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Reduced Basis Model Reduction of Parametrized Two-Phase Flow in Porous Media

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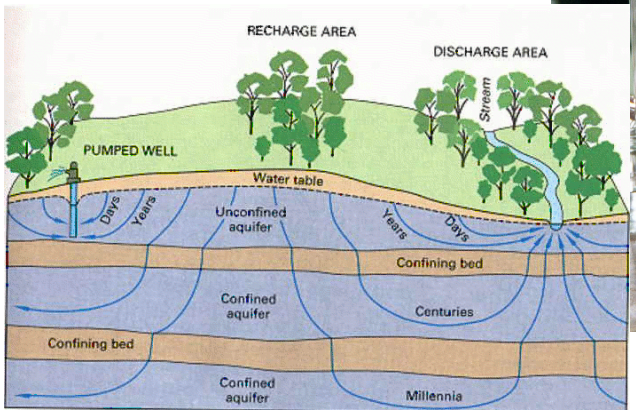
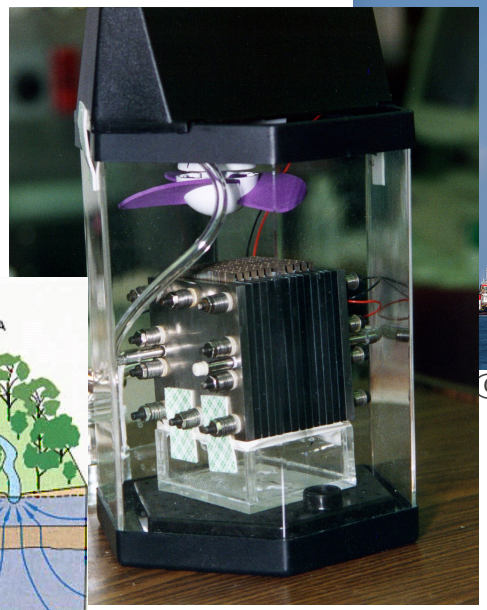
Tow–Phase–Flow: Applications





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- Convection or diffusion dominant problems

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Tow–Phase–Flow

Analytical problem formulation

Given a porous medium $\Omega \in \mathbb{R}^2$, find $(s, \mathbf{u}, \psi) : \Omega \times [0, T] \to \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}$, such that

$$\partial_t s + \nabla \cdot (f(s)\mathbf{u} - \upsilon(s)\nabla s) = q_1 \quad \text{in } \Omega \times [0, T], \tag{1}$$

$$\nabla \cdot (M(s)\nabla\psi) = q_1 + q_2 \quad \text{in } \Omega \times [0, T] \text{ and} \tag{2}$$

$$\mathbf{u} = -M(s)\nabla\psi \quad \text{in } \Omega \times [0, T]. \tag{3}$$

Model magnitudes:

- $f(s)$ fractional flow
- $M(s)$ mobility rate
- $p_c(s)$ capillary pressure
- $\upsilon(s) = k_w(s)f(s)p_c(s)$ diffusion coefficient
- $k_w(s)$ relative permeability of wetting fluid
- q_1, q_2 source terms

Unknowns:

- s saturation
- $\mathbf{u}$ velocity
- ψ global pressure

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Tow–Phase–Flow

Analytical problem formulation (Closed form)

Given a porous medium $\Omega \in \mathbb{R}^2$, find $(s, \mathbf{u}, \psi) : \Omega \times [0, T] \to \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}$, such that

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$$\mathbf{u} = -M(s)\nabla\psi \quad \text{in } \Omega \times [0, T], \tag{3}$$

$$\int_\Omega \psi(x, \cdot)\mathrm{d}x = 0 \quad \text{in } \Omega \times [0, T], \tag{4}$$

$$s(\cdot, 0) = s_0 \quad \text{in } \Omega \times \{0\}, \tag{5}$$

$$\nabla s \cdot \mathbf{n} = 0 \text{ and } \nabla\psi \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega \times [0, T] \tag{6}$$

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Finite volume discretization

Discrete solution

Given two Hilbert spaces $\mathcal{W}_h^{\mathcal{T}}$ and $\mathcal{W}_h^{\mathcal{E}}$ of piecewise constant functions on cells respectively edges of an admissable mesh of Ω, find solutions $u_h^k := (s_h^k, \mathbf{u}_h^k, \psi_h^k) \in \mathcal{W}_h^{\mathcal{T}} \times \mathcal{W}_h^{\mathcal{E}} \times \mathcal{W}_h^{\mathcal{T}}$ for $k = 0, \ldots, K$, such that

$$\left(s_h^0, \mathbf{u}_h^0, \psi_h^0\right) = \left(\mathcal{P}_h^{\mathcal{T}}[s_0], 0, 0\right), \tag{7}$$

$$\mathcal{L}_h\left[u_h^k\right] := \begin{pmatrix} -\mathcal{L}_h^s\left[s_h^{k+1}, \mathbf{u}_h^{k+1}\right] \\ \mathcal{L}_h^{\mathbf{u}}\left[s_h^{k+1}, \psi_h^{k+1}\right] \\ \mathcal{L}_h^{\psi}\left[\mathbf{u}_h^{k+1}\right] \end{pmatrix} = \begin{pmatrix} \frac{1}{\Delta t}\left(s_h^k - s_h^{k+1}\right) + \mathcal{P}_h^{\mathcal{T}}\left[q_1\right] \\ \mathbf{u}_h^{k+1} \\ \mathcal{P}_h^{\mathcal{T}}\left[q_1 + q_2\right] \end{pmatrix} =: \mathcal{R}_h[u_h^k] \tag{8}$$

$$\int_\Omega p_h^{k+1} = 0 \tag{9}$$

with Galerkin projection operator $\mathcal{P}_h^{\mathcal{T}} : L^2(\Omega) \to \mathcal{W}_h^{\mathcal{T}}$ and non–linear finite volume operators.

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Finite volume operators

Saturation operator ($\mathcal{L}_h^s$)

On each mesh cell $e_i \in \mathcal{T}$, the saturation operator is defined by

$$\left(\mathcal{L}_h^s\left[s_h, \mathbf{u}_h\right]\right)_i = \sum_{\sigma \in \mathcal{E}(e_i)} g_\sigma(\mathbf{u}_h, s_h) - \sum_{e_j \in \mathcal{N}(e_i)} \{v(s_h)\}_{ij} \frac{m(\sigma_{ij})}{d_{ij}} (s_{h,j} - s_{h,i}) \tag{10}$$

where $\{\cdot\}_{ij}$ is the harmonic mean on the edge σ_{ij} and $g_\sigma : \mathcal{W}_h^{\mathcal{E}} \times \mathcal{W}_h^{\mathcal{T}} \to \mathbb{R}$ is an upwind finite volume flux

$$g_{\sigma_{ij}}(\mathbf{u}_h, s_h) := \begin{cases} \mathbf{u}_{h,\sigma_{ij}} f(s_{h,i}) & \text{if } \mathbf{u}_{h,\sigma_{ij}} > 0 \\ \mathbf{u}_{h,\sigma_{ij}} f(s_{h,j}) & \text{if } \mathbf{u}_{h,\sigma_{ij}} \leq 0. \end{cases} \tag{11}$$

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Finite Volume operators

Velocity operator ($\mathcal{L}_h^{\mathbf{u}}$)

On each edge $\sigma_{ij} \in \mathcal{E}$, the operator is defined by

$$\left(\mathcal{L}_h^{\mathbf{u}}\left[s_h, \psi_h\right]\right)_{\sigma_{ij}} = \{M(s_h)\}_{\sigma_{ij}} \frac{m(\sigma_{ij})}{d_{ij}}(\psi_{h,j} - \psi_{h,i}). \tag{12}$$

Pressure operator ($\mathcal{L}_h^{\psi}$)

$$\left(\mathcal{L}_h^{\psi}\left[\mathbf{u}_h\right]\right)_i = \sum_{\sigma \in \mathcal{E}(e_i)} \mathbf{u}_{h,\sigma}. \tag{13}$$

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Newton scheme

The non–linear equation system is resolved by the Newton–Raphson method resulting in finding defects $\delta^{k+1,\nu+1}$ by solving

$$\mathbf{D}\mathcal{L}_h\big|_{u_h^{k+1,\nu}} \left[\delta^{k+1,\nu+1}\right] = -\mathcal{L}_h\left[u_h^{k+1,\nu}\right] + \mathcal{R}_h\left[u_h^k + 1\right], \tag{14}$$

with

- Newton updates $u_h^{k+1,0} := u^k$ and $u_h^{k+1,\nu+1} := u_h^{k+1,\nu} + \delta^{k+1,\nu+1}$,
- and solutions $u_h^{k+1} := u_h^{k+1,\nu_{\max}(k)}$ at the final Newton steps $\nu_{\max}(k)$

for $k = 0, \ldots, K-1$.

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Empirical interpolation: What is linear interpolation?

- Interpolation operator $\mathcal{I} : \mathcal{X} \to \mathcal{X}^{\mathcal{I}}$ mapping from arbitrary function space $\mathcal{X} \subset \{f : \mathbb{R}^d \to \mathbb{R}\}$ onto subset $\mathcal{X}^{\mathcal{I}} \subset \mathcal{X}$
- Interpolation points $T^{\mathcal{I}} \subset \mathbb{R}^d$ s.t. $\mathcal{I}[f](x) = f(x)$ for all $x \in T^{\mathcal{I}}$

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Example: polynomial interpolation

- $\mathcal{X} := L^\infty([0,1])$
- $\mathcal{X}^{\mathcal{I}_p} := \Pi_p$
- $T^{\mathcal{I}_p}$ e.g. Chebyshev points.

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Empirical interpolation[Barrault et al, 2004]

Empirical interpolation

- $\mathcal{X} := \{f(\boldsymbol{\mu}); \boldsymbol{\mu} \in \mathcal{M}\} \subset L^\infty(\Omega)$
- $\mathcal{X}^{\mathcal{I}_M} := \text{span}\{q_m\}_{m=1}^M$ "empirically" determined.
- $T^{\mathcal{I}_M}$ "empirically" determined.

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- $\mathcal{X}^{\mathcal{I}_M} := \text{span}\{q_m\}_{m=1}^M$ "empirically" determined.
- $T^{\mathcal{I}_M}$ "empirically" determined.

