# Error Estimation for Nonlinear Reduced Basis Methods based on Empirical Operator Interpolation

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# Error estimates for model order reduction methods

Model reduction implies error to high fidelity model: $e := u_h - u_{\mathrm{red}}$.

## Error bounds

Error bounds $\eta$ should be

- **rigorous**: $\eta \geq \|e\|$,
- **effective**: $\frac{\eta}{\|e\|} \leq C_{\mathrm{eff}}$ or "not too big"

  and
- **efficient**, i.e. quickly computable (as fast as reduced model)

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Often there is a trade-off between **efficiency** and **rigor/effectivity**.

# Outline


1. Reduced basis method for non–linear problems
2. A posteriori error estimators
3. Numerical experiments
4. Summary and future work

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# Reduced Basis Method

**RB Scenario:**

- Parametrized PDEs (e.g. evolution problem "$\partial_t u(\boldsymbol{\mu}) - \mathcal{L}(\boldsymbol{\mu})[u] = 0, u(0; \boldsymbol{\mu}) = u_0(\boldsymbol{\mu})$")
- **time-critical** applications
- **many-query** applications

**Goals:**

- **Offline**-/Online decomposition
- Efficient reduced simulations
- **A posteriori error control**

**References:** [Patera&Rozza, 2006], [Haasdonk et al., 2008]

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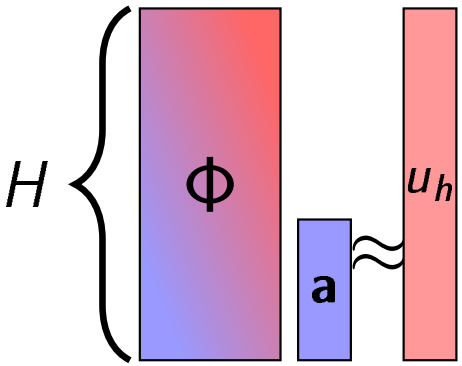
# High–dimensional model

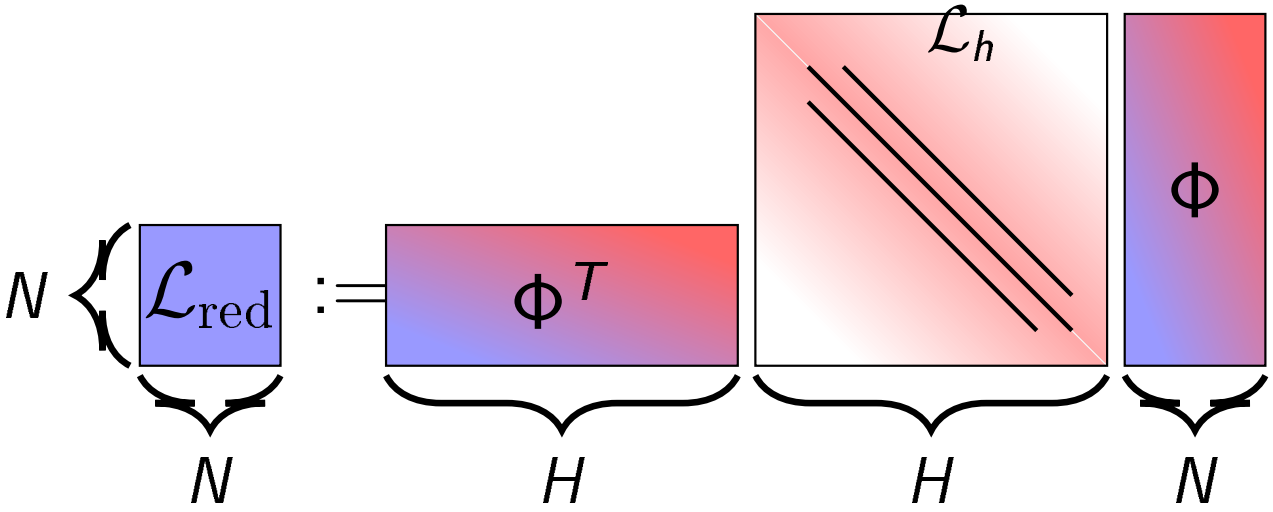
$$(\partial_t u_h) + \quad H \left\{ \underbrace{\begin{bmatrix} \mathcal{L}_h \\ \\ \\ \end{bmatrix}}_{H} \right. \begin{bmatrix} u_h \end{bmatrix} = \begin{bmatrix} b_h \end{bmatrix} \quad \cdot$$

