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Adaptive Reduced Basis Methods for Nonlinear Convection–Diffusion Equations

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Reduced Basis Scenario

Desired: Many solution-*snapshots* $u_h(\cdot;t^k,\boldsymbol{\mu}) \in \mathcal{W}_h$ in a discrete function space $\mathcal{W}_h$ of high dimension H (for $0 = t^0 \leq \ldots \leq t^K = T$). These snapshots are gained from a parametrized numerical scheme (e.g. finite volume discretization) of the following type:

$$u_h(t^0,\boldsymbol{\mu}) = \mathcal{P}_h[u_0(\boldsymbol{\mu})] \quad \text{(1a)}$$

$$\left(\mathrm{Id} + \Delta t \mathcal{L}_h^I\right)\left[u_h(t^{k+1},\boldsymbol{\mu})\right] - \left(\mathrm{Id} + \Delta t \mathcal{L}_h^E\right)\left[u_h(t^k,\boldsymbol{\mu})\right] = 0 \quad \text{(1b)}$$

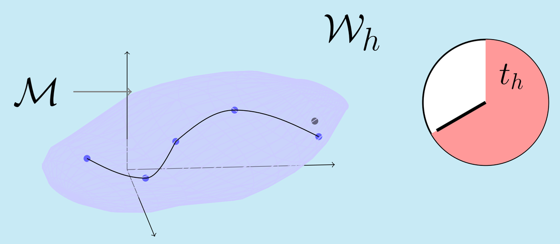
Here, $\mathcal{P}_h$ is a projection operator onto $\mathcal{W}_h$ for an initial data function and $\mathcal{L}_h^* : \mathcal{W}_h \to \mathcal{W}_h$ are (non-linear) discretization operators.
If (1b) includes non-linear terms, a Newton-Raphson method is applied.

The main ingredients for reduced basis model reduction are:

- Parameter vectors $\boldsymbol{\mu} \in \mathcal{P}$ in parameter space $\mathcal{P} \subset \mathbb{R}^p$,
- A manifold $\mathcal{M} := \{u_h(\cdot;t^K,\boldsymbol{\mu}) | k = 0,\ldots,K, \boldsymbol{\mu} \in \mathcal{P}\} \subset \mathcal{W}_h$ of interesting solution snapshots.

High-Dimensional Discrete Simulation

Sketch of a single trajectory of solution snapshots $\{u_h(\cdot;t^k,\boldsymbol{\mu})\}_{k=0}^K$ for a certain parameter $\boldsymbol{\mu}$ embedded in the manifold $\mathcal{M}$, long computation time t_h and high memory consumption depending on dimension H.



Idea of Reduced Basis Method

Suppose:

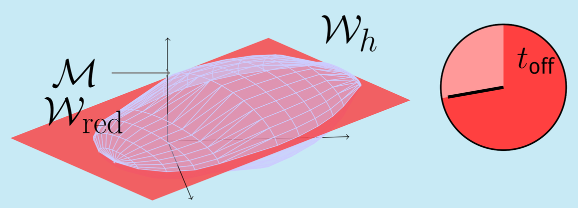
- Solutions for many parameters need to be computed or
- A solution for a single parameter needs to be computed fast, in less than a critical amount of time t_{crit}.

Then: Reduced Basis Method

- Find a linear subspace $\mathcal{W}_{\text{red}}$ of the manifold $\mathcal{M}$ by a greedy search algorithm that minimizes the deviation $E(\mathcal{W}_{\text{red}},\mathcal{M}) = \sup_{x\in\mathcal{M}} \inf_{y\in\mathcal{W}_{\text{red}}} \|x-y\|_{\mathcal{W}_h}$.
- The subspace $\mathcal{W}_{\text{red}} \subset \mathcal{W}_h$ is called *reduced basis space* and has low dimension. $N \ll H$
- Introduce two phases:

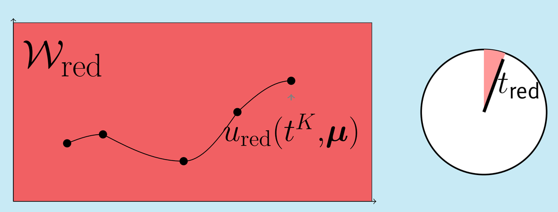
Offline Phase: Reduced Basis Generation

Sketch of the approximation of the manifold $\mathcal{M}$ by a reduced basis space $\mathcal{W}_{\text{red}}$, very long computation time t_{off} and very high memory consumption depending on dimension N and H.



Online Phase: Reduced Simulations

Sketch of a single trajectory of solution snapshots $\{u_{\text{red}}(\cdot;t^k,\boldsymbol{\mu})\}_{k=0}^K$ for a certain parameter $\boldsymbol{\mu}$. Very fast computation time t_{red} and very low memory consumption depending on N.



Error Estimation

Approximation error can be controlled efficiently offline and online by *a posteriori* error estimates $\|u_h(\cdot;t^k,\boldsymbol{\mu}) - u_{\text{red}}(\cdot;t^k,\boldsymbol{\mu})\| \leq \eta^k(\boldsymbol{\mu})$.

Amortization

Assuming, an application needs to calculate simulations for M different parameters. Then model reduction with the reduced basis method pays off if

- $Mt_{\text{red}} + t_{\text{offline}} \leq Mt_h$ or $t_{\text{red}} \leq t_{\text{crit}}$

Abstract

Many applications from science and engineering are based on parametrized evolution equations and depend on time-consuming parameter studies or need to ensure critical constraints on the simulation time. For both settings, model order reduction by the reduced basis methods is a suitable means to reduce computational time. In this proceedings, we show the applicability of the reduced basis framework to a finite volume scheme of a parametrized and highly nonlinear convection-diffusion problem with discontinuous solutions. The complexity of the problem setting requires the use of several new techniques like parametrized empirical operator interpolation, efficient a posteriori error estimation and adaptive generation of reduced data. The latter is usually realized by an adaptive search for base functions in the parameter space. Common methods and effects are shortly revised in this presentation and supplemented by the analysis of a new strategy to adaptively search in the time domain for empirical interpolation data.

Results

Example: Buckley–Leverett problem

Find $u(t;\boldsymbol{\mu}) \in BV(\Omega) \cap L^\infty(\Omega) \subset L^2(\Omega)$ fulfilling

$$\partial_t u(t;\boldsymbol{\mu}) + \nabla\cdot(\mathbf{v}(u;\boldsymbol{\mu})u(t;\boldsymbol{\mu})) - \nabla\cdot(d(u;\boldsymbol{\mu})\nabla u(t;\boldsymbol{\mu})) = 0 \quad \text{in } \Omega\times\mathbb{T} \quad \text{(2a)}$$

$$u(0;\boldsymbol{\mu}) = u_0(\boldsymbol{\mu}) \quad \text{in } \Omega\times\{0\} \quad \text{(2b)}$$

$$u(t;\boldsymbol{\mu}) = u_{\text{dir}}(\boldsymbol{\mu}) \quad \text{on } \Gamma\times\mathbb{T} \quad \text{(2c)}$$

with $\Omega := [0,1]^2$, $\mathbb{T} := [0,0.3]$
and $\boldsymbol{\mu} = (\mu_1,\mu_2,\mu_3) \in \mathcal{P} := [1,2]\times[0,0.1]\times[0.1,0.4]$

Data functions:

- initial data $u_0(\boldsymbol{\mu}) = c_{low} + (1-c_{low})\chi_{[0.2,0.6]\times[0.25,0.75]}$,
- velocity vector $\mathbf{v}(u;\boldsymbol{\mu}) = (0,1)^t f(u;\boldsymbol{\mu})$,
- diffusion $d(u;\boldsymbol{\mu}) = \mu_1 \frac{(1-u)^3}{\mu_2} f(u;\boldsymbol{\mu}) p_c'(u;\mu)$,
- fractional flow rate $f(u;\boldsymbol{\mu}) = \frac{u^3}{\mu_1}\cdot\left(\frac{u^3}{\mu_1} + \frac{(1-u)^3}{\mu_2}\right)^{-1}$,
- and capillary pressure $p_c(u;\boldsymbol{\mu}) = u^{-\lambda}$.