- Lebesgue constant assures: $\|\mathcal{I}_M[f] - f\|_{L^\infty} \leq (\Lambda_M + 1)\left\|f_M^* - f\right\|_{L^\infty}$
- $\Lambda_M \leq 2^M - 1$

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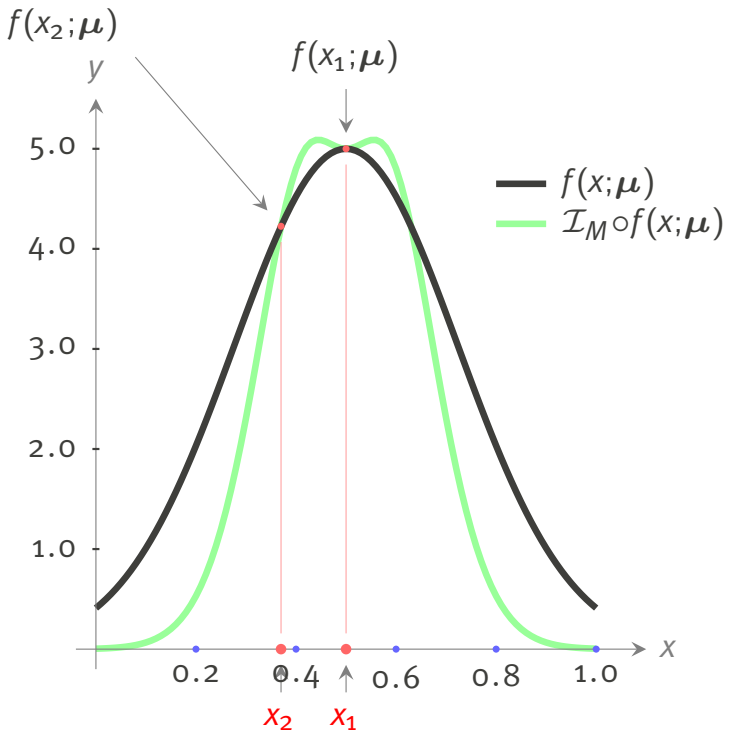
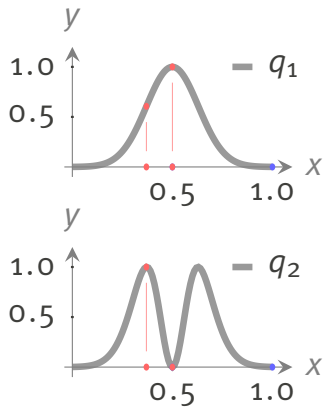
Empirical interpolation[Barrault et al, 2004]

Empirical interpolation for parametrized functions $f(\boldsymbol{\mu}) : \mathbb{R} \to \mathbb{R}$.



Base functions:



- "magic points" $\{x_m\}_{m=1}^{M}$
- basis functions $\{q_m\}_{m=1}^{M}$

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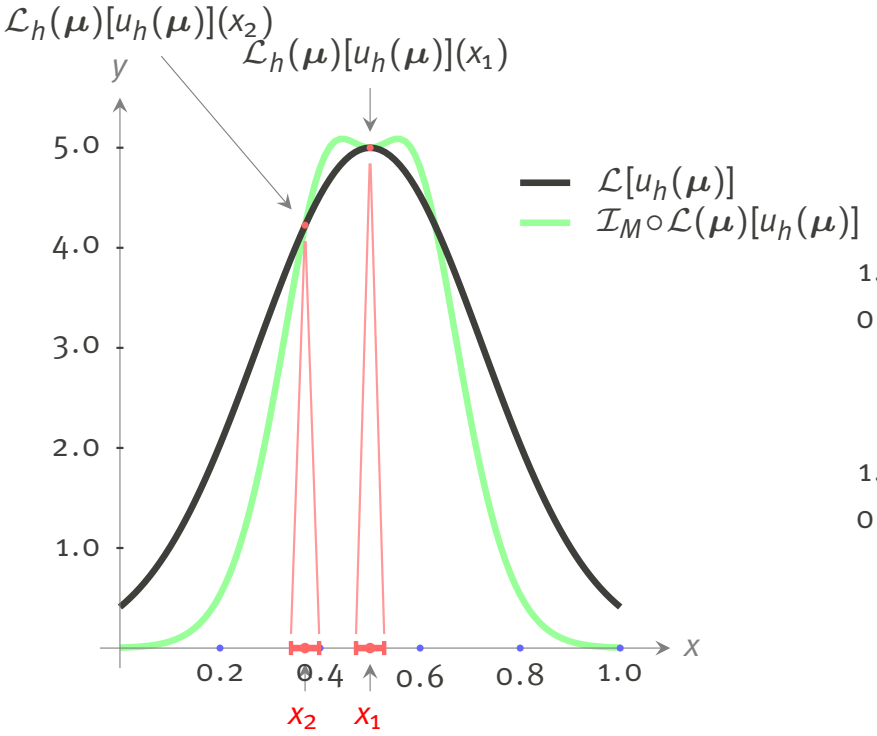
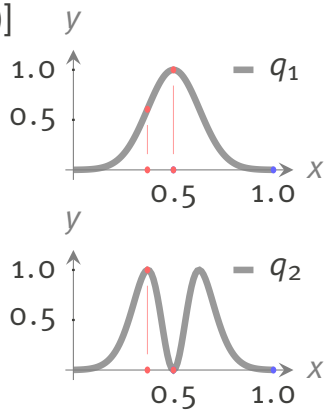
Empirical operator interpolation[DHO11]

Empirical interpolation for parametrized discrete operators $\mathcal{L}_h \in \mathcal{W}_h$.



Base functions:



- "magic points" $\{x_m\}_{m=1}^{M}$
- basis functions $\{q_m\}_{m=1}^{M}$

Discrete operators need to have "H-independent Dof dependence".

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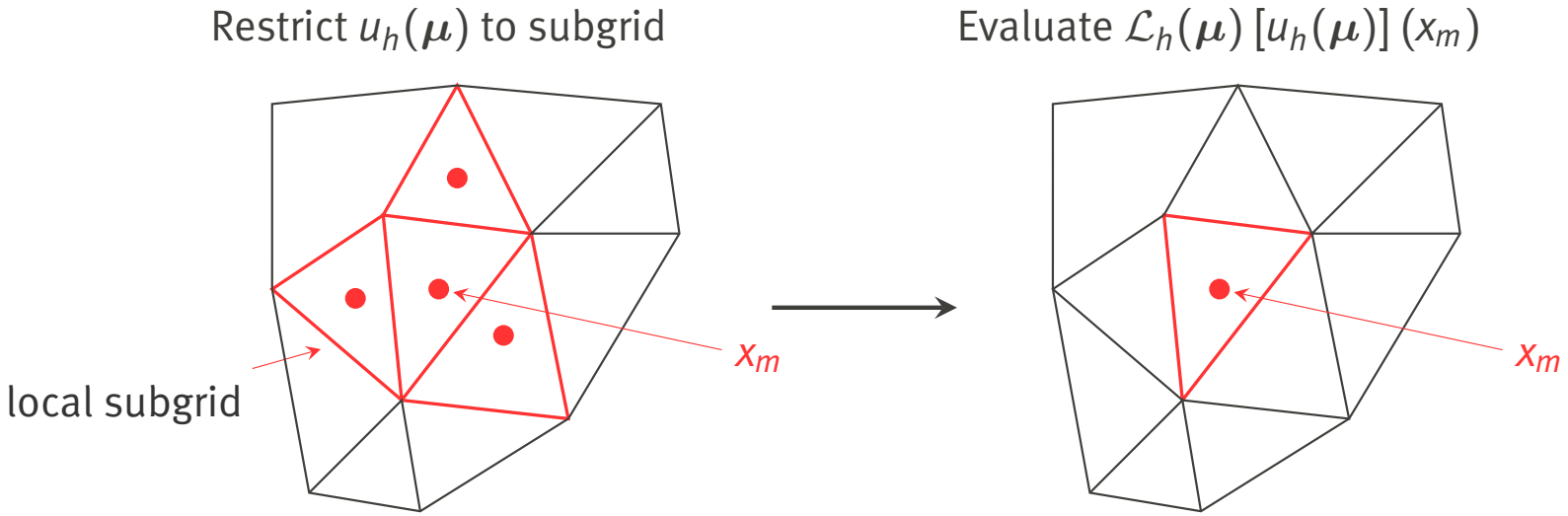
Empirical operator interpolation: Subgrids



Efficient evaluations

The operator evaluations in interpolation points $\mathcal{L}_h(\mu)[\cdot](x_m)$ can be computed efficiently during online phase, if

- the operator has a localized structure (small stencil) and
- the local geometry information is precomputed during offline phase.

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Empirical operator interpolation: Details

General operator approximation

Interpolate operator evaluations at "magic points":

parameter dependent

parameter independent

$$\mathcal{L}_h(\boldsymbol{\mu})\left[u_h^k(\boldsymbol{\mu})\right](x_m) = (\mathcal{I}_M \circ \mathcal{L}_h(\boldsymbol{\mu}))\left[u_h^k(\boldsymbol{\mu})\right](x_m) = \sum_{m=1}^{M} \sigma_m(\boldsymbol{\mu})\; q_m(x_m).$$

Empirical interpolation

- Basis functions q_m are directly computed from operator evaluations
- for selected parameters $\boldsymbol{\mu}_m$ and time steps k_m.
- They are computed by a greedy search algorithm and span a collateral reduced basis space $\mathcal{W}_M \subset \mathcal{W}_h$.

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Model reduction I: Empirical operator interpolation

Substitute $\mathcal{L}_h^s$ and $\mathcal{L}_h^{\mathbf{u}}$ by their empirical operator interpolants $\mathcal{I}_{M^s}\left[\mathcal{L}_h^s\right]$ and $\mathcal{I}_{M^{\mathbf{u}}}\left[\mathcal{L}_h^{\mathbf{u}}\right]$.

