Efficient structure-preserving model reduction for nonlinear mechanical systems with application to structural dynamics

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Time-critical applications

- real-time applications
  - structural health monitoring
  - embedded control
- many-query applications
  - design optimization
  - uncertainty quantification

\[
\text{inputs } \mu \rightarrow \text{high-fidelity model} \rightarrow \text{outputs } y
\]

- barrier: simulation can take days on supercomputers

- model reduction

\[
\text{inputs } \mu \rightarrow \text{reduced-order model} \rightarrow \text{outputs } y
\]

- offline (expensive): ‘training’ analyses
- online (cheap): deploy low-dimensional model
Main idea

- high-fidelity model
  - *parameterized* simple mechanical system
  - nonlinear potential energy
  - Rayleigh damping
  - external force

- existing reduced-order models
  1. preserve structure, but remain expensive
  2. destroy structure, but are cheap

- our proposed reduced-order model
  - preserves structure and is cheap
Outline

1. Motivation

2. Problem formulation

3. Existing model-reduction techniques
   - preserves structure, but expensive
   - cheap, but destroys structure

4. Proposed method

5. Numerical example
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Lagrangian description of structural dynamics

- equations of motion from finite-element discretization

\[ M(\mu)\ddot{q} + C(\mu)\dot{q} + \nabla_q V(q; \mu) = f^{\text{ext}}(t; \mu). \]

- can be derived via Lagrangian dynamics with five ‘ingredients’:
  1. configuration space \( Q = \mathbb{R}^N \)
  2. Riemannian metric \( g(v, w; \mu) = v^T M(\mu) w \)
  3. potential-energy function \( V(q; \mu) \)
  4. dissipation function \( \mathcal{F}(\dot{q}, \mu) = \frac{1}{2} \dot{q}^T C(\mu) \dot{q} \)
  5. external force derived from the Lagrange–D’Alembert principle \( f^{\text{ext}}(t; \mu) \)

- properties 1–3 define a simple mechanical system
- properties 4–5 characterize non-conservative forces
Equations of motion: derived from five ingredients

- configuration space: \( q \in Q = \mathbb{R}^N \)
- kinetic energy: \( T(\dot{q}; \mu) = \frac{1}{2} g(\dot{q}, \dot{q}; \mu) = \frac{1}{2} \dot{q}^T M(\mu) \dot{q} \)
- Lagrangian:
  \[
  L(q, \dot{q}; \mu) = T(\dot{q}; \mu) - V(q; \mu) \\
  = \frac{1}{2} \dot{q}^T M(\mu) \dot{q} - V(q; \mu).
  \]
- non-conservative forces
  \[
  F(t, q, \dot{q}; \mu) = f^{\text{ext}}(t; \mu) - \nabla_\dot{q} F(\dot{q}; \mu)
  \]
- apply forced Euler–Lagrange equations
  \[
  \frac{d}{dt} \nabla_\dot{q} L(q, \dot{q}; \mu) - \nabla_q L(q, \dot{q}; \mu) = F(t, q, \dot{q}; \mu) \\
  M(\mu) \ddot{q} + C(\mu) \dot{q} + \nabla_q V(q; \mu) = f^{\text{ext}}(t; \mu)
  \]
Key properties

- conservative mechanical systems \( (F = 0) \)
  - energy conservation
  - momentum conservation
  - dynamics satisfy variational principle
  - symplectic time-evolution maps

- structure-preserving time integration
  [Marsden and West, 2001, Hairer et al., 2006]
  - discrete system preserves some of the above properties
  - leads to improved long-time behavior

**reduced-order models should preserve these properties**
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5 Numerical example
Galerkin: structure-preserving model reduction \cite{Lall2003}