# Offline phase

- Find reduced basis functions $\Phi := \{\varphi_1, \ldots, \varphi_N\}$,
- such that




and



# Offline phase

- Find reduced basis functions $\Phi := \{\varphi_1, \ldots, \varphi_N\}$,
- interpolation points $X := \{x_1, \ldots, x_M\}$ and functions $\Xi := \{\xi_1, \ldots, \xi_M\}$,
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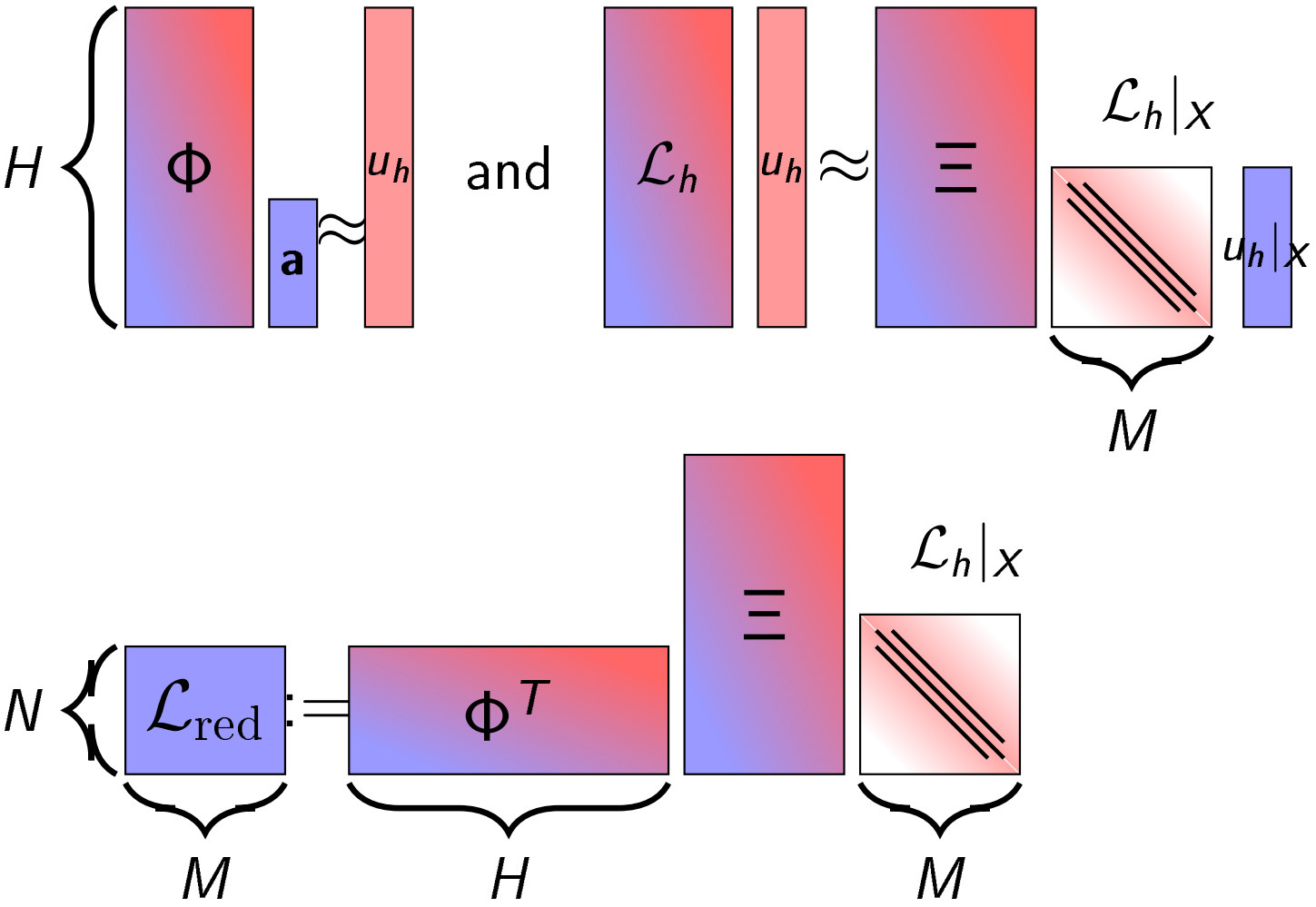