Discrete solution (empirical interpolation)

Find solutions $u_h^k := (s_h^k, \mathbf{u}_h^k, \psi_h^k) \in \mathcal{W}_h^{\mathcal{T}} \times \mathcal{W}_h^{\mathcal{E}} \times \mathcal{W}_h^{\mathcal{T}}$ for $k = 0, \dots, K$, such that

$$\left(s_h^0, \mathbf{u}_h^0, \psi_h^0\right) = \left(\mathcal{P}_h^{\mathcal{T}}[s_0], 0, 0\right), \tag{15}$$

$$\begin{pmatrix} -\mathcal{I}_{M^s}\left[\mathcal{L}_h^s\right]\left[s_h^{k+1}, \mathbf{u}_h^{k+1}\right] \\ \mathcal{I}_{M^{\mathbf{u}}}\left[\mathcal{L}_h^{\mathbf{u}}\right]\left[s_h^{k+1}, \psi_h^{k+1}\right] \\ \mathcal{L}_h^{\psi}\left[\mathbf{u}_h^{k+1}\right] \end{pmatrix} = \begin{pmatrix} \frac{1}{\Delta t}\left(s_h^k - s_h^{k+1}\right) + \mathcal{P}_h^{\mathcal{T}}\left[q_1\right] \\ \mathbf{u}_h^{k+1} \\ \mathcal{P}_h^{\mathcal{T}}\left[q_1 + q_2\right] \end{pmatrix} \tag{16}$$

$$\int_\Omega p_h^{k+1} = 0. \tag{17}$$

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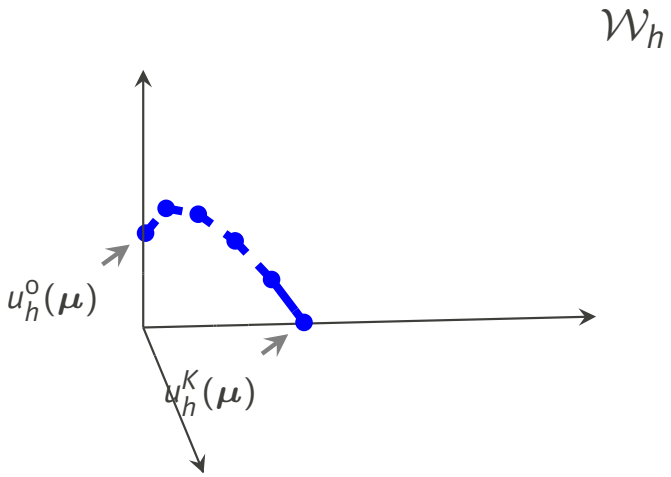
Model reduction II: Reduced basis method

Parametrized PDE

Find
$u : [0, T_{\max}] \to \mathcal{W} \subset L^2(\Omega)$, s.t.

$$\begin{aligned} u(0) &= u_0 \\ \partial_t u(t) - \mathcal{L}[u(t)] &= 0 \end{aligned}$$

plus boundary conditions.



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Model reduction II: Reduced basis method

Parametrized PDE

Find, for $\boldsymbol{\mu} \in \mathcal{M} \subset \mathbb{R}^p$,
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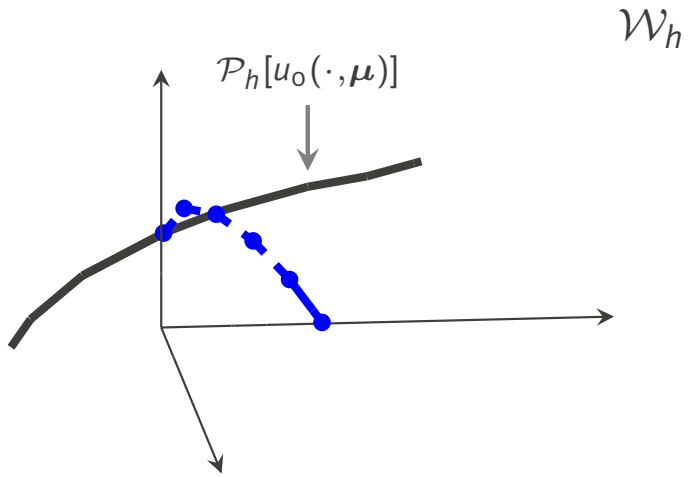

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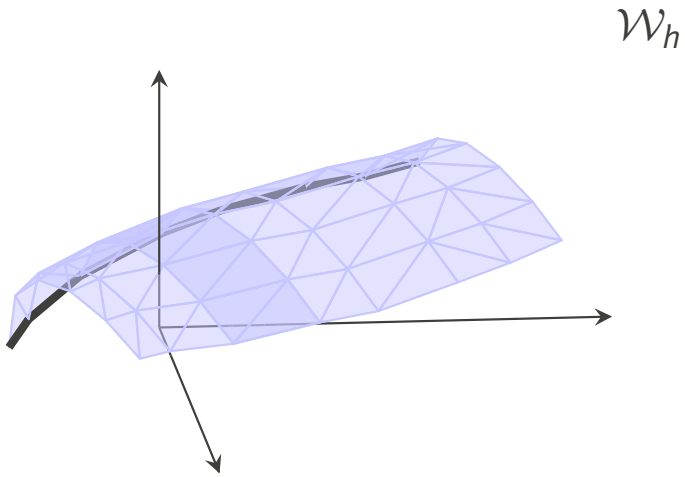

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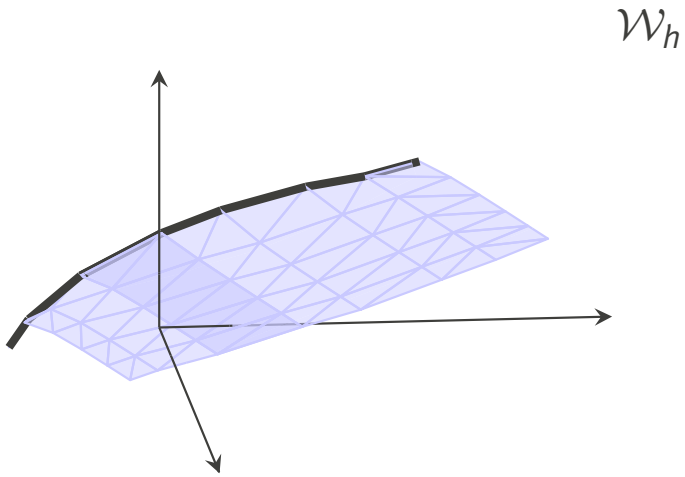

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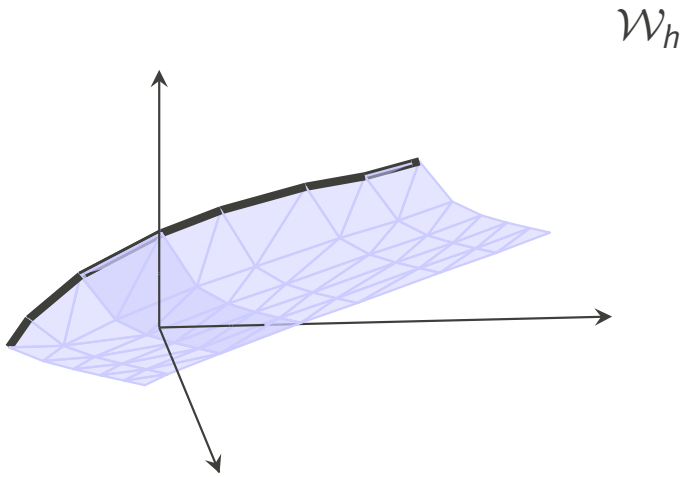

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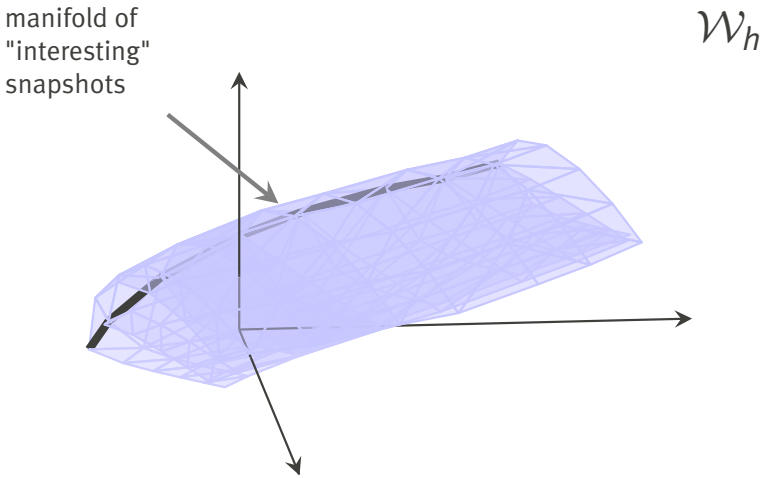

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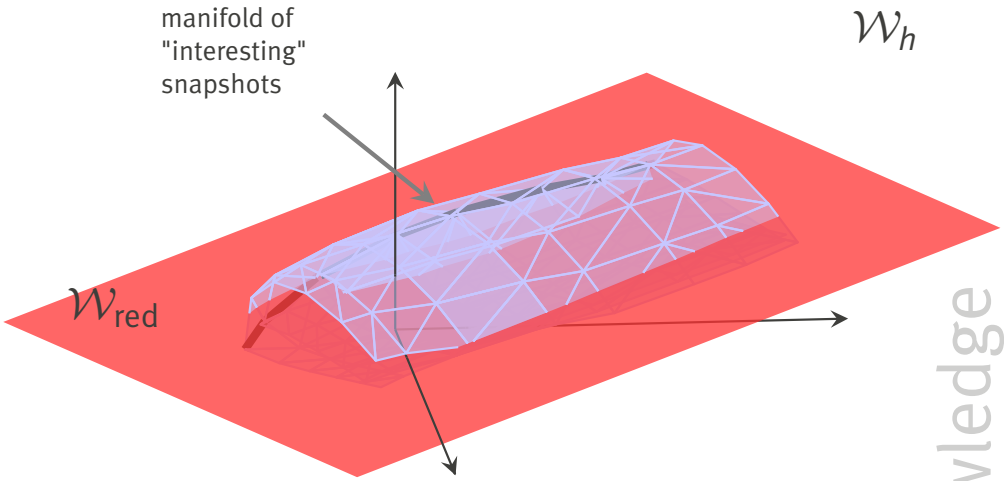

