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Reduced Basis Method for Finite Volume Approximation of Evolution Equations on Parametrized Geometries

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Martin Drohmann, Bernard Haasdonk, Mario Ohlberger

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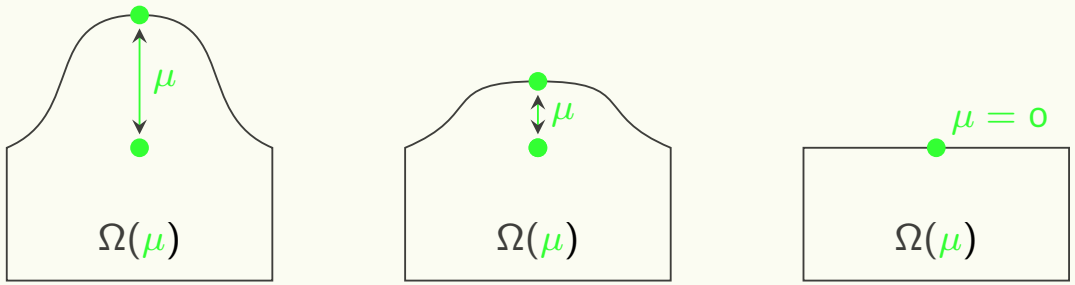
Outline

Model reduction of

- evolution problems
- on parametrized geometries



- with Reduced Basis (RB) methods.

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Outline

Model reduction of

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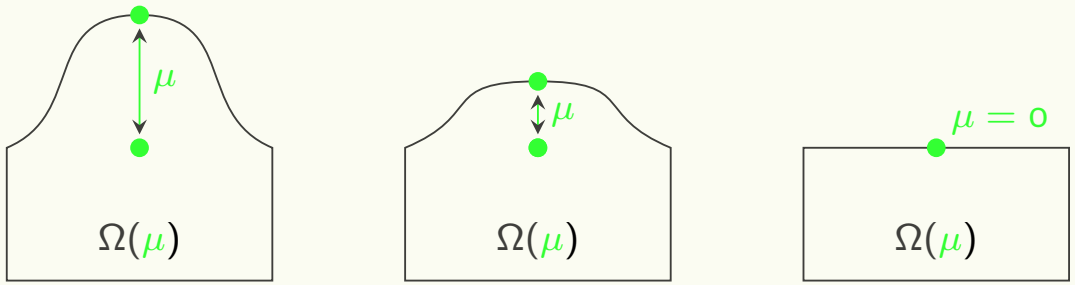

- with Reduced Basis (RB) methods.

Test problem: non-stationary heat-equation

For every $\mu \in \mathcal{P}$ we search for a solution $u(x, t; \mu)$ satisfying

$$\begin{aligned} \partial_t u(x,t;\mu) - a(\mu)\Delta u(x,t;\mu) &= 0 && \text{in } \Omega(\mu) \times [0, T_{\max}] \\ u(x,0;\mu) &= u_0(x;\mu) && \text{in } \Omega(\mu) \end{aligned}$$

and boundary conditions.

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Model reduction with a Reduced Basis

Reduced Basis methods are

- applicable to problems modelled by parametrized PDEs with parameters from a space $\mathcal{P} \subset \mathbb{R}^p$,
- where solutions need to be computed rapidly, repeatedly (and reliably).

Parametrized evolution problem

Find a solution $u \in \mathcal{W}$ satisfying the equations

$$\begin{aligned} \partial_t u(\mu) - \mathcal{L}(\mu)[u(\mu)] &= 0 && \text{in } \Omega \times [0, T_{\max}], \\ u(x, 0; \mu) &= u_0(x; \mu) && \text{in } \Omega \end{aligned}$$

and boundary conditions.

