Empirical Operator Interpolation and Reduced Basis Methods

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Abstract

Many important applications from physics, chemistry, economics or other life sciences are modeled by non-linear partial differential equations. Often, these applications depend on time-consuming **parameter studies** or the response needs to be available rapidly. **Reduced basis methods** are an approach to reduce the computation time notably. They have gained popularity over the last few years for finite element approximations of elliptic and instationary parabolic equations [1].
For the method's central concept - the decomposition in an offline and an online phase - a separation of the parameter dependent and the space dependent contributions in the discretization operators, is necessary. This is quite simple, if the space operators are linear and depend affinely on the parameter. In order to extend the concept for settings where these conditions are not fulfilled, especially for non-linear problems, the operators need to be approximated somehow.
The device presented here, to make the operators fit into the method's setting, is called **empirical operator interpolation** and adapts ideas from [2].

Model reduction with Reduced Basis Methods

Scenario:

- Parametrized partial differential equations: shape, material or control parameters $\boldsymbol{\mu} \in \mathcal{P} \subset \mathbf{R}^p$

$$\begin{aligned} u(x,0;\boldsymbol{\mu}) &= u_0(x;\boldsymbol{\mu}) && \text{in } \Omega \\ \partial_t u(x,t;\boldsymbol{\mu}) - \mathcal{L}\left[u(x,t;\boldsymbol{\mu})\right] &= 0 && \text{in } \Omega \times [0, T_{\max}] \end{aligned} \tag{1}$$

+ boundary conditions. Solutions $u(\cdot,t;\boldsymbol{\mu})$ live in a (Sobolev) space $\mathcal{W}$ for each $t \in [0, T_{\max}]$.

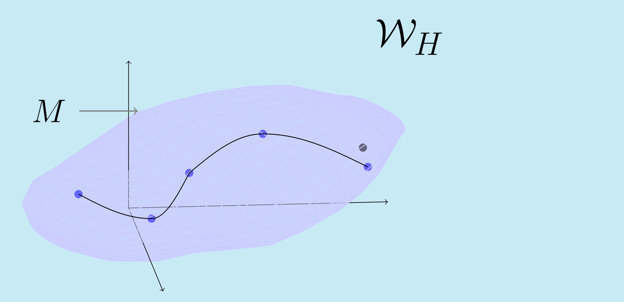
- Simulation requests need to be answered **rapidly** or **repeatedly** for many different parameters, e.g. design optimization, control, parameter estimation, real-time applications.

Reduced basis recipe:

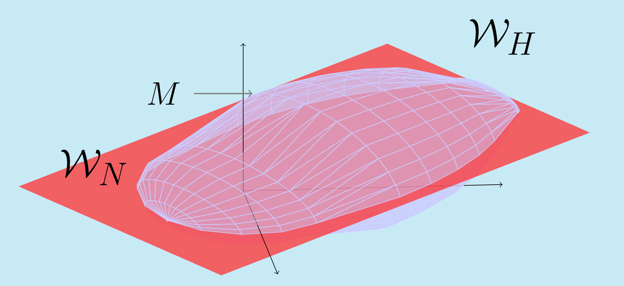
1. Discretize the problem in a H dimensional discrete function space $\mathcal{W}_H$ e.g. in a finite volume space.

$$\begin{aligned} u_h^0(\boldsymbol{\mu}) &= P[u_0(x;\boldsymbol{\mu})] \\ L_I(\boldsymbol{\mu}, \Delta t^k)[u_h^{k+1}(\boldsymbol{\mu})] - L_E(\boldsymbol{\mu}, \Delta t^k)[u_h^k(\boldsymbol{\mu})] &= 0 \end{aligned} \tag{2}$$