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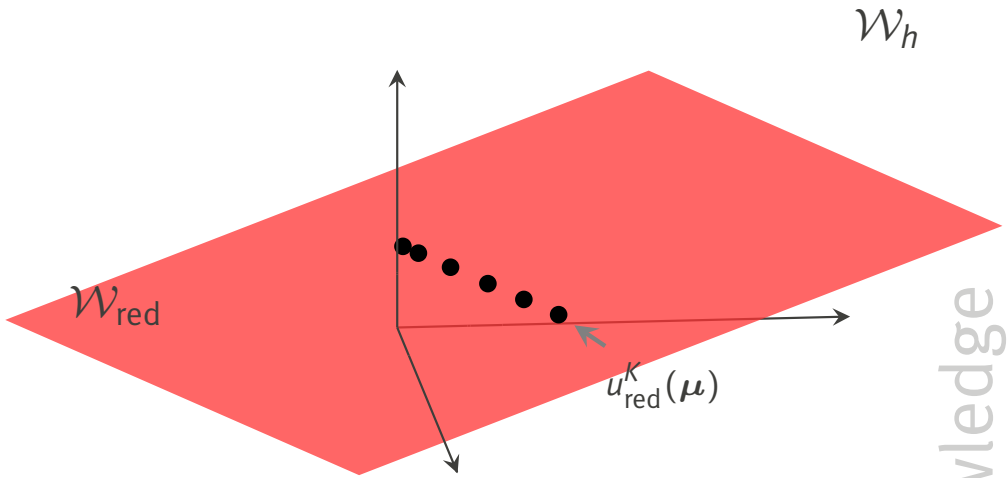

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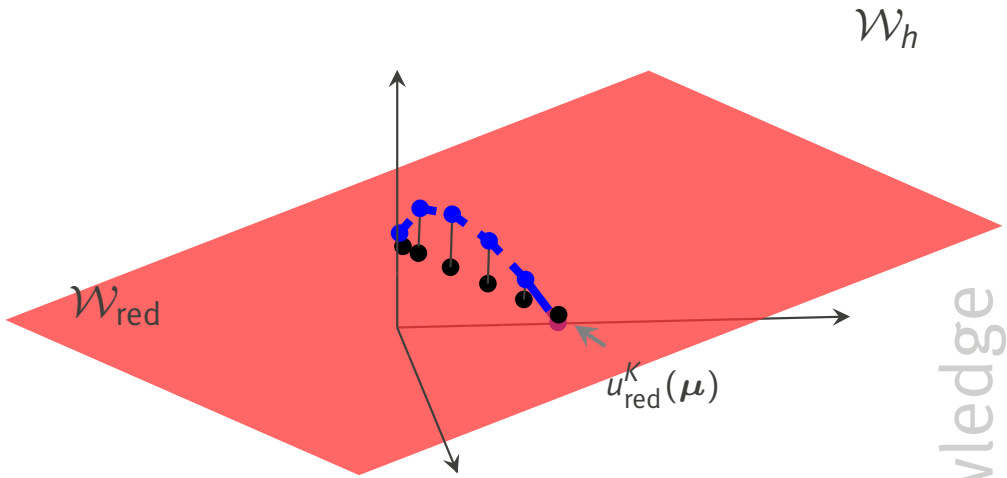

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Reduced basis scheme

Reduced solution

Given reduced Hilbert spaces $\mathcal{W}^s_{\text{red}}, \mathcal{W}^\psi_{\text{red}} \subset \mathcal{W}^\mathcal{T}_h$ and $\mathcal{W}^\mathbf{u}_{\text{red}} \subset \mathcal{W}^\mathcal{E}_h$ find solutions $u^k_{\text{red}} := (s^k_{\text{red}}, \mathbf{u}^k_{\text{red}}, \psi^k_{\text{red}}) \in \mathcal{W}^s_{\text{red}} \times \mathcal{W}^\mathbf{u}_{\text{red}} \times \mathcal{W}^\psi_{\text{red}}$ for $k = 0, \dots, K$, such that

$$\left(s^0_{\text{red}}, \mathbf{u}^0_{\text{red}}, \psi^0_{\text{red}}\right) = \left(\mathcal{P}^s_{\text{red}} \circ \mathcal{P}^\mathcal{T}_h[s_0], 0, 0\right), \tag{18}$$

$$\begin{pmatrix} -\mathcal{P}^s_{\text{red}} \circ \mathcal{L}^s_h\left[s^{k+1}_h, \mathbf{u}^{k+1}_h\right] \\ \mathcal{P}^\mathbf{u}_{\text{red}} \circ \mathcal{L}^\mathbf{u}_h\left[s^{k+1}_h, \psi^{k+1}_h\right] \\ \mathcal{P}^\psi_{\text{red}} \circ \mathcal{L}^\psi_h\left[\mathbf{u}^{k+1}_h\right] \end{pmatrix} = \begin{pmatrix} \mathcal{P}^s_{\text{red}}\left[\frac{1}{\Delta t}\left(s^k_h - s^{k+1}_h\right) + \mathcal{P}^\mathcal{T}_h[q_1]\right] \\ \mathcal{P}^\mathbf{u}_{\text{red}}\left[\mathbf{u}^{k+1}_h\right] \\ \mathcal{P}^\psi_{\text{red}} \circ \mathcal{P}^\mathcal{T}_h[q_1 + q_2] \end{pmatrix} \tag{19}$$

with Galerkin projection operators $\mathcal{P}^s_{\text{red}} : \mathcal{W}^\mathcal{T}_h \to \mathcal{W}^s_{\text{red}}, \mathcal{P}^\mathbf{u}_{\text{red}} : \mathcal{W}^\mathcal{E}_h \to \mathcal{W}^\mathbf{u}_{\text{red}}$ and $\mathcal{P}^\psi_{\text{red}} : \mathcal{W}^\mathcal{T}_h \to \mathcal{W}^\psi_{\text{red}}$.

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Reduced basis scheme

Reduced solution

Given reduced Hilbert spaces $\mathcal{W}^s_{\text{red}}, \mathcal{W}^\psi_{\text{red}} \subset \mathcal{W}^\mathcal{T}_h$ and $\mathcal{W}^\mathbf{u}_{\text{red}} \subset \mathcal{W}^\mathcal{E}_h$ find solutions $u^k_{\text{red}} := (s^k_{\text{red}}, \mathbf{u}^k_{\text{red}}, \psi^k_{\text{red}}) \in \mathcal{W}^s_{\text{red}} \times \mathcal{W}^\mathbf{u}_{\text{red}} \times \mathcal{W}^\psi_{\text{red}}$ for $k = 0, \dots, K$, such that

$$\left(s^0_{\text{red}}, \mathbf{u}^0_{\text{red}}, \psi^0_{\text{red}}\right) = \left(\mathcal{P}^s_{\text{red}} \circ \mathcal{P}^\mathcal{T}_h[s_0], 0, 0\right), \tag{18}$$

$$\begin{pmatrix} -\mathcal{P}^s_{\text{red}} \circ \mathcal{L}^s_h\left[s^{k+1}_h, \mathbf{u}^{k+1}_h\right] \\ \mathcal{P}^\mathbf{u}_{\text{red}} \circ \mathcal{L}^\mathbf{u}_h\left[s^{k+1}_h, \psi^{k+1}_h\right] \\ \mathcal{P}^\psi_{\text{red}} \circ \mathcal{L}^\psi_h\left[\mathbf{u}^{k+1}_h\right] \end{pmatrix} = \begin{pmatrix} \mathcal{P}^s_{\text{red}}\left[\frac{1}{\Delta t}\left(s^k_h - s^{k+1}_h\right) + \mathcal{P}^\mathcal{T}_h[q_1]\right] \\ \mathcal{P}^\mathbf{u}_{\text{red}}\left[\mathbf{u}^{k+1}_h\right] \\ \mathcal{P}^\psi_{\text{red}} \circ \mathcal{P}^\mathcal{T}_h[q_1 + q_2] \end{pmatrix} \tag{19}$$

with Galerkin projection operators $\mathcal{P}^s_{\text{red}} : \mathcal{W}^\mathcal{T}_h \to \mathcal{W}^s_{\text{red}}$, $\mathcal{P}^\mathbf{u}_{\text{red}} : \mathcal{W}^\mathcal{E}_h \to \mathcal{W}^\mathbf{u}_{\text{red}}$ and $\mathcal{P}^\psi_{\text{red}} : \mathcal{W}^\mathcal{T}_h \to \mathcal{W}^\psi_{\text{red}}$.
No parametrization! Reduced spaces obtained via POD of trajectories

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Offline/online decomposition

We introduce the DOF vectors

$$\mathbf{a}^k := \left(a_n^k\right)_{n=1}^{N_s}, \mathbf{b}^k := \left(b_n^k\right)_{n=1}^{N_\mathbf{u}} \text{ and } \mathbf{c}^k := \left(c_n^k\right)_{n=1}^{N_\psi},$$

such that $s_{\text{red}}^k = \sum_{n=1}^{N_s} a_n^k \varphi_n^s$, $\mathbf{u}_{\text{red}}^k = \sum_{n=1}^{N_\mathbf{u}} b_n^k \varphi_n^\mathbf{u}$ and $\psi_{\text{red}}^k = \sum_{n=1}^{N_\psi} c_n^k \varphi_n^\psi$.