For details in elliptic case: [Patera&Rozza, 2006]

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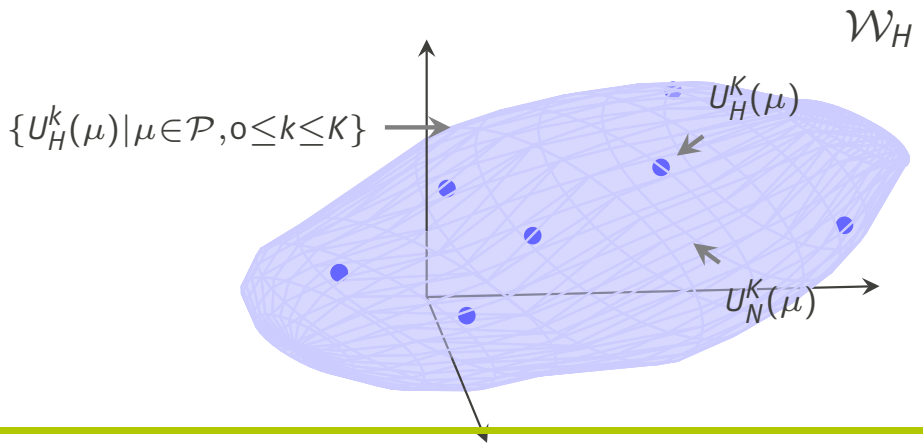
Idea of RB method

First step

Choose a parametrized numerical scheme (FV,FE,DG,...), e.g. the explicit scheme

$$U_H^0(\mu) := P[u_0(x;\mu)]$$
$$U_H^k(\mu) := U_H^{k-1}(\mu) + \Delta t L(\mu)[U_H^{k-1}(\mu)],$$

such that we get for each discrete timestep $0 \leq t^k \leq T_{\max}$ an approximation $u(x, t^k; \mu) \approx U_H^k(x;\mu)$ in a *"high-dimensional"* space $\mathcal{W}_H$.



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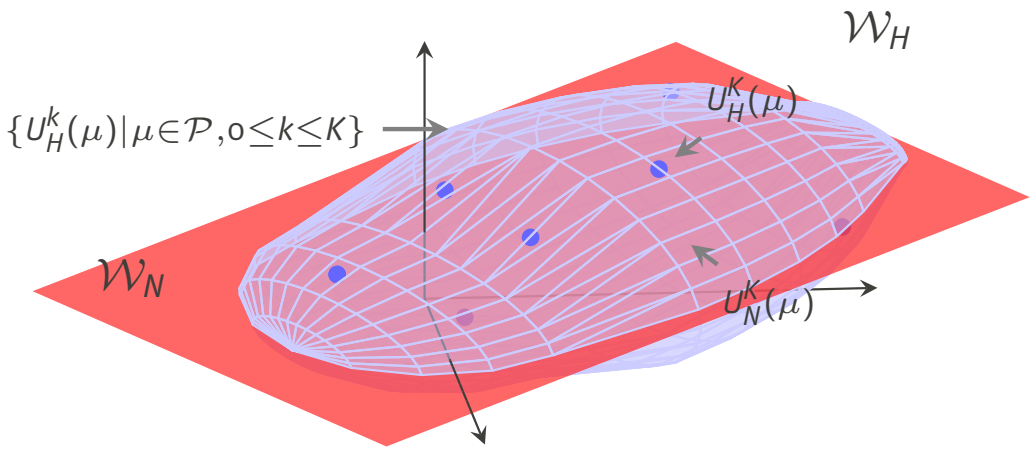
Idea of RB method

Second step – "offline phase"

- Approximate the set $\{U_H^k(\mu) \mid k = 0, \ldots, K, \mu \in \mathcal{P}\}$ by a *"low-dimensional"* reduced basis space $\mathcal{W}_N$.
- Its basis $\Phi := \{\varphi_i, \ldots, \varphi_N\}$ is called a reduced basis.



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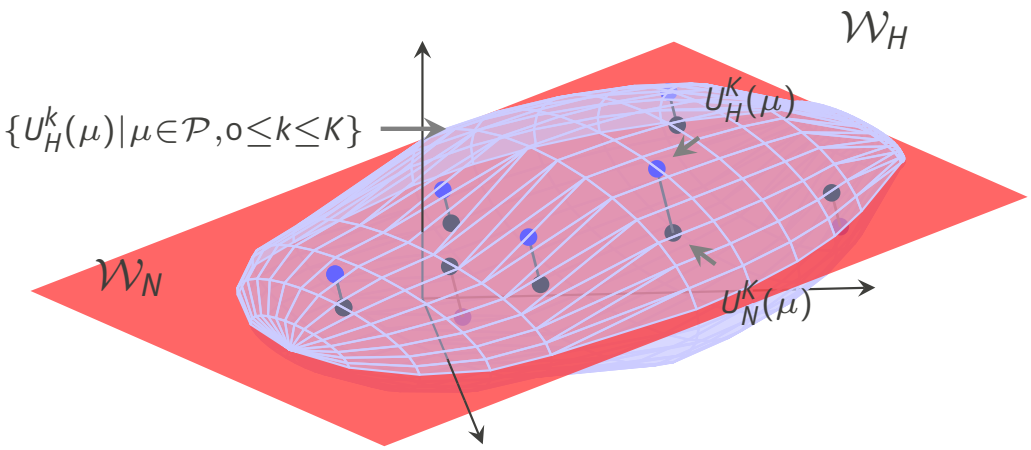
Idea of RB method