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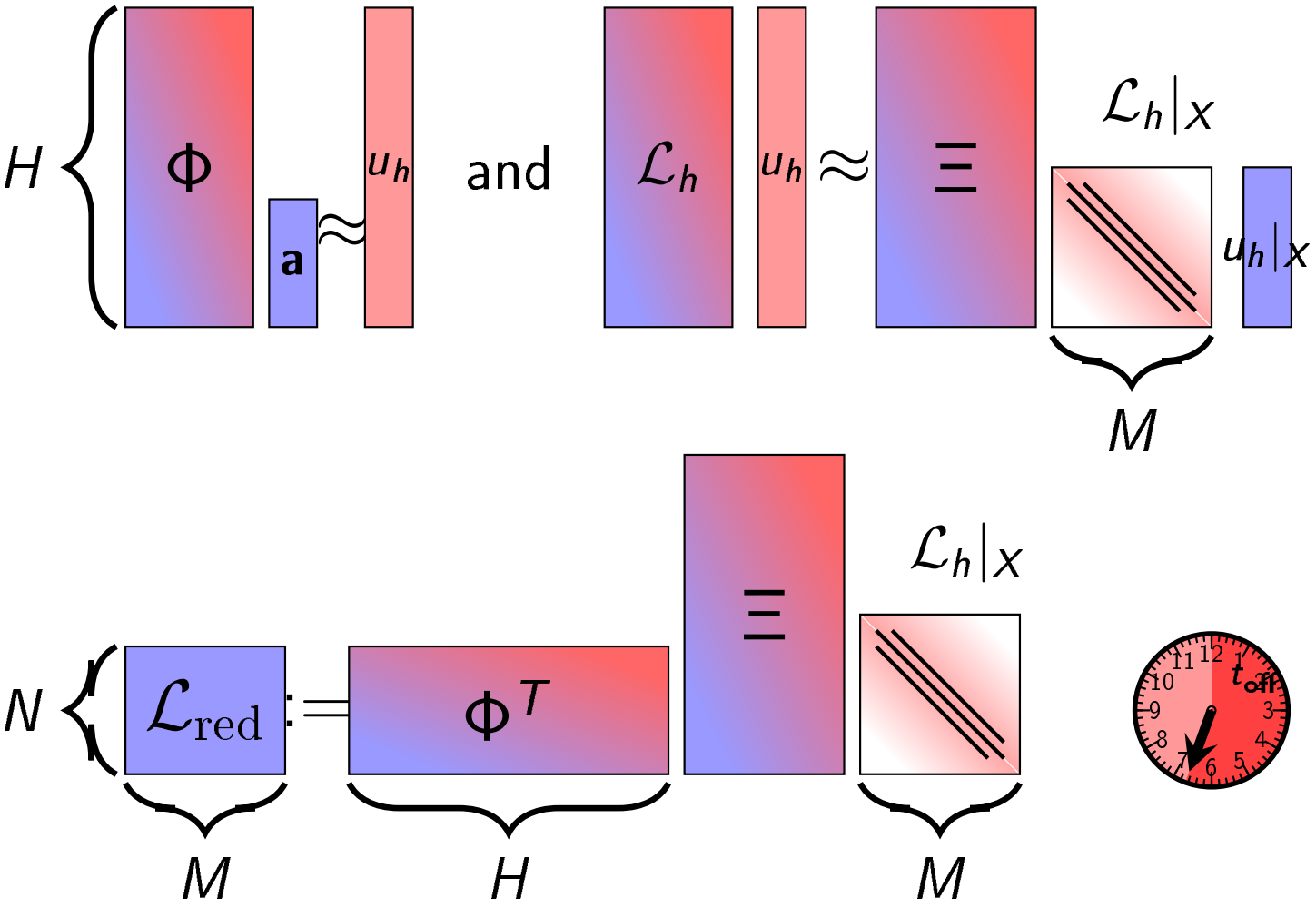
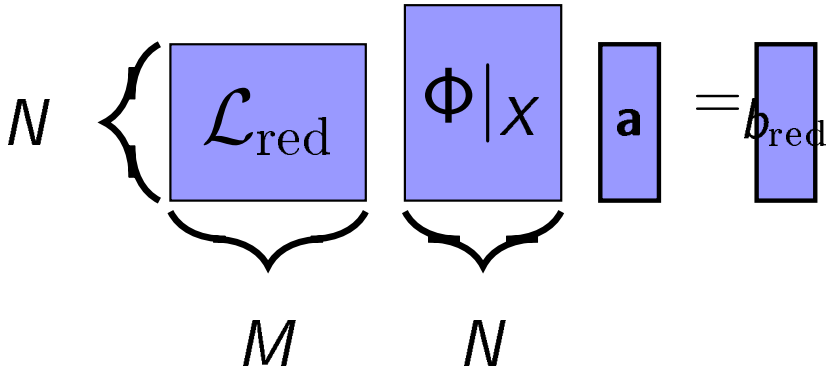


# Reduced basis model

$(\partial_t u_{\mathrm{red}} +)$



Error: $e := u_h - u_{\mathrm{red}}$ with $u_{\mathrm{red}} := \Phi \mathbf{a}$

# Theoretical complexities

| | detailed simulation | reduced simulation |
|---|---|---|
| Initial data projection | $\mathcal{O}(H)$ | $\mathcal{O}(QN)$ |
| Assembling of operators and derivatives in Newton step | $\mathcal{O}(H)$ with large constant | $\mathcal{O}(N^2M^2)$ |
| Solving Newton step | approximately $\mathcal{O}(H^2)$ (depending on solver) | $\mathcal{O}(N^3)$ |
| Computing residual | $\mathcal{O}(H)$ | $\mathcal{O}(NM^2)$ |

Table: Comparison of theoretical run–time complexities between detailed and reduced simulations.