- determine low-dimensional basis $\Phi \in \mathbb{R}^{N \times m}$
  - modal decomposition, proper orthogonal decomposition
- substitute $q = \Phi q_r$ to obtain ‘reduced ingredients’
  1. configuration space $Q_r = \mathbb{R}^m$ with $Q_r \equiv \{\Phi q_r \mid q_r \in Q_r\}$
  2. Riemannian metric $g_r(v_r, w_r; \mu) \equiv g(\Phi v_r, \Phi w_r; \mu)$
  3. potential-energy function $V_r(q_r; \mu) \equiv V(\Phi q_r; \mu)$
  4. dissipation function $\mathcal{F}_r(\dot{q}_r; \mu) \equiv \mathcal{F}(\Phi q_r; \mu)$
  5. external force $f_r^{\text{ext}} = \Phi^T f^{\text{ext}}$
- forced Euler–Lagrange equations yield

$$
\Phi^T M(\mu) \Phi \ddot{q}_r + \Phi^T C(\mu) \Phi \dot{q}_r + \Phi^T \nabla_q V(\Phi q_r; \mu) = \Phi^T f^{\text{ext}}(t; \mu)
$$

+ preserves Lagrangian structure
- remains expensive for parameterized, nonlinear systems
Computational bottleneck

\[ \Phi^T M(\mu) \Phi \dot{q}_r + \Phi^T C(\mu) \Phi \dot{q}_r + \Phi^T \nabla_q V(\Phi q_r; \mu) = \Phi^T f^{\text{ext}}(t; \mu) \]

- when \( \mu \) changes, must recompute \( \Phi^T M(\mu) \Phi \) and \( \Phi^T C(\mu) \Phi \)
  - \( \mathcal{O}(Nm^2) \) operations: scales with large dimension \( N \)

\[ \Phi^T = \begin{bmatrix} \end{bmatrix} \]
\[ M(\mu) = \begin{bmatrix} \end{bmatrix} \]
\[ \Phi = \begin{bmatrix} \end{bmatrix} = \Phi^T M(\mu) \Phi \]

- when \( q_r \) changes, must recompute \( \Phi^T \nabla_q V(\Phi q_r; \mu) \)
  - \( \mathcal{O}(Nm) \) operations
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\[ \Phi^T M(\mu)\Phi \ddot{q}_r + \Phi^T C(\mu)\Phi \dot{q}_r + \Phi^T \nabla_q V(\Phi q_r; \mu) = \Phi^T f^{\text{ext}}(t; \mu) \]

- compute subset of equations before performing Galerkin projection

\[ \Phi^T Z^T Z M(\mu)\Phi \ddot{q}_r + \Phi^T Z^T Z C(\mu)\Phi \dot{q}_r + \Phi^T Z^T Z \nabla_q V(\Phi q_r; \mu) = \Phi^T Z^T Z f^{\text{ext}}(t; \mu). \]

‘sampling matrix’ \( Z \): \( n_Z \ll N \) rows of identity matrix

- destroyed properties:
  2. mass matrix not symmetric: does not define a metric
  3. stiffness matrix not symmetric: does not derive from a potential-energy function
  4. dissipation matrix not symmetric: does not derive from a dissipation function
Empirical interpolation/least-squares approximation

[Grepl et al., 2007, Nguyen and Peraire, 2008, Chaturantabut et al., 2010, Carlberg et al., 2011]

\[
\Phi^T M(\mu)\Phi \ddot{q}_r + \Phi^T C(\mu)\Phi \dot{q}_r + \Phi^T \nabla_q V(\Phi q_r; \mu) = \Phi^T f_{\text{ext}}(t; \mu)
\]

- Interpolate functions before performing Galerkin projection
  \[
  \Phi^T \tilde{f}_1(q_r; \mu) + \Phi^T \tilde{f}_2(\dot{q}_r; \mu) + \Phi^T \tilde{f}_3(q_r; \mu) = \Phi^T \tilde{f}_{\text{ext}}(t; \mu)
  \]

\[\tilde{f} = \Phi_f [Z \Phi_f]^+ Z f\]: least-squares approximation of \( f \)

- Destroyed properties:
  2. Mass matrix not symmetric: does not define a metric
  3. Stiffness matrix not symmetric: does not derive from a potential-energy function
  4. Dissipation matrix not symmetric: does not derive from a dissipation function
Existing complexity-reduction methods

- Reduced Lagrangian ingredients
  - Apply Euler–Lagrange equations
  - Reduced-order equations of motion

  \[ \text{leads to } N\text{-independent cost} \]
  - Destroys Lagrangian structure
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Proposed complexity-reduction method