Third step – "online phase"

Project the numerical scheme onto the reduced basis space, i.e. we solve

$$\mathbf{a}^0(\mu) := (\langle P[u_0(\mu)], \varphi_1\rangle, \ldots, \langle P[u_0(\mu)], \varphi_N\rangle)^t$$
$$\mathbf{a}^{k+1}(\mu) := \mathbf{a}^k(\mu) + \Delta t \mathbf{L}(\mu)[\mathbf{a}^k(\mu)]$$

and get approximations $U_N^k(\mu) := \sum_{n=1}^N a_n^k(\mu)\varphi_n \approx U_H^k(\mu)$.



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RB method for linear schemes

In [Haasdonk&Ohlberger, 2007] it is shown that the RB method can efficiently be applied on numerical schemes with

- linear operators (implicit and explicit) and
- affinely parameter dependence of operators and data functions.

Affine parameter dependence

An operator or a data function is said to be affinely parameter dependent, if it can be written as an affine combination of parameter independent parts with parameter dependend coefficients, e.g.

$$L(\mu)[U_H^k(\mu)] = \sum_{q=0}^{Q} \underbrace{L^q[U_H^k]}_{\text{parameter independent}} \; \underbrace{\sigma_L^q(\mu)}_{\text{parameter dependent}} \quad \Rightarrow \quad \mathbf{L}(\mu)[\mathbf{a}^k(\mu)] = \boxed{\sum_{q=1}^{Q} \underbrace{\mathbf{L}^q}_{\text{offline}}[\mathbf{a}^k(\mu)] \underbrace{\sigma_L^q(\mu)}_{\text{online}}}$$

with matrices $(\mathbf{L}^q)_{n,m} := \int_\Omega L[\varphi_m]\varphi_n \mathrm{d}x$.

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Back to the heat equation

We look again at the non-stationary heat equation

$$\partial_t u(x,t;\mu) - a(\mu)\Delta u(x,t;\mu) = 0 \quad \text{in } \Omega(\mu) \times [0, T_{\max}].$$

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Back to the heat equation

The reduced basis space must not depend on the parameter.

Therefore, we introduce a reference geometry $\hat{\Omega}$ and a diffeomorphic mapping $\Phi(\mu) : \hat{\Omega} \to \Omega(\mu)$ for every parameter.

We look again at the non-stationary heat equation

$$\partial_t u(x,t;\mu) - a(\mu)\Delta u(x,t;\mu) = 0 \quad \text{in } \Omega(\cancel{\mu}) \times [0, T_{\max}].$$

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Geometry transformation

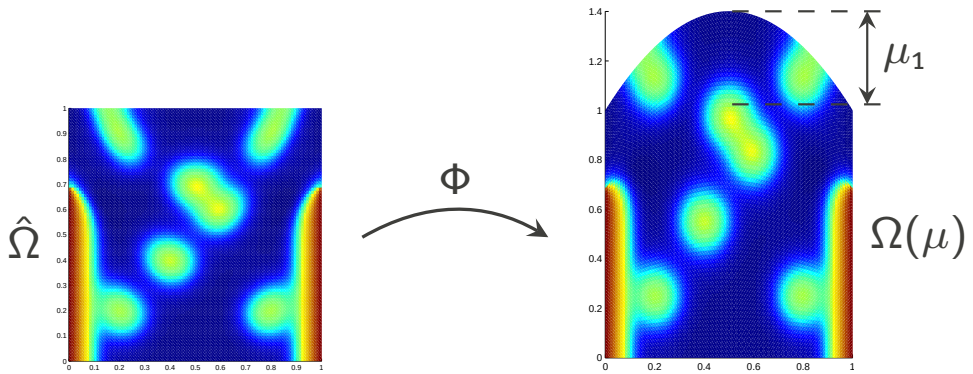
Transformed heat equation

The result is a PDE with (anisotropic) diffusion, convection and a reaction term:

$$\partial_t \hat{u} - a(\mu)\nabla \cdot (GG^t \nabla \hat{u}) + a(\mu)\nabla \cdot (v\hat{u}) - a(\mu)(\nabla \cdot v)\hat{u} = 0 \qquad \text{in } \hat{\Omega} \times [0, T_{\max}],$$

with notations

$$\hat{x} := \Phi^{-1}(x), \qquad \hat{u}(\hat{x}, t) := u(\Phi(\hat{x}), t),$$
$$G(\hat{x}) := D\Phi^{-1}|_{\Phi(\hat{x})}, \qquad v(\hat{x}) := G(\hat{x})\left(\nabla_{\hat{x}} \cdot G(\hat{x})\right).$$



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Finite volume scheme

Challenges

- The geometry transformation introduces a diffusion tensor.
- For non-affine geometry transformations the discrete operators do not depend affinely on the parameter. $\Rightarrow$ No easy projection onto RB space.

Numerical scheme for transformed heat equation

Discretization with an explicit finite volume scheme on a structured grid with gradient reconstruction. [Drblíková&Mikula, 2007]

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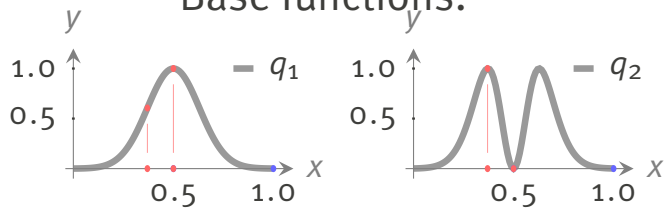
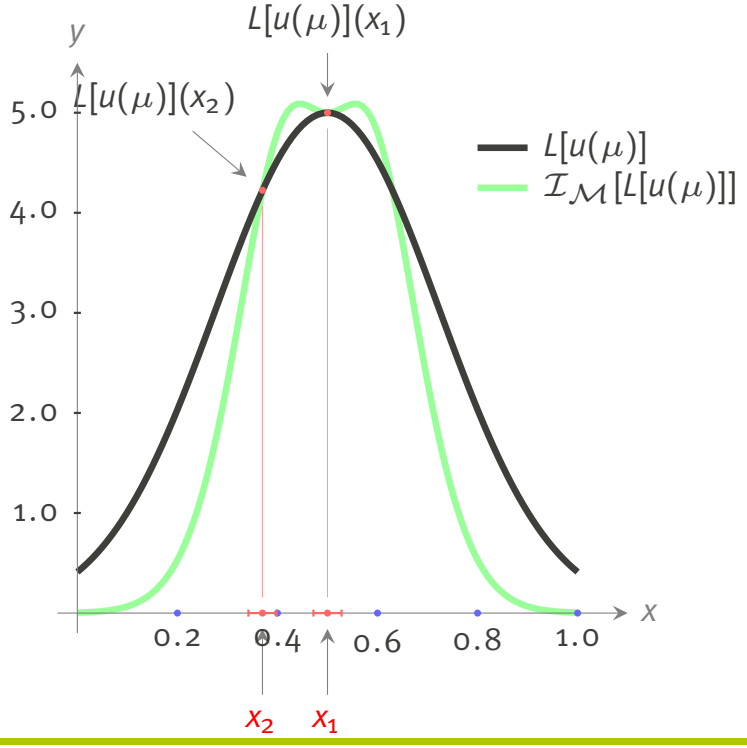
Dealing with non-affine parameter dependence

Idea

Approximate an operator $L(\mu)[U^k(\mu)]$ with few point evaluations.

Base functions:





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Solution: Empirical interpolation

Idea

Approximate an operator $L(\mu)[U^k(\mu)]$ with few point evaluations.