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# A posteriori error estimator (DHO, 2012)

**Estimator**

$$\left\| u_h^k(\boldsymbol{\mu}) - u_{\mathrm{red}}^k(\boldsymbol{\mu}) \right\| \leq \eta_{N,M,M'}^k(\boldsymbol{\mu})$$

# A posteriori error estimator

## Theorem (A posteriori error estimator )

**Assumptions:**

- *Operator(s) fulfill "Lipschitz" properties:*
  - $\|u - v + \Delta t \mathcal{L}_I [u] - \Delta t \mathcal{L}_I [v]\| \geq \frac{1}{C_{I,\Delta t}} \|u - v\|_{\mathcal{W}_h}$
  - $\|u - v - \Delta t \mathcal{L}_E [u] + \Delta t \mathcal{L}_E [v]\| \leq C_{E,\Delta t} \|u - v\|_{\mathcal{W}_h}$
- *$M'$-trick: Empirical interpolations exact for larger number of interpolation points $M + M'$ and the initial data is $u_0(\boldsymbol{\mu}) \in \mathrm{span}\Phi$*

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## Theorem (A posteriori error estimator cont.)

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- *$M'$-trick: Empirical interpolations exact for larger number of interpolation points $M + M'$ and the initial data is $u_0(\boldsymbol{\mu}) \in \operatorname{span}\Phi$*

**Then:**

$$\left\| u_{\mathrm{red}}^k(\boldsymbol{\mu}) - u_h^k(\boldsymbol{\mu}) \right\| \leq \eta_{N,M,M'}^k(\boldsymbol{\mu})$$

*with*

$$\eta_{N,M,M'}^k(\boldsymbol{\mu}) := \sum_{i=0}^{k-1} C_{I,\Delta t}^{k-i+1} C_{E,\Delta t}^{k-i} \left( \left\| R_{I+E,M,M'}^{k+1}(\boldsymbol{\mu}) \right\| + \left\| \Delta t R^{k+1}(\boldsymbol{\mu}) \right\| + \varepsilon^{New} \right)$$

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*The residuals $R_{*,M}$ measure the empirical interpolation error, e.g.*

$$R_{*,M,M'}^{k+1,\nu} := \sum_{m=M}^{M+M'} I_m^* \left[ u_{\mathrm{red}}^{k+1,\nu} \right] \xi_m = \mathcal{I}_{M+M'}[\mathcal{L}] - \mathcal{I}_M[\mathcal{L}]$$

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| | |
|---|---|
| empirical interpolation error | |
| Galerkin projection error | |
| time evolution | |
| Newton step error | |

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| | |
|---|---|
| empirical interpolation error | (quasi-) rigorous, $M'$ affects efficiency and effectivity |
| Galerkin projection error | rigorous and efficient |
| time evolution | selection affects effectivity |
| Newton step error | neglectable |

# Rigorous error bound

- Compute many interpolation points $M'$.
- "Rigorous" Lipschitz–constants

## Computation of Lipschitz constants

- For separable problems $\mathcal{L}(\boldsymbol{\mu}) = \sum_{q=1}^{Q} \sigma(\boldsymbol{\mu})\mathcal{L}^q$, use SCM method [Huynh et al., 2010] (can be very inefficient)
- Otherwise use matrix approximation algorithms as in [Wirtz et al., 2012]
- $C_{I,low} \leq C_I(\boldsymbol{\mu})$, $C_{E,upper} \geq C_E(\boldsymbol{\mu})$ for all $\boldsymbol{\mu} \in \mathcal{M}$.