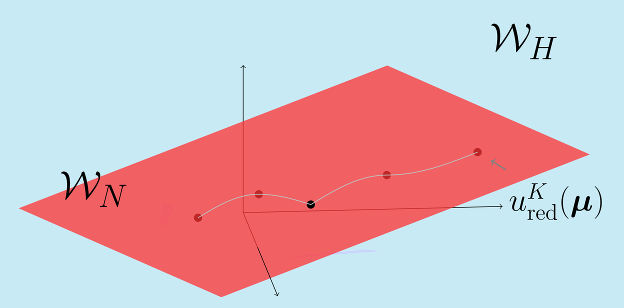
2. Generate an $N \ll H$ dimensional reduced basis space $\mathcal{W}_N \subset \mathcal{W}_H$ by approximating the manifold $M := \left\{u_h^k(\boldsymbol{\mu}) | \boldsymbol{\mu} \in \mathcal{P}, k = 0, \ldots, K\right\}$. The generating algorithm iteratively chooses badly approximated snapshots from a training subset of M and adds them to an orthonormal basis of $\mathcal{W}_N$.



3. Project the numerical scheme (**??**) on the lower dimensional reduced basis space space and **precompute** parameter independent parts. This allows low-dimensional and therefore fast reduced simulations. The concept of separating parameter and space dependent computations is called **offline/online decomposition**.



4. Reconstruct the low dimensional solution or evaluate an output functional. Note, that the latter is preferable, because it makes the reduced simulations independent of any high-dimensional data.



Remarks:

- If the implicit operator is non-linear, a Newton scheme needs to be applied requiring an efficient evaluation of the Fréchet derivative of the discrete operators.
- Rigorous and efficiently computed **a posteriori error estimators** are available for verification and for efficiently searching for new snapshots during the reduced basis generation process.

Empirical operator interpolation

Efficient decomposition into offline and online computations for simulation data is simple, if the discretization operators and the problem data

- are linear and
- depend affinely on the parameter $\boldsymbol{\mu}$.

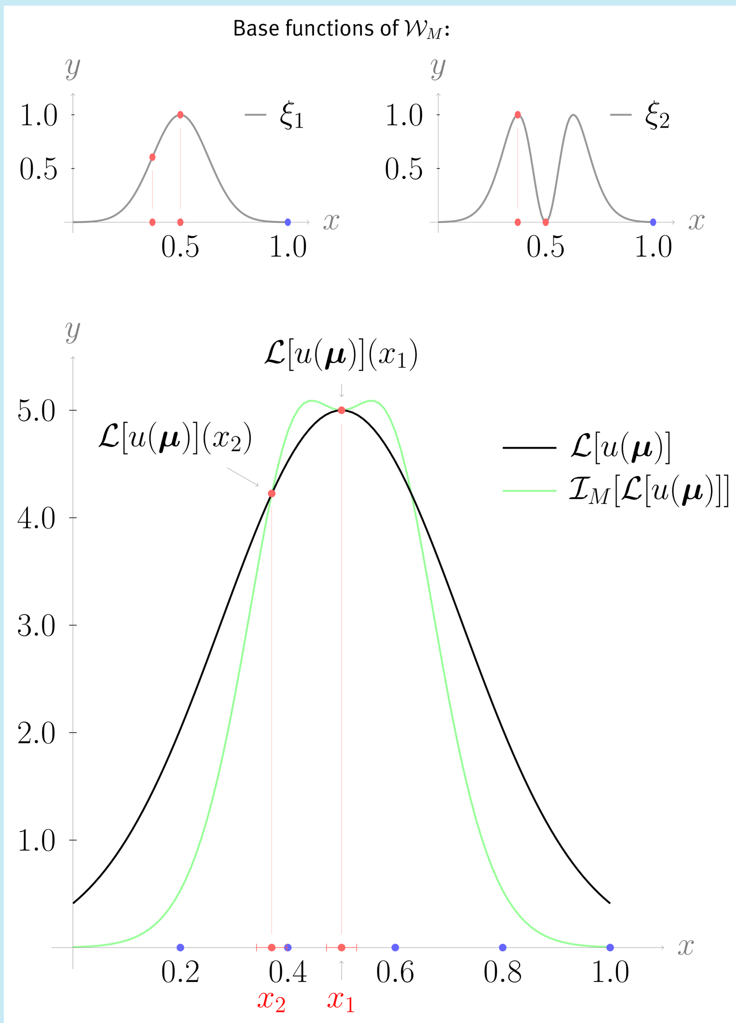
For non-linear problems or problems with complex parameter functions, we approximate the operators with **empirical operator interpolation**.