Discretization: The problem is discretized with a standard finite volume scheme comprising an explicitly computed Engquist–Osher flux for the convective terms and an implicit discretization of the diffusive terms. This leads to a scheme as in (1a-b) with a Newton-Raphson method.

Figure 1: Detailed simulation solution snapshots at time instants $t = 0.0$ $t = 0.1$, $t = 0.3$ and for different parameters $\boldsymbol{\mu} = (0, 0.1, 0.4)$ and $\boldsymbol{\mu} = (2, 0.1, 0.4)$. The last column shows the reduced solution on cross-sections at $y = 0.5$ for the time instants $t = 0.0$ (solid line), $t = 0.1$ (dotted line), $t = 0.3$ (dashed line).

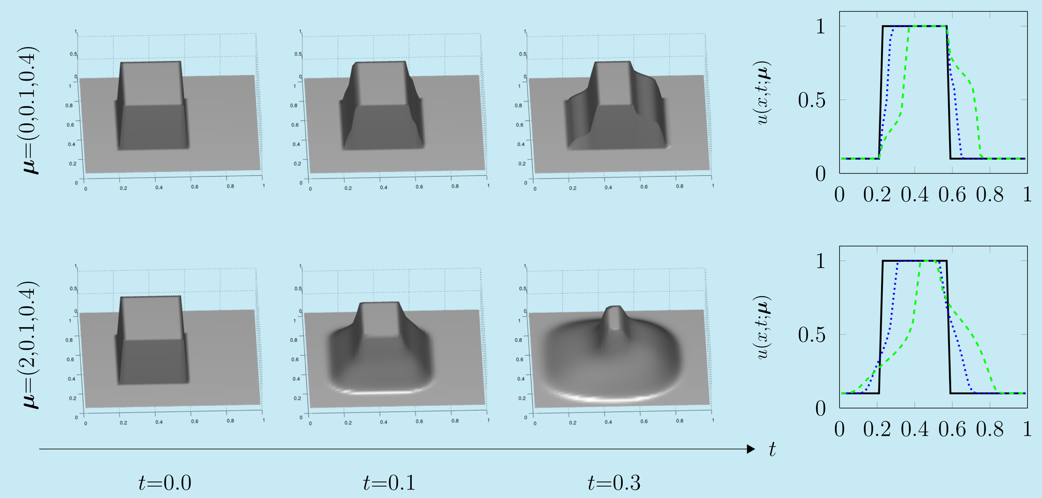
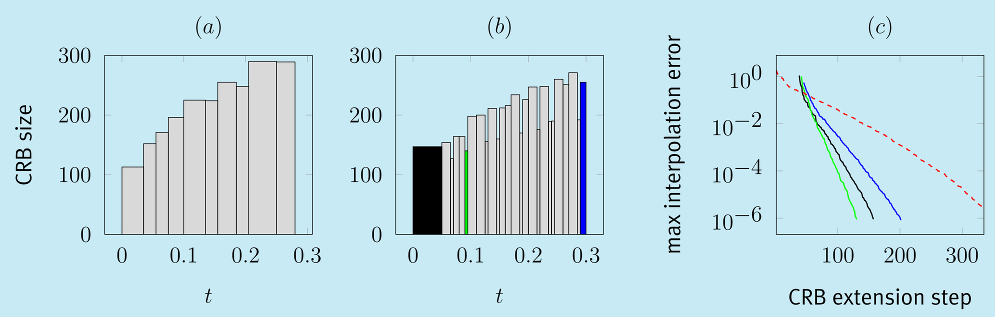