# Effective error bound

- (Compute many interpolation points $M'$.)
- Define Lipschitz–constants as tight a s possible by approximation of $C_{I,\Delta t} = \sup_{k=0,\dots,K} \left\| \left( D\left(\mathrm{Id} + \mathcal{L}_{I,\Delta t}\right)|_{u_h^k} \right)^{-1} \right\|$, e.g. as

## Computation of Lipschitz constants

- Operator–norm approximation:

$$C_{I,\Delta t} \approx \inf_{v \in X_{\mathbf{test}}} \frac{\left\| D\left(\mathrm{Id} + \mathcal{L}_{I,\Delta t}\right)|_{u_h^k}[v] \right\|}{\|v\|}$$

$$C_{E,\Delta t} \approx \sup_{v \in X_{\mathbf{test}}} \frac{\left\| D\left(\mathrm{Id} + \mathcal{L}_{E,\Delta t}\right)|_{u_h^k}[v] \right\|}{\|v\|},$$

  where $X_{\text{text}}$ is set of intermediate Newton step solutions.
- $C_{I,low} \approx C_I(\boldsymbol{\mu})$, $C_{E,upper} \approx C_E(\boldsymbol{\mu})$ for all $\boldsymbol{\mu} \in \mathcal{M}$

# Remark: Output of interests

- If application has output of interest: $s(u_h)$ ($s$ linear functional),
- the error bound $\eta^s \geq |s(u_h) - s(u_{\mathrm{red}})|$
- can be improved by solving bounds for adjoint problems.

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# Porous Medium Equation

## Nonlinear Diffusion

**Problem definition:**

$$\partial_t u - m\Delta u^p = 0 \quad \text{in } \Omega \times [0,1], \qquad u(\cdot,0) = c_0 + u_0 \quad \text{on } \Omega \times \{0\}$$

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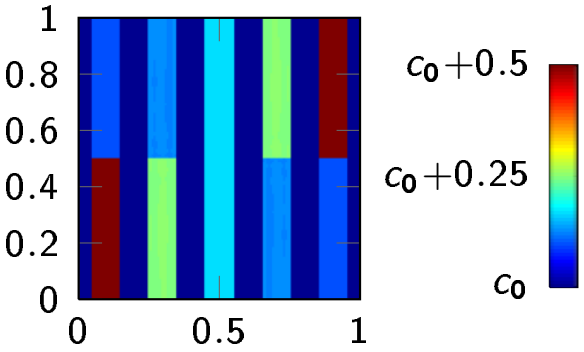
- Parametrization $\boldsymbol{\mu} = (p, m, c_0) \in [1,5] \times [0, 0.01] \times [0, 0.2]$
- $\Omega = [0,1]^2$ with homogeneous boundary conditions
- rectangular 100x100 grid with $K = 80$ time steps.