Idea: Approximate operator with few point evaluations

1. Build a collateral reduced basis space spanned by operator evaluations $\mathcal{W}_M := \text{span}\left\{L(\boldsymbol{\mu}_i)[u_h^{k_i}(u_i)\right\}_{i=1}^M$. Each of the base functions is associated with an interpolation point x_m.
2. If operator evaluations $y_m(\boldsymbol{\mu}; u_h(\boldsymbol{\mu})) := L\left[u_h(\boldsymbol{\mu}\right](x_m)$ in the interpolation points depend only on a small number of the argument function's degrees freedom, we call the operator **localised**. In that case the interpolation

$$\mathcal{I}_M\left[L(\boldsymbol{\mu})[u_h(\boldsymbol{\mu})]\right] := \sum_{m=1}^{M} y_m(\boldsymbol{\mu}; u_h(\boldsymbol{\mu}))\xi_m,$$

can be computed efficiently and provide us with an admissible operator decomposition. Here, $(\xi_m)_{m=1}^M$ denote nodal base functions of the space $\mathcal{W}_M$.



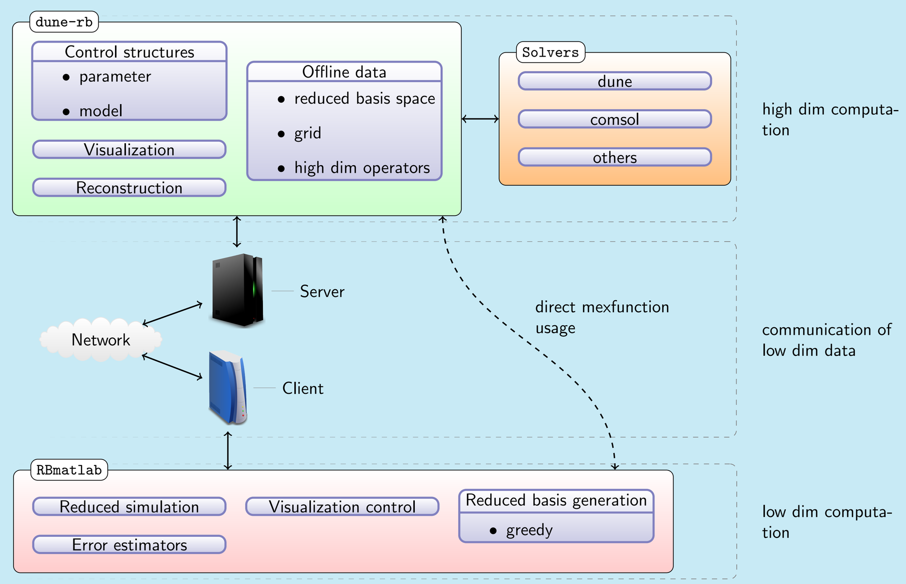
Remarks:

- Most discretization operators, like e.g. finite volume or finite element discretization operators, are localised.
- The **Fréchet derivative** $\mathrm{D}\mathcal{I}_M[L]$ of an empirical operator interpolation can also be computed efficiently allowing us to to extend the reduced basis method to nonlinear problems depending on the Newton method.

