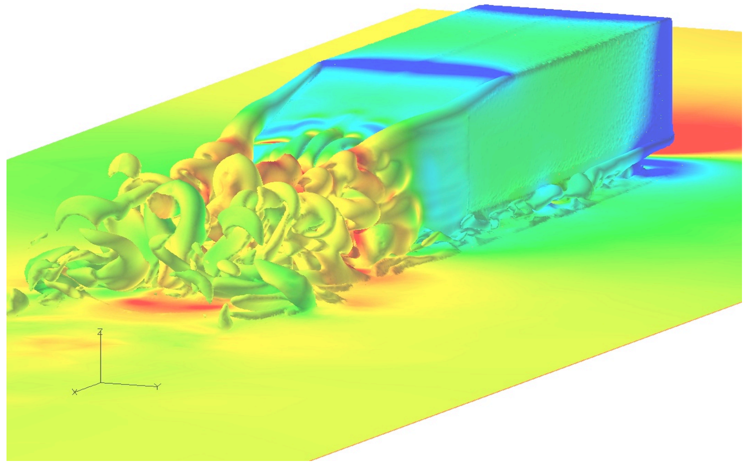
NONLINEAR MODEL REDUCTION USING PETROV-GALERKIN PROJECTION AND DATA RECONSTRUCTION



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TECHNICAL ISSUES

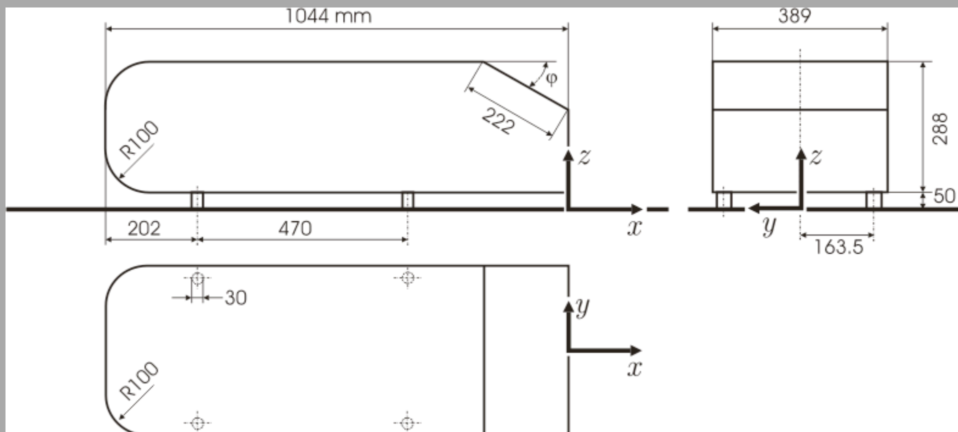
- Two main problems arise in nonlinear model reduction:

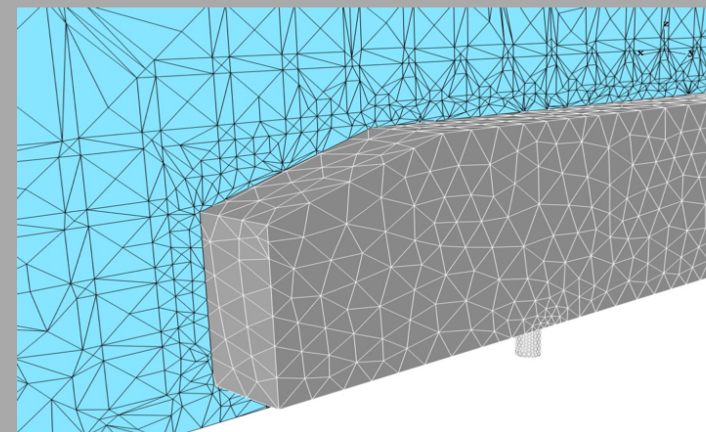
1. Accuracy/Stability: projection does not always preserve stability or capture nonlinearities