- leads to $N$-independent cost
- preserves Lagrangian structure
directly approximate reduced Lagrangian ingredients

1. configuration space $Q_r = \mathbb{R}^m$ with $Q_r \equiv \{ \Phi q_r \mid q_r \in Q_r \}$
2. Riemannian metric $\tilde{g}_r \approx g_r$
3. potential-energy function $\tilde{V}_r \approx V_r$
4. dissipation function $\tilde{F}_r \approx F_r$
5. external force $\tilde{f}^\text{ext}_r \approx f^\text{ext}_r$
Approximated reduced Lagrangian ingredients

1. configuration space \( Q_r = \mathbb{R}^m \) with \( Q_r \equiv \{ \Phi q_r \mid q_r \in Q_r \} \)
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5. external force \( \tilde{f}_{r}^\text{ext} \approx f_{r}^\text{ext} \)
External-force approximation $\tilde{f}_{r}^{\text{ext}}$

- least-squares approximation of external force

$$\tilde{f}^{\text{ext}} = \Phi_f [Z\Phi_f]^+ Zf^{\text{ext}} \approx f^{\text{ext}}$$

- apply Lagrange–D’Alembert principle to $\tilde{f}^{\text{ext}}$ with variations in reduced configuration space:

$$\tilde{f}_{r}^{\text{ext}} = \Phi^T \tilde{f}^{\text{ext}} = \Phi^T \Phi_f [Z\Phi_f]^+ Zf^{\text{ext}}$$

- Offline (expensive)
  1. collect snapshots of the external force and compute basis $\Phi_f$
  2. determine sampling matrix $Z$
  3. compute small-scale matrix $A = \Phi^T \Phi_f [Z\Phi_f]^+$

- Online (cheap)
  1. compute a few entries of the external force $Zf^{\text{ext}}$
  2. compute small-scale product $A[Zf^{\text{ext}}]$
configuration space $Q_r = \mathbb{R}^m$ with $Q_r \equiv \{\Phi q_r \mid q_r \in Q_r\}$

Riemannian metric $\tilde{g}_r \approx g_r$

potential-energy function $\tilde{V}_r \approx V_r$

dissipation function $\tilde{F}_r \approx F_r$

external force $\tilde{f}^\text{ext}_r \approx f^\text{ext}_r$
Riemannian-metric and dissipation-function approximations

\[ g_r(v_r, w_r; \mu) = v_r^T \left[ \Phi^T M(\mu) \Phi \right] w_r \]

\[ \mathcal{F}_r(\dot{q}_r; \mu) = \dot{q}_r^T \left[ \Phi^T C(\mu) \Phi \right] \dot{q}_r \]

- approximated quadratic ingredients:

\[ \tilde{g}_r(v_r, w_r; \mu) = v_r^T \tilde{M}_r(\mu) w_r \]

\[ \tilde{\mathcal{F}}_r(\dot{q}_r; \mu) = \dot{q}_r^T \tilde{C}_r(\mu) \dot{q}_r \]

- relies on approximating low-dimensional matrices

\[ \tilde{M}_r(\mu) \approx \left[ \Phi^T M(\mu) \Phi \right] > 0 \]

\[ \tilde{C}_r(\mu) \approx \left[ \Phi^T C(\mu) \Phi \right] \geq 0 \]
Mass-matrix approximation (similar for $C$)

- Offline (expensive)
  1. collect matrix snapshots $\{M_i\}$ and corresponding $\{\Phi^T M_i \Phi\}$
  2. determine ‘sample entries’

- Online (cheap)
  1. compute only sample entries of $M(\mu)$
  2. solve cheap optimization problem for $\alpha_i$:

\[
\begin{align*}
\text{minimize} & \quad \alpha_1, \alpha_2 \\
\text{subject to} & \quad \alpha_1 \Phi^T M_1 \Phi + \alpha_2 \Phi^T M_2 \Phi > 0
\end{align*}
\]

3. set $\tilde{M}_r(\mu) = \sum_i \alpha_i \Phi^T M_i \Phi$

Efficient, structure-preserving model reduction

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Approximated reduced Lagrangian ingredients

