ finite element pair does not satisfy the inf-sup condition.

In all experiments Ω is the unit square in $\mathbb{R}^2$. We consider the fully discrete formulation (21)-(22) with $\theta = 1$ (implicit Euler) and $\gamma = 0$ (Pressure-Poisson spatial stabilization). Then we take one time-step using the exact steady state solution

$$\mathbf{u} = \begin{pmatrix} \sin(\pi x - 0.7)\sin(\pi y + 0.2) \\ \cos(\pi x - 0.7)\cos(\pi y + 0.2) \end{pmatrix};$$

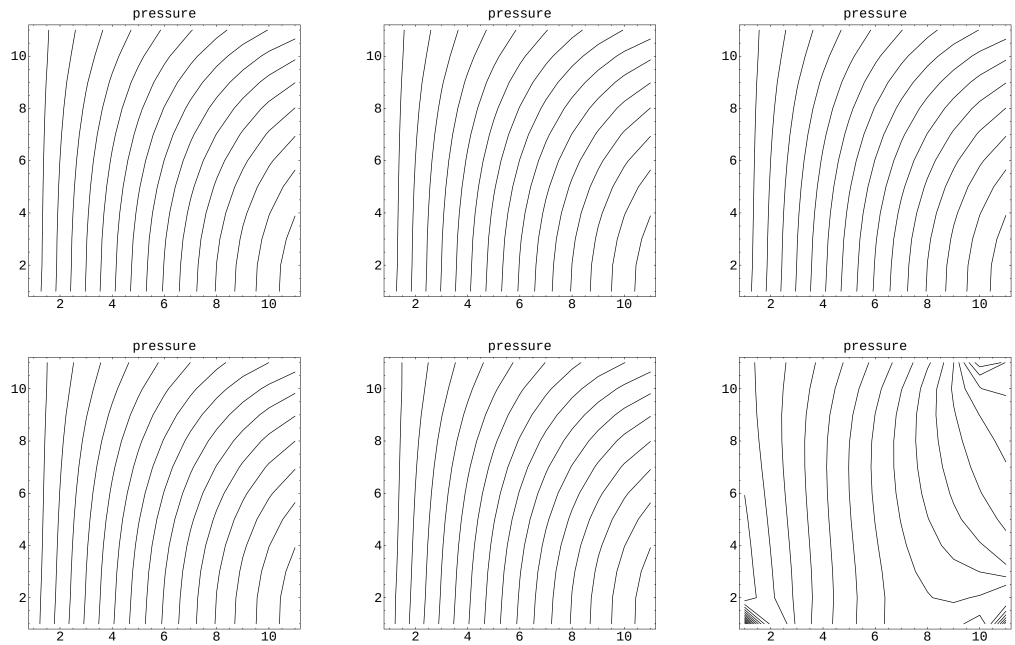

Figure 2: The $\Delta t \to 0$ limit: Taylor-Hood element. $n = 1, 2, 3$ (top) and $n = 4, 5, 6$ (bottom).