2. Computational complexity: reduced-order operators are expensive to assemble, even though they have small dimension

ILLUSTRATION: AHMED BODY ROM

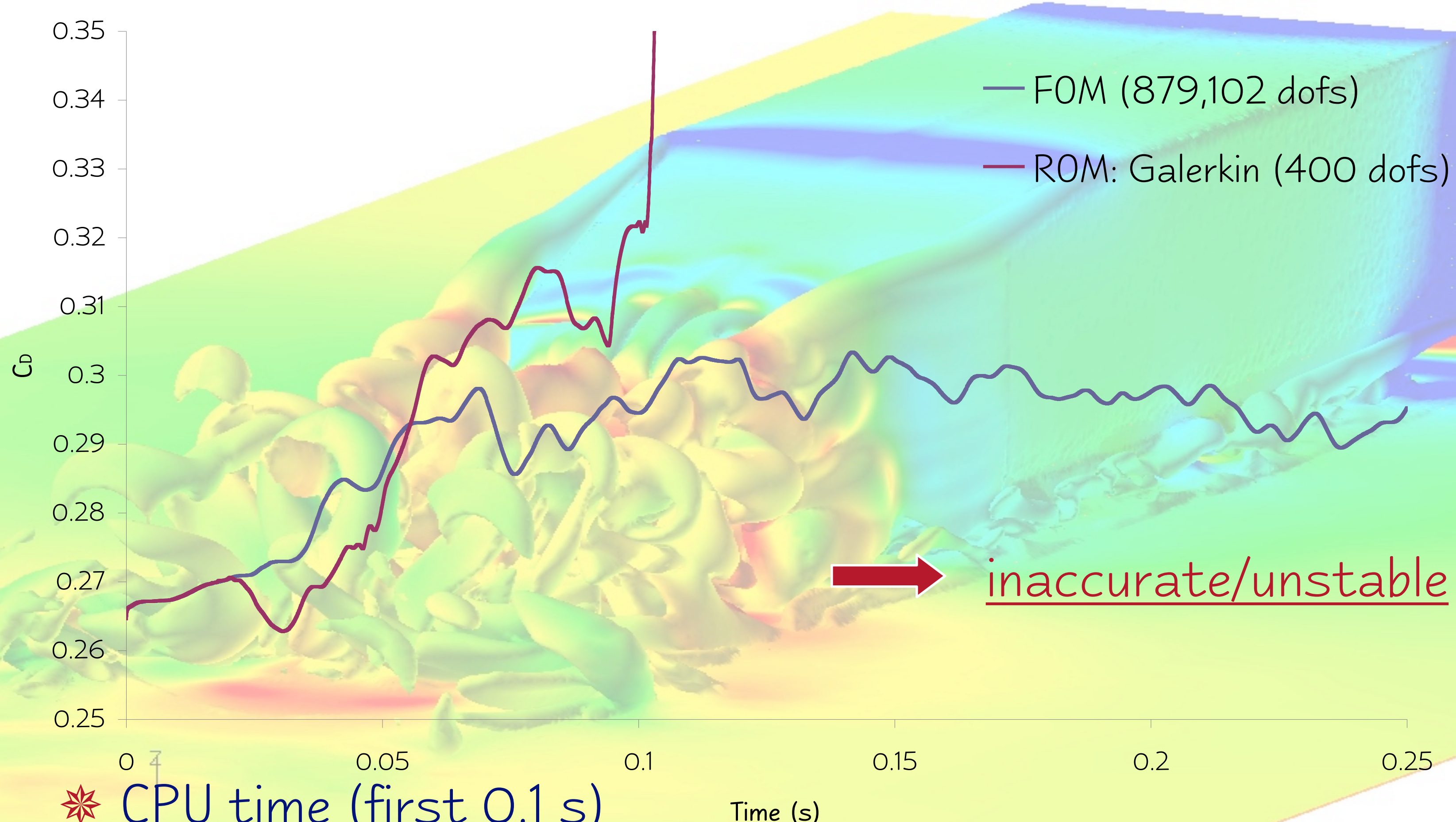
- Ahmed body





- Navier-Stokes Simulation
  - φ = 20 deg
  - V = 60 m/s
  - Re = 4.29×10^6
  - DES turbulence model
  - 146,517 nodes
  - 837,894 tetrahedra
- FOM: 879,102 dofs
- POD/Galerkin ROM (typical): 400 dofs (0.046% size of FOM)