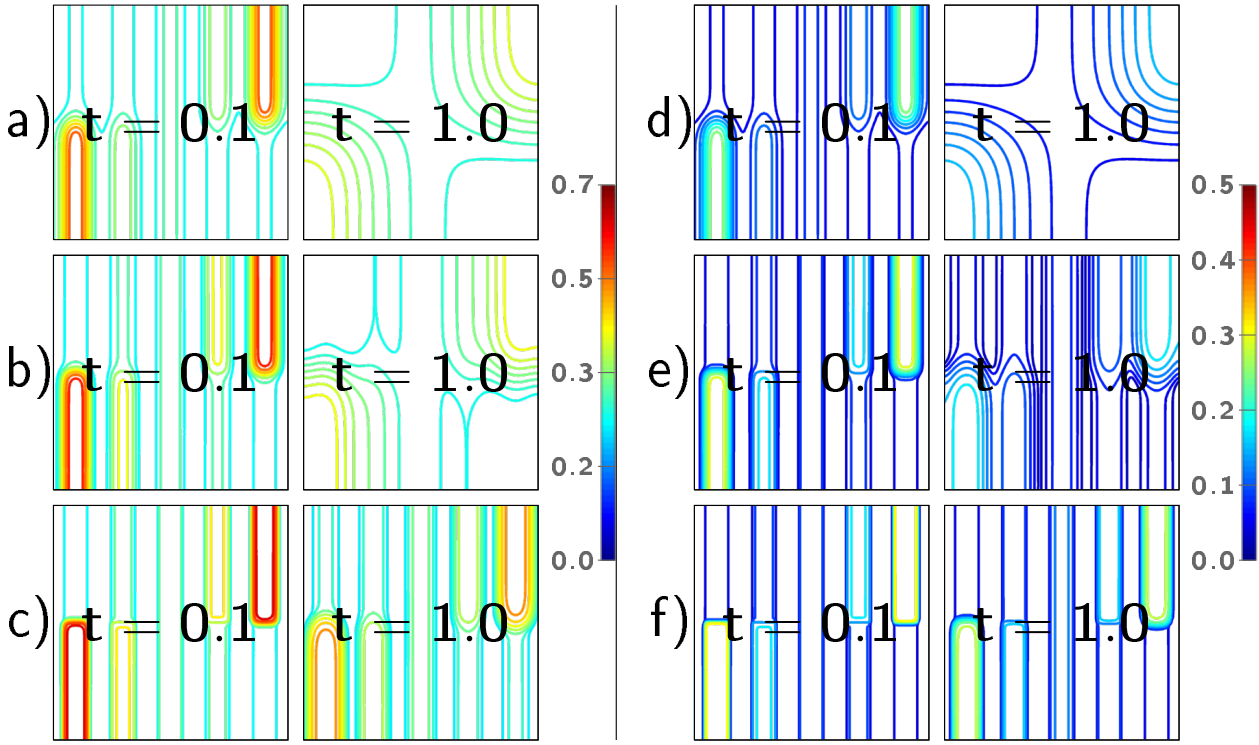
# PME: Solution snapshots

Initial data $u_0(\boldsymbol{\mu})$:




- density–dependent diffusivity
- degenerates at $u = 0$ for $p > 1$

Contour plots of concentration snapshots $u(\cdot, t; \boldsymbol{\mu})$:

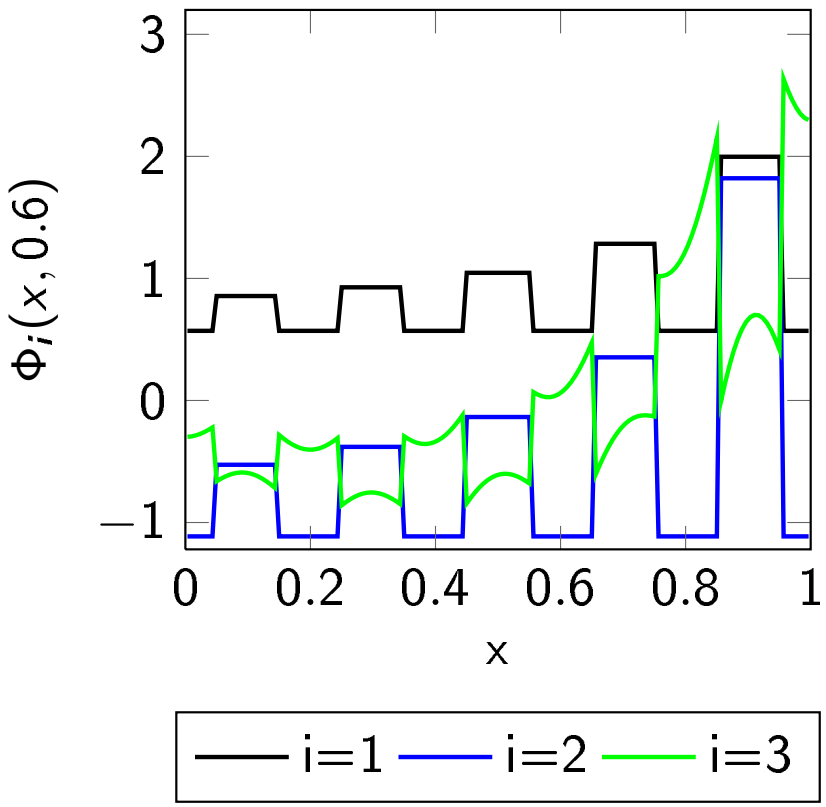


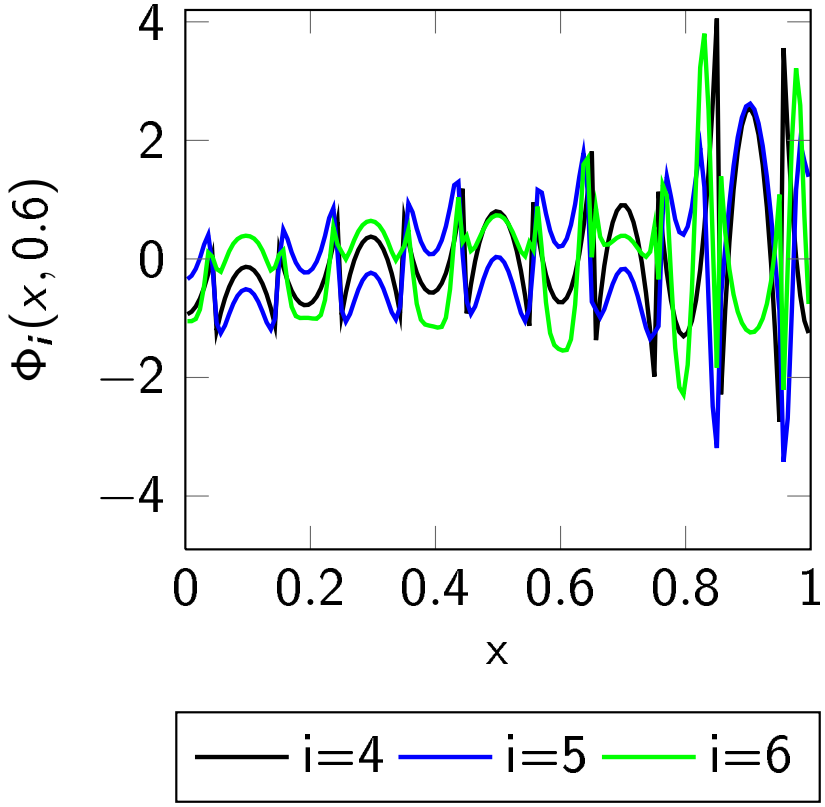

$\boldsymbol{\mu} = (p, m, c_0)$

a) $\boldsymbol{\mu} = (1, 0.01, 0.2)$, b) $\boldsymbol{\mu} = (1, 0.01, 0.0)$,
c) $\boldsymbol{\mu} = (2, 0.01, 0.2)$, d) $\boldsymbol{\mu} = (2, 0.01, 0.0)$,
e) $\boldsymbol{\mu} = (4, 0.01, 0.2)$, f) $\boldsymbol{\mu} = (4, 0.01, 0.0)$

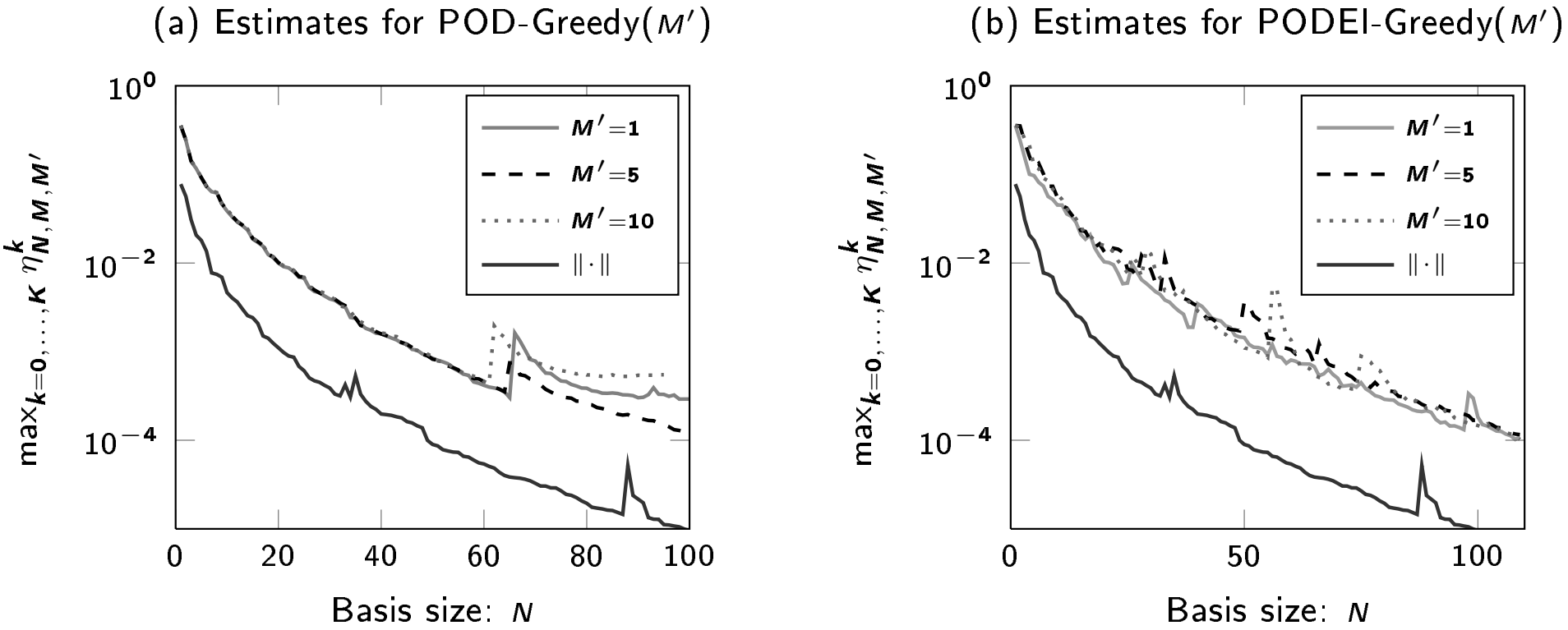
# PME: Reduced basis functions

**Slices of first six reduced basis functions at** $y = 0.6$



# PME: Greedy error convergence



# PME: Run–times

- $\dim(\mathcal{W}_h)$ 10000
- $\nu_{\max} \approx 1 - 20$
- $\#M_{\text{train},0}$ 27.
- $\#M_{\text{train}}$ 305