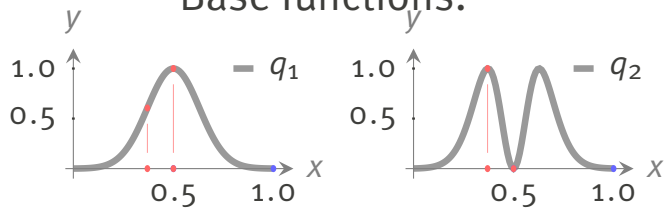
Empirical interpolation

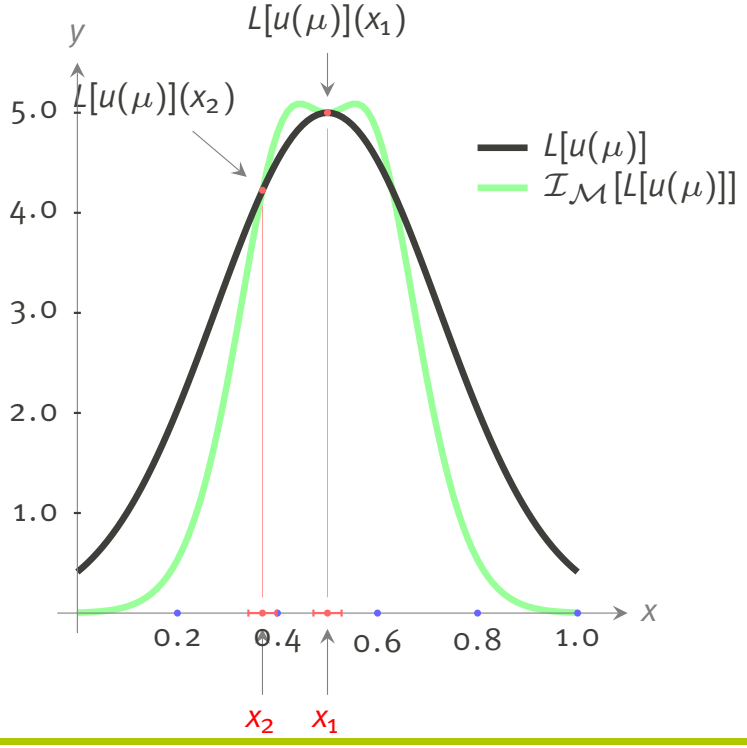
- Collateral Reduced Basis space of operator evaluations

$$\mathcal{W}_M := \operatorname{span}\{L(\mu_i)[U_H^{k_i}(\mu_i)]\}_{i=1}^M$$

- Collateral reduced basis $\Xi_M := \{\xi_i\}_{i=1}^M$ of nodal base functions
- Interpolation points $x_1, \ldots, x_M$.

Base functions:





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Solution: Empirical interpolation

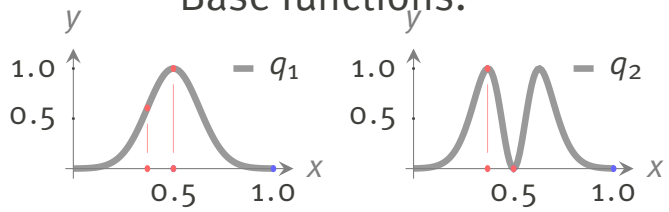
Operator evaluations

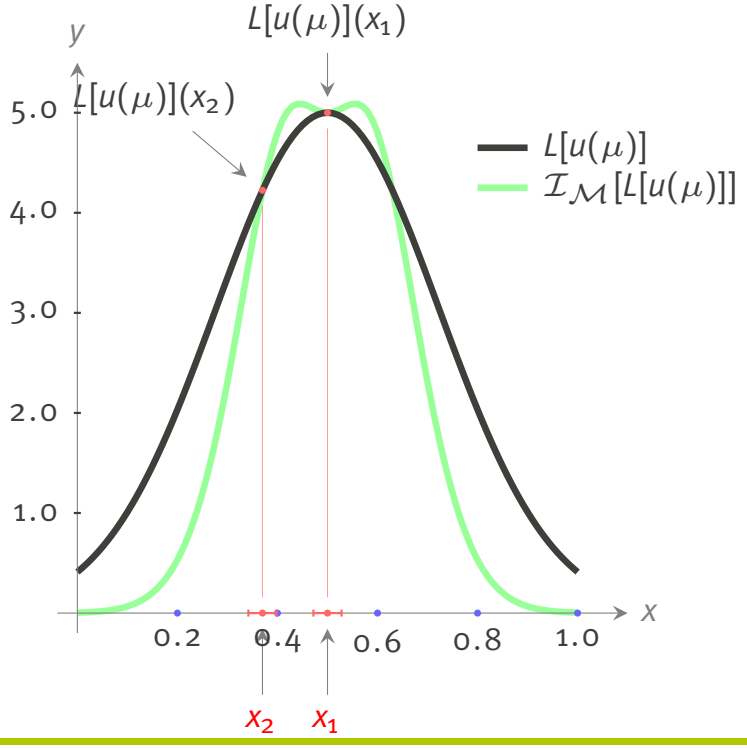
Interpolation are gained through exact operator evaluations at the interpolation points such that