AHMED BODY POD/GALERKIN RESULTS



→ inaccurate/unstable

- CPU time (first 0.1 s)
  - FOM. 9,292 s
  - ROM: Galerkin. 24,984 s → computationally expensive

METHODOLOGY

- Consecutively introduce approximations that satisfy 2 properties

offline

I. Full-order model → data collection → compression

projection (dimension reduction)

II. Reduced-order model → data collection → compression

system approximation (complexity reduction)

online

III. Reduced-order model + system approximation

1. Consistency: In the limit of no compression, reproduce exactly the solution of the previous model for sampled problems

2. Optimality: Minimize an error measure with respect to the previous model (a priori convergence)

MATHEMATICAL FRAMEWORK

- (Nonlinear) partial differential equation

$$\mathcal{L}(u; x, t) = 0$$

- Semi-discretized partial differential equation (ODE)

$$\mathcal{L}^d(y; t) = 0 \text{ (dimension} = N)$$

- Fully-discretized PDE with implicit time integration

$$R(y^{n+1}; y^n, \ldots, y^n; t^n) = 0 \text{ (dimension} = N)$$

  - Full-order model: sequence of N-dimensional nonlinear problems

$$R(y) = 0 \text{ (dimension} = N)$$

PROJECTION

- Projection leads to "Model II"

I. Full-order model → data collection → compression

projection (dimension reduction)

II. Reduced-order model → data collection → compression

system approximation (complexity reduction)

III. Reduced-order model + system approximation

PETROV-GALERKIN PROJECTION

- To decrease the dimensionality, search for a solution in the affine subspace: $y^{(0)}+\text{range}(\Phi_y)$ of dimension $n_y \ll N$

$$y \approx y^{(0)}+\Phi_y y_r \quad R(y^{(0)}+\Phi_y y_r)=0$$

- Enforce residual orthogonal to n_y-dimensional subspace range(Ψ)

$$\Psi^T R(y^{(0)}+\Phi_y y_r)=0$$

- Solve via Newton's method for k=1,…,K (until converged)

$$\Psi^T J^{(k)} \Phi_y p^{(k)} = -\Psi^T R^{(k)}$$

$$y_r^{(k+1)} = y_r^{(k)} + \alpha^{(k)} p^{(k)}$$

  - $R^{(k)} \equiv R(y^{(0)}+\Phi_y y_r^{(k)})$, $\quad J^{(k)} \equiv dR/dy(y^{(0)}+\Phi_y y_r^{(k)})$

COMPRESSION VIA POD

- Proper orthogonal decomposition (POD) method
  - Given n_x "snapshots" x^i, the first k POD vectors satisfy:

Find $k \le n_x$ orthonormal vectors minimizing

$$J(\varphi^1, \varphi^2, \ldots, \varphi^k) = \sum_{j=1}^{n_x} \| x^j - \sum_{i=1}^{k} (x^j, \varphi^i)\, \varphi^i \|_2^2$$

  - $\Phi = [\varphi^1 \cdots \varphi^{n_x}]$; $\Phi^T \Phi = I$
- Properties
  - "No compression": $k = n_x$ and range (Φ) = range $([x^1 \cdots x^{n_x}])$
  - Efficient computation by singular value decomposition (SVD)
- Consistency issue: what should the snapshots be?