Software concepts (Current and future work)

Goals:

- Implementation of a framework that cleanly separates the low-dimensional and high-dimensional computations where the fast simulations computations and the communication is provided in the Matlab based software package **RBmatlab**. [7]
- Plug-in concept for efficient high-dimensional solvers, like Dune
- Parallelisation of high-dimensional solvers during reduced basis generation.

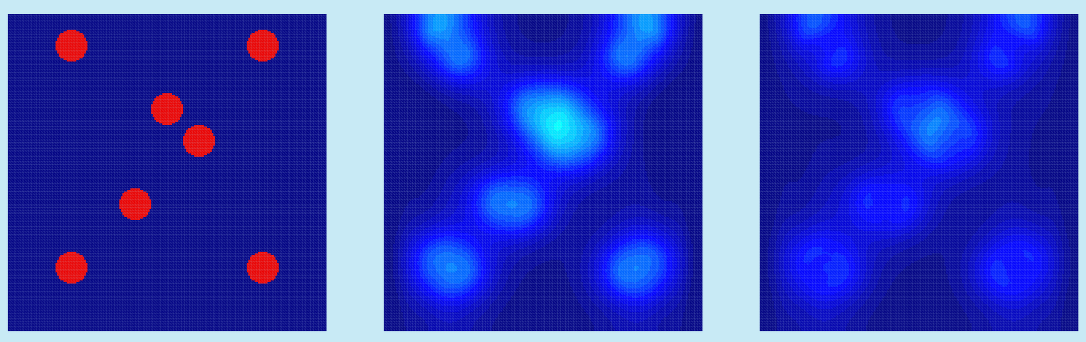


Status:

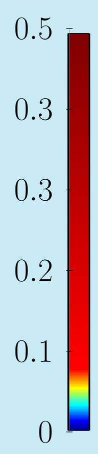
- High dimensional computations with Dune module **dune-rb** (linear finite volume schemes with affinely decomposed operators.)
- Low dimensional computations with RBmatlab
- TCP/IP communication between high-dimensional solver and RBmatlab

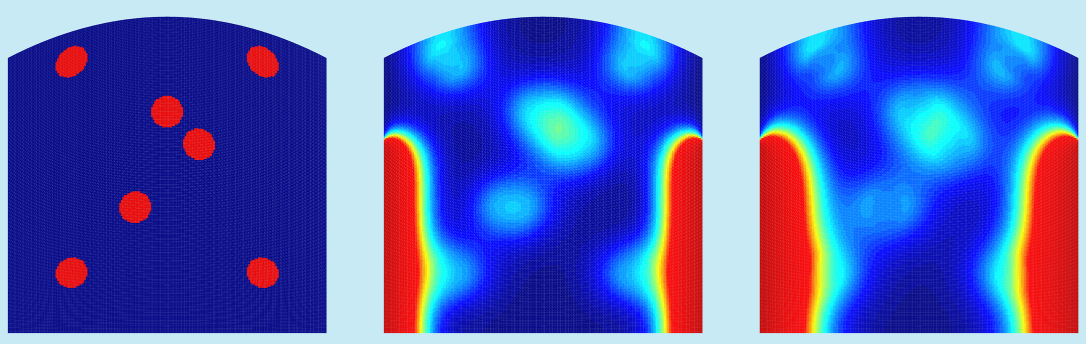
Results

As a test case, we present reduced basis simulations for a convection-diffusion problem on a parametrized geometry. The setting allows many applications like e.g. groundwater flow via the Richard's equation.
The implemented numerical scheme is a semi-implicit finite volume discretization on a structured grid and a first order time discretization. The numerical scheme includes a gradient reconstruction as proposed in [5], because through the geometry parametrization, we get a diffusion tensor in the partial differential equation on the reference domain.
It can be observed that only few reduced basis vectors suffice to approximate the high dimensional solutions $u_h(\boldsymbol{\mu})$, but the implicit operator needs to be approximated fairly accurate by the empirical operator interpolation.