$$\mathcal{I}_M[L(\mu)[U]] := \sum_{m=1}^{M} y_m(\mu)\xi_m(x)$$

$$y_m(\mu) := L(\mu)[U](x_m)$$

Base functions:





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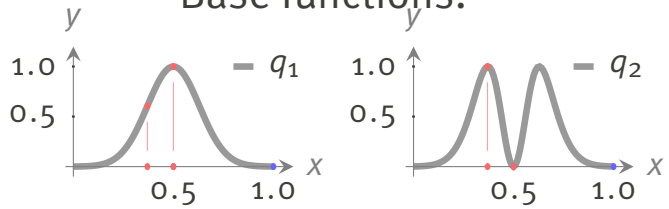
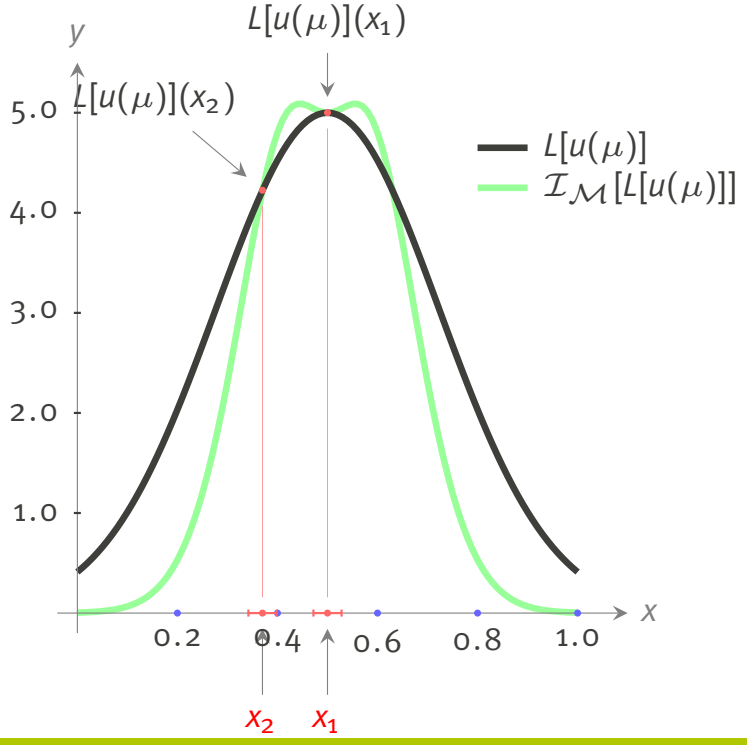
$$\mathcal{I}_M[L(\mu)[U]] := \sum_{m=1}^{M} y_m(\mu)\xi_m(x)$$

$$y_m(\mu) := L(\mu)[U](x_m)$$

- The coefficients can be computed efficiently, if the operator is localised.
- So far we only applied empirical interpolation to explicit numerical schemes.

Base functions:





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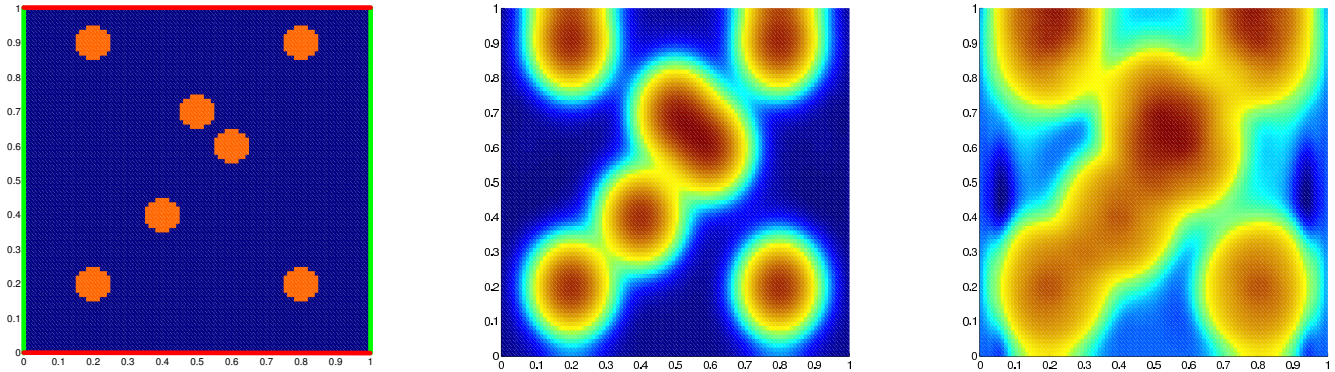
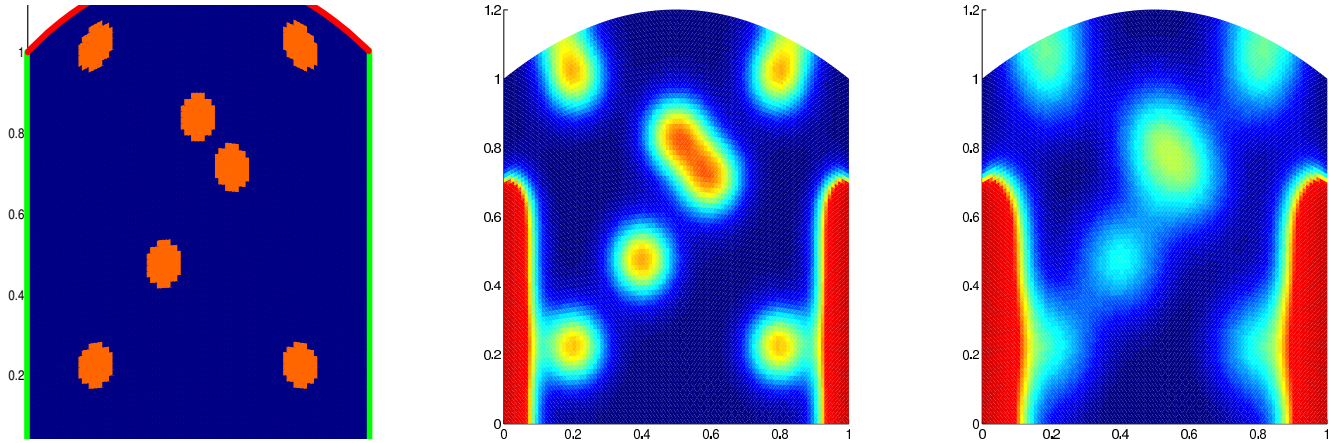
Results



Solutions for $\mu = (0, 0)$ at timesteps $t = 0.0$, $t = 0.45$, $t = 0.9$.



Solutions for $\mu = (0.2, 0.2)$ at timesteps $t = 0.0$, $t = 0.45$, $t = 0.9$.