PROJECTION CONSISTENCY

Proposition

If Φ_y is a POD basis computed with snapshots $(y-y^{(0)})$ collected during the evaluation of Model I, and y is sufficiently close to $y^{(0)}$, then the projection approximation is consistent

→ Defines data and procedure for computing right reduced-order basis Φ_y

PROJECTION OPTIMALITY

- Optimality: would like solution $p^{(k)}$ to satisfy for some norm Θ

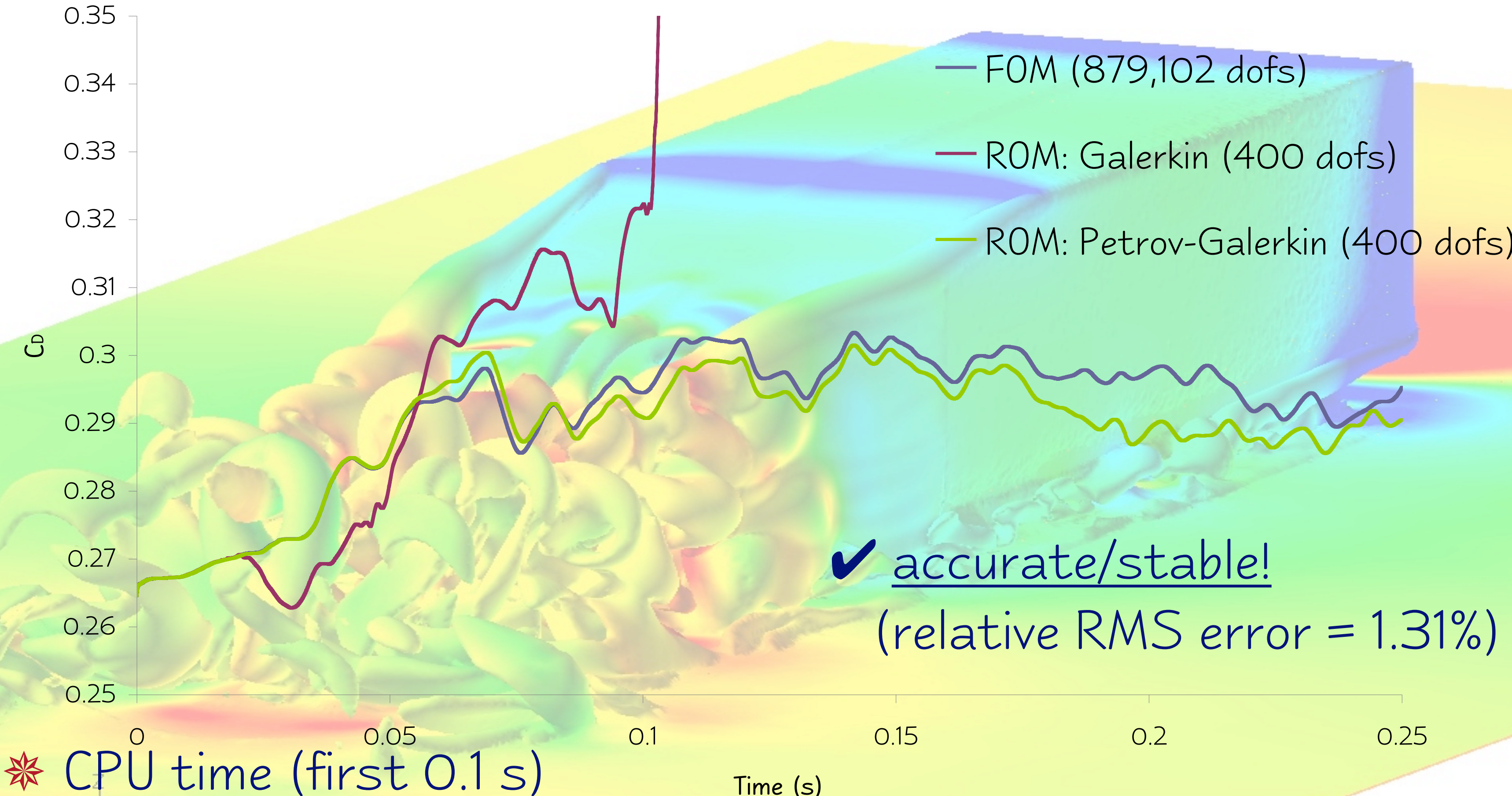
$$p^{(k)} = \arg\min_{p \in \mathfrak{R}^{n_y}} \| \Phi_y p - (J^{(k)})^{-1} R^{(k)} \|_\Theta \qquad (1)$$