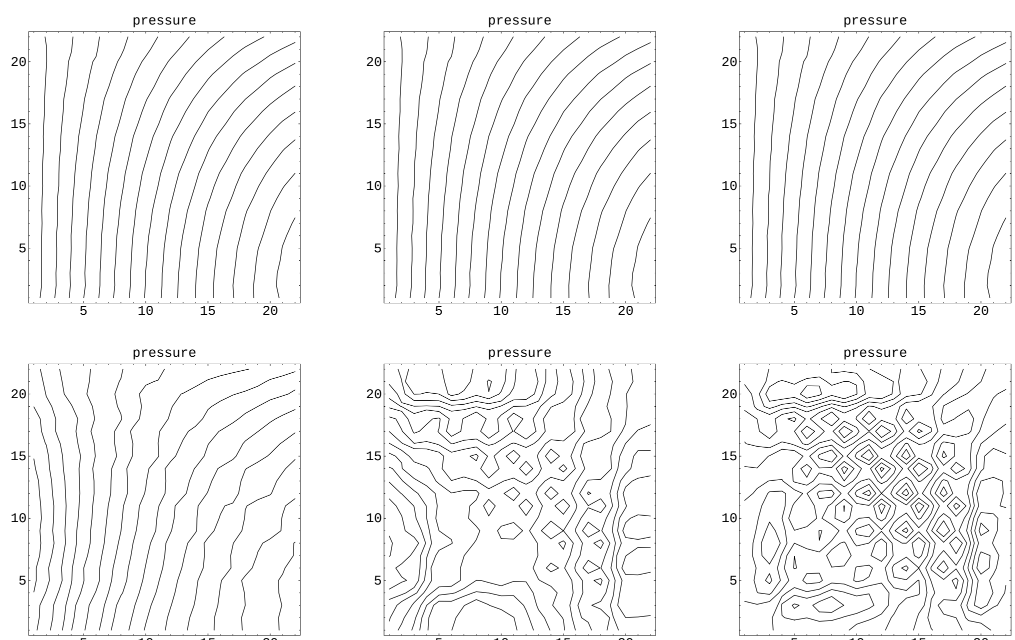

Figure 3: The $\Delta t \to 0$ limit: Stabilized method with $\delta = 0.05$. $n = 1, 2, 3$ (top) and $n = 4, 5, 6$ (bottom).

Table 4: Stabilized method with varying δ and $\Delta t = (\Delta t)_1$.

	velocity		pressure
δ	L^2	H^1	L^2
5	0.23160E-02	0.39364E-01	0.96711E-01
0.5	0.22787E-03	0.46992E-02	0.13234E-01
0.05	0.54734E-04	0.36186E-02	0.44419E-02
0.005	0.35735E-04	0.36392E-02	0.23466E-02

Table 5: Stabilized method with varying δ and $\Delta t = (\Delta t)_6$.

	velocity		pressure
δ	L^2	H^1	L^2
5	0.85777E-04	0.41886E-02	0.81182E+01
0.5	0.55806E-04	0.40932E-02	0.19642E+01
0.05	0.46465E-04	0.40049E-02	0.10385E+01
0.005	0.38037E-04	0.40331E-02	0.14324E+01

$$p(x, y) = \sin(x)\cos(y) + (\cos(1) - 1)\sin(1)$$

to generate the initial condition in (4). The right hand side $\mathbf{f}$ is computed by evaluating the momentum equation (1) for the exact solution. The new velocity approximation is computed by (25) with U^0 initialized by the finite element interpolant of the exact velocity field. The new pressure is computed by (26). Note that an initial condition for the pressure is not required for $\theta = 1$. We use a uniform triangulation of Ω consisting of 96 triangles. Each P_3 triangle has 10 local nodes for a total of 484 different nodes in the mesh. For this triangulation $h \approx 0.14$ and $h^2 \approx 0.02$. All matrices in (25)-(26) are assembled using a thirteen-point quadrature rule; see [8, p.343], and the linear system is solved using a direct solver.

Our first experiment illustrates the $\tau \to 0$ limit discussed in Section 4.2. The objective is to demonstrate an unstable mixed discretization of the Stokes problem as $\delta \to 0$. We set $\Delta t = 0.1$ and consider $\delta_k = 5 \times 10^{-k} \quad \text{for } k = 4, 5, 6$. Figure 1 shows increasing node-to-node oscillations in the pressure approximations as δ approaches 0. The H^1-seminorm of the pressure is reported in Table 1 and also confirms the loss of stability in the $\delta \to 0$ limit.

The second experiment is designed to test the $\Delta t \to 0$ limit for fixed h and δ. We fix $\delta = 0.05$ and take $(\Delta t)_n = 10^{-n} \quad \text{for } n = 1, 2, 3, 4, 5, 6$. This value of δ was found to give the best pressure errors in the steady-state method (13); see [1]. The time-step selection ensures that there are enough time-steps smaller than $O(h^2)$. To asses the impact of small time-steps on the stabilized method we compute a reference finite element solution