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Results II

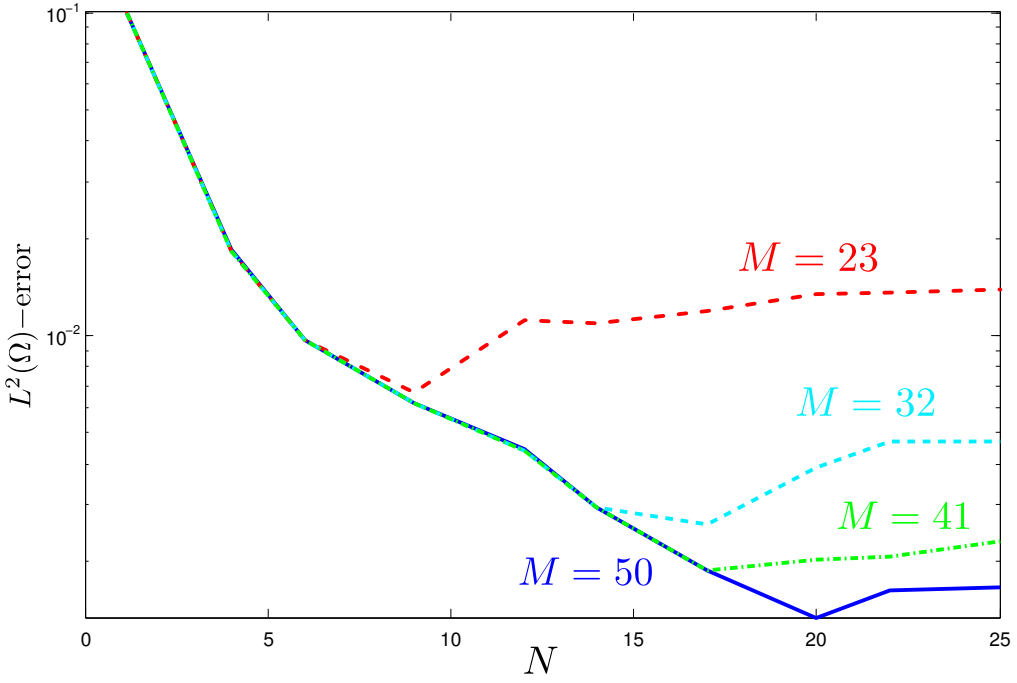

Figure: RB error convergence on 100 test samples for growing basis size N at different CRB dimensions M.

Number of base functions in $\mathcal{W}_H$:	8000
Number of detailed simulations in offline-phase:	16
Time gain factor for online phase:	5

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Summary and Outlook

Summary

- Reduced bases allow model reduction for parametrized evolution problems
- Geometry parametrization is possible for diffeomorphic reference mappings
- Nonlinear problems can be modelled with empirical interpolation
- Reliability through a posteriori estimators in many cases

Outlook

- Improvement of methodical parts (error estimators, RB/EI generation, stability)
- Expansion to nonlinear schemes with implicit operators
- Application to PDE systems

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Proceedings of ALGORITMY 2009, 111–120

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RB scheme

Precondition for applying RB method: RB space $\mathcal{W}_N$ is known and the operator $L^k(\mu; t^k)$ is affine decomposable, i.e it can be written as

$$\mathcal{L}(\mu; t^k)[U] = \sum_{q=1}^{Q_l} L^q[U]\sigma_l^q(\mu), \tag{1}$$

$$\tag{2}$$

where the operators L^q are linear and strictly independent from the parameter μ, and the coefficient functionts $\sigma_{l/E}^q(\mu)$ can be computed with a complexity of $\mathcal{O}(N)$ for a $N \ll H$.

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RB scheme II

The functions $U_N^k(x;\mu) := \sum_{n=1}^N a_n^k(\mu)\varphi_n$ produce for every $\mu \in \mathcal{P}$ a *Reduced Basis solution*, if the coefficient vectors $\mathbf{a}^k := (a_n^k)_{n=1}^N$ satisfy for $k = 1, \dots, K$ and $\mu \in \mathcal{P}$ the equations

$$\frac{1}{\Delta t}(a^k - a^{k-1}) + \sum_{q=1}^{Q_I} \sigma_I^q(\mu)\mathbf{L}_I[a^k] + \sum_{q=1}^{Q_E} \sigma_E^q \mathbf{L}_E[a^{k-1}] = 0 \tag{3}$$

and fullfil an initial condition, where the matrices are given by

$$(\mathbf{L}_I^q)_{m,n} := \int_\Omega L_I^q(\varphi_n)\varphi_m, \qquad (\mathbf{L}_E^q)_{m,n} := \int_\Omega L_E^q(\varphi_n)\varphi_m. \tag{4}$$

Hence $U_H(x,t;\mu) \approx U_N(x,t;\mu) := \sum_{k=0}^{K-1} U_N^k(x;\mu)\chi_{\{t_k \le t < t^{k+1}\}}$

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Generation of the Reduced Basis [offline phase]

- Start with an initial reduced basis $\mathcal{W}_{\tilde{N}}$, which can be e.g. constructed from the initial data function.
- Expand the basis step by step by a further base function
  - Compute $U_{\tilde{N}}(t^k; \mu)$ for all $k = 0, \ldots, K$ and $\mu \in \mathcal{M} \subset \mathcal{P}$
  - Find $\mu_{\max} \in \mathcal{M}$, which maximizes $\|U_H - U_N\|$. Best done with an a posteriori error estimator.
  - Construct the next base function $\varphi_{\tilde{N}+1}$ out of the RB functions $U_H(x, t; \mu_{\max})$.

  until: maximum error drops beneath given bound ε_{tol}.

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