  - Galerkin: $\Psi = \Phi_y$ satisfies (1) with $\Theta = J^{(k)}$ only if $J^{(k)}$ SPD
  - Petrov-Galerkin: $\boxed{\Psi = J^{(k)}\Phi_y}$ satisfies (1) with $\Theta = J^{(k)T} J^{(k)}$ if $J^{(k)}$ nonsingular
- Least-squares Petrov-Galerkin projection
  - Equivalent to globally convergent Gauss-Newton method for

$$\min \|R(y + \Phi_y y_r)\|_2$$

  - Ψ is state-dependent, changes each Newton iteration
  - Cannot be introduced at ODE level
  - Improved accuracy, stability over Galerkin in the general case

AHMED BODY ROM RESULTS



✔ <u>accurate/stable!</u>

(relative RMS error = 1.31%)

- CPU time (first 0.1 s)
  - FOM. 9,292 s
  - ROM: Galerkin. 24,973 s
  - ROM: Petrov-Galerkin. 24,984 s ➡ <u>remains expensive</u>

SYSTEM APPROXIMATION

- System approximation leads to "Model III"

I. Full-order model ⟶ data collection ⟶ compression

↓ projection (dimension reduction)

II. Reduced-order model ⟶ data collection ⟶ compression

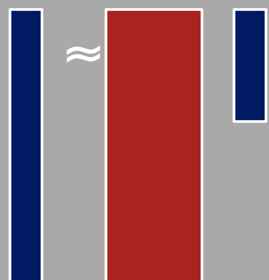
↓ system approximation (complexity reduction)

III. Reduced-order model + system approximation

SYSTEM APPROXIMATION

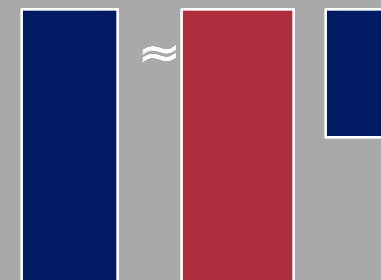
- Tensor approximations: $R^{(k)} \in \mathrm{range}(\Phi_R)$, $J^{(k)}\,\Phi_y \in \mathrm{range}(\Phi_J)$

$$R^{(k)} \approx \Phi_R\, R_r^{(k)} \qquad\qquad J^{(k)}\,\Phi_y \approx \Phi_J\, J_r^{(k)}$$



invariant

iteration-dependent



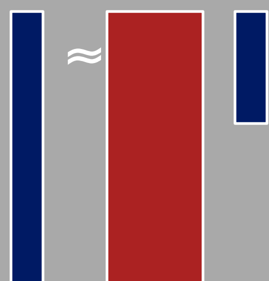
- Newton iterations become

$$[J^{(k)}\,\Phi_y]^T[J^{(k)}\,\Phi_y]\,p^{(k)} = -[J^{(k)}\,\Phi_y]^T R^{\,k)}$$

SYSTEM APPROXIMATION

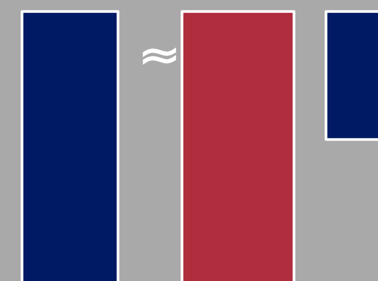
- Tensor approximations: $R^{(k)} \in \text{range}(\Phi_R)$, $J^{(k)}\,\Phi_y \in \text{range}(\Phi_J)$

$$R^{(k)} \approx \Phi_R\, R_r^{(k)} \qquad\qquad J^{(k)}\,\Phi_y \approx \Phi_J\, J_r^{(k)}$$