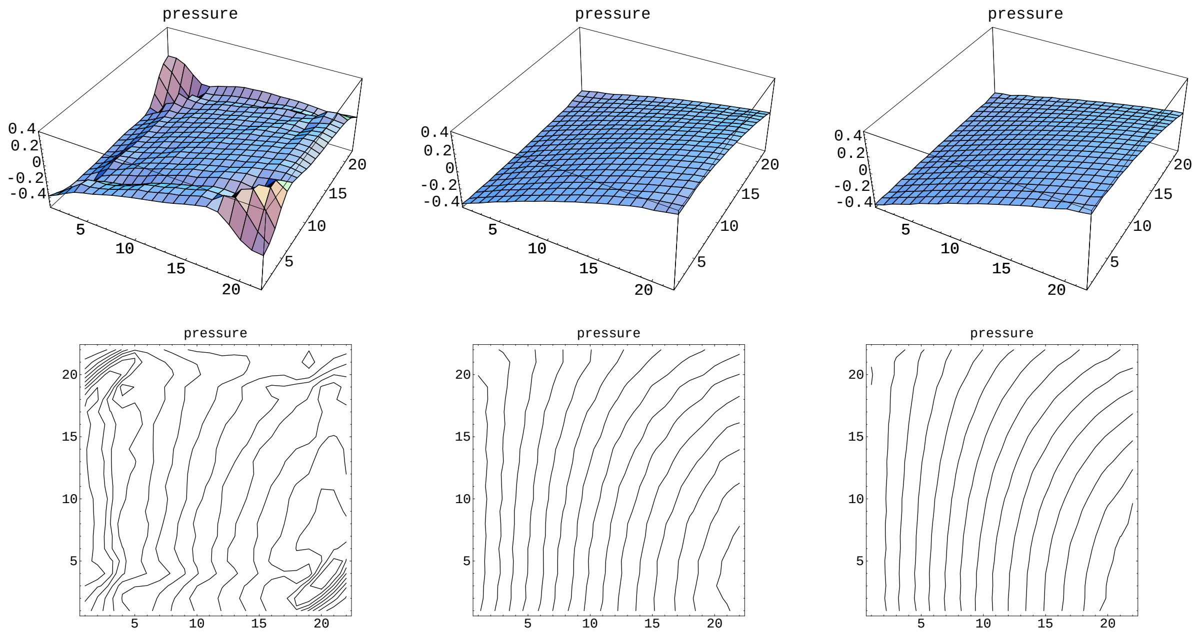

Figure 4: Stabilized method with $\delta = 5.0, 0.5$ and 0.005. $\Delta t = (\Delta t)_1$.

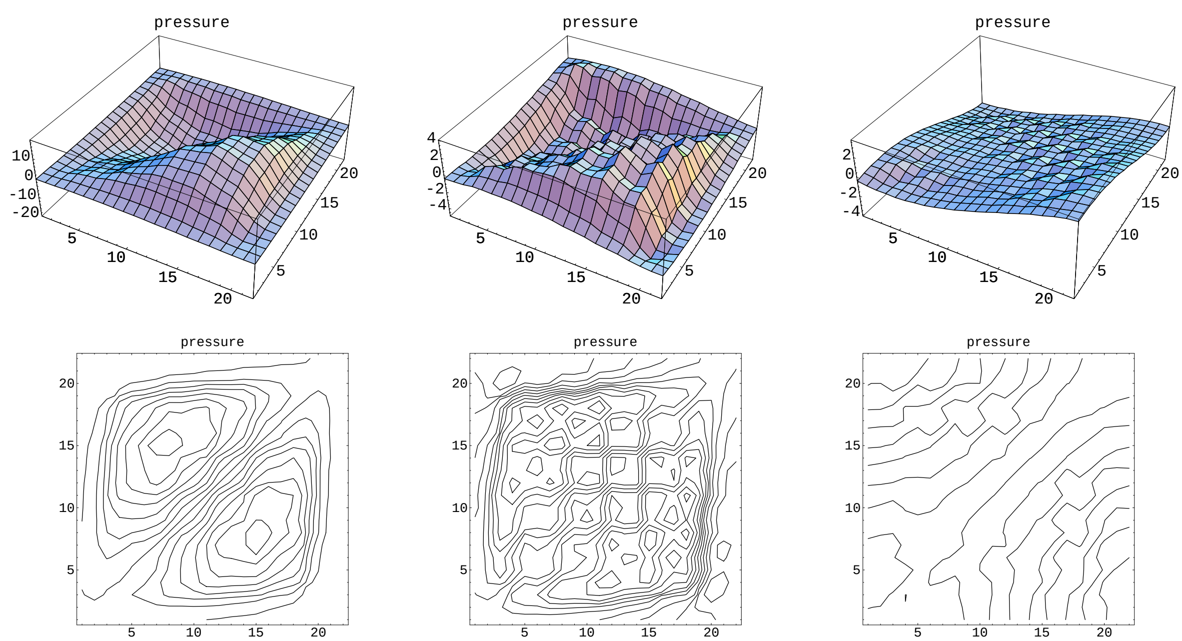

Figure 5: Stabilized method with $\delta = 5.0, 0.5$ and 0.005. $\Delta t = (\Delta t)_6$.