Numerical solutions for $\mu = (0.0, 0.0)$ at timesteps $t = 0.0$, $t = 0.75$, $t = 1.5$.





Numerical solutions for $\mu = (0.2, 0.2)$ at timesteps $t = 0.0$, $t = 0.75$, $t = 1.5$.

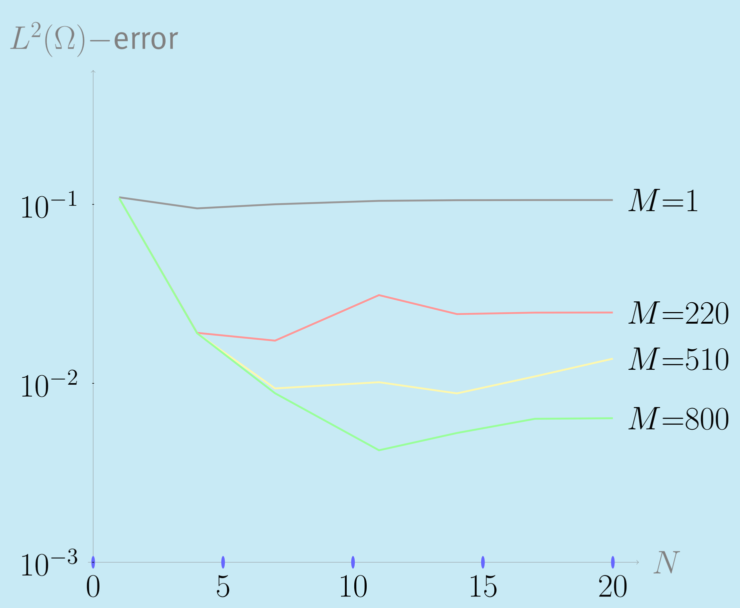


Figure 1: RB error convergence on 100 test samples

dimension	time [s]
$H = 40000$	24.3675
$N = 7, M = 267$	1.2224
$N = 7, M = 800$	2.0501
$N = 14, M = 267$	1.246
$N = 14, M = 800$	2.104
$N = 20, M = 267$	1.2707
$N = 20, M = 800$	2.1127

Table 1: average time measurements on 100 test samples

Dimension of $\mathcal{W}_H$: 40000
Time gain factor for online phase: ≈ 10

References

- [1] A. Patera and G. Rozza. *Reduced Basis Approximation and A Posteriori Error Estimation for Parametrized Partial Differential Equations.* Version 1.0, Copyright MIT 2006, to appear in (tentative rubric) MIT Pappalardo Graduate Monographs in Mechanical Engineering
- [2] M. Barrault, M., Y. Maday, N. Nguyen and A. Patera, *An 'empirical interpolation' method: application to efficient reduced-basis discretization of partial differential equations* C. R. Math. Acad. Sci. Paris Series I, 2004, 339, 667-672
- [3] B. Haasdonk und M. Ohlberger *Reduced Basis Method for Finite Volume Approximations of Parametrized Linear Evolution Equations* M2AN, Math. Model. Numer. Anal. 42, 277–302, 2008
- [4] B. Haasdonk, M. Ohlberger und G. Rozza. *A Reduced Basis Method for Evolution Schemes with Parameter-Dependent Explicit Operators* ETNA, Electronic Transactions on Numerical Analysis, vol. 32, 2008
- [5] O. Drblíková and K. Mikula, *Convergence Analysis of Finite Volume Scheme for Nonlinear Tensor Anisotropic Diffusion in Image Processing.* SIAM J. Numer. Anal., 46, 27–60, 2007
- [6] M. Drohmann, B. Haasdonk and M. Ohlberger, *Reduced Basis Method for Finite Volume Approximation of Evolution Equations on Parametrized Geometries* Proceedings of ALGORITMY 2009, 111–120
- [7] `http://www.morepas.org`

http://www.uni-muenster.de/math/num/ag_ohlberger

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