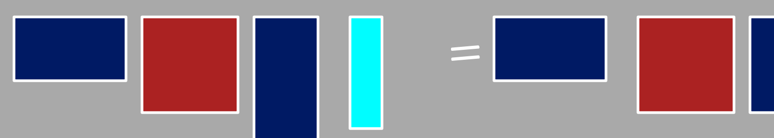
invariant

iteration-dependent



- Newton iterations become

$$(J_r^{(k)})^T\, \Phi_J^T \Phi_J J_r^{(k)}\, p^{(k)} = -(J_r^{(k)})^T \Phi_J^T \Phi_R R_r^{(k)}$$

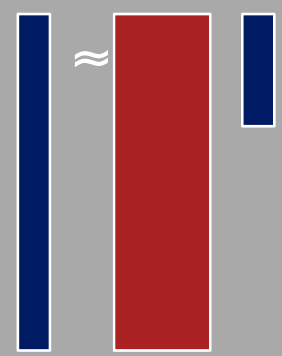


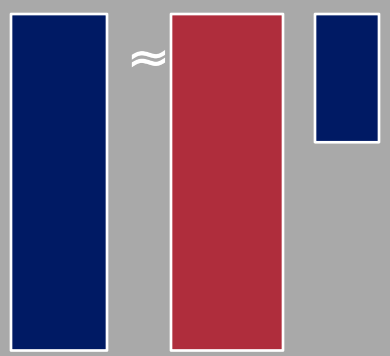
- Form $\Phi_J^T\Phi_J$, $\Phi_J^T\Phi_R$ offline: online operations independent of N!
- Similar: Barrault et al., 2004; Grepl et al., 2007; Nguyen & Peraire, 2008; Chatarantabut & Sorensen, 2009; Galbally et al., 2010

SYSTEM APPROXIMATION

- Tensor approximations

$$R^{(k)} \approx \Phi_R \, R_r^{(k)} \qquad J^{(k)} \Phi_y \approx \Phi_J \, J_r^{(k)}$$





- Property 1: Optimality
  - $R_r^{(k)}$, $J_r^{(k)}$ computed online by gappy data reconstruction
- Property 2: Consistency
  - Φ_R, Φ_J computed offline by POD with specific snapshots

GAPPY DATA RECONSTRUCTION

- **Goal**: accurately reconstruct vector $F \in \Re^N$
- **Given**: 1) a basis $Z = [z^1, \ldots, z^p]$

  2) some sampled entries of the vector

  F_i, $i \in \mathcal{I}$ with $|\mathcal{I}| = q \geq p$ and $q << N$

[Everson & Sirovich, 1995]

- **Define restriction**:

  $\underline{F} \equiv [F]_i$, $i \in \mathcal{I}$

  $\underline{Z} \equiv [\underline{z}_1, \ldots, \underline{z}_p]$

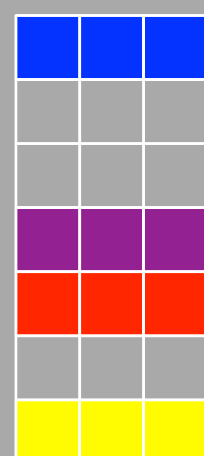
Example

$N=7$

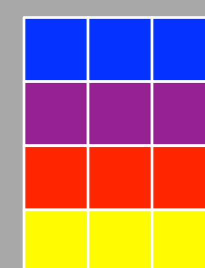
$q=4$

$\mathcal{I} = \{1,4,5,7\}$

$F=$ 

$Z=$ 

$\underline{F}=$ 

$\underline{Z}=$ 

GAPPY DATA RECONSTRUCTION

- Least squares minimization on sampled entries

$$F_r = \arg\min_x \| \underline{Z} x - \underline{F} \|_2 = Z (\underline{Z}^T \underline{Z})^{-1} \underline{Z}^T \underline{F}$$

- Reconstructed vector

$$F \approx Z F_r$$

- Optimality
  - error $\| \underline{Z} F_r - \underline{F} \|_2$ monotonically decreases as p increases
- Apply at each Newton iteration to compute
  - $R_r^{(k)}$, with $Z = \Phi_R$
  - $J_r^{(k)}$, with $Z = \Phi_J$