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Offline/online decomposition

Reduced solution

Find solutions $(\mathbf{a}^k, \mathbf{b}^k, \mathbf{c}^k)$ such that

$$\frac{1}{\Delta t}\left(a_n^{k+1} - a_n^k\right) - \sum_{m=1}^{M_s} \mathcal{L}_h[s_{\text{red}}^{k+1}, \psi_{\text{red}}^{k+1}](x_m^s) \int_\Omega \xi_m^s \varphi_n^s - \int_\Omega q_1 \varphi_n^s = 0 \qquad \text{for } n = 1, \dots, N_s,$$

$$\sum_{m=1}^{M_{\mathbf{u}}} \mathcal{L}_{\text{red}}\left[s_{\text{red}}^{k+1}, \psi_{\text{red}}^{k+1}\right](x_m^{\mathbf{u}}) \int_\Omega \xi_m^{\mathbf{u}} \varphi_{\hat{n}}^{\mathbf{u}} - b_{\hat{n}}^{k+1} = 0 \qquad \text{for } n = 1, \dots, N_{\mathbf{u}},$$

$$\sum_{i=1}^{N_u} b_i^{k+1} \langle \varphi_i^{\mathbf{u}}, [\varphi_n^{\psi}] \rangle_{\mathcal{W}_h^{\mathbf{u}}} - \int_\Omega (q_1 + q_2) \varphi_n^{\psi} = 0 \qquad \text{for } n = 1, \dots, N_\psi.$$

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Offline/online decomposition

Reduced solution

Find solutions $(\mathbf{a}^k, \mathbf{b}^k, \mathbf{c}^k)$ such that

$$\frac{1}{\Delta t}\left(a_n^{k+1} - a_n^k\right) - \sum_{m=1}^{M_s} \mathcal{L}_h[s_{\text{red}}^{k+1}, \psi_{\text{red}}^{k+1}](x_m^s) \int_\Omega \xi_m^s \varphi_n^s - \int_\Omega q_1 \varphi_n^s = 0 \qquad \text{for } n = 1, \dots, N_s,$$

$$\sum_{m=1}^{M_{\mathbf{u}}} \mathcal{L}_{\text{red}}\left[s_{\text{red}}^{k+1}, \psi_{\text{red}}^{k+1}\right](x_m^{\mathbf{u}}) \int_\Omega \xi_m^{\mathbf{u}} \varphi_{\hat{n}}^{\mathbf{u}} - b_{\hat{n}}^{k+1} = 0 \qquad \text{for } n = 1, \dots, N_{\mathbf{u}},$$

$$\sum_{i=1}^{N_u} b_i^{k+1} \langle \varphi_i^{\mathbf{u}}, [\varphi_n^{\psi}] \rangle_{\mathcal{W}_h^{\mathbf{u}}} - \int_\Omega (q_1 + q_2) \varphi_n^{\psi} = 0 \qquad \text{for } n = 1, \dots, N_\psi.$$

High–dimensional offline computations

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Numerical results: visualization

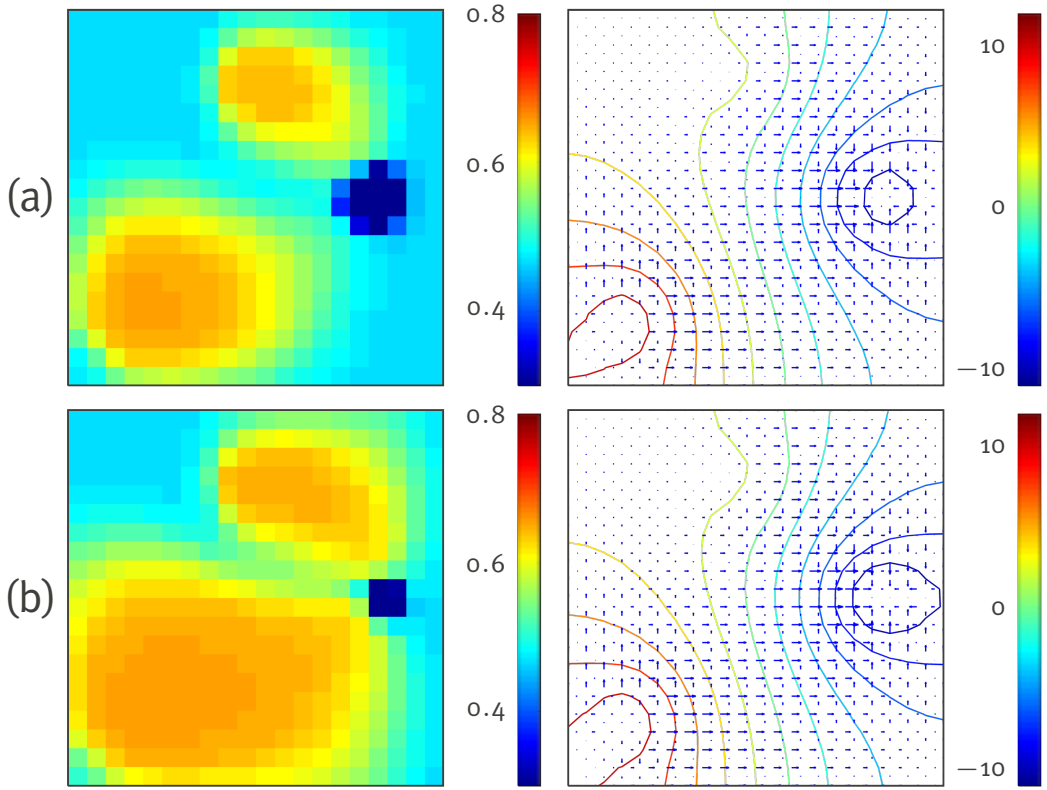

Figure: Illustration of saturation concentration, and contour plot of pressure field with velocity flux at time instances (a) $t = 0.25$ and (b) $t = 0.5$. The snapshots are reconstructed from a reduced simulation.

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Numerical results: Empirical operator interpolation

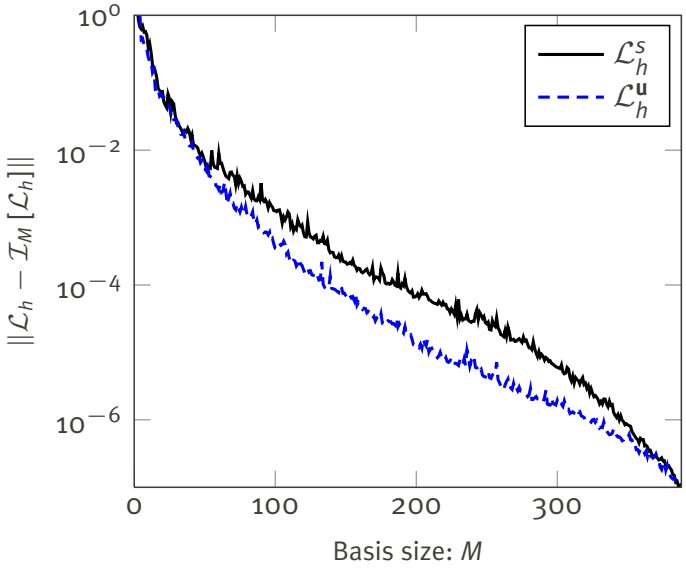

Figure: Illustration of error convergence of EI-GREEDY algorithm for operators $\mathcal{L}_h^s$ and $\mathcal{L}_h^{\mathbf{u}}$.

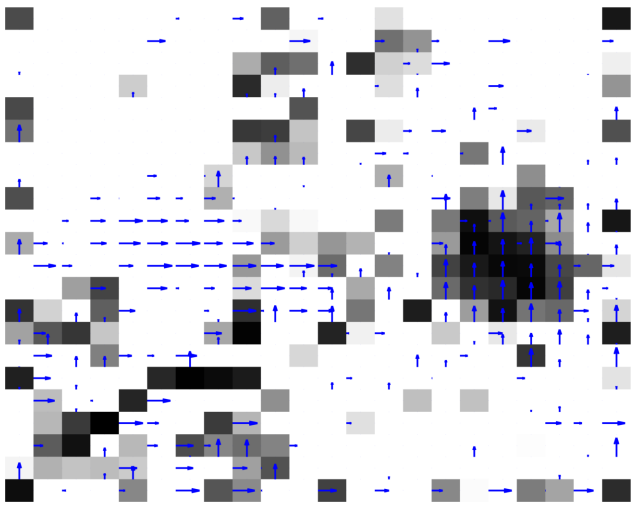

Figure: Illustration of selected interpolation DOFs for operators $\mathcal{L}_h^s$ (control volumes) and $\mathcal{L}_h^{\mathbf{u}}$ (fluxes over edges). Darker shades of marked control volumes and longer arrows represent earlier selected DOFs.

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Numerical results: error and run–times

Detailed simulation time: $\approx$ **52 s**

Table: Error and timings of reduced simulations with different basis sizes.

$(N_s, N_{\mathbf{u}}, N_\psi)$	$(M_s, M_{\mathbf{u}})$	$\|s_h - s_{\text{red}}\|$	$\|\psi_h - \psi_{\text{red}}\|$	time
(28,72,34)	(387,386)	$6.2 \cdot 10^{-5}$	$4.11 \cdot 10^{-4}$	30.15
(28,72,34)	(75,75)	$1.03 \cdot 10^{-4}$	$2.11 \cdot 10^{-3}$	21.56
(28,72,34)	(75,125)	$7.59 \cdot 10^{-5}$	$8.69 \cdot 10^{-4}$	20.61
(23,58,28)	(75,125)	$2.47 \cdot 10^{-4}$	$2.55 \cdot 10^{-3}$	18.24

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Example II: Nonlinear Diffusion

Nonlinear Diffusion

Problem definition:

$$\partial_t u - m\Delta u^p = 0 \quad \text{in } \Omega \times [0,1], \qquad u(\cdot,0) = c_0 + u_0 \quad \text{on } \Omega \times \{0\}$$

with (implicit) finite volume discretization with Engquist–Osher flux

- $\Omega = [0,1]^2$ with homogeneous boundary conditions
- rectangular 100x100 grid with $K = 80$ time steps.
- Parametrization $\boldsymbol{\mu} = (p, m, c_0) \in [1,5] \times [0, 0.01] \times [0, 0.2]$