Table 1: Comparison of the number of bases, the reduced basis sizes averaged over subintervals, offline time, averaged online reduced simulation times and maximum errors. The average online run-times and maximum errors are obtained from 20 simulations with randomly selected parameters $\boldsymbol{\mu}$.

adaptation	no. of bases	ø-dim(CRB)	offline time[h]	ø-runtime[s]	max. error
no	1	350	1.47	6.79	$5.88\cdot10^{-4}$
yes, $c_{\min}=5$	11	223.09	2.08	4.06	$5.80\cdot10^{-4}$
yes, $c_{\min}=1$	26	198.42	8.40	3.38	$5.75\cdot10^{-4}$

Figure 2: Illustration of basis sizes on time intervals after adaptation with (a) $c_{\min} = 5$ and (b) $c_{\min} = 1$. Plot (c) illustrates the error decrease during generation of bases on three intervals and for a single basis without adaptation.



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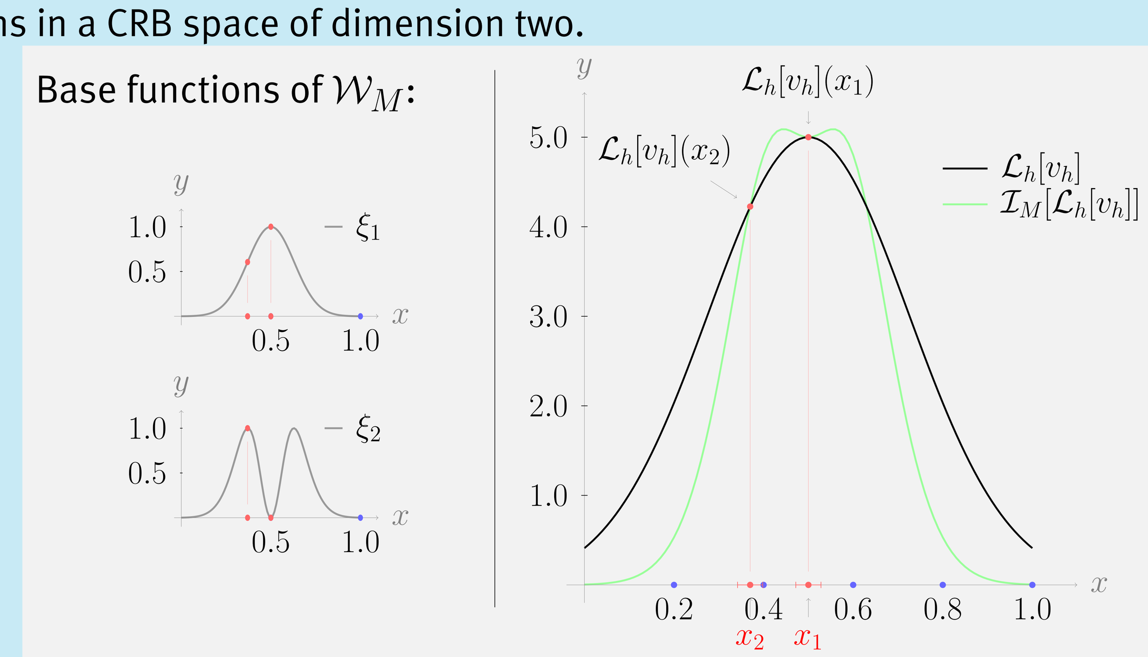
Empirical Operator Interpolation

In the online phase, the discretization operators need to be efficiently computable, i.e. with complexity independent of the high dimension H. Therefore, we want to interpolate operator evaluations in a low dimensional *collateral reduced basis (CRB) space* $\mathcal{W}_M \subset \mathcal{W}_h$ with few point evaluations at the so-called "magic points" as proposed in [2].
So for each operator $\mathcal{L}_h$, we need a further interpolation operator $\mathcal{I}_M : \mathcal{W}_h \to \mathcal{W}_M$ computing exact evaluations $\mathcal{I}_M[\mathcal{L}_h[v_h]](x_m) = \mathcal{L}_h[v_h](x_m)$ at interpolation points $x_m, m = 1,\ldots,M$.
The following illustration shows how the interpolation with two local operator evaluations in a CRB space of dimension two.