SYSTEM APPROXIMATION CONSISTENCY

Proposition

If Φ_R and Φ_J are POD bases computed with snapshots satisfying the following conditions:

1. $R^{(k)}$ from the Model II simulation is a snapshot used for Φ_R
2. $J^{(k)}\Phi_y p^{(k)}$ from the Model II simulation is a snapshot used for Φ_J
3. Each column of $J^{(k)}\Phi_y$ from the Model II simulation is a snapshot used for Φ_J

then the system approximation is consistent

- Leads to hierarchy of snapshot collection procedures characterized by tradeoffs between consistency and offline cost/storage

SNAPSHOT COLLECTION HIERARCY

ID	0	1	2	3
Snapshots for y	$y_I - y_I^{(0)}$	$y_I - y_I^{(0)}$	$y_I - y_I^{(0)}$	$y_I - y_I^{(0)}$
Snapshots for $R^{(k)}$	$R_I^{(k)}$	$R_{II}^{(k)}$	$R_{II}^{(k)}$	$R_{II}^{(k)}$
Snapshots for $J^{(k)}\Phi_y$	$R_I^{(k)}$	$R_{II}^{(k)}$	$[J^{(k)}\Phi_y p^{(k)}]_{II}$	$[J^{(k)}\Phi_y]_{II}$
# simulations	1	2	2	2
# snapshots per Newton iteration	1	1	2	$1+n_y$
consistency conditions satisfied	none	1	1, 2	1, 2, 3

- $()_I$, $()_{II}$: snapshot saved during Model I, II
- $()^{(k)}$: snapshot saved at each Newton iteration

SNAPSHOT COLLECTION HIERARCY

ID	0	1	2	3
Snapshots for y	$y_l - y_l^{(0)}$	$y_l - y_l^{(0)}$	$y_l - y_l^{(0)}$	$y_l - y_l^{(0)}$
Snapshots for $R^{(k)}$	$R_l^{(k)}$	$R_{ll}^{(k)}$	$R_{ll}^{(k)}$	$R_{ll}^{(k)}$
Snapshots for $J^{(k)}\boldsymbol{\Phi}_y$	$R_l^{(k)}$	$R_{ll}^{(k)}$	$[J^{(k)}\boldsymbol{\Phi}_y p^{(k)}]_{ll}$	$[J^{(k)}\boldsymbol{\Phi}_y]_{ll}$
# simulations	1	2	2	2
# snapshots per Newton iteration	1	1	2	$1+ n_y$
consistency conditions satisfied	none	1	1, 2	1, 2, 3

- Procedure 0: most common, yet satisfies no consistency conditions
- Procedure 3: consistent, but prohibitive cost for most problems
- ✓ Procedure 2: similar cost as 1, more consistency conditions

PROCEDURE

- The approximated Newton iterations are

$$[\underline{J_{(k)}\Phi_y}]^T A\,[\underline{J_{(k)}\Phi_y}]\,p^{(k)} = -[\underline{J_{(k)}\Phi_y}]^T B\,\underline{R_{(k)}}$$

- $A = (\underline{\Phi_J}^T\underline{\Phi_J})^{-T}\Phi_J^T\Phi_J(\underline{\Phi_J}^T\underline{\Phi_J})^{-1}$, $B = (\underline{\Phi_J}^T\underline{\Phi_J})^{-T}\Phi_J^T\Phi_R(\underline{\Phi_R}^T\underline{\Phi_R})^{-1}$