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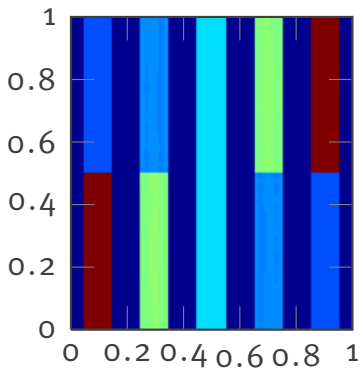
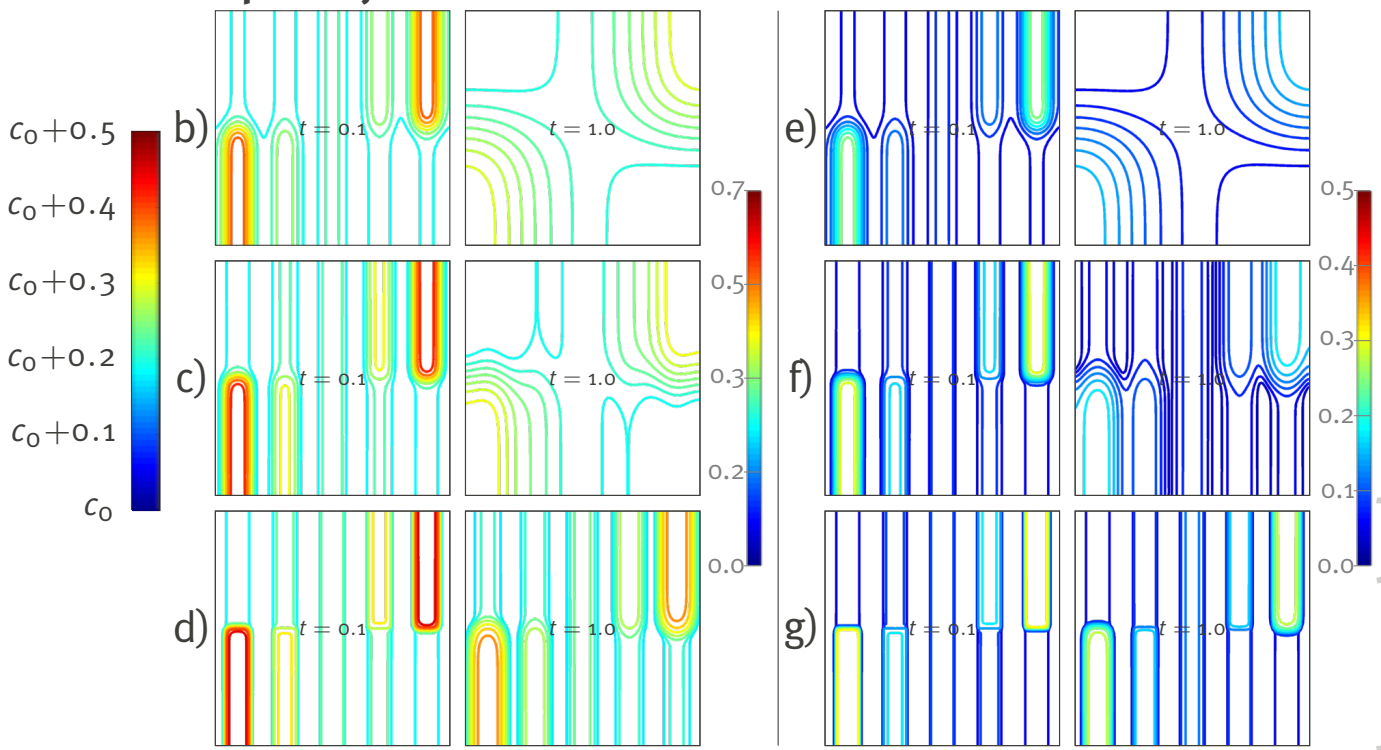
Example II: Solution snapshots

Initial data:



Sample trajectories:



a) $\boldsymbol{\mu} = (1, 0.01, 0.2)$, b) $\boldsymbol{\mu} = (1, 0.01, 0.0)$,
c) $\boldsymbol{\mu} = (2, 0.01, 0.2)$, d) $\boldsymbol{\mu} = (2, 0.01, 0.0)$,
e) $\boldsymbol{\mu} = (4, 0.01, 0.2)$, f) $\boldsymbol{\mu} = (4, 0.01, 0.0)$

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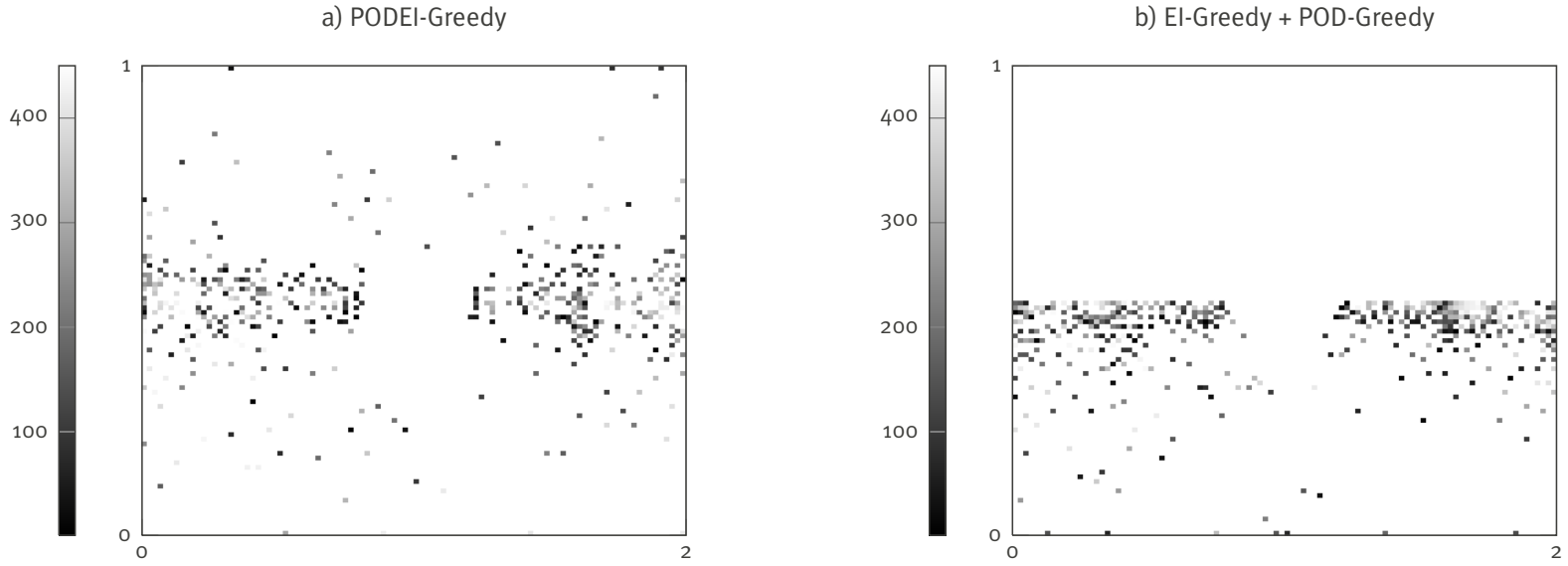
Example II: Empirical interpolation of $\mathcal{L}_h I$



Subgrid at interpolation points

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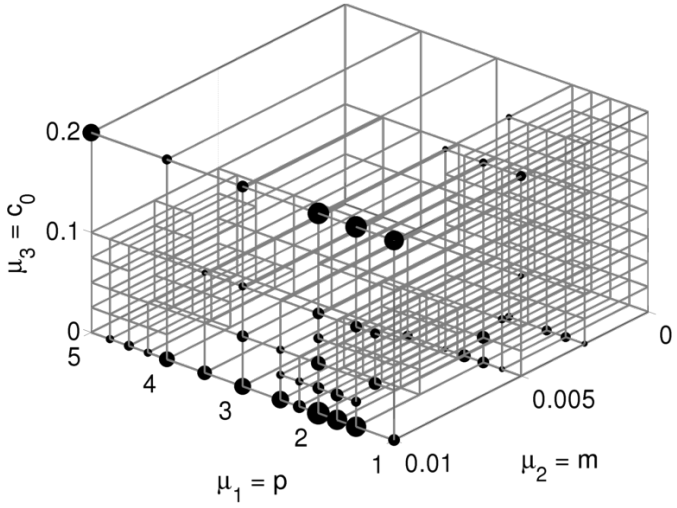
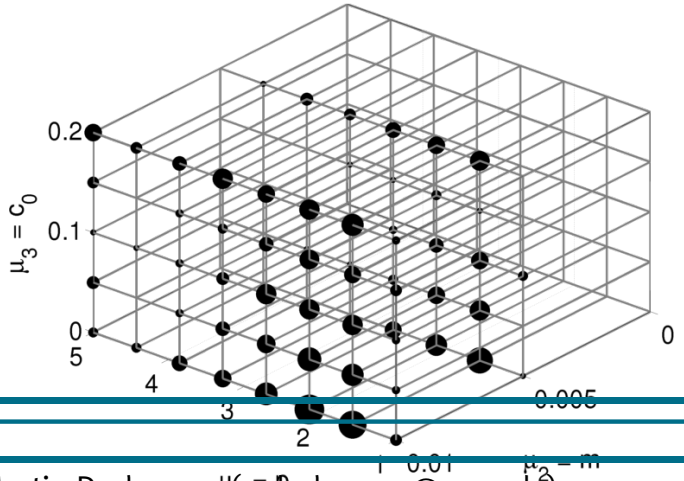
Example II: Empirical interpolation of $\mathcal{L}_h I$