of the time-dependent Stokes problem (1)-(4) using the stable Taylor-Hood pair, the same six values of Δt, and $h = 0.1$. Figure 2 shows that with the exception of the smallest time-step, all pressure contours are virtually identical. Deterioration of the finite element solution in this case can be explained by observing that for $(\Delta t)_6$, the fully discrete equations can be interpreted as discretization of a singularly perturbed elliptic equation; see [12]. The L^2 pressure errors in Table 2 also experience only a slight deterioration for the first five values of Δt. From these results we can conclude that in a stable mixed method the limit $\Delta t \to 0$ is not accompanied by serious degradation of the pressure approximation, beyond what can be normally expected in the singularly perturbed limit.

The first row in Figure 3 shows that with the first three time-steps the stabilized method also gives accurate pressure approximations. However, as the time-step becomes smaller and smaller, Figure 3 shows a significant deterioration of the pressure approximation. The L^2 and H^1 errors of the finite element solution for all six time-steps are reported in Table 3. This table shows that, as expected, velocity approximation does not suffer at all when the time-step is being reduced. However, the L^2 error in the pressure increases over several orders of magnitude. We note that the most severe deterioration in the pressure occurs for $n = 4, 5, 6$ when Δt is orders of magnitude smaller than h^2. In contrast, for $n = 1, 2$ the time-step is of the same order as h^2 and the pressure approximation is relatively stable. These observations are consistent with the conclusions in Section 4.3 that trouble might occur if h^2 is large compared to Δt.

Our last experiment investigates pressure behavior for fixed values of h and Δt and a varying stabilization parameter δ. We take $\delta_k = 5 \times 10^{-k}$ for $k = 0, 1, 3$, and compute the finite element solutions first with $\Delta t = 0.1$ and then with $\Delta t = 0.000001$. Pressure approximations for the first time-step are shown in Figure 4. While for $\delta = 5$ the pressure approximation is not very good, all three solutions are qualitatively similar. This assessment is confirmed by Table 4 which shows that L^2 pressure errors improve when δ is close to the optimal steady state value of 0.05.

However, repeating the same experiments with $\Delta t = 0.000001$ shows a dramatic change in the pressure behavior. Figure 5 shows three qualitatively different pressure approximations, and so confirms the conjecture from Section 5 that the discrete pressure will change if one changes the stabilization parameter.

Dependence of discrete pressures on the value of δ would not be troublesome if not for the fact that some of the pressure profiles in Figure 5 appear completely “legitimate”. That is, they do not exhibit the strong node-to-node oscillations of the $\delta \to 0$ limit that would have allowed us to rule them out as spurious. This will pose a problem in simulations of, e.g., chemically reacting flows, where the goal is to accurately resolve the non-equilibrium chemical reaction rather than to compute a steady state solution.

7 CONCLUSIONS

We have shown that fully discrete methods that employ finite difference discretization in time and consistent inf-sup stabilized finite elements in space may experience difficulties in the small time-step limit. Problems in the small time-step limit for separated discretizations of parabolic equations have been reported and analyzed in [12]. There, the difficulties are caused by the singularly perturbed limit as $\Delta t \to 0$, and can be remedied by application of stabilization of the same kind as used in problems with strong reaction terms. In contrast, for the equations considered here, the origin of the difficulties in the small Δt limit is in the type of stabilization employed. In particular, we have demonstrated that they are caused by the coupling of the velocity time derivative, needed to fulfill the consistency requirement, with the weighting functions that define the spatial stabilization.

These difficulties are manifested by pressure approximations that become dependent on the values of the stabilization parameter δ as $\Delta t \to 0$. The absence of clear indicators for spurious modes such as strong node-to-node oscillations, makes this type of behavior especially difficult to identify. One possible remedy may be to use formulations based on space-time finite elements. However, considering that the inf-sup condition is a purely spatial constraint, a better and simpler strategy may be to employ direct modification of the continuity equation as in [9], [17], or [18].