<u>Offline</u>

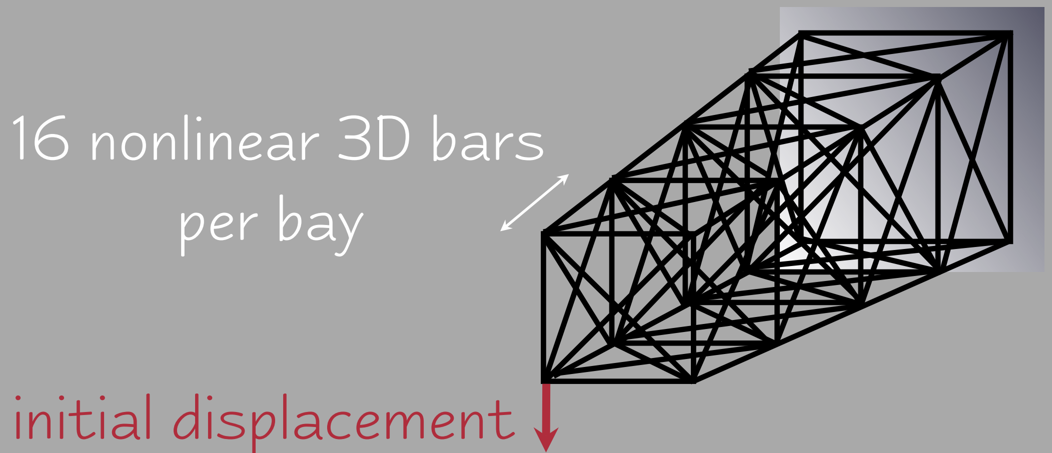
1. Collect snapshots by evaluating Model I, Model II
2. Compute POD bases Φ_y, Φ_R, Φ_J
3. Determine sample indices $\mathcal{I}$ for gappy reconstruction
4. Compute matrices A, B

<u>Online</u> (each Newton step)

1. Compute r<<n required state vector entries
2. Compute restricted quantities by evaluating $R^{(k)}$ and $J^{(k)}\Phi_y$ at q << n indices
3. Solve for $p^{(k)}$

PERFORMANCE ASSESSMENT

- Geometrically nonlinear structural dynamics



- Full-order model: 12,000 degrees of freedom (1000 bays)
- Output of interest: downward velocity at tip
- Truncation of POD bases
  - Φ_y: n_y=26 (99% of y_l-$y_l^{(0)}$ snapshot energy)
  - Φ_R, Φ_y: n_R=n_J=28 (99% of $R_l^{(k)}$ snapshot energy)

COMPARISON WITH TRUNCATION

consistency conditions satisfied

Model	e_I (%)	Total Newton it	Speed-up over I
II	3.44	601	1.03
III.0	unstable	–	–
III.1	6.31	603	158
III.2	6.01	891	122

- Model II: Good accuracy, but insufficient speed-up
- Model III.0: inaccurate, perhaps because not consistent
- Models III.1, III.2: accurate, promising speed-ups

→ Meeting more consistency conditions improves accuracy/stability

LEAST SQUARES V. INTERPOLATION

- $n_y=26$, $n_R=n_J=28$

	$\lvert\mathcal{I}\rvert$	Relative error e_I (%)			Speed-up over Model I		
		III.0	III.1	III.2	III.0	III.1	III.2
interpolation	28	unstable	6.31	6.01	120	158	122
least squares	42	4.95	3.77	3.63	137	152	119
least squares	56	4.03	3.50	3.44	129	136	116

- Increasing $\lvert\mathcal{I}\rvert$ improves accuracy, even stabilizing III.0

→ Least-squares reconstruction seems better than interpolation

CONCLUSIONS

- Approximations are consecutively introduced and satisfy consistency and optimality conditions:
  1. Least-squares Petrov-Galerkin projection
  2. System approximation

- Numerical experiments indicate
  - Petrov-Galerkin projection improves accuracy/stability
  - Tensor approximations are accurate/stable when they satisfy at least some consistency conditions
  - Orders of magnitude speed-ups achieved

QUESTIONS?

- For more details, see forthcoming paper:

  K. Carlberg, C. Bou-Mosleh, and C. Farhat, "Efficient Nonlinear Model Reduction via a Least-Squares Petrov-Galerkin Projection and Compressive Tensor Approximations", International Journal of Numerical Methods in Engineering, submitted.