With empirical interpolation, the numerical scheme (1a-b) can be reduced to a low dimensional one for online simulations:

$$u_{\text{red}}(t^0,\boldsymbol{\mu}) = \mathcal{P}_{\text{red}}[u_0(\boldsymbol{\mu})] \quad \text{(3a)}$$

$$\left(\mathrm{Id} + \Delta t \mathcal{L}_{\text{red}}^I\right)\left[u_{\text{red}}(t^{k+1},\boldsymbol{\mu})\right] - \left(\mathrm{Id} + \Delta t \mathcal{L}_{\text{red}}^E\right)\left[u_{\text{red}}(t^k,\boldsymbol{\mu})\right] = 0 \quad \text{(3b)}$$

with projection operator $\mathcal{P}_{\text{red}} : \mathcal{W}_h \to \mathcal{W}_{\text{red}}$
and reduced operators $\mathcal{L}_{\text{red}}^{I/E} = \mathcal{P}_{\text{red}} \circ \mathcal{I}_M \circ \mathcal{L}_h^{I/E}$

Adaptive Basis Generation

Both the RB and CRB spaces are generated during the offline phase by greedy algorithms iteratively adding basis functions. The PodGreedy algorithm for RB generation and the EIDetailed algorithm for are described in the proceedings paper.
For complex manifolds $\mathcal{M}$ the basis spaces can become very big reducing the model reduction effects. Therefore, it is sometimes desirable to produce different bases for subsets of $\mathcal{M}$.
In this presentation we propose the following algorithm producing different CRB spaces $\mathcal{W}_M^{\mathcal{K}}$ and magic points for sub-intervals $\mathcal{K} \subset \mathbb{T}$ of the time interval.

procedure TimeSlicedEI($\mathcal{W}_{\text{init}}, \mathcal{K}, L_{\text{train}}^{\mathcal{K}}$)
  $\mathcal{W}_M \leftarrow$ EIDetailed($\mathcal{W}_{\text{init}}, L_{\text{train}}^{\mathcal{K}}, M_{\max}, \varepsilon_{tol}$)
  if ε_{tol} reached **then**
    $M^k \leftarrow M$ and $\mathcal{W}_{M^k}^k \leftarrow \mathcal{W}_M$ for all $k \in \mathcal{K}$.
  else if $\mathrm{card}(\mathcal{K}) \leq 2c_{\min}$ **then**
    $\mathcal{W}_{M^k}^k \leftarrow$ EIDetailed($\mathcal{W}_M, L_{\text{train}}^{\mathcal{K}}, \infty, \varepsilon_{tol}$) for all $k \in \mathcal{K}$.
  else % *maximum number of extensions* $M_{\max}$ *reached*
    $\mathcal{K}_1, \mathcal{K}_2 \leftarrow$ SplitTimeInterval($\mathcal{K}, \mathcal{W}_M$)
    TimeSlicedEI($\mathcal{W}_M^{\mathcal{K}_1}, L_{\text{train}}^{\mathcal{K}_1}$)
    TimeSlicedEI($\mathcal{W}_M^{\mathcal{K}_2}, L_{\text{train}}^{\mathcal{K}_2}$)
  end if
end procedure

The number of different bases produced can be controlled by constant $c_{\min}$ for the miminum size of sub-intervals $\mathcal{K}$.

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