The problems and issues discussed in this paper are typical only for the small time-step limit. For standard incompressible flow applications where excessively small time-steps are not needed, the use of consistent spatial stabilization, in conjunction with an implicit time integration, remains an attractive and viable alternative to mixed Galerkin methods. This is particularly true if the main goal of the simulation is to compute a steady state solution. However, for applications that require very small time-steps, such as chemically reacting flows, one has to exercise an extreme caution when the time-step becomes smaller than $O(h^2)$.

REFERENCES

[1] T. Barth, P. Bochev, M. Gunzburger, J. Shadid, A taxonomy of consistently stabilized finite element methods for the Stokes problem, to appear in *SIAM J. Sci. Comput.*

[2] J. Blasco and R. Codina, Stabilized finite element method for the transient Navier-Stokes equations based on a pressure gradient projection, *Comp. Meth. Appl. Mech. Engrg.*, **182**, pp. 277–300, 2000.

[3] J. Blasco and R. Codina, Space and time error estimates for a first-order, pressure stabilized finite element method for the incompressible Navier-Stokes equations, *Appl. Numer. Math.*, **38**, pp. 475–497, 2001.

[4] P. Bochev, M. Gunzburger and R. Lehoucq, On stabilized finite element methods for the Stokes problem in the small time-step limit, *In preparation.*

[5] P. Bochev, M. Gunzburger, J. Shadid, On inf-sup stabilized finite element methods for transient problems, to appear in *Comp. Meth. Appl. Mech. Engrg.*

[6] F. Brezzi, On existence, uniqueness and approximation of saddle-point problems arising from Lagrangian multipliers, *RAIRO Model. Math. Anal. Numer.*, **21**, pp. 129–151, 1974.

[7] F. Brezzi and J. Pitkaranta, On the stabilization of finite element approximations of the Stokes problem. In: *Efficient solution of elliptic systems.* Notes on Numerical Fluid Dynamics, Vol **10**, W. Hackbush, ed., pp. 11-19, Braunschweig, Wiesbaden, Viewig, 1984.

[8] G. Carey, J.T. Oden, *Finite elements. Computational aspects*, Prentice Hall, 1984.

[9] C. Dohrmann and P. Bochev, A stabilized finite element method for the Stokes problem based on polynomial pressure projections, submitted to *Int. J. Numer. Meth. Fluids.*

[10] J. Douglas and J. Wang, An absolutely stabilized finite element method for the Stokes problem, *Math. Comp.*, **52** pp. 495–508, 1989.

[11] V. Girault and P. Raviart, *Finite Element Methods for Navier-Stokes Equations,* Springer, Berlin, 1986.

[12] I. Harari, Stability of semidiscrete formulations for parabolic problems at small time-steps, *Comp. Meth. Appl. Mech. Engrg.*, to appear.

[13] P. Gresho and R. Sani, *Incompressible Flow and the Finite Element Method*, Wiley, Chichester, 1998.

[14] M. Gunzburger, *Finite Element Methods for Viscous Incompressible Flows,* Academic, Boston, 1989.

[15] T. Hughes and L. Franca, A new finite element formulation for computational fluid dynamics: VII. The Stokes problem with various well-posed boundary conditions: symmetric formulations that converge for all velocity pressure spaces, *Comput. Meth. Appl. Mech. Engrg.*, **65**, pp. 85–96, 1987.

[16] T. Hughes, L. Franca, and M. Balestra, A new finite element formulation for computational fluid dynamics: V. Circumventing the Babuska-Brezzi condition: A stable Petrov-Galerkin formulation of the Stokes problem accommodating equal-order interpolations, *Comput. Meth. Appl. Mech. Engrg.* **59**, pp. 85–99, 1986.

[17] D.J. Silvester, Optimal low order finite element methods for incompressible flow, *Comp. Meth. Appl. Mech. Engrg.* **111**, pp. 357–368, 1994.

[18] D. J. Silvester and N. Kechkar, Stabilized bilinear-constant velocity-pressure finite elements for the conjugate gradient solution of the Stokes Problem, *Comp. Meth. Appl. Mech. Engrg.* **79**, pp.71-86, 1990.