1. configuration space $Q_r = \mathbb{R}^m$ with $Q_r \equiv \{\Phi q_r \mid q_r \in Q_r\}$
2. Riemannian metric $\tilde{g}_r \approx g_r$
3. potential-energy function $\tilde{V}_r \approx V_r$
4. dissipation function $\tilde{F}_r \approx F_r$
5. external force $\tilde{f}_{r}^{\text{ext}} \approx f_{r}^{\text{ext}}$
Potential-energy function approximation

\[ V_r(q_r; \mu) \equiv V(\Phi q_r; \mu) \]

- replace \( \Phi \) with a sparse matrix \( \Psi \) \((n_Z \ll N\) nonzero rows\)

\[ \tilde{V}_r(q_r; \mu) \equiv V(\Psi q_r; \mu). \]

- cost reduction
  - \( \nabla_{q_r} V_r(q_r; \mu) = \Phi^T \nabla_q V(\Phi q_r; \mu) \) incurs \( O(Nm) \) flops
  - \( \nabla_{q_r} \tilde{V}_r(q_r; \mu) = \Psi^T \nabla_q V(\Psi q_r; \mu) \) incurs \( O(n_Z m) \) flops

- compute \( \Psi \) by matching \( \Psi^T \nabla_q V(\Psi q_r; \mu) \) and \( \Phi^T \nabla_q V(\Phi q_r; \mu) \) for ‘training’ values of \( q_r \) and \( \mu \)
Potential-energy function approximation

- **Offline (expensive)**
  1. collect snapshots of $\nabla_{q_r} V_r(q_r; \mu)$ for ‘training’ values of $q_r$, $\mu$
  2. determine nonzero rows of $\Psi$
  3. solve optimization problem

$$\min_{\Psi} \sum_{j=1}^{J} \| \Psi^T \nabla_q V(\Psi q_r^j; \mu^j) - \Phi^T \nabla_q V(\Phi q_r^j; \mu^j) \|_2^2.$$

- **Online (cheap):** replace $V_r$ with $\tilde{V}_r$
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Simple example: conservative clamped-free truss

\[ M(\mu)\ddot{q} + \nabla_q V(q; \mu) = 0 \]

- \( V \): potential energy, *high-order nonlinearity in q*
- density \( \rho = 1 + \mu_1 \)
- bar cross-sectional area \( A = 1 + \mu_2 \)
- modulus of elasticity \( E = 1 + \mu_3 \)
- \( \mu_i \in [-1, 1], \ i = 1, \ldots, 6 \)
- 120 dofs in ‘high-fidelity’ model
- time integrator: implicit midpoint rule (symplectic)
Reduced-order models

1. Galerkin projection
   + preserves structure
   - expensive

2. Galerkin projection + collocation
   - destroys structure
   + cheap

3. Galerkin projection + gappy POD approximation of residual
   - destroys structure
   + cheap

4. proposed method
   + preserves structure
   + cheap

- reduced-order-model parameters
  - $\Phi \in \mathbb{R}^{N \times m}$: POD, $m = 18$ chosen via $99\%$ ‘energy criterion’
  - sample indices $n_Z = 30$
  - $\Phi_f \in \mathbb{R}^{N \times m_f}$: POD, $m_f = m = 10$
  - train at 3 configurations, test at a new configuration
Galerkin + collocation (structure-preserving)
Galerkin + LS recon.
Galerkin training time:
high-fidelity model

Tip displacement $d$

<table>
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<th>Galerkin</th>
<th>Galerkin + collocation</th>
<th>Galerkin + LS recon.</th>
<th>proposed method</th>
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<td>error</td>
<td>6.85%</td>
<td>18.7%</td>
<td>690%</td>
<td>7.0%</td>
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<td>speedup</td>
<td>0.41</td>
<td>1.77</td>
<td>2.06</td>
<td>1.82</td>
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Conclusions

- directly approximate reduced Lagrangian ingredients
  - Lagrangian-structure preservation
  - computational efficiency
- only reduced-order model delivering accuracy and speedup!
- future work
  - deploy on more realistic (larger, more highly nonlinear) problem
  - apply framework to preserve structure for other systems
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