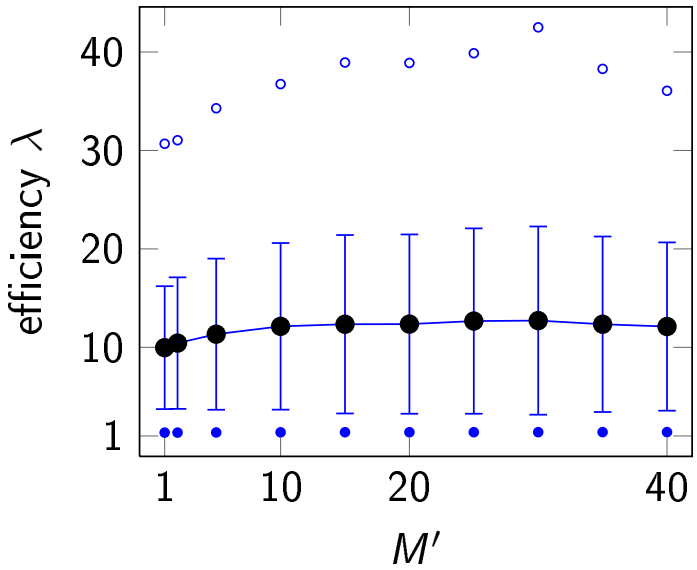
- Time gain: $9 - 50$
- Efficient error bound computation

| N | M | ø-runtime[s] | ø-runtime($\eta$)[s] | max. error | ø-offline time[h] |
|---|---|---|---|---|---|
| 0 | 0 | 55.38 | $-1.00$ | 0.00 | 0 |
| 19 | 72 | 1.61 | $5.37 \cdot 10^{-2}$ | $3.01 \cdot 10^{-3}$ | 0.16 |
| 37 | 143 | 2.07 | $5.72 \cdot 10^{-2}$ | $7.90 \cdot 10^{-4}$ | 0.45 |
| 56 | 215 | 2.67 | $9.27 \cdot 10^{-2}$ | $1.66 \cdot 10^{-4}$ | 1.01 |
| 74 | 286 | 3.6 | 0.12 | $6.36 \cdot 10^{-5}$ | 1.69 |
| 93 | 358 | 4.83 | 0.14 | $3.54 \cdot 10^{-5}$ | 2.72 |
| 111 | 429 | 6.55 | 0.18 | $1.96 \cdot 10^{-5}$ | 4.02 |

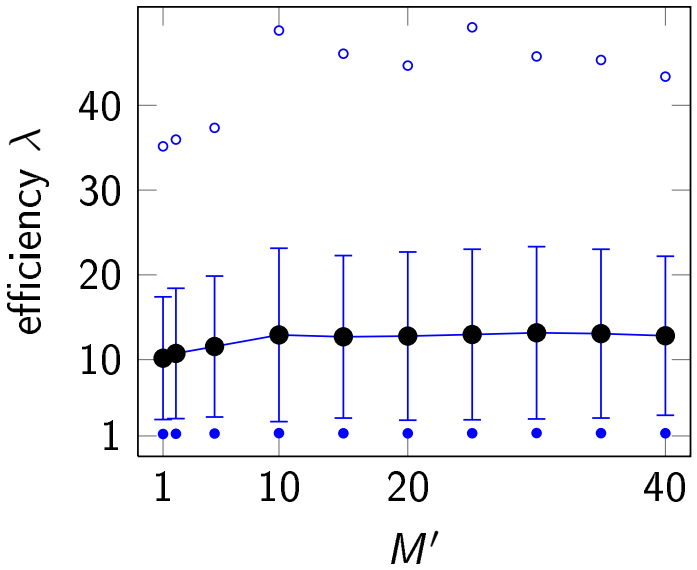
# PME: A posteriori error

Efficiency of error estimator $\eta$: $\lambda(\boldsymbol{\mu}) := \frac{\eta(\boldsymbol{\mu})}{\|\boldsymbol{u_h}(\boldsymbol{\mu}) - \boldsymbol{u}_{\mathrm{red}}(\boldsymbol{\mu})\|}$

(a) $N = 50, M = 200$




(b) $N = 75, M = 290$




- $C_I, C_E = 1$
- 100 random parameters
- minimum (•) and maximum (○)

- $\lambda \geq 1$ for all $\boldsymbol{\mu}$
- Larger $M'$ **does not** improve efficiency

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- Rigorous, effective and efficient
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## Future work

- Improve / implement computation of Lipschitz-constant
- Compare effectivity results

# References


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