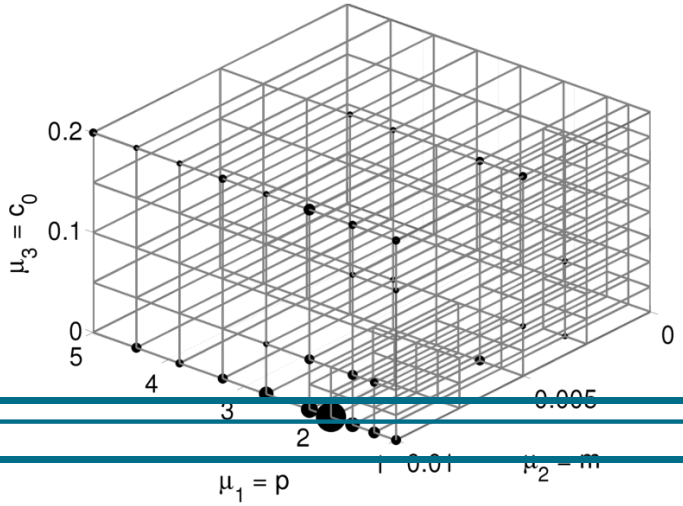
(a) Parameter selection PODEI-Greedy



(b) Parameter selection EI-Greedy



(c) Parameter selection POD-Greedy



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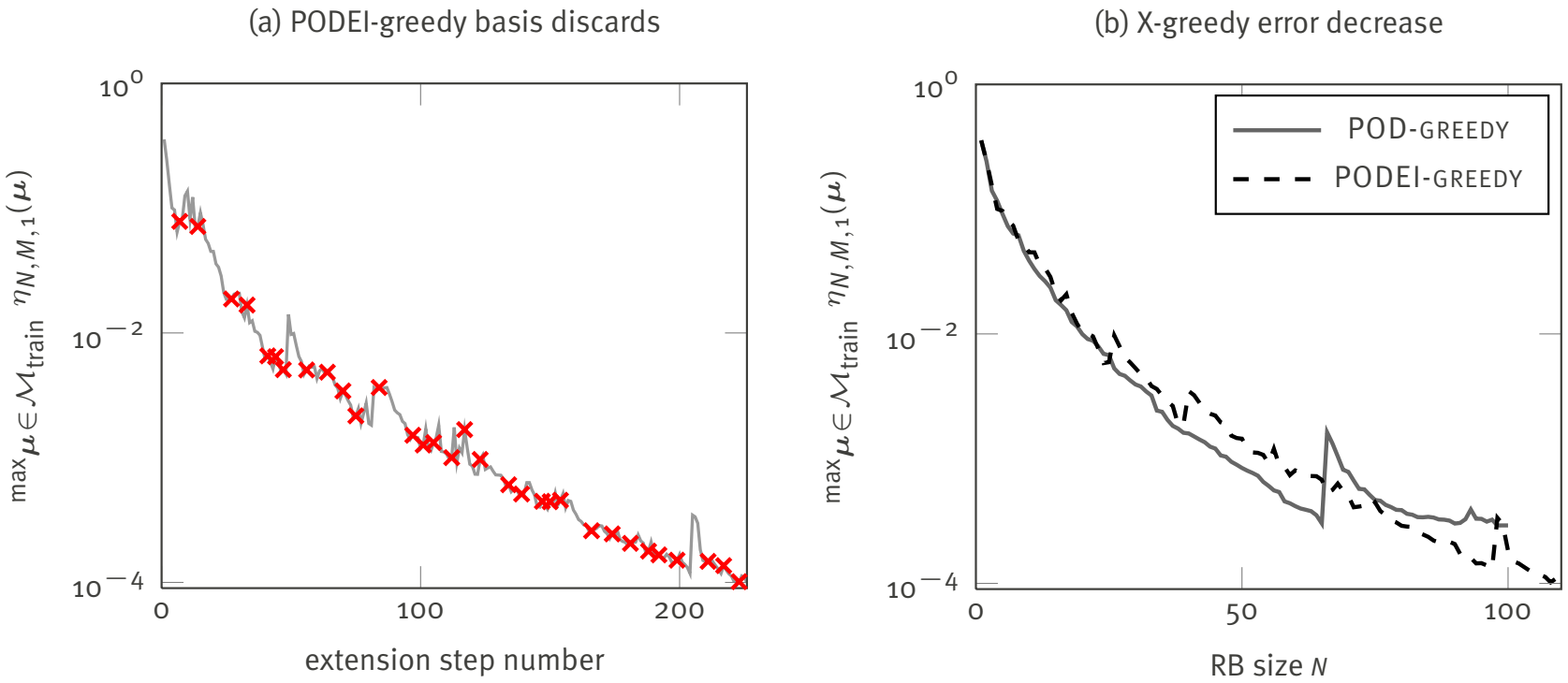
Example II: Greedy error convergence



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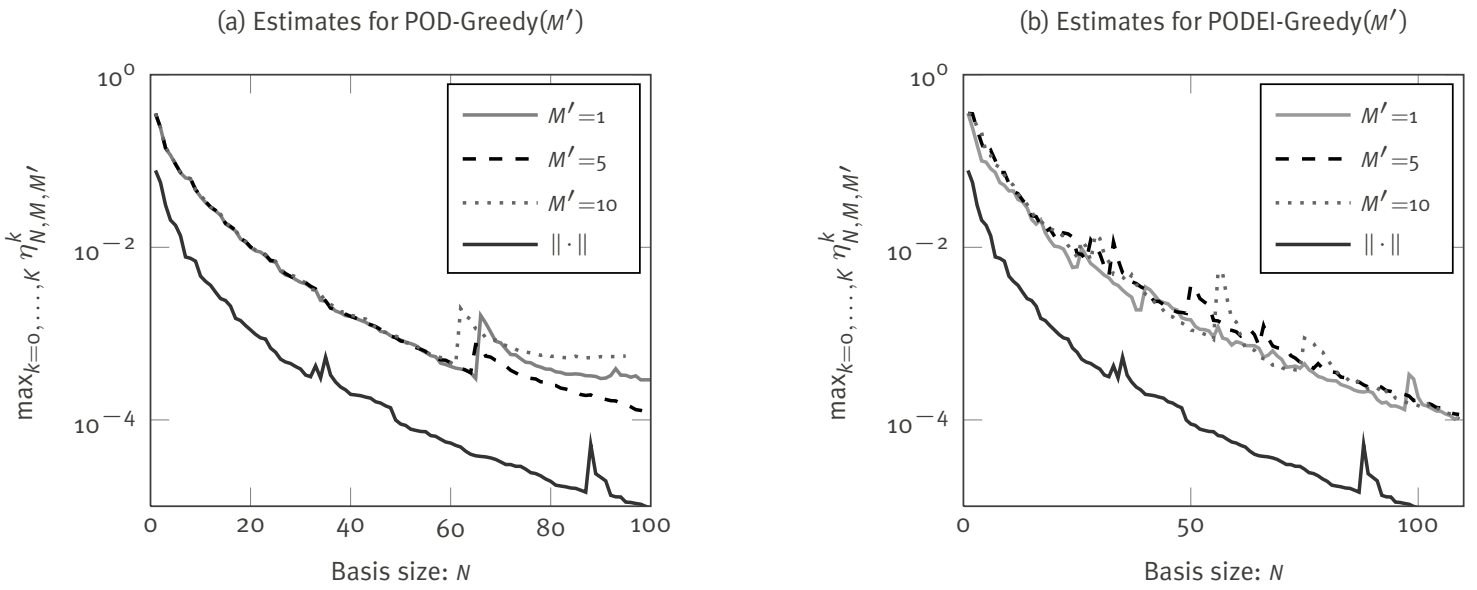
Example II: Greedy error convergence



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Example II: Numerical results

- $\dim(\mathcal{W}_h)$ 10000
- $\nu_{\max} \approx 1 - 20$
- $\#M_{\text{train},0}$ 27.
- $\#M_{\text{train}}$ 305

N	M	ø-runtime[s]	max. error	ø-offline time[h]
0	0	55.38	0.00	0
17	71	1.57	$3.56 \cdot 10^{-3}$	1.35
33	142	1.95	$8.33 \cdot 10^{-4}$	1.67
50	213	2.51	$2.08 \cdot 10^{-4}$	2.07
66	283	3.19	$5.88 \cdot 10^{-5}$	2.43
83	354	4.07	$5.55 \cdot 10^{-5}$	2.88
99	425	5.3	$4.06 \cdot 10^{-5}$	3.3

N	M	ø-runtime[s]	max. error	ø-offline time[h]
0	−1	55.38	0.00	0
19	72	1.61	$3.01 \cdot 10^{-3}$	0.16
37	143	2.07	$7.90 \cdot 10^{-4}$	0.45
56	215	2.67	$1.66 \cdot 10^{-4}$	1.01
74	286	3.6	$6.36 \cdot 10^{-5}$	1.69
93	358	4.83	$3.54 \cdot 10^{-5}$	2.72
111	429	6.55	$1.96 \cdot 10^{-5}$	4.02

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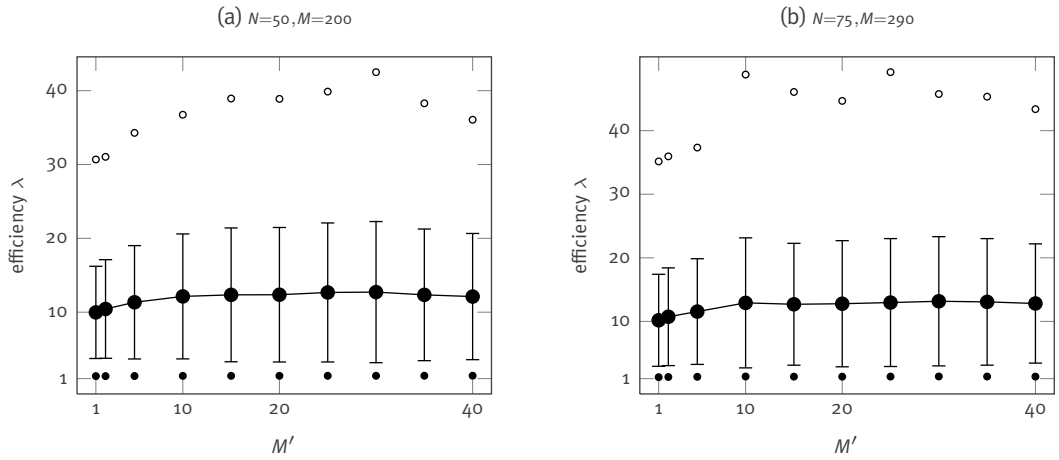
Example II: A posteriori error estimator

Efficiency of error estimator η: $\lambda(\boldsymbol{\mu}) := \frac{\eta(\boldsymbol{\mu})}{\|u_h(\boldsymbol{\mu}) - u_{\text{red}}(\boldsymbol{\mu})\|}$



Error bar plot showing mean and standard deviation of error estimator efficiency over a sample of 100 random parameters for different values of M'. The dots indicate the minimum (●) and maximum (○) efficiency.

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Outlook

Conclusion

- Model order reduction of general parametrized evolution schemes
- with reduced basis methods and empirical interpolation for discrete operators

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Outlook

Conclusion

- Model order reduction of general parametrized evolution schemes
- with reduced basis methods and empirical interpolation for discrete operators

Future work

- Parametrization of Two–Phase–Flow example
- Rigorous error control of reduced data

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X-greedy algorithm

X-GREEDY($\mathcal{M}_{\text{train}}, \varepsilon_{\text{tol}}, \Upsilon_{\max}$)

– *Initialize reduced basis of dimension* Υ_0*:*

$\mathcal{D}_{\Upsilon_0} \leftarrow$ X-INITBASIS()

$\Upsilon \leftarrow \Upsilon_0$

repeat

– *Find worst approximated parameter:*

$(\boldsymbol{\mu}_{\max}, t_{\max}) \leftarrow \arg\max_{\boldsymbol{\mu} \in \mathcal{M}_{\text{train}}}$ X-ERRORESTIMATE($\mathcal{D}_\Upsilon, \boldsymbol{\mu}, t^k$)

– *Extend reduced basis by snapshot:*

$\mathcal{D}_{\Upsilon+1} \leftarrow$ X-EXTENDBASIS($\mathcal{D}_\Upsilon, \boldsymbol{\mu}_{\max}, t_{\max}$)

$\Upsilon \leftarrow \Upsilon + 1$

until $\max_{\boldsymbol{\mu} \in \mathcal{M}_{\text{train}}}$ X-ERRORESTIMATE($\mathcal{D}_\Upsilon$) $\leq \varepsilon_{\text{tol}}$ *or* $\Upsilon > \Upsilon_{\max}$

return *reduced data:* $\mathcal{D}_\Upsilon$

- Extension: Adaptive extension of Parameter sampling $\mathcal{M}_{\text{train}}$.

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EI-greedy methods (1/2)

EI-InitBasis()

return *empty initial basis:* $\mathcal{D}_0 \leftarrow \{\}$

EI-ErrorEstimate$((\mathbf{Q}_M, \Sigma_M), \boldsymbol{\mu}, t^k)$

– *Compute exact operator evaluation*

$$v_h \leftarrow \mathcal{L}_h[u_h^k(\boldsymbol{\mu})]$$

– *Compute empirical interpolated operator evaluation*

$$v_M \leftarrow \mathcal{I}_M \circ \mathcal{L}_h[u_h^k(\boldsymbol{\mu})]$$

return *approximation error:* $\|v_h - v_M\|$.

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EI-greedy methods (1/2)

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– *Compute empirical interpolated operator evaluation*

$$v_M \leftarrow \mathcal{I}_M \circ \mathcal{L}_h[u_h^k(\boldsymbol{\mu})]$$

return *approximation error:* $\|v_h - v_M\|$.

Detailed simulation for all trainings parameters needed. Expensive!

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EI-greedy methods (2/2)

EI-ExtendBasis$((\mathbf{Q}_M, \Sigma_M), \boldsymbol{\mu}, t^k)$

– *Compute exact operator evaluation.*

$$v_h \leftarrow \mathcal{L}_h[u_h^k(\boldsymbol{\mu})]$$

– *Compute empirical interpolated operator evaluation.*

$$v_M \leftarrow \mathcal{I}_M \circ \mathcal{L}_h[u_h^k(\boldsymbol{\mu})]$$

– *Compute the residual.*

$$r_M \leftarrow v_h - v_M.$$

– *Find "magic point" maximizing the residual.*

$$x_{M+1} \leftarrow \arg\sup_{x \in \Sigma_h} |r_M(x)|$$

– *Normalize to obtain a new basis function.*

$$q_{M+1} \leftarrow r_M \cdot \frac{1}{r_M(x_{M+1})}$$

return *extended basis data:* $\mathcal{D}_{M+1} \leftarrow \left(\{q_m\}_{m=1}^{M+1}, \{x_m\}_{m=1}^{M+1}\right)$

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POD-greedy methods

POD-InitBasis()

return initial reduced basis functions: $\{\varphi_n\}_{n=1}^{N_0}$

POD-ErrorEstimate($\{\varphi_n\}_{n=1}^{N}, \boldsymbol{\mu}, t^k$)

return *error estimate:* $\eta_{N,M}^k(\boldsymbol{\mu}) \geq \left\| u_{\text{red}}^k(\boldsymbol{\mu}) - u_h^k(\boldsymbol{\mu}) \right\|$

POD-ExtendBasis($\{\varphi_n\}_{n=1}^{N}, \boldsymbol{\mu}_{\text{max}}, \tilde{}$)

- *Compute trajectory* $\{u_h^k(\boldsymbol{\mu}_{\text{max}})\}_{k=0}^{K}$.
- *Compute new basis function with POD and Galerkin projection* $\mathcal{P}_{\text{red}}$ *projecting onto* span $\{\varphi_n\}_{n=1}^{N}$:

$$\varphi_{N+1} \leftarrow \text{POD}\left(\left\{u_h^k(\boldsymbol{\mu}_{\text{max}}) - \mathcal{P}_{\text{red}}\left[u_h^k(\boldsymbol{\mu}_{\text{max}})\right]\right\}_{k=0}^{K}\right)$$

return *extended reduced basis:* $\{\varphi_n\}_{n=1}^{N+1}$

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POD-greedy methods

POD-InitBasis()

return initial reduced basis functions: $\{\varphi_n\}_{n=1}^{N_0}$

POD-ErrorEstimate($\{\varphi_n\}_{n=1}^{N}, \boldsymbol{\mu}, t^k$)

return *error estimate:* $\eta_{N,M}^k(\boldsymbol{\mu}) \geq \left\| u_{\text{red}}^k(\boldsymbol{\mu}) - u_h^k(\boldsymbol{\mu}) \right\|$

POD-ExtendBasis($\{\varphi_n\}_{n=1}^{N}, \boldsymbol{\mu}_{\text{max}}, \tilde{}$)

– *Compute trajectory* $\{u_h^k(\boldsymbol{\mu}_{\text{max}})\}_{k=0}^{K}$.
– *Compute new basis function with POD and Galerkin projection* $\mathcal{P}_{\text{red}}$ *projecting onto* span $\{\varphi_n\}_{n=1}^{N}$:

$$\varphi_{N+1} \leftarrow \text{POD}\left(\left\{u_h^k(\boldsymbol{\mu}_{\text{max}}) - \mathcal{P}_{\text{red}}\left[u_h^k(\boldsymbol{\mu}_{\text{max}})\right]\right\}_{k=0}^{K}\right)$$

return *extended reduced basis:* $\{\varphi_n\}_{n=1}^{N+1}$

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Remarks: EI-greedy + POD-greedy

- "EI-greedy" has to be computed before "POD-greedy" with detailed simulations for all trainings parameters.
- A priori it is unknown, how to choose the error tolerance ε_{tol} for the "EI-greedy".
- Number of EI base functions might be too large.

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Alternative: "PODEI-greedy"

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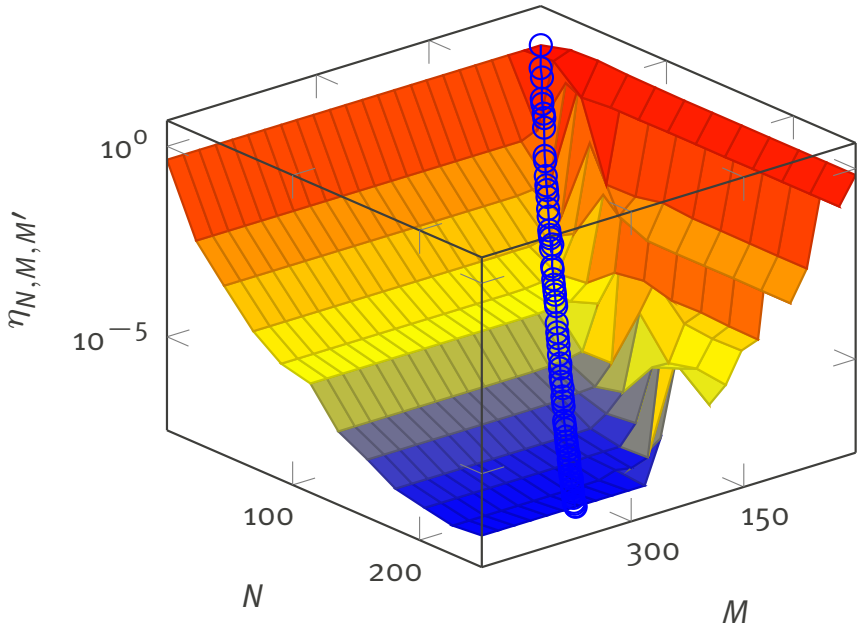
Motivation: PODEI-greedy



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PODEI-greedy methods (1 of 2)

PODEI-InitBasis()

– *Generate small empirical interpolation basis:*

$$(\mathbf{Q}_{M_{\text{small}}}, \Sigma_{M_{\text{small}}}) \leftarrow \text{EI-GREEDY}(\mathcal{M}_{\text{train}}^{\text{coarse}}, \varepsilon_{\text{tol,small}}, M_{\text{small}})$$

– *Compute initial reduced basis:*

$$\{\varphi_n\}_{n=0}^{N_0} \leftarrow \text{RB-InitBasis}()$$

return *initial bases data:* $\mathcal{D}_1 \leftarrow \{\varphi_n\}_{n=1}^{N_0} \cup (\mathbf{Q}_{M_{\text{small}}}, \Sigma_{M_{\text{small}}})$

PODEI-ErrorEstimate($\mathcal{D}_\Upsilon, \boldsymbol{\mu}, t^k$)

return *reduced basis error estimate:* $\eta_{N,M}^k(\boldsymbol{\mu})$

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PODEI-greedy methods (2 of 2)

PODEI-ExtendBasis($\mathcal{D}_\Upsilon, \boldsymbol{\mu}_{\max}, t^k$)

Reduced data $\mathcal{D}_\Upsilon$ comprises $\mathcal{D}_N^{RB} := \{\varphi_n\}_{n=1}^N$ and $\mathcal{D}_M^{EI} := (\mathbf{Q}_M, \Sigma_M)$

– Extend EI basis: $\mathcal{D}_{M+1}^{\mathrm{EI}} \leftarrow$ EI-ExtendBasis($\mathcal{D}_M^{\mathrm{EI}}, \boldsymbol{\mu}_{\max}, t^k$)

– Extend RB basis: $\mathcal{D}_{N+1}^{\mathrm{RB}} \leftarrow$ RB-ExtendBasis($\mathcal{D}_N^{\mathrm{RB}}, \boldsymbol{\mu}_{\max}, t^k$)

– Discard extended RB if error increases:

if $\eta_{N-1,M-1}^k(\boldsymbol{\mu}_{\max}) \geq \max_{(\boldsymbol{\mu},t)\in\mathcal{M}_{\mathrm{train}}} \eta_{N,M}^k(\boldsymbol{\mu})$ **then**

  return *extended basis data:* $\mathcal{D}_{\Upsilon+1} \leftarrow (\mathcal{D}_N^{\mathrm{RB}}, \mathcal{D}_{M+1}^{\mathrm{EI}})$

else

  return *extended basis data:* $\mathcal{D}_{\Upsilon+1} \leftarrow (\mathcal{D}_{N+1}^{\mathrm{RB}}, \mathcal{D}_{M+1}^{\mathrm{EI}})$

end if

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